TOPOLOGY WITH UNCOUNTABLY MANY TEARS

TAUGHT BY
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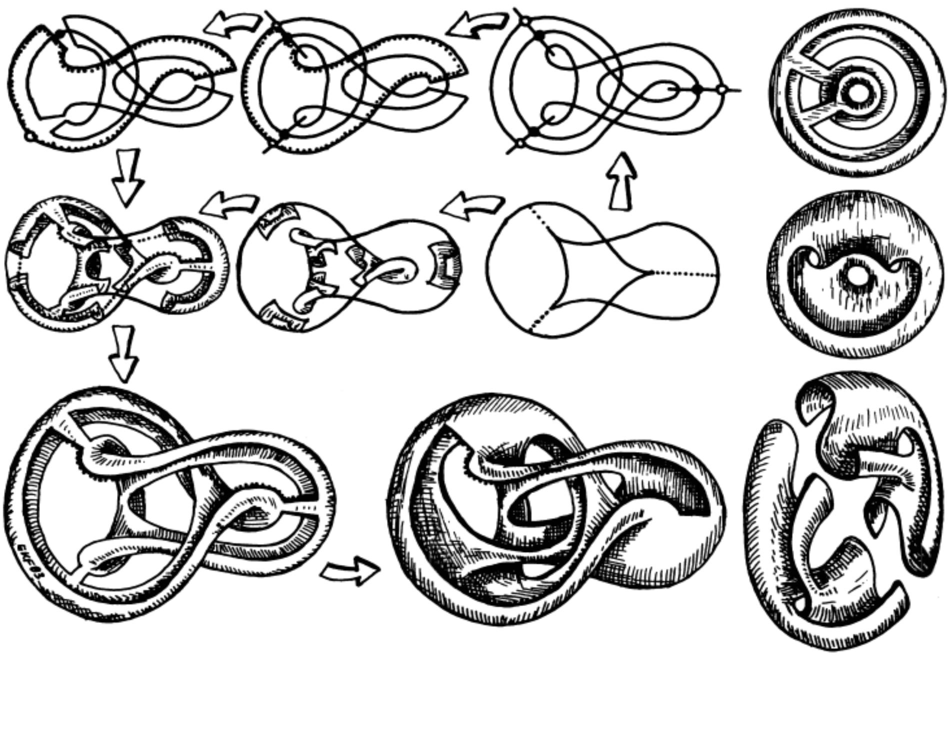
"THIS IS HOW I QUIT TOPOLOGY"



COVER FROM: GEORGE K. FRANCIS
A TOPOLOGICAL PICTUREBOOK

Preface

So I was at lunch with two of my classmates, who were also taking the graduate topology course here at UCSB. One told me that he recently saw a book called *Topology Without Tears* (by the way, good book). Then the other classmate suggested me to write a book, called *Topology with Uncountably Many Tears*, which is a perfect description for what we were going through on that class. This is how this notes originated.

The graduate topology class (221) is a three-quarter sequence, covering point-set topology, a little bit of algebraic topology (Hatcher chapter 1), and differential topology, respectively. I will try my best to finish this project (which will probably take at least a year).

For point-set topology, here are the references we use:

- Jänich [**?**]: A concise but beautifully written book on point-set topology. But it has no exercise! This is our textbook, and will be our main reference for the first part of this notes. In fact, the structure of the first part is almost identical to that of Jänich's book.

- Munkres [**?**]: An encyclopedia of point-set topology. Most concepts can be found in here. Many weird examples. There is also a portion on algebraic topology, which we may use later.

More to come...

I owe a huge debt to the two professors who taught this course, Fedya Manin and Daryl Cooper at UCSB. I wouldn't have been able to appreciate the beauty of this subject without their patient and riveting instruction. Also, Prof. Manin came up with all those interesting exercises in part I of this notes. This notes is dedicated to them.

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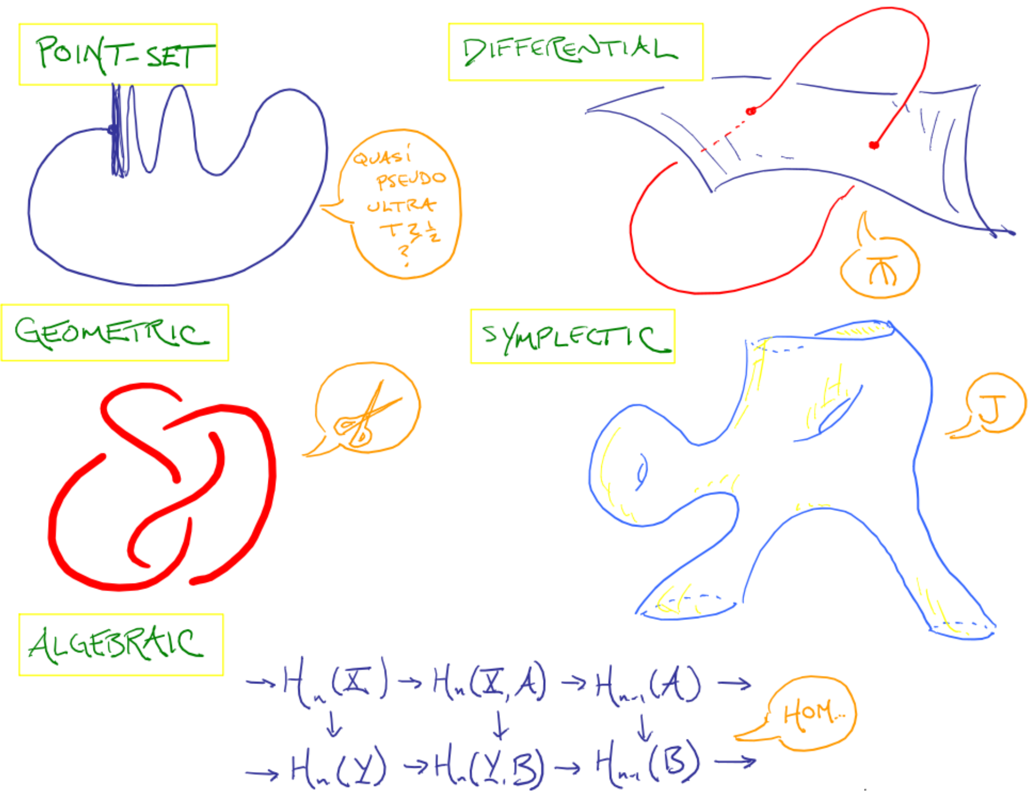
Part I

Point-Set Topology



Chapter 1

A Review of Basic Concepts

Since this is a notes for a graduate topology class, we assume the reader is already prettey familiar with the basic definition of topological spaces, open sets, closed sets, continuous maps, homeomorphisms, compact, connected... So in this chapter we quickly review some basic concepts and move to the next stage as soon as possible, since the real fun begin in the next chapter.

1.1 Topological Spaces

A set $\mathscr{T}$ becomes a topological space if we declear certain subsets of $\mathscr{T}$ to be open:

Definition 1.1. A topological space $(\mathscr{T},\mathscr{O})$ is a set $\mathscr{T}$ and a collection of subsets $\mathscr{O}$ of $\mathscr{T}$, which we call open sets, such that the following holds:

- $\emptyset, \mathscr{T} \in \mathscr{O}$.
- Arbitrary unions of elements in $\mathscr{O}$ is still in $\mathscr{O}$.
- Finite intersections of elements in $\mathscr{O}$ is still in $\mathscr{O}$.

Usually, if $U \in \mathscr{O}$, we say that U is open (in $\mathscr{T}$, which can be omitted when the context is clear). Now we add more structure on a topological space:

Definition 1.2. Let X be a topological space.

- $A \subset X$ is called closed if $X \setminus A = A^c$ is open.
- $U \subset X$ is called a neighborhood of $x \in X$ if there exists an open set V such that $x \in V \subset U$.
- Let $B \subset X$ be any subset. A point $x \in X$ is called in interior, exterior, or boundary point of B, respectively, according to whether $B, X \setminus B$ or neither is a neighborhood of B.
- The set of interior points of B is called the interior of B, usually denoted by B^o. The set of points of X which are not exterior points of B is called the closure of B, usually denoted by $\overline{B}$. The set $\overline{B} \setminus B^o$ is called the boundary of B, usually denoted by ∂B.

The most familiar topological spaces that we know is $\mathbb{R}^n$. We have an intuitive notion of what open and closed sets look like in the Euclidean space. But this is only an example of a particular type of topological spaces, called metric spaces.

Definition 1.3. A metric space (X, d) is a set X, and a function $d : X \times X \to \mathbb{R}$, such that

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (called the triangle inequality).

The metric d induces the topology on space X. We can define an r-ball centered at a point $x \in X$ to be $B_r(x) = \{y \in X : d(x, y) < r\}$. We declare a set $U \subset X$ to be open, if for every $x \in U$, there exists $\varepsilon > 0$, such that $B_\varepsilon(x) \subset U$. We call this topology the topology induced by metric d, and we denote it by $\mathscr{O}(d)$. Different metrics can induce the same metric topology. That is, given metric d, d', we may have $\mathscr{O}(d) = \mathscr{O}(d')$. Usually we show this by showing for every open ball in d metric, it contains an open ball in d' metric, so every d-ball is d'-open, and vice versa. An example is in $\mathbb{R}^2$, we have two metrics

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

$$d'(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|).$$

It is easy to see that d-balls are open disks and d'-balls are open squares.

One of the big questions in point-set topology is whether a topological space is metrizable:

Definition 1.4. A topological space $(\mathscr{T}, \mathscr{O})$ is metrizable if there is a metric d on $\mathscr{T}$ such that $\mathscr{O}(d) = \mathscr{O}$.

We leave this questions for readers to think about, and we will discuss later in the following chapters.

1.2 New Spaces from Old, Continuous Maps

Suppose now you have a topological space, or multiple topological spaces, can you construct new ones from them? The answer is yes, and we introduce three constructions: subspace, disjoint union, and product.

Definition 1.5. Given a topological space X, and a subset $Y \subset X$, we declare a set $V \subset Y$ to be open (in Y) if there is some open set U in X such that $V = U \cap Y$. This topology is called the subspace topology.

Definition 1.6. Let X, Y be topological spaces. The disjoint union $X \sqcup Y$ (or sometimes $X + Y$) to be:

$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\}).$$

We view X, Y as subsets of $X \sqcup Y$, in the obvious way. We declare the open sets in $X \sqcup Y$ to be those given by $U \sqcup V$, where $U \subset X$ open and $V \subset Y$ open.

We can follow the construction of $X \sqcup Y$ from the definition above, and think of $X \sqcup Y$ as the set of elements of the form $(x,0)$ and $(y,1)$, but this is just a notation. What we really want to say is that $X \sqcup Y$ contains two disjoint copies of X and Y, where the label 0,1 in $(x,0),(y,1)$ is just to make sure we know where an element in $X \sqcup Y$ is coming from. This may sound silly, but an example demonstrates the point: Consider $X \sqcup X$. If we were just doing union $X \cup X$, this is still X. But for $X \sqcup X$, we have two disjoint copies of X, the first copy is the collection $(x,0)$ and the second is the collection $(x,1)$, where $x \in X$. We should think of the second coordinate as some sort of labels. So a picture would be:

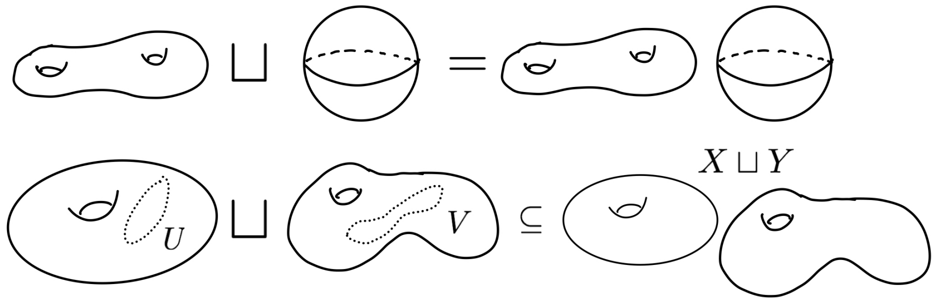


Figure 1.1: Disjoint union gives two disjoint copies of spaces. An open set in $X \sqcup Y$ is a disjoint union of open sets $U \subset X$ and $V \subset Y$.

Now, we will introduce products. Finite products are easy, infinite products will be trickier. We first do the simple case:

Definition 1.7. Let X, Y be topological spaces. The product space $X \times Y$ cosists of points of the form $(x, y), x \in X, y \in Y$, and we say a subset $W \subset X \times Y$ is open, if for every $(x, y) \in W$ there exists $U \subset X, V \subset Y$ open, such that $(x, y) \in U \times V \subset W$.

First we need to introduce the concepts of basis and subbasis.

Definition 1.8. Let $(X,\mathscr{O})$ be a topological space. A set $\mathscr{B}$ of open sets in X is called a basis for the topology, if every open set U is a union of elements in $\mathscr{B}$. A collection of open sets $\mathfrak{S}$ is called a subbasis if every open set is a union of finite intersections of elements in $\mathfrak{S}$:

$$U = \bigcup_{O_i \in \mathfrak{S}} O_1 \cap \cdots \cap O_n.$$

You should check that the subspace topology, disjoint union topology, and topology generated by basis and subbasis are indeed topologies. This is partially done in [**?**].

Definition 1.9. Let $f : X \to Y$ be a function between two topological spaces. f is called continuous if every preimage of open sets in Y under f is open in X. If f is bijective and f^{-1} is continuous, then f is called a homeomorphism, and the space X, Y are said to be homeomorphic, which we denote by $X \cong Y$.

Homeomorphism preserves topological properties (those that can be formulated by open sets) of spaces, since $U \subset X$ is open if and only if $f(U) \subset Y$ is open. Therefore, if there exists a homeomorphism between X, Y, they have the same topological properties. So topologically speaking, they are "almost identical", in the sense of isomorphic groups and linearly isomorphic vector spaces are algebraically "almost identical".

Now we can talk about infinite product: The product

$$\prod_{\lambda\in\Lambda} X_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} : x_\lambda \in X_\lambda\}$$

where Λ is an arbitrary index set (could be uncountable). We can define a projection map $\pi_\mu : \prod_{\lambda\in\Lambda} X_\lambda \to X_\mu$ for each $\mu \in \Lambda$, by $(x_\lambda)_{\lambda\in\Lambda} \mapsto x_\mu$. We need to give this product space a topology. Of course, we can say that a basis for this topology is given by the sets of the form $\prod_{\lambda\in\Lambda} U_\lambda$ where $U_\lambda \subset X_\lambda$ is open. This topology is called the **box topology**. But as we will see (and in exercises later), this is not the best topology for the product space. We can also give the space *the coarsest topology relative to which the projections on the individual components are continuous.* So $\pi_\lambda^{-1}(U_\lambda)$ needs to be open for each $U_\lambda \subset X_\lambda$ open. Each set of the form $\pi_\lambda^{-1}(U_\lambda)$ are "open cylinders" in the product space, as the figure below shows. And the intersection of them are "open boxes". Guided by the intuition that any open sets in the product space can be written as unions of open boxes (think of $\mathbb{R}^n$), we conclude that sets the of the form $\pi_\lambda^{-1}(U_\lambda)$ is a subbasis for the product space, and a basis for the this topology is given by

$$\left\{\pi_{\lambda_1}^{-1}(U_1) \cap \cdots \cap \pi_{\lambda_r}^{-1}(U_r) \mid \lambda_1, \ldots, \lambda_r \in \Lambda, U_{\lambda_i} \subset X_{\lambda_i} \text{ open}\right\}.$$

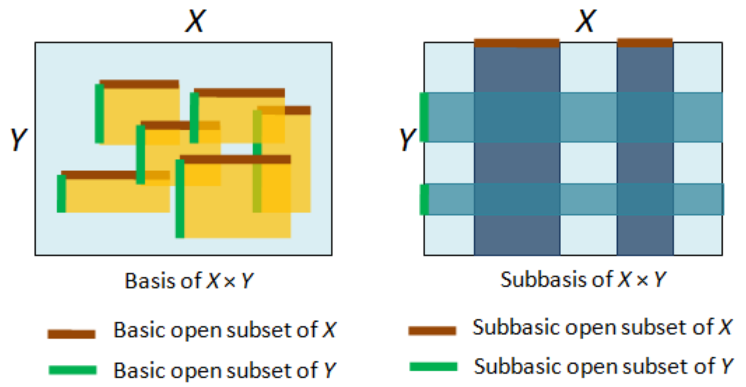


Figure 1.2: Consider $X \times Y$. Basis of $X \times Y$ are given by open boxes, while subbasis are given by open cylinders.

One final remark: When all the factors are the same: $X_\lambda = X$ for all $\lambda \in \Lambda$, then $\prod_{\lambda\in\Lambda} X_\lambda = \prod_{\lambda\in\Lambda} X$. In this case, the elements are of the form $(x_\lambda)_{\lambda\in\Lambda}$ where $x_\lambda \in X$. Then for every coordinate λ we have a value of $x_\lambda \in X$. So this is really the set of maps $f : \Lambda \to X$. And we write $\prod_{\lambda\in\Lambda} X = X^\Lambda$. For example, $\mathbb{R}^\mathbb{N}$ is the set of all real sequences a_n (starting from a_0), and $\mathbb{R}^\mathbb{R}$ is the set of all real functions.

1.3 Connected, Compact, and Hausdorff Spaces

Definition 1.10. Let X be a topological space.

- X is called connected, if the only subsets which are clopen are $\emptyset$ and X.
- X is called compact, if every open cover $\mathscr{U}$ of X has a finite subcover.
- X is called Hausdorff, if for any two different points there exists disjoint neighborhoods.

Proposition 1.11. Continuous images of compact spaces are compact.

Proof. Let $f : X \to Y$ where X compact and f continous. Let $\mathscr{U} = \{U_\lambda\}$ be an open cover of $f(X)$. Then $\left\{f^{-1}(U_\lambda)\right\}$ is an open cover of X. Hence $X = f^{-1}\left(U_{\lambda_1}\right) \cup \cdots \cup f^{-1}\left(U_{\lambda_n}\right)$ and $f(X) = U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$. □

Proposition 1.12. Closed subspace of compact subspaces are compact.

Proof. Let $A \subset X$ and A closed, X compact. Let $\{U_\lambda\}$ be an open cover of A. Then by subspace topology, there exists V_λ open in X such that $U_\lambda = A \cap V_\lambda$. Then $X \setminus A$ and V_λs gives an open cover of X, so there exists finite subcover $(X \setminus A) \cup V_{\lambda_1} \cup \cdots \cup V_{\lambda_r} = X$, i.e. $U_{\lambda_1} \cup \cdots \cup U_{\lambda_r} = A$. □

Proposition 1.13. X and Y are both compact iff $X \sqcup Y$ is, iff $X \times Y$ is.

Proof. Exercise. See [**?**], pg 21. □

Proposition 1.14. If X is Hausdorff and $X_0 \subset X$ is compact, then X_0 is closed in X.

Proof. Every point $p \in X \setminus X_0$ has a neighborhood that does not intersect X_0: for each $x \in X_0$ choose U_x containing p and V_x containing x that are disjoint, which can be done by the Hausdorff property of X. Then $V_x \cap X_0$ is an open cover for X_0, hence choose $\left(V_{x_1} \cap X_0\right) \cup \cdots \cup \left(V_{x_n} \cap X_0\right) = X_0$, and let $U := U_{x_1} \cap \cdots \cap U_{x_n}$. Then U is a neighborhood of p disjoint from $X \setminus X_0$, which shows that $X \setminus X_0$ is open. □

Corollary 1.15. *In a compact Hausdorff space, closed and compact are equivalent.*

Proof. This is immediate from Prop 1.12 and Prop 1.14. □

Theorem 1.16. *A continous bijection $f : X \to Y$ from a compact space X into a Hausdorff space Y is always a homeomorphism.*

Proof. To show that f^{-1} is continuous, we need to show that $(f^{-1})^{-1}(A) = f(A)$ is closed for any A closed in X. Let A be closed in X, then A is compact since X is compact. Then $f(A)$ is compact, hence $f(A)$ is closed in Y by Prop 1.14. □

1.4 Exercises

1. Let X and Y be topological spaces.

(a) Show that for any topological space T, a function $f : T \to X \times Y$ is continuous if and only if the compositions $p_X f : T \to X$, $p_Y f : T \to Y$ are continuous. Here p_X and p_Y are the obvious projection maps.

(b) Let Z be a topological space with continuous maps $g_X : Z \to X$ and $g_Y : Z \to Y$. Suppose that

> for every space T and pair of continuous functions $f_X : T \to X$ and $f_Y : T \to Y$, there is a unique continuous function $f : T \to Z$ such that $f_X = g_X f$ and $f_Y = g_Y f$.

Show that Z must be homeomorphic to $X \times Y$ with the product topology and g_X and g_Y are p_X and p_Y.

This shows that $X \times Y$ with the product topology is the (categorical) product of X and Y in the category of topological spaces.

2. Similarly to the previous problem, suppose that Z is a space equipped with continuous maps $q_X : X \to Z$ and $q_Y : Y \to Z$, and that

> for every space T and pair of functions $f_X : X \to T$ and $f_Y : Y \to T$, there is a unique continuous function $f : Z \to T$ such that $f_X = f q_X$ and $f_Y = f q_Y$.

Show that Z is homeomorphic to $X \sqcup Y$ and q_X and q_Y are the obvious inclusions. This shows that $X \sqcup Y$ is the coproduct of X and Y in the category of topological spaces.

3. Let X be a topological space and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous. Show that $\min(f, g)$ is a continuous function.

4. Let X and Y be metric spaces. Show that the metrics

$$d_\infty((x, y), (x', y')) = \max(d(x, x'), d(y, y'))$$
$$d_1((x, y), (x', y')) = d(x, x') + d(y, y')$$

both induce the product topology on $X \times Y$.

5. A space X is locally (path-) connected at x if for every neighborhood N of x, there is a (path-) connected neighborhood $N' \subseteq N$ of x. It is locally (path-) connected if it is locally (path-) connected at every point.

(a) Show that every countable metric space is totally disconnected.

(b) Show that if $A \subset \mathbb{R}^2$ is countable, then $\mathbb{R}^2 \setminus A$ is path-connected.

6. Give an example of a subset $A \subset \mathbb{R}$ such that the following sets are all different:

$$A, \quad \bar{A}, \quad \operatorname{int} A, \quad \operatorname{int}(\bar{A}), \quad \overline{\operatorname{int} A}, \quad \overline{\operatorname{int}(\bar{A})}, \quad \operatorname{int}(\overline{\operatorname{int} A}).$$

7. Let X be a metric space and A a subset. For a point $x \in X$, we define

$$d(x, A) = \inf_{a \in A} d(x, a).$$

(a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define the *ϵ-neighborhood* of A in X to be the set

$$U(A, \epsilon) = \{x : d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ is the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

(d) If A is compact and U is an open set containing A, show that some ϵ-neighborhood of A is contained in U.

(e) Show that the result of (d) need not hold if A is closed but not compact.

8. Given a nice enough path-connected subset $X \subseteq \mathbb{R}^n$, we can define the *intrinsic distance* between two points of X by

$$d(x, y) = \inf\left\{\int_0^1 |\dot{\gamma}(t)|\,dt : \gamma \text{ is an almost everywhere differentiable path from } x \text{ to } y\right\}.$$

Give an example of an $X \subset \mathbb{R}^2$ which is compact and path-connected, yet the intrinsic distance takes every positive real value (and hence is infinite for some pairs of points).

The moral is that a path being "infinitely long" is not a topological property and does not prevent you from being path-connected!

9. Prove or disprove:

(a) The arbitrary product of path-connected spaces is path-connected.

(b) The arbitrary product of locally path-connected spaces is locally path-connected.

10. A metric space is proper if every closed ball in it is compact.

(a) Show that every proper metric space is complete.

(b) Show that every open set in a proper metric space is a union of a countable sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets.

11. A continuous map is called *proper* if the preimage of every compact set is compact. Show that there is no surjective proper map $f : \mathbb{R}^2 \to \mathbb{R}$.

Chapter 2

Topological Vector Spaces

In this chapter we study a particular kind of topological spaces, topological vector spaces. In this chapter we will see some real applications of topology in other fields of mathematics, functional analysis in particular.

Definition 2.1. A topological vector space over a field K is a vector space V, with a topology in which addition $+ : V \times V \to V$ and scalar multiplication $\bullet : K \times V \to V$ of vectors are continuous.

Usually we will be dealing with the case where $K = \mathbb{R}$. But we can use other fields, like $\mathbb{C}, \mathbb{Q}$, or even p-adic fields.

2.1 Finite Dimensional Topological Vector Space

What are some examples of topological vector spaces? Well, the easiet example is $\mathbb{R}^n$ with the usual topology. This is a finite dimensional topological vector space. Are there any more finite dimensional topological spaces? Sadly, no. To see this we first need a lemma:

Lemma 2.2. *Let E, F be two topological vector spaces. If $T : E \to F$ is a linear map, and E is finite dimensional, then T is continuous.*

Proof. Let $(e_1, \ldots, e_n)$ be a basis of E. Then by rank-nullity theorem, $\dim T(E) = k \leq n$. Then let $(v_1, \ldots, v_k)$ be a basis for $T(E)$. Then for any $x \in E$, we can write $T(x) = \sum_{i=1}^{k} a_i(x) v_i$. Since scalar multiplication and addition are continous in topological vector space, it suffices to show that the map $x \mapsto a_i(x)$ is continuous, for each $1 \leq i \leq k$.

We use the fact that a finite dimensional vector space can be equipped with a norm (in fact a unique one), by letting

$$N\left(\sum_{j=1}^{n} \alpha_j x_j\right) := \sum_{j=1}^{n} |\alpha_j|.$$

Then we can let $x = \sum_{j=1}^n x_j e_j$ and $y = \sum_{j=1}^n y_j e_j$, and we have

$$\begin{aligned}\left|a_i(x) - a_i(y)\right| &\le \sum_{j=1}^n \left|a_i\left(\left(x_j - y_j\right)e_j\right)\right| \\ &= \sum_{j=1}^n \left|x_j - y_j\right| \cdot \left|a_i\left(e_j\right)\right| \le N(x-y)\sum_{j=1}^n |a_i(e_j)|.\end{aligned}$$

This proves the continuity of the function $a_i : E \to K$. Therefore T is continuous. □

Corollary 2.3. *Any topological vector space of dimension n over K is homeomorphic to K^n.*

Proof. By the previous lemma, and by the fact that there exists a linear isomorphism $T : V \to K^n$ for every vector space of dimension n over K. □

Now we give the following theorem without proof:

Theorem 2.4. *The usual topology is the only topology that makes $\mathbb{R}^n$ into a finite dimensional and Hausdorff topological vector space.*

Sometimes the proof is not as important as the lesson behind the statement. The point is, finite dimensional topological vector spaces are boring. So if we want to find anything interesting, we should look for infinite dimensional topological vector spaces.

2.2 Function Spaces

Let X be a topological space, we can define

$$C_B(X) = \{f : X \to \mathbb{R} : f \text{ is continuous and bounded}\}.$$

We can also define a norm on this space, given by $\|f\| = \sup_{x\in X} |f(x)|$. This is called the sup norm. Sometimes we also write $\|f\|_\infty$ for the same thing. The sup norm has some nice properties: $\|f\| > 0$ iff $f \neq 0$, and $\|af\| = |a|\|f\|$, where $a \in \mathbb{R}$. Finally it satisfies the triangle inequality: $\|f + g\| \le \|f\| + \|g\|$. We can define a metric $d(f,g) = \|f - g\|$ on $C_B(X)$. This makes $C_B(X)$ into a complete metric space (i.e. every Cauchy sequence converges).

Definition 2.5. A complete normed vector space is called a Banach space.

Of course we can consider more generally

$$C(X) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}.$$

We can make $C(X)$ into a topological space. Recall that for every compact set $K \subset X$, $f|_K$ is bounded. So we can define a seminorm (which is like a norm, but it can be zero even when f is not identically zero)

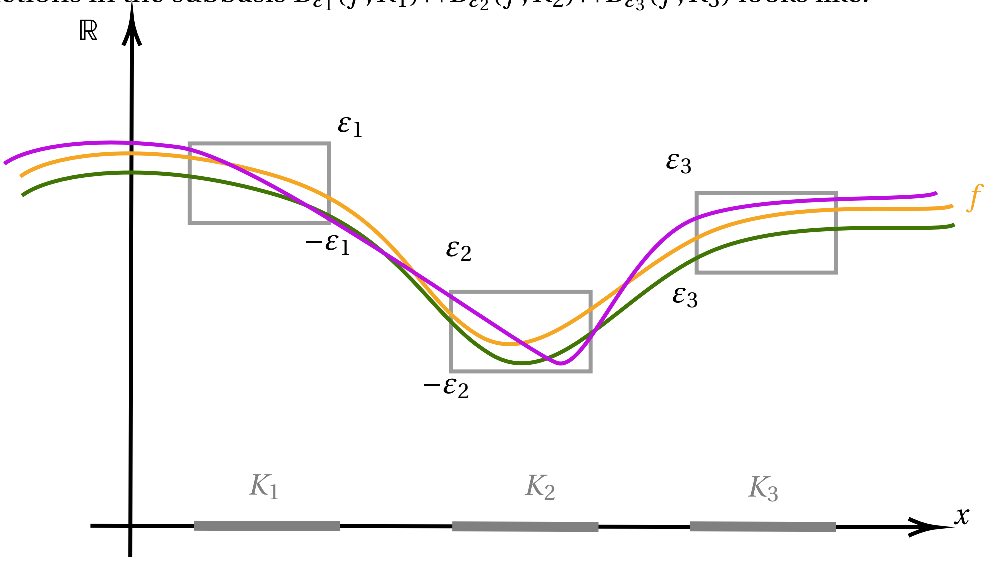
$$\|f\|_K = \sup_{x\in K} |f(x)| \text{ for each compact } K.$$

Then we can define the **compact-open topology** on $C(X)$ to be the topology generated by $\|\cdot\|_K$-balls for every compact K, that is, these balls are subbasis.

So, given this topology, what does a neighborhood of f look like? Let us denote $B_\varepsilon(f,K) = \{g \in C(X) : \|f-g\|_K < \varepsilon\}$. Then an open neighborhood of f is given by

$$\bigcup B_{\varepsilon_1}(f,K_1) \cap B_{\varepsilon_2}(f,K_2) \cap \cdots \cap B_{\varepsilon_n}(f,K_n).$$

This basically says that if $g \in C(X)$ is in this neighborhood of f, then outside of $K_1, K_2, ..., K_n$, g can do whatever it wants, but inside each K_i, it needs to stay within ε_i distance from f. So on these compact domains $K_1, ..., K_n$, the function g is close to f. And we are taking the unions over some compact domains, so the neighborhood consists of functions that are locally close to f on compact domains. The following demonstrates what functions in the subbasis $B_{\varepsilon_1}(f,K_1) \cap B_{\varepsilon_2}(f,K_2) \cap B_{\varepsilon_3}(f,K_3)$ looks like.



There is also another subbasis for the compact open topology. Define $\Delta(K,U) = \{f \in C(X) : f(K) \subset U, K \text{ compact and } U \text{ open}\}$. Then a subbasis is a finite intersection of such elements, and an open set is given by

$$\bigcup \Delta(K_1,U_1) \cap \cdots \cap \Delta(K_n,U_n).$$

The picture is more or less the same. If f is contained in this open set, then another element g in this neighborhood is "close" to f in the sense that $g(K_i) \subset U_i$, so both the value of f and g on K_i lie in the open set U_i.

Proposition 2.6. $C(X)$ with compact open topology is Hausdorff.

Proof. If $f \neq g$, then we can pick a point x such that $f(x) \neq f(y)$. Since a single point is always compact (why?), take $K = \{x\}$, and take

$$U_f = \{h : h(x) \in (f(x)-\varepsilon, f(x)+\varepsilon)\} = \Delta(\{x\}, B_\varepsilon(f(x))),$$
$$U_g = \{h : h(x) \in (g(x)-\varepsilon, g(x)+\varepsilon)\} = \Delta(\{x\}, B_\varepsilon(g(x)))$$

where $\varepsilon \le \frac{1}{2}|f(x)-g(x)|$. Then U_f, U_g are disjoint open sets containing f, g. □

2.3 Completeness of Topological Vector Spaces

We first start with completeness of metric space. Most result here should be familiar to our audiences, so we just do a quick recap.

Definition 2.7. Let X be a metric space. A cauchy sequence (x_n) in X is a sequence with the property that for every $\varepsilon > 0$, there exists $N > 0$, such that for all $m, n \geq N$, we have $d(x_n, x_m) < \varepsilon$.

A metric space X is called complete if every Cauchy sequence in X converges.

Lemma 2.8. *If (x_n) is Cauchy and has a convergent subsequence, then it converges.*

Proof. Let (x_{n_i}) be the subsequence converges to some $x \in X$. For every $\varepsilon > 0$, we can find N such that for all $m, n \geq N$ we have $d(x_m, x_n) < \varepsilon/2$, and $n_i \geq N$ implies $d(x_{n_i}, x) < \varepsilon/2$, then for all $n \geq N$ we have $d(x_n, x) \leq \varepsilon$ by triangle inequality. □

Corollary 2.9. *$\mathbb{R}$ is complete.*

Proof. Every sequence in $\mathbb{R}$ has a convergent subsequence. □

On the other hand, $(0, 1)$ is not complete, since the Cauchy sequence $(\frac{1}{n})$ does not converge in $(0, 1)$. But $(0, 1)$ is homeomorphic to $\mathbb{R}$. So we conclude that completeness is not a topological property.

Proposition 2.10. A subspace of $\mathbb{R}^n$ is complete iff it is closed.

Proof. If A is not closed, then we can find a sequence $(x_n) \subset \mathbb{R}^n$ converging to a point outside A. Then (x_n) is Cauchy but does not converge in A. Conversely, if A is closed and (x_n) is a Cauchy sequence in A, then it is contained in a n-box $[-B, B]^n$ for some B (why?). Then (x_n) is contained in a compact set $A \cap [-B, B]^n$. We claim that (x_n) has a convergent subsequence. Otherwise, there is no limit point of (x_n) in $A \cap [-B, B]^n$. So for every point in $A \cap [-B, B]^n$, take an open set containing this point, this open set will only contain finitely many x_ns. Then all open sets containing points in $A \cap [-B, B]^n$ form a open cover and has a finite subcover, which implies (x_n) is finite, a contradiction. Since (x_n) is Cauchy and has a convergent subsequence, then it converges by the previous lemma. □

Corollary 2.11. *Any compact metric space is complete.*

Proof. Exactly the same argument as above when we prove the Cauchy sequence (x_n) has a convergent subsequence, but this time change $[-B, B]^n$ to a open ball. □

Proposition 2.12. A closed subspace of a complete metric space is complete.

Proof. Duh... □

But now we move into general topological space, and the notion of "distance" or "open balls" may not be defined. So how do we define a Cauchy sequence in a general topological space? We notice that

$$\begin{aligned}
&(x_n) \text{ in a normed space is Cauchy} \\
\Leftrightarrow\ & \forall \varepsilon > 0, \exists N \text{ s.t. } \forall n, m \geq N, \|x_n - x_m\| < \varepsilon \\
\Leftrightarrow\ & \forall \text{neighborhoods } U \text{ of } 0 \text{ there exists } N \text{ s.t. } \forall n, m \geq N, x_n - x_m \in U.
\end{aligned}$$

Motivated by this observation, we give the following definition:

Definition 2.13. Let X be a topological vector space, a sequence $(x_n) \subset X$ is Cauchy if for every neighborhood U of 0, there exists N such that for all $m, n \geq N$ we have $x_n - x_m \in U$.

A topological vector space is complete if every Cauchy sequence (in this sense) converges.

2.4 More on $C_B(X)$ and Kuratowski Embedding

In this section, we show the completeness of $C_B(X)$ with the sup norm and we show that every metric space embeds isometrically in $C_B(X)$.

Proposition 2.14. For any space X, the space $C_B(X)$ of continous bounded function on X with the sup norm is complete.

Proof. Let (f_n) be a Cauchy sequence. Then f_n converges pointwise to a function f. We want to show that f is continous: that is, for all $x \in X$ and $\varepsilon > 0$, there is a neighborhood U containing x such that $f(U) \subset B_\varepsilon(f(x))$.

So let N be such that for all $m, n \geq N$ we have $\|f_m - f_n\|_\infty < \varepsilon/4$, and $|f(x) - f_N(x)| < \varepsilon/4$. Let U be a neighborhood of x such that $f_N(U) \subset B_{\varepsilon/4}(f_N(x))$. Then for all $m \geq N$ we have $f_m(U) \subset B_{\varepsilon/2}(f_N(x)) \subset B_\varepsilon(f(x))$. This proves the claim. □

Proposition 2.15. If X is a metric space, it always embeds isometrically in $C_B(X)$ with sup norm.

Proof. Fix a base point $x_0 \in X$. Define the **Kuratowski embedding**

$$e_{x_0} : X \to C_B(X) \qquad (e_{x_0}(x))(y) = d(x, y) - d(x_0, y).$$

This is a continous function, and for each x, $e_{x_0}(x)$ is bounded by the triangle inequality:

$$|e_{x_0}(x)(y)| = |d(x, y) - d(x_0, y)| \leq d(x, x_0).$$

So this map e_{x_0} is well-defined, and it is clearly injective. We next show e_{x_0} preserves distances, thus an isometric embedding. First notice that $e_{x_0}(x)(x) = -d(x_0, x)$ and $e_{x_0}(x')(x) = d(x', x) - d(x_0, x)$ implies that

$$\|e_{x_0}(x) - e_{x_0}(x')\|_\infty \geq d(x', x).$$

Conversely, for any y, we have

$$|e_{x_0}(x)(y) - e_{x_0}(x')(y)| = |d(x, y) - d(x', y)| \leq d(x, x').$$

This proves the inequality in other direction, hence

$$\|e_{x_0}(x) - e_{x_0}(x')\|_\infty = d(x', x).$$

This proves the proposition. □

2.5 Exercises

1. If (X,d) is a metric space, a map $f : X \to X$ is a *contraction* if there is a number $\alpha < 1$ such that

$$d(f(x), f(y)) \le \alpha d(x, y).$$

Show that if f is a contraction of a complete metric space, then there is a unique point $x \in X$ such that $f(x) = x$.

2. In this question, we use the following definition of completion: the metric space Y is a completion of X if it contains an isometrically embedded copy of X whose closure is Y.

(a) A map $f : X \to Z$ between metric spaces is *Lipschitz* if there's a constant L such that

$$d(f(x), f(y)) \le L d(x, y).$$

Show that if Z is a complete metric space, then any Lipschitz map $f : X \to Z$ extends uniquely to the completion.

(b) Show that the completion of a metric space is unique. That is, if Y and Z are two completions of X, show that the identity map $X \to X$ extends to an isometry $Y \to Z$.

3. Show that the closed unit ball in $C_B([0,1])$ with the norm topology is not compact.

4. Recall that we say that a sequence (v_n) in a topological vector space X is *Cauchy* if for every neigborhood U of 0 there is an N such that for $m, n \ge N$, $v_m - v_n \in U$. The topological vector space X is *complete* if every Cauchy sequence converges.

(a) Show that if X is a *locally compact* space (every point has a compact neigborhood) then $C(X)$ with the compact-open topology is complete.

(b) Show that $C_B(\mathbb{R})$ is not complete when given the compact-open topology.

(c) (Optional, only if you're familiar with Lebesgue integration) Show that $C_B([0,1])$ is not complete when given the *weak-* topology*, which is generated by inverse images of open sets under maps $\int_M : C_B([0,1]) \to \mathbb{R}$ which send f to its integral over a measurable set M. (The completion is called $L^\infty([0,1])$.)

5. This problem will guide you to show that for locally compact Hausdorff spaces X, Z, we have the homeomorphism

$$C(Z, C(X,Y)) \approx C(Z \times X, Y).$$

Convince yourself that if X is locally compact and Hausdorff, and Y, Z be any spaces, a bijection between two spaces (where $C(X,Y)$ is given the compact open topology) is given by

$$\begin{aligned} g : C(Z, C(X,Y)) &\to C(Z \times X, Y) \\ f &\mapsto \mathrm{ev}(f \times \mathrm{id}_X) := F \end{aligned}$$

where

$$\mathrm{ev}(f \times \mathrm{id}_X)(z,x) = \mathrm{ev}(f(z),x) = f(z)(x).$$

Or in better notation, we send $f \to F$ such that

$$f(z)(x) = F(z,x).$$

We will show that g is a homeomorphism in two steps:

(a) Let X and Z be locally compact Hausdorff spaces. Suppose $K \subseteq X \times Z$ is a compact set contained in an open set U. Show that K is covered by a finite number of compact boxes $A_i \times B_i \subseteq U$.

(b) We showed that if X is a locally compact Hausdorff space, then for any spaces Z and Y, $C(Z, C(X,Y))$ and $C(Z \times X, Y)$ are in bijection as sets, where $C(X,Y)$ is given the compact-open topology. Show that if Z is also locally compact Hausdorff, then this bijection is a homeomorphism if both sets are considered as spaces with the compact-open topology.
Hint: As always, a good strategy to show two spaces are homeomorphic is to show that a subbasis for the topology of one is open in the other and vice versa.

6. Are the following subspaces closed? Prove it or give a counterexample.

(a) The set of compactly supported functions (ones which are zero outside a compact set) in $\mathscr{C}_B(\mathbb{R})$ with the sup norm.

(b) The set of functions in $\mathscr{C}(\mathbb{R})$ with the compact-open topology which are zero on the set $[0,1]$.

Chapter 3

Quotient Topology

3.1 Quotient Spaces

We denote $\sim$ as an equivalence relation on a set X, and write $X/\sim$ for the set of equivalence classes. The map $\pi : x \to X/\sim$ is called the projection map that sends x to its equivalence class in $X/\sim$. Sometimes we write the equivalence class of x in $X/\sim$ as $[x]$.

Definition 3.1. Let X be an topological space and $\sim$ and equivalence relation. We give the space $X/\sim$ a topology by declaring that a set U is open on the *quotient topology* on $X/\sim$ iff $\pi^{-1}(U) \subset X$ is open.

So the quotient topology is the finest topology such that π is continuous.

Here is an example: Suppose X, Y are two topological spaces and we consider $X \sqcup Y$. Then pick two points $x \in X$ and $y \in Y$. We can define our equivalence relation $\sim$ by: $x \sim y$, and no other points are identified. Then $X \sqcup Y/\sim$ is the space obtained by guling X, Y on at one point, namely, the identification point is $x \in X, y \in Y$.

Now we study maps from quotient spaces. We have the following:

Proposition 3.2. Let X, Y be a topological space and $X/\sim$ be the quotient space. Then $f : X/\sim \to Y$ is continuous iff $f \circ \pi : X \to Y$ is continuous.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ X/\sim \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. Since π is continous, if f is continous, then clearly $f \circ \pi$ is continous. If $f \circ \pi$ is continous, then take an open set $V \subset Y$, we need to show that $f^{-1}(V)$ is open in $X/\sim$. Since $f^{-1}(V)$ is open iff $\pi^{-1}(f^{-1}(V)) = (f \circ \pi)^{-1}(V)$ is open, we are done. $\square$

What about maps into the quotient spaces? Consider the following diagram. Observe that, if there is a continuous map $\phi : Y \to X$ such that $\varphi = \pi \circ \phi$, then φ is continuous:

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Y \arrow[ur, "\phi"] \arrow[r, "\varphi"'] & X/\sim \end{tikzcd}$$

Of course, even if this diagram is only true locally, φ is still continuous. I.e, if for every $y \in Y$ there is a neighborhood $y \in U$ such that

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Y \arrow[ur, "\phi|_U"] \arrow[r, "\varphi|_U"'] & X/\sim \end{tikzcd}$$

then φ is continuous. This is because if we take an open set $W \subset X/\sim$, then

$$\varphi^{-1}(W) = \bigcup_{y \in Y} \varphi^{-1}(W) \cap U_y.$$

But by definition each $\varphi^{-1}(W) \cap U_y = (\varphi|_{U_y})^{-1}(W)$ is open, and the claim follows.

Some nice properties of topological spaces are inherited to quotients:

Proposition 3.3. If X is a connected/path-connected/compact topological space, then $X/\sim$ is also connected/path-connected/compact.

Proof. $\pi : X \to X/\sim$ is continous and the properties listed above are preserved by continuous maps. □

But notice that if X is Hausdorff, $X/\sim$ need not be Hausdorff! To conclude this section, we present one last easy lemma that should really be proven when we were talking about Hausdorff spaces (but since we were on the subject, so...)

Lemma 3.4. *In a Hausdorff space, points are closed.*

Proof. Given $x \in X$ Hausdorff, for every $y \neq x$, we can find $U_y \ni y$ such that $x \notin U_y$. Then

$$X \setminus \{x\} = \bigcup_{X \setminus \{x\}} U_y$$

is open. This proves the claim. □

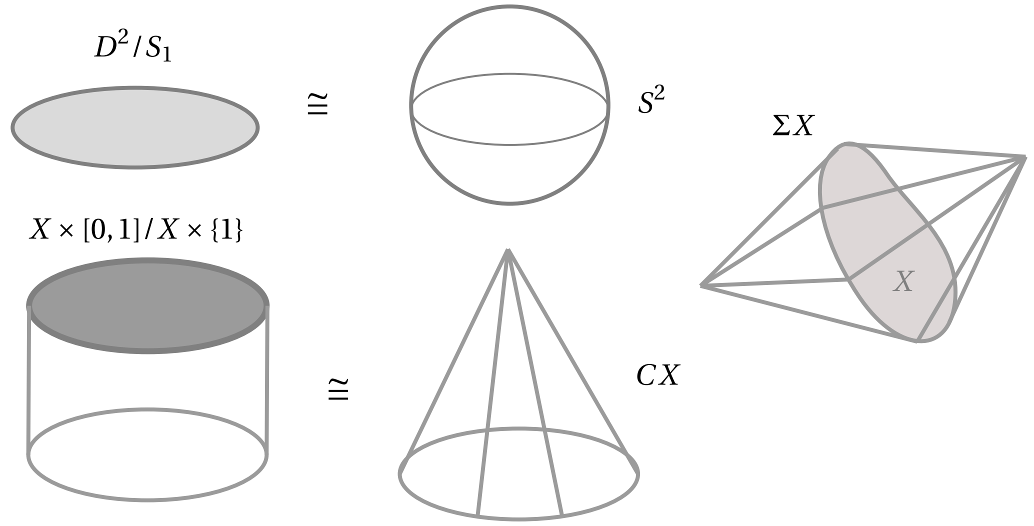
So for $X/\sim$ be Hausdorff, each equivalence classes must be closed! But this is not a sufficient condition! [**?**] gives a nice example in his book pg 34.

3.2 Collapsing and Gluing

In this section we give a few examples of quotient spaces. The first example is a collapsing space. So if X is a topological space and $A \subset X$ is a subspace, then define

$$X/A = X/\sim$$

where the relation $\sim$ is defined by $x \sim y$ if $x \in A$ or $x = y$, for $x, y \in X$. That is, we "collapse" the subspace A into a single point. Here are some examples:



The space $\Sigma X = X \times [-1,1]/X \times \{-1\}, X \times \{1\}$ is called the suspension of X. For X/A to be Hausdorff, A has to be closed. This is because X/A is Hausdorff implies each equivalence class is closed, in particular, the equivalence class of $a \in A$, say $[a] \in X/A$, is closed. Then $\pi^{-1}([a]) = A$ is closed by the definition of quotient topology. We now present a converse to the claim above, i.e. the condition when X/A is Hausdorff.

Proposition 3.5. If X is a metric space and A is closed, then X/A is Hausdorff.

Proof. We want to show that for any $[x] \neq [y] \in X/A$, we have disjoint open sets containing them. Consider two cases. First, if $[y]$ is the equivalent classes of A in X, then in X, since A is closed, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \emptyset$. Then $B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(A) = \emptyset$. Then these project to open sets in the quotient, since $\pi^{-1}(\pi(B_{\varepsilon/2}(x))) = B_{\varepsilon/2}(x)$ is open in X and $\pi^{-1}(\pi(B_{\varepsilon/2}(A))) = B_{\varepsilon/2}(A)$ is open in X.

Now suppose neither of $[x], [y]$ is the equivalence class of A. Then choose U, V containing x, y disjoint in X, we then have $U \setminus A, V \setminus A$ are disjoint neighborhoods of x, y, which projects down to open sets containing $[x], [y]$, for the same reason as in the last paragraph. $\square$

Now we give a second example of quotient spaces.

Definition 3.6. If we have a map $f : A \to Y$, where A is a subspace of X, then we can build a quotient space

$$X \cup_f Y = X \sqcup Y / a \sim f(a)$$

for $a \in A$. We can view this as an gluing map that glues X, Y along A.

For example, we can consider a point x, and a map $f : S^1 \to \{x\}$ by sending every element in S^1 to x. Then

$$S^2 = D^2 \sqcup \{x\} / \sim_f .$$

Also, we can construct S^2 by guling to copies of disks along boundary. So suppose $\mathbf{1} : S^1 \to S^1$ is the identity map on S^1, then

$$S^2 = D^2 \sqcup D^2 / \sim_{\mathbf{1}} .$$

3.3 Simplicial Complex

Definition 3.7. The standard k-simplex is the set

$$\Delta^k = \left\{ (x_0, \ldots, x_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} x_i = 1, 0 \le x_i \le 1 \right\} \subset \mathbb{R}^{k+1}.$$

An l-face of a k-simplex is a subset (an l-simplex) where the other $k - l$ variables are zero.

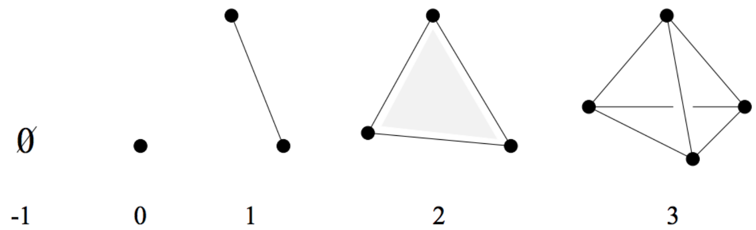


Figure 3.1: a 0-simplex is a point, 1-simplex is a line, 2-simplex is a triangle, and 3-simplex is a tetrahedron. A 0-face is a point, a 1-face is an edge, a 2-face is a side.

Definition 3.8. A finite simplicial complex on k vertices is a union of faces of Δ^k.

Definition 3.9. A simplicial complex K is a collection of simplices such that

(1) If $\sigma \in K$, then for any face σ' of σ we have $\sigma' \in K$.

(2) For two simplices σ_1, σ_2, the intersection $\sigma_1 \cap \sigma_2$ is either empty or a common face of both σ_1 and σ_2.

The dimension of K is the highest dimension of any of K's simplices. The j-skeleton of K, denoted by $K^{(j)}$, is the set

$$K^{(j)} = \{\sigma \in K : \dim(\sigma) \le j\}.$$

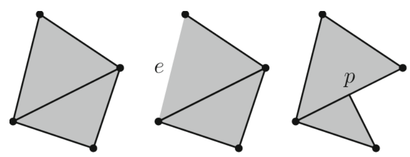


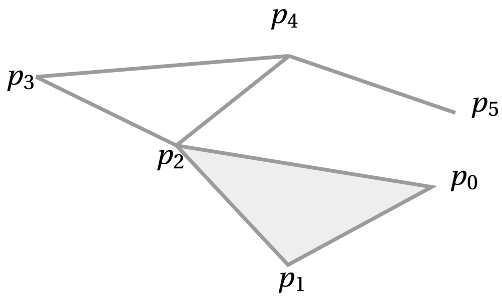
Figure 3.2: The first two are simplicial complexes, but the third one is not.

When we talk about a simplicial complex, we always imagine the underlying geometric realization of the simplicial complex. But we can also write a simplicial complex K in a completely set theoretic way, namely use brackets and the elements will be each simplices. So something like $K = \{\sigma_1, \ldots, \sigma_n\}$. There is also a abstract definition for a simplicial complex.

Definition 3.10. An **abstract simplicial complex** consists of

- A set V, called vertices (could be infinite)
- And a family K of finite subsets of V, called simplices, that is closed under subsets. That is, if any subsets $V' \subset V$ is contained in K, then any subset $V'' \subset V'$ should also be contained in K.

For example, we can consider the following geometric picture of a simplicial complex:



Then the vertices is given by $V = \{p_0, p_1, p_2, p_3, p_4, p_5\}$. And K should contain all the simplices. For example, it contains the triangle shaded in the picture, so it contains $\{p_0, p_1, p_2\}$. Since the simplices are closed under subsets (meaning any sub-simplex of a simplex in K is also in K), then sub-simplices like the lines $\{p_0, p_1\}, \{p_1, p_2\}, \{p_0, p_2\}$ is also in K. So if we explicitly write out this simplicial complex, we get

$$K = \left\{ \begin{array}{l} \langle p_0, p_1, p_2 \rangle, \langle p_0, p_1 \rangle, \langle p_0, p_2 \rangle, \langle p_1, p_2 \rangle, \langle p_2, p_3 \rangle, \langle p_2, p_4 \rangle, \langle p_3, p_4 \rangle, \langle p_4, p_5 \rangle \\ \langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_3 \rangle, \langle p_4 \rangle, \langle p_5 \rangle \end{array} \right\}.$$

We used $\langle \cdot \rangle$ instead of braket for the simplices in order to stress that these are not merely sets (which can be quite useful sometimes as they can indicate orientation of each simplex).

Now given a simplicial complex X, we can always write

$$X = \bigcup_{k=0}^{\infty} X^{(k)}$$

where $X^{(0)} = V$, thought of as a discrete space, $X^{(1)}$ is the set of vertices plus edges, $X^{(2)}$ is the set of vertices, edges, and 2-simplices... and so on. We want to make X into a topological space, so we give X a topology, called the *direct limit topology*.

Definition 3.11. A set U is open in the direct limit topology on X if $U \cap X^{(k)}$ is open for every k.

for example, we can consider the infinite dimensional Euclidean space, which we view as

$$\mathbb{R}^\infty = \bigcup_{k=1}^{\infty} \mathbb{R}^k = \{(x_1, x_2, \ldots, x_k, 0, 0, \ldots) : x_1, \ldots, x_k \in \mathbb{R}\}.$$

Then we define

$$\Delta^\infty = \bigcup_{k=0}^{\infty} \Delta^k.$$

Consider the point $(1, 0, 0, \ldots) \in \Delta^\infty$. What does an open set around this point look like? By direct limit topology, it has to contain an open piece of each Δ^k.

3.4 CW complex

Definition 3.12. A CW complex X is defined inductively as follows:

- $X^{(0)}$ is a discrete set of points (called the vertices or 0-cells);
- $X^{(n+1)}$ is obtained from $X^{(n)}$ by attaching $n+1$-cells: for each $i \in I$ (some index set), take maps $\varphi_i : S^n \to X^{(n)}$, and define

  $$X^{(n+1)} = X^{(n)} \sqcup D_i^{n+1} / x \sim \varphi(x), x \in \partial D_i^n.$$

  That is, we glue each cell to $X^{(n)}$ along its boundary. The map φ_i are called characteristic maps of the corresponding cells to X, since it maps to a subset of X.
- Finally, take $X = \bigcup_{n=0}^{\infty} X^{(n)}$ with the direct limit topology.

We now construct two CW complexes that serves nice examples, one example is the sphere S^2 and the other is the torus T.

(I) For the sphere S^2, there are two ways to do this. First, we can start with a 0-cell, just a point, and we take $X^{(0)} = X^{(1)}$. That is, we add no 1-cells. Or you can also think of it as we take the index set $I = \emptyset$ in the definition above for $X^{(1)}$. Then we add in a 2-cell. Let the attaching map be the unique map that sends $\partial S^2 = S^1$ to the 0-cell, and we "glue" the entire boundary of S^2 to the 0-cell.

   Or we can do it by starting with 2 points, which are our zero cells, and we glue the boundaries of two copies of S^1 (our 1-cells), which are homeomorphic to $[0,1]$, to the two points. Finally, glue two copies of S^2 (2-cells) to $X^{(1)}$ along the boundary.

(II) For the torus T, we start with a single 0-cell, and we attach 2 1-cells to it. The boundary of each one cells are all mapped to the single 0-cell. Then we take a 2-cell (which is homeomorphic to a rectangle, as the figure suggests), and glue the opposite sizes of the square to the corresponding 1-cells, as the figure suggests.

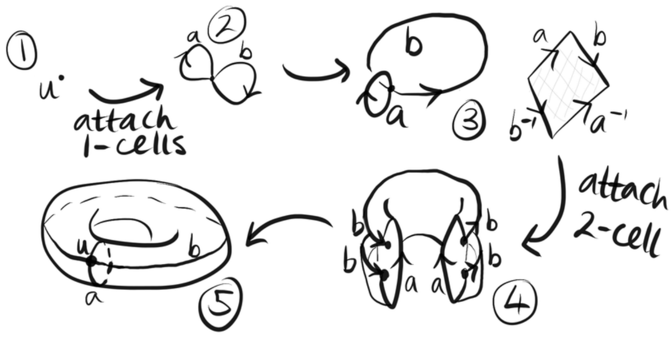


Figure 3.3: Construction of a CW torus.

The remaining of this section will be devoted to proving CW complexes are Hausdorff. To prove this statement, we need a lemma:

Lemma 3.13. *Let X be a CW-complex. Then $U \subset X$ is open iff $F^{-1}(U)$ is open for every characteristic map $F : D^n \to X$ of a cell.*

Proof. We defined the direct limit topology on X by saying $U \subset X$ is open iff $U|_{X^{(n)}} = U \cap X^{(n)}$ is open for every n. So it suffices to check that the statement holds for each $X^{(n)}$. We proceed by induction. For $X^{(0)}$, which is just a set of discrete points, every set is open. So the statement holds trivially. Now suppose the statement holds for $X^{(n-1)}$. Then by the construction of $X^{(n)}$, we have that

$$U|_{X^{(n)}} \text{ open in } X^{(n)} \Leftrightarrow \pi^{-1}\left(U|_{X^{(n)}}\right) \text{ is open in } X^{(n-1)} \sqcup \bigsqcup_{i \in I} D_i^n$$

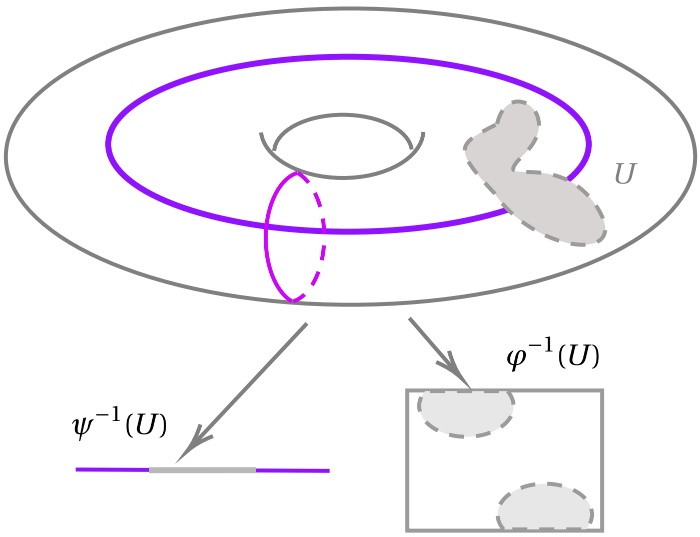
where π is the quotient map of $\sim$ in the definition, where $x \sim F_i(x)$ if $x \in \partial D_i^n$. And since

$$\pi^{-1}\left(U|_{X^{(n)}}\right) \cap X^{(n-1)} = U|_{X^{(n-1)}}$$

And by the definition of disjoint union topology

$$\begin{aligned} U|_{X^{(n)}} \text{ open in } X^{(n)} \Leftrightarrow U|_{X^{(n-1)}} &\text{ open in } X^{(n-1)} \text{ and} \\ &F_i^{-1}(U) \text{ open for every } F_i : D_i^n \to X. \end{aligned}$$

Then the statement follows by the induction hybothesis. □

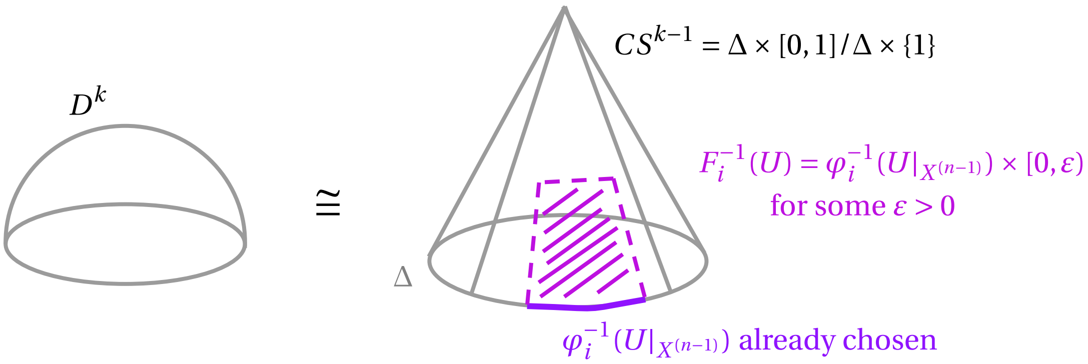


The picture above provides an example of this lemma. Suppose we have an open set U on a CW torus T. Then we look at the characteristic map $\psi : S^1 \to T$ which sends the 1-cell (i.e. the purple line) to T, and the characteristic map φ that sends the 2-cell to T. Take the preimages $\psi^{-1}(U), \varphi^{-1}(U)$ of U under these maps. Then we see that these preimages are open subsets of the cells.

Theorem 3.14. *CW complexes are Hausdorff.*

Proof. First, choose a point $x \in X$ in our CW complex. We build a neighborhood of x inductively. Let n be such that x is in the interior of an n-cell. Then since x is away from the boundary of the n-cell, we can choose U such that $U \cap X^{(n)}$ contains an ε-ball that contains x inside U, and $U \cap X^{(n-1)} = \emptyset$ (Notice then $U \cap X^{(k)} = \emptyset$ for any $k < n$). This is our base step. Now the inductive step. For $k > n$, let D_i^k be any cell, notice that we can take $D_i^k \cong CS^{n-1}$, the cone defined before.

Now, Specifying the preimage of the open set U at every stage is equivalent to specifying U itself. Here is the procedure to construct an open neighborhood inductively:



Let $\varphi_i : D_i^k \to X^{(k)}$ (it attaches ∂D_i^k to $X^{(k-1)}$, but it maps the whole thing to $X^{(k)}$). Suppose we have already chosen an open neighborhood $U|_{X^{(k-1)}}$ of x in $X^{(k-1)}$, then take the preimage $\varphi_i^{-1}(U|_{X^{(k-1)}})$, and for some $0 < \varepsilon < 1$, take

$$F_i^{-1}(U) = \varphi_i^{-1}(U|_{X^{(k-1)}}) \times [0,\varepsilon).$$

This is an open set, since $\varphi_i^{-1}(U|_{X^{(k-1)}})$ is open in $X^{(k-1)}$, and $[0,\varepsilon)$ is open in $[0,1]$ by subspace topology, so $F_i^{-1}(U)$ is open in D_i^k. By the previous lemma,

$$\begin{aligned} U|_{X^{(k)}} \text{ open in } X^{(k)} \Leftrightarrow U|_{X^{(k-1)}} &\text{ open in } X^{(n-1)} \text{ and} \\ &F_i^{-1}(U) \text{ open for every } F_i : D_i^k \to X. \end{aligned}$$

Thus $U|_{X^{(k)}}$ is open in $X^{(k)}$, for $k > n$. and the statement also holds for $k = n$, in which case $U|_{X^{(k)}} = U$, and $k < n$, in which case $U|_{X^{(k)}} = \emptyset$. This shows that U is indeed open.

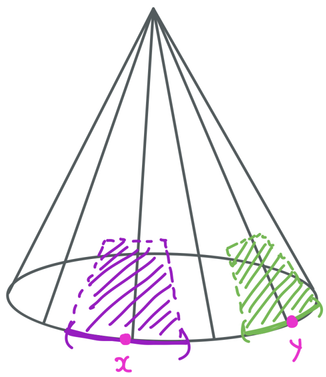


Figure 3.4: We can always choose two disjoint neighborhood U, V at stage n.

Now that we know how to construct open neighborhood of a point x, we move on to prove that X is Hausdorff. Take two points x, y, and m, n such that x is in the interior of an m-cell and y is in the interior of an n-cell. WLOG assume $m \le n$. Then at the stage m of our construction of the CW complex X, pick the open neighborhood U of x, and extend it inductively as discussed before. Then at stage n, we can pick a neighborhood V of y disjoint from U, and we proceed the inductive construction. Then we will obtain two disjoint open sets containing x, y. $\square$

3.5 Exercises

1. If X' is a compact Hausdorff space and X is a proper subspace of X' whose closure equals X', then X' is said to be a *compactification* of X. If $X' - X$ equals a single point, then X' is called the *one point compactification* of X.

Let X be a noncompact, locally compact Hausdorff space, Y be any Hausdorff compactification, and X_∞ be the one-point compactification $X \cup \{\infty\}$. Show that the map $Y \to X_\infty$ which is the identity on X and collapses all points of $Y \setminus X$ to ∞ is a quotient map.

2. Consider the quotient space $X = \mathbb{N} \times [0,1]/\mathbb{N} \times \{0\}$. We can give $\mathbb{N} \times [0,1]$ a metric by embedding it in $\mathbb{R}^2$.

(a) A space is *first-countable* if for every point x there is a countable set $\{U_i\}$ of open sets containing x such that every neighborhood of x contains some U_i. Show that X is not first-countable.

(b) Show that every metric space is first-countable. Conclude that X is not metrizable.

3. Let X be a metric space and A a closed subset. As in Homework 2, for a point $x \in X$, we define

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Define the *quotient metric* on X/A by

$$d_{X/A}(x, y) = \min\{d_X(x, y), d(x, A) + d(y, A)\}.$$

Show that this defines a metric.

4. Say a quotient map $X \to X/\sim$ satisfies the *local lifting property* if for every space Z and every map $\varphi : Z \to X/\sim$ and every point $y \in Z$, there is a neighborhood U of y in Z such that $\varphi|_U$ lifts to a map $\Phi_U : Z \to X$.

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Z \arrow[ur, "\Phi|_U"] \arrow[r, "\varphi|_U"'] & X/\sim \end{tikzcd}$$

Give an example of a collapse map $X \to X/A$ that doesn't have the local lifting property (and an example map to show this).

5. Construct a simplicial complex and a CW complex homeomorphic to

$$\mathbb{R}P^2 = S^2/x \sim -x.$$

Chapter 4

Countability Axioms

4.1 First-Countable Spaces

Definition 4.1. Let X be a topological space, a neighborhood basis for a point $x \in X$ is a family of neighborhoods $\{U_i\}$ of x such that every neighborhood of x contains one of the U_i.

A topological space X is **first-countable** if every point has a countable neighborhood basis.

In Exercise 3.2 you have shown that every metric space is first countable, and the space $\mathbb{N} \times [0,1]/\mathbb{N} \times \{0\}$ is not first countable.

Proposition 4.2. If $\{X_\lambda\}_{\lambda\in\Lambda}$ is a uncountable family of nontrivial topological spaces, then $\prod_{\lambda\in\Lambda} X_\lambda$ is not first-countable.

By nontrivial we mean the open sets in X_λ is not just $\emptyset, X_\lambda$, for every λ.

Proof. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a point such that each x_λ is contained in some open $U_\lambda \subset X_\lambda$. Then $\pi_\lambda^{-1}(U_\lambda)$ is a neighborhood of (x_λ). Suppose there is a countable neighborhood basis. WLOG, we can assume it consists of basic open sets in the product topology (subbasis):

$$\pi_{\lambda_1}^{-1}(V_{\lambda_1}) \cap \cdots \cap \pi_{\lambda_n}^{-1}(V_{\lambda_n}),$$

for we can always replace an element U in a neighborhood basis for x with any basic open set $x \in V \subset U$. Now if you take countably many of them, each of which looks like $\pi_{\lambda_1}^{-1}(V_{\lambda_1}) \cap \cdots \cap \pi_{\lambda_n}^{-1}(V_{\lambda_n})$, then since all the index that appear in the countable neighborhood basis is countable. So we can choose an index $\lambda \in \Lambda$ that does not appear in any of the neighborhood basis. Then all the neighborhood basis will contain X_λ in the λth entry thus $\pi^{-1}(U_\lambda)$ does not contain any neighborhood basis, since for each neighborhood basis there exists a point (p) such that $\pi_\lambda(p) \in X_\lambda \setminus U_\lambda$. This is a contradiction. □

The first-countability axiom is related to sequences.

Proposition 4.3. If $f : X \to Y$ is continuous, and $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.

Proof. If U is a neighborhood of $f(x)$ in Y, then all but finitely many x_n lie in $f^{-1}(U)$, hence all but finitely many $f(x_n)$ lie in U. □

What about the converse? We know the converse is true in $\mathbb{R}^n$. We say

Definition 4.4. f is sequentially continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.

Then is sequentially continuous equivalent to continuity? Answer: NO. For example, take the set X of continuous function $[0,1] \to [0,1]$, with subspace topology of $[0,1]^{[0,1]}$. Every continous function on $[0,1]$ is square integrable, so we have a map $X \to L^2[0,1]$, $\varphi \to \varphi$. This map is sequentially continuous, by Lebesgue convergence theorem. But it is not continuous. Otherwise, for every $\varepsilon > 0$, there is an open box

$$K = \pi_{\lambda_1}^{-1}(V_{\lambda_1}) \cap \cdots \cap \pi_{\lambda_n}^{-1}(V_{\lambda_n}) \qquad \lambda_i \in [0,1], V_i \subset [0,1] \text{ a neighborhood of } 0$$

such that $\int_0^1 \varphi^2 dx < \varepsilon$ for all $\varphi \in K \cap X$. But being in K means that $\varphi(\lambda_i) \subset V_{\lambda_i}$ is small, but only for finitely many points, which does not gaurantee the integral to be small.

Proposition 4.5. If X is first countable and Y is an arbitrary topological spaces, then $f : X \to Y$ is continous iff it is sequentially continous.

Proof. One direction is already shown in the beginning, it remains to show that sequentially continuity implies continuity. If f is sequentially continous, we want to show that for all $a \in X$, $U \ni f(a)$ a neighborhood, there is a neighborhood of V such tthat $f(V) \subset U$ [neighborhood definition of continuity]. Suppose not. Then there is a countable neighborhood basis $\{V_i\}$ such that $f(V_i) \not\subset U$. WLOG we assume $V_1 \supset V_2 \supset \ldots$ (otherwise we can use $\bigcap_{j=1}^n V_j$ as a countable neighborhood basis instead). Choose $x_i \in V_i \setminus f^{-1}(U)$. Then $x_n \to a$ since each V_i contains all but finitely many of these points. So $f(x_n) \to f(a)$. But $f(x_n) \notin f(U)$ for any n. A contradiction. □

Definition 4.6. A topological space X is called sequentially compact if every sequence in X has a convergent subsequence.

Proposition 4.7. Every first-countable compact space is sequentially compact.

Proof. First, suppose (x_n) is a sequence such that for all $x \in X$, there's a neighborhood U_x which meets (x_n) only finitely many times, then there cannot be a finite cover by such U_x, since (x_n) is infinite. Thus X is not compact (This part is true for all spaces, not just first-countable).

Now, suppose X is compact, and (x_n) is a sequence. Then there is some $x \in X$ such that (x_n) meets every neighborhood of X infinitely many times. Choose a countable neighborhood basis $V_1 \supset V_2 \supset \ldots$ (if they are not nested, choose $V_n' = \bigcup_{j=1}^n V_j$ as our countable basis instead). Then choose $n_1 < n_2 < n_3 < \ldots$ such that $x_{n_i} \in V_i$. This subsequence converges to x. □

Proposition 4.8. If X is a sequentially compact metric space, then it is compact.

Proof. Suppose X is sequentially compact but there is an open cover $(U_\lambda)_{\lambda \in \Lambda}$ which does not have a finite subcover. Then for every $x \in X$, choose $\lambda(x)$ so that x is "deeply inside" $U_{\lambda(x)}$, i.e. $U_{\lambda(x)}$ contains $B_r(x)$ where

$$r \geq \min\left\{1, \frac{1}{2} \sup\{\text{max radius of a ball around } x \text{ in } U_\lambda : \lambda \in \Lambda\}\right\}.$$

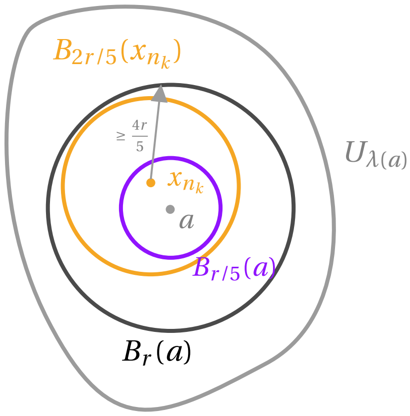
That is, we choose $\lambda(x)$ such that the radius r of biggest open ball around x contained in $U_{\lambda(x)}$ is either greater than 1 or so big that the ball or radius $2r$ around x is not contained in any of the covering (U_λ).

Since this cover does not have a finite subcover, we can choose a sequence (x_n) such that

$$x_{n+1} \notin U_{\lambda(x_1)} \cup \cdots \cup U_{\lambda(x_n)}$$

for each n, and let a be the limit of a subsequence (x_{n_k}) of (x_n).

Then choose $0 < r < 1$ such that $B_r(a) \subset U_{\lambda(a)}$. For k sufficiently large, we must have $x_{n_k} \in B_{r/5}(a)$. But we know $B_{2r/5}(x_{n_k}) \subset U_{\lambda(x_{n_k})}$, since at least a $4r/5$-ball around x_{n_k} fits in $U_{\lambda(a)}$ (since by the definition of $U_{\lambda(x_{n_k})}$, if $B_{2r/5}(x_{n_k})$ is not contained in it, then $B_{4r/5}(x_{n_k})$ will not be contained in any of the covering U_λs).



But then $x_{n_{k+1}} \in B_{r/5}(a) \subset U_{\lambda(x_{n_k})}$. This is a contradiction since we explicitly chose (x_n) such that $x_{n_{k+1}}$ is not contained in $U_{\lambda(x_{n_k})}$. □

Corollary 4.9. *A metric space is compact iff it is sequentially compact.*

4.2 Second-Countable Spaces

Definition 4.10. A topological space is second-countable if it has a countable basis.

For example, $\mathbb{N} \times [0,1]/\mathbb{N} \times \{0\}$ with the quotient metric is first-countable but not second-countable. $C([0,1])$ with sup norm is second-countable but $C_B(\mathbb{R})$ with the sup norm is not. The verification is left to readers as exercises.

Definition 4.11. A topological space is called separable if it has a countable dense set, i.e. a countable set whose closure is the whole space.

4.3 Exercises

1. A space is *Lindelöf* if every open cover has a countable subcover. Show that a Lindelöf metric space is second-countable.

2. Show that $\mathscr{C}([0,1])$ with the sup norm is separable (and therefore second-countable).
Hint: This is equivalent to showing that every continuous function $[0,1] \to \mathbb{R}$ can be approximated arbitrarily closely by functions from some countable set.

3. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of connected spaces. In this problem you will show that their product $X = \prod_{\lambda\in\Lambda} X_\lambda$ is connected.

(a) Fix a point $\mathbf{a} = (a_\lambda)_{\lambda\in\Lambda} \in X$. Given a finite subset $K \subset \Lambda$, let X_K denote the subspace of points in X whose coordinates are all a_λ except perhaps for $\lambda \in K$. Show that X_K is connected.

(b) Show that the union Y of all the X_K is connected.

(c) Show that the closure of a connected subset of any space is connected.

(d) Show that the closure of Y is X. Conclude that X is connected.

4. Let X be a topological space.

(a) Show that if X is a first-countable space, then for every $A \subset X$, every point in the closure of A is the limit of a sequence in A.

(b) Using the previous problem, show that this is not true for spaces that aren't first-countable.

5. Let X be a compact metric space and $f : X \to X$ an isometric embedding, i.e. for every $x, y \in X$, $d(f(x), f(y)) = d(x, y)$. Show that f is surjective.
Hint: Suppose not, find a point x_0 not in the image, and consider the sequence

$$x_0, f(x_0), f(f(x_0))\ldots$$

Chapter 5

Construction of Continuous Functions

In this chapter, we want to construct nice continous functions that lives in general topological spaces. This can be quite difficult, but constructing continous functions in topological spaces have lots of advantages, as we shall see. In analysis, we have seen a nice example called a "bump function":

Definition 5.1. A bump function is a continuous function $X \to [0,1]$ that is 0 on some closed set A and 1 on some closed set B disjoint from A.

In analysis, we often want the bump functions to be smooth. In a metric space X, if $A \subset X$ is compact, and $B \subset X$ is closed, $A \cap B = \emptyset$, then there is a bump function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$. To see this, we know from Exercise 1.8 that there is some $\varepsilon > 0$, such that the ε-neighborhood $U(A,\varepsilon)$ of A is contained in the open set $X \setminus B$. So we can let our bump function be

$$f(x) = \min\left\{1, \frac{1}{\varepsilon} d(x, A)\right\}.$$

One can check that this is a continuous function with the desired property. However, as we said, this procedure of constructing a continuous bump function is quite difficult for general metric spaces. Urysohn's lemma, the key result in this chapter, shows that it is possible to do this for a normal space.

5.1 Normal Spaces

Definition 5.2. A space X is normal if any two disjoint closed sets A, B can be separated by disjoint open neighborhoods $U \supset A, V \supset B, U \cap V = \emptyset$.

Before we go into the proof of Urysohn's lemma, we first give two examples of what spaces are normal.

Proposition 5.3. Compact Hausdorff spaces are normal.

Proof. Let A, B be disjoint closed (hence compact) subsets. We divide the proof intwo several steps:

- **Step 0:** For every $a \in A, b \in B$, we choose disjoint open sets $U_{a,b} \ni a, V_{a,b} \ni b$.

- **Step 1:** For every $a \in A$, choose a finite subcover $V_{a,b_1},\dots,V_{a,b_n}$ of B. Then $U_a = \bigcap_{i=1}^n U_{a,b_i}$ is an open neighborhood of a disjoint from the open neighborhood $V_a = \bigcup_{i=1}^n V_{a,b_i}$ of B, hence from B.
- **Step 2:** we choose a finite subcover $U_{a_1},\dots,U_{a_n}$ of A. Then $U = \bigcup_{i=1}^m U_{a_i}$ is an open neighborhood of A, and $V = \bigcap_{i=1}^m V_{a_i}$ is an open neighborhood of B, and they are disjoint.

This completes the proof. □

Proposition 5.4. Metric spaces are normal.

Proof. Let A,B be disjoint closed subsets. Given $a \in A$, choose $\varepsilon(a)$ such that $B_{\varepsilon(a)}(a)$ does not intersect B. Similarly, given $b \in B$, choose $\varepsilon(b)$ such that $B_{\varepsilon(b)}(b)$ does not intersect A. Define

$$U = \bigcup_{a\in A} B_{\varepsilon(a)/2}(a) \qquad V = \bigcup_{b\in B} B_{\varepsilon(b)/2}(b).$$

Then U,V are open sets containing A and B. We claim that $U \cap V = \emptyset$. If not, let $z \in U \cap V$, then

$$z \in B_{\varepsilon(a)/2}(a) \cap B_{\varepsilon(b)/2}(b)$$

for some $a \in A, b \in B$. The triangle inequality shows that $d(a,b) < (\varepsilon(a)+\varepsilon(b))/2$. If $\varepsilon(a) \le \varepsilon(b)$, then $d(a,b) < \varepsilon(b)$. Hence $a \in B_{\varepsilon(b)}(b)$, a contradiction, since the ball is chosen to be disjoint from A, similarly for the case where $\varepsilon(a) \ge \varepsilon(b)$. □

5.2 Urysohn's Lemma

Some background information: Pavel Samuilovich Urysohn was a Soviet mathematician who is best known for his contributions in dimension theory, and for developing Urysohn's metrization theorem and Urysohn's lemma, both of which are fundamental results in topology. At the age of 23, Urysohn was already an assistant professor at Moscow university. He drowned in 1924 (at the age of 26) while swimming off the coast of Brittany, France. So all of Urysohn's accomplishments were done before he was 26.

Lemma 5.5. *If X is normal, then for any two subsets M,N with $\overline{M} \subset N^o$, there is a third subset L such that $\overline{M} \subset L^o \subset \overline{L} \subset N^o$.*

Proof. Take $A = \overline{M}$ and $B = X \backslash N^o$ be our two disjoint closed sets. Separate them by open neighborhood $U \supset A, V \supset B$ and put $L := U$. Then $X\backslash V$ is a closed set containing U, hence $\overline{L} \subset X\backslash V$. And $X\backslash V \subset X\backslash B = N^o$. So $\overline{L} \subset N^o$. □

Theorem 5.6 (**Urysohn's Lemma**). *Let X be a normal space, and let A,B be disjoint closed subsets of X. Then there exists a continous map $f : X \to [0,1]$ such that $f|_A = 1$ and $f|_B = 0$.*

Proof. We construct f as a limit of a sequence of step functions (f_n). Let A,B be disjoint closed subsets of X. An incrasing chain $\mathfrak{U} = (A_0, A_1, \dots, A_r)$ of subsets of X with

$$A = A_0 \subset A_1 \subset \cdots \subset A_r \subset X\backslash B$$

is called **addmissible** if $\overline{A_{i-1}} \subset A_i^o$. The function $X \to [0,1]$ that takes the constant value 1 on A, the value $1 - k/r$ on $A_k \backslash A_{k-1}$, and 0 outside of A_r will be called the **uniform step function of the chain** $\mathfrak{U}$. The open sets $A_{k+1}^o \backslash \overline{A_{k-1}}$, $k = 0, ..., r$, $A_{-1} = \emptyset$, $A_{r+1} = X$ will be called the **step domains of** $\mathfrak{U}$. Notice that the step domains of an addmissible chain cover the whole space, because

$$\overline{A_k} \backslash \overline{A_{k-1}} \subset A_{k+1}^o \backslash \overline{A_{k-1}}.$$

Also notice that the uniform step function does not fluctuate by more than $1/r$ on each step domain. A **refinement** of an addmissible chain $(A_0, ..., A_r)$ is an addmissible chain $(A_0, A_1', A_1, ..., A_r', A_r)$. By the lemma above, every addmissible chain can be refined.

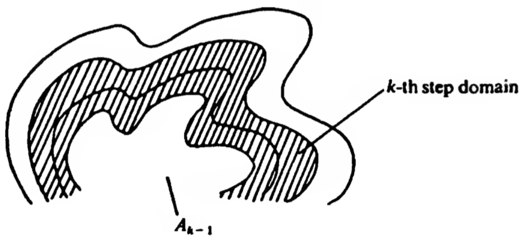


Figure 5.1: Step domains.

Now, let $\mathfrak{U}_0 = (A, X \backslash B)$. Since A is contained in $(X \backslash B)^o = X \backslash B$, this is an addmissible chain. Let $\mathfrak{U}_{n+1}$ be a refinement of $\mathfrak{U}_n$ for each n. Call f_n the uniform step function of $\mathfrak{U}_n$. Then several observations:

- The function sequence (f_n) is pointwise monotonically increasing and bounded by 1, so it pointwise converges to a limit function $f = \lim_n f_n$.
- Since each f_n is 1 on A and 0 on B, so is f.

It remains to prove that f is continous. Everytime we refine the chain $\mathfrak{U}_n$, the descend in each step decreases by a factor of $1/2$ (see the figure on the next page!), by construction. So we have for each x,

$$|f(x) - f_n(x)| \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

And f_n itself does not fluctuate by more than $1/2^n$ on each step domain of $\mathfrak{U}_n$. This implies continuity: For every $\varepsilon > 0$, choose n large such that $3/2^n < \varepsilon$. Then the whole step domain of $\mathfrak{U}_n$ containing x (which is an open neighborhood of x by definition!) will be mapped to $B_\varepsilon(f(x))$, for if y is in the step domain of $\mathfrak{U}_n$ containing x, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} = \frac{3}{2^n} < \varepsilon. \end{aligned}$$

This proves the continuity of f, and concludes the proof of the theorem. $\square$

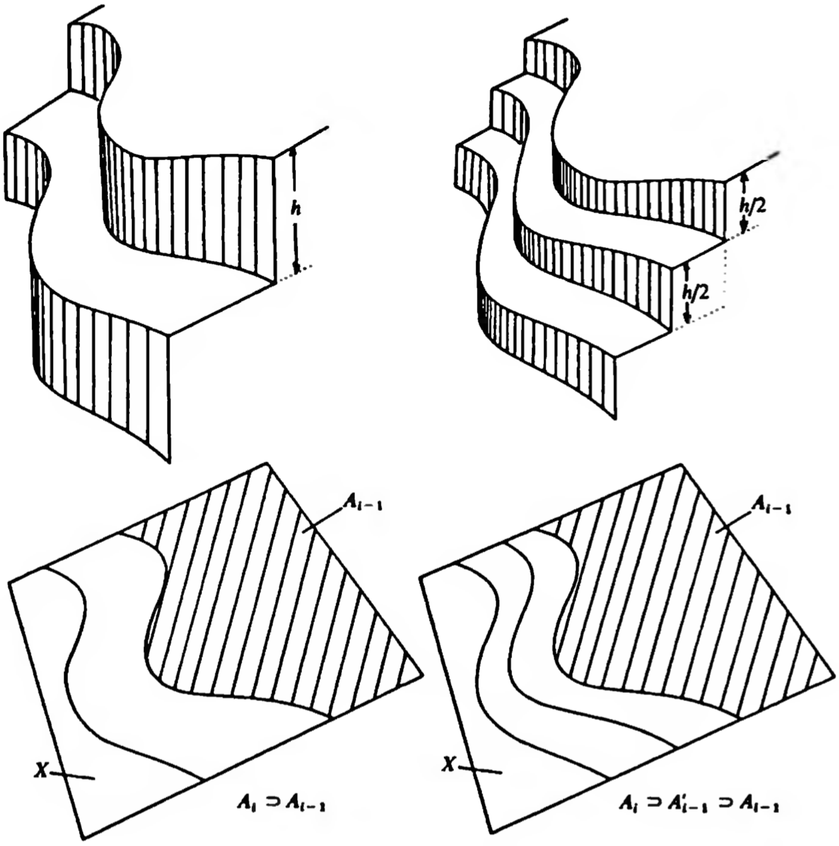


Figure 5.2: Everytime we refine $\mathfrak{U}_n$, the height of each descend decreases by a factor of $1/2$.

Corollary 5.7. *Let X be a normal space, and let A, B be disjoint closed subsets of X. Let $[a,b] \subset \mathbb{R}$ be a closed interval. Then there exists a continous map $f : X \to [a,b]$ such that $f|_A = a$ and $f|_B = b$.*

Proof. By Urysohn's lemma there exists a function $h : X \to [0,1]$ such that $h(A) = 1$ and $h(B) = 0$. Then consider the continuous function $g : [0,1] \to [a,b]$ by $g(x) = ax + b(1-x)$. Then $g \circ h : X \to [a,b]$ is the desired continuous function, for $g(h(A)) = g(1) = a$ and $g(h(B)) = g(0) = b$. □

5.3 The Urysohn Metrization Theorem

This is the first theorem that allows us to determine what kind of topological spaces is metrizable. The original statement more more general, and a bit strange, so we will only give a modified version. But this is already general enough.

Theorem 5.8 (**Urysohn Metrization Theorem**)**.** *Every second-countable normal Hausdorff space is metrizable.*

Proof. Suppose X is a second-countable normal Hausdorff space. We will embed X into $C_B(\mathbb{N})$ with the sup norm, sometimes also called the ℓ^∞ space, which is just the space of bounded real sequence.

We take a countable basis $\mathscr{B}$ of X, and consider the set of pairs

$$\left\{(U,V)\in\mathscr{B}\times\mathscr{B}:\overline{U}\subset V\right\}.$$

This is a countable set, which we can index by positive integers n. For every n (every such pair), apply Urysohn's lemma to the disjoint closed set $\overline{U}, V^c$ and get a bump function $f_n : X \to [0,1]$. Then we define

$$f : X \to C_B(\mathbb{N}) \qquad f(x) = \left(\frac{1}{n} f_n(x)\right).$$

Now we need to show that f is an embedding (a homeomorphism to a subspace). That means we need to show 3 things:

(1) f is injective.

(2) f is continuous.

(3) The inverse of f, say $g : f(X) \to X$ is continous. This means that for every $U \subset X$, the preimage $g^{-1}(U) = f(U) \subset f(X)$ is open. Now, since $f(X)$ is a subspace of the metric space, we can give it the subspace topology. If we can show that for every $x \in U$, there exists $\varepsilon > 0$, such that an open ball in the subspace topology $B_\varepsilon(f(x)) \cap f(X) \subset f(U)$, then $f(U)$ is open by the definition of openness in a metric space.

Now we check these statements one by one:

(1) To show that f is injective, let $x, y \in X$ be distinct. It suffices to show $f_n(x) \neq f_n(y)$ for some n. Choose an open set $W \ni x$ such that $y \notin W$. This contains a basic open set V. Since X is Hausdorff, $\{x\}$ is closed, so we can choose $U \ni x$ such that $\overline{U} \subset V$. WLOG U is also basic (every open set contains a basic open set). Then there is a suitable n such that $f_n(x) = 0$ and $f_n(y) = 1$. This proves that f is injective.

(2) To show that f is continuous, given $x \in X$ and $\varepsilon > 0$, we want to show that $f^{-1}(B_\varepsilon(f(x)))$ is open. Since for $n > 2/\varepsilon$, the coordinates $|\frac{1}{n} f_n(x)| \le \frac{1}{n} < \frac{\varepsilon}{2}$, so $|\frac{1}{n+k} f_{n+k}(y) - \frac{1}{n+k} f_{n+k}(x)| < \varepsilon$. Now $f^{-1}(B_\varepsilon(f(x)))$ is all the point $y \in X$ such that the sequence $f(y)$ is no more than ε apart from x. Then this gives

$$f^{-1}(B_\varepsilon(f(x))) = \bigcap_{j=1}^{\infty} f_j^{-1}(B_\varepsilon(f_j(x))) = \bigcap_{j=1}^{n} f_j^{-1}(B_\varepsilon(f_j(x))),$$

where the second equality is because for all $y \in X$, $\frac{1}{n+k} f_{n+k}(y)$ will be no more than ε apart from $\frac{1}{n+k} f_{n+k}(x)$, so $f_{n+k}^{-1}(B_\varepsilon(f_{n+k}(x))) = X$. Thus they do not contribute in the intersection. Therefore $f^{-1}(B_\varepsilon(f(x)))$ is a finite intersection of open sets, hence is open.

(3) To show that $f^{-1} = g$ is continuous, as discussed above, we want to show that given $x \in U \subset X$, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \cap f(X) \subset f(U)$. To do this, we can choose basic open sets V_1, V_2 such that $x \in V_1 \subset V_2 \subset U$ and $\overline{V_1} \subset V_2$. (By

normality, we can choose an open neighborhood V_2 of x, which is closed since the space is Hausdorff, disjoint from the closed set U^c. WLOG V_2 is a basic open set. Then we can choose an open neighborhood V_1 of x disjoint from V_2^c, and $\overline{V_1} \subset V_2$.) Then there is a corresponding n such that $f_n(x) = 0$ and $f_n(y) = 1$ for $y \notin V_2$. Then we claim that

$$B_{1/n}(f(x)) \cap f(X) \subset f(V_2) \subset f(U).$$

If not, then there exists $f(y) \in B_{1/n}(f(x)) \cap f(X)$ such that $f(y) \notin f(V_2)$. That is, $y \notin V_2$. Then $f_n(y) = 1$ and $f_n(x) = 0$ implies that

$$\left|\frac{1}{n} f_n(y) - \frac{1}{n} f_n(x)\right| = \frac{1}{n}.$$

But $f(y) \in B_{1/n}(f(x)) \cap f(X)$ implies $\left|\frac{1}{n} f_n(y) - \frac{1}{n} f_n(x)\right|$ is strictly less than $1/n$, since the open ball does not contain the boundary, this is a contradiction. Therefore, we have proven the claim, and the theorem follows.

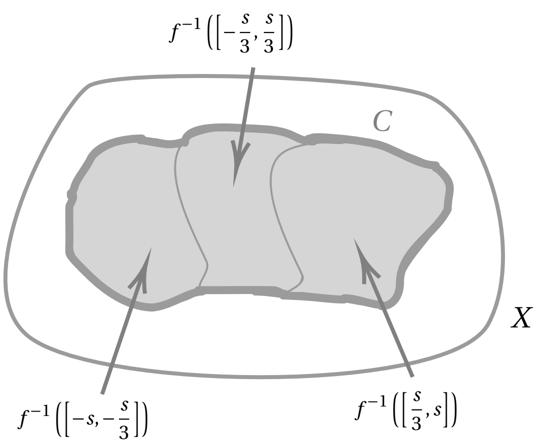
□

But since there are many metric spaces that are not second-countable, Urysohn metrization theorem is not a necessary and sufficient condition. For readers interested in exploring the necessary and sufficient condition for a space to be metrizable, look into Nagata-Smirnov metrization theorem and Smirnov metrization theorem, which can be found in Chapter 6 of [**?**].

5.4 Tietze Extension Theorem

Theorem 5.9. *Let X be a normal space, $C \subset X$ closed. Then every continous function $f : C \to [a, b]$ can be extended to a continous function $F : X \to [a, b]$.*

Proof. We can assume (by adding a constant) that $[a, b] = [-s, s]$ for some s. Now we use Urysohn's lemma (actually the general Urysohn's lemma, i.e. the Corollary in the end of that section) to construct a sequence of continuous functions $F_n : X \to [-s, s]$ which converges everywhere, and to f on C.



First, we consider the function

$$\varphi_1(x) = \begin{cases} -s/3 & x \in f^{-1}\big([-s, -\frac{s}{3}]\big) \\ s/3 & x \in f^{-1}\big([\frac{s}{3}, s]\big) \\ \text{in } [-\frac{s}{3}, \frac{s}{3}] & \text{otherwise.} \end{cases}$$

Such a φ_1 exists, since $A = f^{-1}\big([-s, -\frac{s}{3}]\big)$ and $B = f^{-1}\big([\frac{s}{3}, s]\big)$ are closed and disjoint sets in C, and hence in X. Thus we can apply Urysohn's lemma and get a continuous $\varphi_1 : X \to [-\frac{s}{3}, \frac{s}{3}]$ such that $\varphi_1|_A = -\frac{s}{3}$ and $\varphi_1|_B = \frac{s}{3}$. Notice that

$$|\varphi_1(x) - f(x)| \leq \frac{2s}{3} \qquad \forall x \in C.$$

So this is a very rough approximation. We let $F_1(x) = \varphi_1(x)$. We apply the same trick to get an approximation of $f - F_1$: We get $\varphi_2(x) : X \to [-\frac{2s}{9}, \frac{2s}{9}]$ such that

$$\varphi_2(x) = \begin{cases} -2s/9 & x \in (f - F_1)^{-1}\big([-s, -\frac{s}{3}]\big) \\ 2s/9 & x \in (f - F_1)^{-1}\big([\frac{s}{3}, s]\big) \\ \text{in } [-\frac{2s}{9}, \frac{2s}{9}] & \text{otherwise.} \end{cases}$$

Then we see that

$$|(\varphi_1(x) + \varphi_2(x)) - f(x)| = |\varphi_2(x) - (f(x) - F_1(x))| \leq \frac{4s}{9} \qquad \forall x \in C.$$

And we let $F_2(x) = \varphi_1(x) + \varphi_2(x)$. And we keep going like this: For every n, φ_n is an approximation of $f - F_{n-1}$, and $F_n = F_{n-1} + \varphi_n = \sum_{j=1}^n \varphi_j$, with

$$|\varphi_n(x)| \leq \frac{2^{n-1}s}{3^n} \qquad |F_n(x) - f(x)| \leq \left(\frac{2}{3}\right)^n s.$$

Thus $F_n = \sum_{j=1}^n \varphi_j$ converges uniformly by the Weierstrass M-test, since $|\varphi_n| \leq \frac{2^{n-1}s}{3^n}$ and $\sum_n \frac{2^{n-1}s}{3^n}$ converges. Since each F_n is continuous, the resulting limit F is continous. And since $|F_n(x) - f(x)| \leq \left(\frac{2}{3}\right)^n s$, we see that F_n converges to f on C, i.e. $F|_C = f$. $\square$

5.5 Partitions of Unity and Paracompactness

We have seen that we can construct continuous function on normal spaces by Urysohn's lemma, and Tietze extension theorem. A partition of unity is another nice family of functions on a topological space, that patches together local properties into global properties.

Definition 5.10. A family of continuous functions $\{\tau_\lambda : X \to [0,1]\}_{\lambda \in \Lambda}$ is a partition of unity if

(1) It is locally finite, in the sense that every point $x \in X$ has a neighborhood in which τ_λ vanish for all but finitely many λs.

(2) for every $x \in X$, we have
$$\sum_{\lambda \in \Lambda} \tau_\lambda(x) = 1.$$
Notice that for every x this is only a finite sum.

The partition of unity is said to be subordinate to a given open cover $\mathscr{U}$ if for every λ the support of τ_λ is entirely contained in one of the sets $U_\alpha \in \mathscr{U}$ of the covering. i.e.
$$\operatorname{Supp} \tau_\lambda := \overline{\{x \in X : \tau_\lambda(x) \neq 0\}} \subset U_\alpha.$$

Proposition 5.11. Let X be a compact Hausdorff space and $\mathscr{U}$ and open cover. Then X has a partition of unity subordinate to $\mathscr{U}$.

Proof. We show that given a finite subcover $\{U_1, ..., U_n\}$ of $\mathscr{U}$, one can always "shrink down" U_i to another finite cover $\{V_1, ..., V_n\}$ such that $\overline{V_i} \subset U_i$. Once this is done, we can use Urysohn's lemma to find $f_i : X \to [0,1]$ such that f_i is supported on U_i and $f_i|_{V_i} = 1$. Then simply define
$$\tau_i(x) = f_i(x)\left(\sum_{i=1}^{n} f_i(x)\right)^{-1}.$$
Then just let $\tau_\lambda(x) = f_\lambda(x)$ for $\lambda = 1, 2, ..., n$, and $\tau_\lambda = 0$ for the rest. This is the desired partition of unity.

To do the shrinking, for every $x \in X$, pick a neighborhood V_x such that $\overline{V_x} \subset U_i$ for some i, which can be done since X is normal. Then take a finite subcover $V_{x_1}, ..., V_{x_m}$ of X, and let
$$V_i = \bigcup \left\{ V_{x_j} : V_{x_j} \subset U_i \right\}.$$
Then
$$\overline{V_i} = \overline{V_{x_{j_1}} \cup \cdots \cup V_{x_{j_r}}} = \overline{V_{x_{j_1}}} \cup \cdots \cup cl V_{x_{j_r}} \subset U_i.$$
Moreover, the set of V_is cover X. □

In the proof above, we have used the fact that for any $A, B \subset X$ a topological space, we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$. This is left as an exercise (Exercise 1).

How do we know whether a space have partitions of unity? We say X has the **partition of unity property** if for every open cover $\{U_\lambda\}$ of X, there is a partition of unity subordinate to $\{U_\lambda\}$.

Definition 5.12. A topological space X is **paracompact** if it is Hausdorff, and every open cover has a **locally finite refinement**, i.e., given an open cover $\mathscr{U}$ there is another open cover $\mathscr{B}$ such that

- $\mathscr{B}$ is locally finite: every $x \in X$ has a neighborhood that intersects only finitely many $B \in \mathscr{B}$.
- $\mathscr{B}$ is a refinement of $\mathscr{U}$: every $B \in \mathscr{B}$ is a subset of some $U \in \mathscr{U}$.

Lemma 5.13. *If X is paracompact, it is normal.*

Proof. Let $A, B \subset X$ be closed and disjoint.

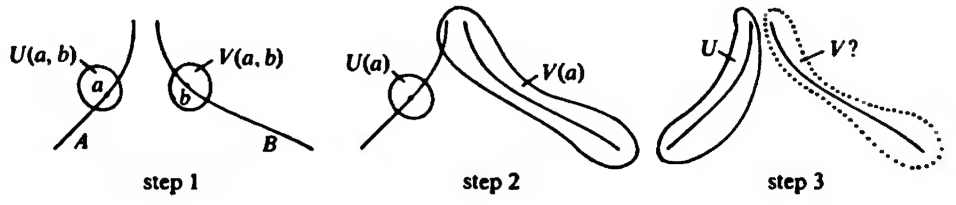


Figure 5.3: Each step of the proof.

- **Step 1:** Then for every $a \in A, b \in B$, we can choose disjoint open sets $U(a,b), V(a,b)$ containing a, b respectively. Now fix $a \in A$.

- **Step 2:** The collection
$$\{V(a,b) : b \in B\} \cup \{X \setminus B\}$$
is an open cover of X, hence it has a locally finite refinement $\mathscr{B}_a$. Now define
$$V(a) = \bigcup \{V \in \mathscr{B}_a : V \subset V(a,b) \text{ for some } b\}.$$
This is an open set containing B, since we are discarding only the ones that is contained in $X \setminus B$. Since $\mathscr{B}_a$ is locally finite, there exists an open neighborhood W of a that intersects finitely many $V_1, \ldots, V_n \in \mathscr{B}_a$. Then define
$$U(a) = W \cap U(a, b_1) \cap \cdots \cap U(a, b_n).$$
This is a open neighborhood of a disjoint from $V(a)$.

- **Step 3:** Now that
$$\{U(a) : a \in A\} \cup \{X \setminus A\}$$
is an open cover of X, so it has a locally finite refinement $\mathscr{B}'$. Define
$$U = \bigcup \left\{U' \in \mathscr{B}' : U' \subset U(a) \text{ for some } a \in A\right\}.$$
Then U is an open set containing A. For every $b \in B$, there is a neighborhood W'_b which intersects finitely many $U'_1, \ldots, U'_m \in \mathscr{B}'$ where $U'_j \subset U(a_j)$. Then
$$V'_b = W'_b \cap V(a_1) \cap \cdots \cap V(a_m)$$
is disjoint from U. And this V'_b contains $b \in B$, since each $V(a_j)$ is a open set containg B disjoint from $U(a_j)$. Then we can take
$$V = \bigcup_{b \in B} V'_b$$
is the desired open neighborhood of B disjoint from U, the open neighborhood of A.

This proves that X is normal. $\square$

Theorem 5.14. *A Hausdorff space has the partition of unity property iff it is paracompact.*

Proof. One direction is easy: Suppose X has the partition of unity property and let $\mathscr{U}$ be an open cover. Then if $\{\tau_\lambda\}_{\lambda\in\Lambda}$ is a partition of unity subordinate to $\mathscr{U}$, then take $\mathscr{B} = \left\{\operatorname{Supp}\tau_\lambda\right\}_{\lambda\in\Lambda}$. By the definition of partitions of unity, $\mathscr{B}$ is a locally finite refinement.

Conversely, let X be paracompact and $\mathscr{U}$ be an open cover, WLOG $\mathscr{U}$ is locally finite. We want to build a partition of unity subordinate to $\mathscr{U}$. Again, suppose we can "shrink" the cover (i.e. for each $U_\lambda \in \mathscr{U}$ find V_λ with $\overline{V_\lambda} \subset U_\lambda$ and $\{V_\lambda\}_{\lambda\in\Lambda}$ is still an open cover of X), then use Urysohn's lemma to construct $f_\lambda : X \to [0,1]$ such that

$$f_\lambda(x) = \begin{cases} 0 & x \in X\backslash U_\lambda \\ 1 & x \in \overline{V_\lambda}. \end{cases}$$

and we set

$$\tau_\lambda(x) = f_\lambda(x)\left(\sum_{\lambda\in\Lambda} f_\lambda(x)\right)^{-1}.$$

Note that the sum is finite for each x, since $\mathscr{U}$ is locally finite. The sum is not 0 for every x since at least one of $f_\lambda(x) = 1$ at every point. This is the desired partition of unity subordinate to $\mathscr{U}$.

Therefore, it remains to show that we can "shrink" the cover $\mathscr{U}$. To construct the shrinking, for each $x \in X$, choose W_x a neighborhood of x such that $\overline{W_x} \subset U_\lambda$ for some λ. Then $\{W_x\}_{x\in X}$ covers X, so we can pick a locally finite refinement $\left\{W'_\mu\right\}_{\mu\in M}$ where M is some index set. Now for each λ let

$$V_\lambda = \bigcup\left\{W'_\mu : \mu \in M \text{ and } \overline{W'_\mu} \subset U_\lambda\right\}.$$

We claim that $\{V_\lambda\}_{\lambda\in\Lambda}$ is the desired shrinking of $\mathscr{U}$.

- It is a cover, since each $W'_\mu \subset U_\lambda$ in some U_λ, and when we union over all $\lambda \in \Lambda$ we are taking a union of all $\left\{W'_\mu\right\}_{\mu\in M}$, which is a cover of X.

- Every $\overline{V_\lambda} \subset U_\lambda$. Let $x \in \overline{V_\lambda}$. Since $\left\{W'_\mu\right\}_{\mu\in M}$ is locally finite, there is a neighborhood $Z \ni x$ which intersects only finitely many $W'_{\mu_1}, \ldots, W'_{\mu_r} \subset V_\lambda$. Again by the same result before (Exercise 1), we have

  $$\overline{W'_{\mu_1} \cup \cdots \cup W'_{\mu_r}} = \overline{W'_{\mu_1}} \cup \cdots \cup \overline{W'_{\mu_r}} \subset U_\lambda.$$

  Since $x \in \overline{V_\lambda}$ and V_λ is the union of all W'_μs whose closure is in U_λ, we have that $x \in \overline{W'_{\mu_1} \cup \cdots \cup W'_{\mu_r}} \subset U_\lambda$.

This completes the proof. □

5.6 Exercises

1. Show that if X is an topological space, and $A, B \subset X$, then we have

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

2. In this problem we are using the notion of "totally disconnected" introduced in class: a space is totally disconnected if there is a separation between every pair of points. This notion is called *totally separated* in other sources.

(a) Show that every totally disconnected space has a continuous injection into $\{0,1\}^J$ for some index set J.

(b) Show that every second-countable totally disconnected space has a continuous injection into $\{0,1\}^{\mathbb{N}}$.

(c) Show that $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the *middle-thirds Cantor set*: the set of numbers in $[0,1]$ which have a base 3 expansion consisting only of 0's and 2's, with the subspace topology inherited from $\mathbb{R}$.

So actually every second-countable totally disconnected space has a continuous injection into $\mathbb{R}$!

3. Let X be a locally compact Hausdorff space.

(a) Show that if K is compact in X and U is an open set containing K, then there is a function $f : X \to [0,1]$ which is supported on U and such that $f(K) = 1$.

Hint: First find a larger compact set whose interior contains K.

(b) Define the subspaces $C_c(X) \subseteq C_0(X) \subseteq C_B(X)$ as follows:

- A function is in $C_c(X)$ if it is *compactly supported*, i.e. it is zero outside a compact set.
- A function $f \in C_0(X)$ if it "vanishes at infinity", i.e. for every $\epsilon > 0$, $\{x \in X : |f(x)| \geq \epsilon\}$ is compact.

Show that $C_0(X)$ is the closure of $C_c(X)$ in the sup norm topology.

4. A set $A \subset X$ is a *retract* if there is a continuous map $X \to A$ which is the identity on A.

- Convince yourself that $\{0,1\}$ is not a retract of $[0,1]$.
- Convince yourself that the two obvious circles are (separately) retracts of the torus.

(a) Show that if A is a retract of a Hausdorff space X, then A is closed in X.

(b) A space Y is an *absolute retract* if whenever X is a normal space which contains a closed set Y_0 homeomorphic to Y, then X retracts to Y_0. Show that $\mathbb{R}^J$ is an absolute retract, for any index set J.

5. Show that a metric space X is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.

6. Let X be a metric space and let r be a constant. You may assume the theorem that metric spaces are paracompact.

(a) Show that there is a simplicial complex P and a map $f : X \to P$ such that the preimage of each simplex has diameter at most r. (The *diameter* of a set in a metric space is the sup distance between points inside that set.)

(b) Let P be as in the previous part and equip $C(X)$ with the sup norm. Construct a map $g : P \to C(X)$ such that $g \circ f(x)$ is at most distance $2r$ from the Kuratowski embedding (defined in Section 2.4).

(c) Deduce that if x and y are two points of X, then

$$|d(x,y) - d(g \circ f(x), g \circ f(y))| \leq 4r.$$

We can think of P as a kind of locally finite approximation to X, though we would have to make more assumptions to say anything more precise.

7. Prove the Remark on p. 124 of Jänich [**?**]: a locally compact Hausdorff space which is a countable union of compact subspaces is paracompact.
Hint. Use the fact that a compact subspace of a locally compact space is contained in the interior of a bigger compact subspace.

8. Show that every connected normal Hausdorff space with at least two points is uncountable.

Chapter 6

Baire Category

Measure theory gives a natural notion of a "negligible" set and a complementary notion of a "generic" set that covers "almost" everything, i.e. measure zero sets and their complements. Baire category is a corresponding topological notion. We us start directly with a definition.

Definition 6.1. Let X be a topological space.

- $A \subset X$ is dense if $\overline{A} = X$. That is, every open set in X intersects nontrivially with A. (example: $\mathbb{Q} \subset \mathbb{R}$ is dense).
- $A \subset X$ is **nowhere dense** if it is not dense on any open subset of A. Equivalently, $\overline{A}$ has **empty interior** (contains no open set of X other than $\emptyset$, or the complement is dense in X).
- $A \subset X$ is called **meager** if it is a countable union of nowhere dense sets.
- If every meager set has empty interior in X, then X is called a **Baire space**. Equivalently, a space is Baire if given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, then $\bigcup_n A_n$ also has empty interior in X.

The reason that having empty interior is equivalent to containing no open sets other than emptyset, is because the interior of Y is the union of all open subsets contained in Y. If Y has empty interior, then every point $y \in Y$ must be a limit point of Y^c. And since $X = Y \cup Y^c$, then $\overline{Y^c} = X$, so Y^c is dense.

Figure 6.1: The Cantor set.

For example, the Cantor set C is nowhere dense. Since in every step of the construction of a Cantor set, we are taking a finite union of closed intervals (hence closed at each step), and we take the countable intersection of finite union of closed intervals to be C, Therefore C is closed. Hence $\overline{C} = C$. Now C contains no interval (it has measure zero), hence has empty interior. Therefore, the Cantor set is nowhere dense.

But there is also the fat cantor set C_r. Now instead of removing the middle thirds, we remove the middle rth interval at every step. Then in the first step we removed 1 interval of length r, in the second step 2 intervals of length r^2, the third step 4 intervals of length r^3... So the total interval removed after infinitely many steps is

$$\sum_{n=1}^{\infty} 2^{n-1} r^n = \frac{r}{1-2r}.$$

Therefore, the fat Cantor set has measure

$$\mu(C_r) = 1 - \frac{r}{1-2r} = \frac{1-3r}{1-2r}.$$

And it is easy to see that $\mu(C_r) > 0$ for some proper choice of r. But by construction, the fat Cantor set still contains no interval. And it is still closed, so still nowhere dense, but it has a positive measure.

Figure 6.2: The fat Cantor set.

But does being topological negligible (meager) implies analytically negligible (measure zero)? Not necessarily. The set

$$B = \bigcup_{n=3}^{\infty} C_{1/n}$$

is a countable union of nowhere dense sets, hence is meager. But $\mu(B) = 1$.

6.1 Baire Catagory Theorem

So why is meager a good definition of "negligible"? Because it is preserved under countable unions: If you take a countable union of meager sets, each a countable union of nowhere dense sets, then you still get a countable union of nowhere dense (meager) set. Moreover, the next theorem tells us that no set is both meager and comeager (complement is meager):

Theorem 6.2 (**Baire Category Theorem**). *If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Proof. Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, we want to show that $\bigcup_n A_n$ also has empty interior in X. So given any nonempty open set U_0 of X, we must find a point $x \in U_0$ that does not lie in any of the sets A_n.

Consider the first set A_1. By hypothesis A_1 does not contain U_0. So there is a point $u \in U_0 \backslash A_1$. Since X is regular (see [?] for regular spaces and their properties), we can find an open neighborhood U_1 of y such that

$$\overline{U_1} \cap A_1 = \emptyset \qquad \overline{U_1} \subset U_0.$$

Then we do so inductively, given the nonempty open set U_{n-1}, choose a point in $U_{n-1} \backslash A_n$, and choose an open neighborhood of this point U_n such that

$$\overline{U_n} \cap A_n = \emptyset \qquad \overline{U_n} \subset U_{n-1}.$$

We claim that in a compact Hausdorff space, we have

$$\bigcap_{n=1}^{\infty} \overline{U_n} \neq \emptyset.$$

Then this proves the theorem, since $\bigcap_{n=1}^{\infty} \overline{U_n} \subset U_0$ and does not intersect any of the A_ns. Suppose it is empty, then $\bigcup_{n=1}^{\infty} \overline{U_n}^c$ is an open cover of X. Since X is compact, there is a finite subcover. But $\overline{U_1}^c \subset \overline{U_2}^c \subset \overline{U_3}^c \ldots$ So we have $X = \overline{U_n}^c$ for some n. But this implies that $U_n = \emptyset$ while we assumed that it is nonempty, a contradiction.

If X is a complete metric space, then choose

$$\operatorname{diam}(U_n) < \frac{1}{n} \qquad \text{for each } n$$

and apply the next lemma below. □

Lemma 6.3. *Let $C_1 \supset C_2 \supset \ldots$ be a nested sequence of nonempty closed sets in a complete metric space X. If $\operatorname{diam}(C_n) \to 0$, then $\bigcap C_n \neq \emptyset$.*

Proof. Choose $x_n \in C_n$ for each n. Because $x_n, x_m \in C_N$ for $n, m \geq N$ and $\operatorname{diam}(C_N)$ can be made less than any $\varepsilon > 0$ for N sufficiently large, this sequence (x_n) is Cauchy. Suppose x_n converges to x. Then for any k, the subsequence $x_k, x_{k+1}, \ldots$ also converges to x. So $x \in \overline{C_k} = C_k$ for all k, hence $x \in \bigcap C_k$. □

6.2 Applications of Baire Category Theorem

The Baire Catagory Theorem has many interesting applications. Here we give three of them.

Proposition 6.4. Every complete metric space with no isolated points (i.e. points that are open) is uncountable.

Proof. If X is countable, it is a countable union of points. If every point is nowhere dense, then X would be meager in itself, i.e. X has empty interior in itself, a contradiction. □

Proposition 6.5. Every infinite-dimensional Banach space (complete normed vector space) has uncountable dimension.

Proof. It is enough to show that every-finite dimensional subspace is closed and nowhere dense, for if this is true, then every countable-dimensional subspace is meager, thus the whole space cannot be countable-dimensional.

To show the claim above, let F be a finite-dimensional subspace. First notice that every finite-dimensional subspace is complete, hence contains all its limit points, therefore F is closed. To show that F is nowhere dense, consider $B_\varepsilon(x)$ for $x \in F$. Since if $x \notin F$ then obviously $B_\varepsilon(x) \not\subset F$. Since the topology and F are invariant under translation $y \mapsto y + x$, WLOG assume $x = 0$. Now take a vector $v \notin F$. Since the scalar multiplication $\alpha \mapsto \alpha v$ is continuous, then $B_\varepsilon(0)$ contains αv for some $\alpha \neq 0$. Thus $B_\varepsilon(0) \not\subset F$. □

Before we enter into the last application of Baire Category theorem, we first need a lemma:

Lemma 6.6. *Any open subspace Y of a Baire space X is Baire.*

Proof. Let A_n be a countable collection of closed subsets of Y that have empty interiors in Y. We show that $\bigcup A_n$ has empty interior in Y.

Let $\overline{A_n}$ be the closure of A_n in X. Then $\overline{A_n} \cap Y = A_n$. The set $\overline{A_n}$ has empty interior in X: If U is a nonempty open set of X contained in $\overline{A_n}$, then U must intersect A_n. Then $U \cap Y$ is a nonempty open set of Y contained in A_n, contradiction.

If $\bigcup A_n$ contains the nonempty open set W of Y, then so does $\bigcup \overline{A_n}$. And W is open in Y which is open in X, so W is open in X. But each $\overline{A_n}$ has empty interior, and X is baire, but $\bigcup \overline{A_n}$ contains open set W, contradiction. □

Corollary 6.7. *Every locally compact Hausdorff space is Baire.*

Proof. It is an open subset of its 1-point compactification. □

Now the last application:

Proposition 6.8. Let X be a Baire space and Y be a metric space. $f_n : X \to Y$ be a sequence of functions that converges pointwise to a function $f : X \to Y$. Then the set of points at which f is continous is dense in X.

Proof. Given a positive integer N and $\varepsilon > 0$, let

$$A_N(\varepsilon) = \left\{x : d\left(f_n(x), f_m(x)\right) \le \varepsilon \text{ for all } n, m \ge N\right\}.$$

Then $A_N(\varepsilon)$ is closed in X, since

$$A_N(\varepsilon) = \bigcap_{m,n \ge N} \left\{x : d\left(f_n(x), f_m(x)\right) \le \varepsilon\right\}$$

which is closed, since each $\left\{x : d\left(f_n(x), f_m(x)\right) \le \varepsilon\right\}$ is closed by the continuity of f_n, f_m. More specifically, let x be a limit point then there exists a sequence (x_n) in that set converging to it, then we can find an M big enough such that x_M is close to x, and such that by continuity $d(f_n(x_M), f_n(x)) < \varepsilon/3$, $d(f_m(x_M), f_m(x)) < \varepsilon/3$. And since $f_n \to f$ pointwise, the sequence $(f(x_n))$ is Cauchy, so WLOG we can assume $d(f_n(x_M), f_m(x_M)) < \varepsilon/3$. Then by triangle inequality $d(f_n(x), f_m(x)) \le \varepsilon$. Therefore this set contains all its limit points, hence is closed.

For fixed ε, consider the sets

$$A_1(\varepsilon) \subset A_2(\varepsilon) \subset A_3(\varepsilon) \subset \ldots$$

Then $\bigcup_n A_n(\varepsilon) = X$, since $f_n \to f$, so $(f_n(x))$ is a Cauchy sequence, and for sufficiently large N, we must have $d(f_n(x), f_m(x)) \leq \varepsilon$ for $m, n \geq N$. Now let $U(\varepsilon)$ be the union of the interiors of $A_N(\varepsilon)$:

$$U(\varepsilon) = \bigcup_{N=1}^{\infty} \operatorname{Int} A_N(\varepsilon).$$

We shall prove two things:

(1) $U(\varepsilon)$ is open and dense in X, for every $\varepsilon > 0$.

(2) The function f is continuous at each point of the set

$$C = U(1) \cap U(1/2) \cap U(1/3) \cap \ldots$$

And once we have shown these two claims, the theorem follows since X is Baire: by definition (take the negation of Def 6.1): If $\{B_n\}$ is a countable collection of open and dense sets in a Baire space X, then the countable intersection $\bigcap B_n$ is dense. Hence we proceed:

(1) To show that $U(\varepsilon)$ is dense in X, it suffices to show that for any nonempty open set V of X, the set $V \cap \operatorname{Int} A_N(\varepsilon)$ is nonempty for some N. We note that by subspace topology, the set $V \cap A_N(\varepsilon)$ is closed in V. By the previous lemma, V is itself a Baire space. Therefore, at least one of these sets, say $V \cap A_M(\varepsilon)$ must contain a nonempty open set W of V. Because if not, then $V \cap A_N(\varepsilon)$ has empty interior for every N, and $V = \bigcup_{N=1}^{\infty} A_N(\varepsilon) \cap V$. Since V is Baire, V itself would have empty interior in V! This is a contradiction. QED for (1).

(2) Now let $x_0 \in C$, we show that f is continuous at x_0. So Given $\varepsilon > 0$, we shall find a neighborhood W of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for $x \in W$. First, choose k such that $1/k < \varepsilon/3$. Since $x_0 \in C$, we have $x_0 \in U(1/k)$. Then there is an N such that $x_0 \in \operatorname{Int} A_N(1/k)$. continuity of f_N allows us to choose a neighborhood $W \subset A_N(1/k)$ of x_0, such that

$$d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}. \qquad x \in W \tag{*}$$

The fact that $W \subset A_N(1/k)$ implies that

$$d(f_n(x), f_N(x)) \leq \frac{1}{k} \qquad \text{for } n \geq N, x \in W.$$

Take $n \to \infty$, we obtain

$$d(f(x), f_N(x)) \leq \frac{1}{k} < \frac{\varepsilon}{3}. \qquad x \in W \tag{**}$$

And since $x_0 \in W$, we have

$$d(f(x_0), f_N(x_0)) < \frac{\varepsilon}{3}. \tag{***}$$

Now apply triangle inequality to (*),(**),(***) gives the desired result.

□

6.3 Exercises

1. A real number r is *badly approximable* if there is some $c > 0$ such that for every rational p/q we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}.$$

A somewhat weaker condition (let's say α*-badly approximable*) is when q^2 is replaced by q^α for some $\alpha > 2$. Show that there are lots of α-badly approximable numbers and lots of not-α-badly approximable numbers (for example, both sets are uncountable).

2. Prove the **Uniform boundedness principle**:

> Let X be a complete metric space and let $\mathscr{F}$ be a subset of $C(X)$ such that for each $x \in X$, the set
>
> $$\mathscr{F}_x = \{f(x) : f \in \mathscr{F}\}$$
>
> is bounded. Then there is a nonempty open set U of X on which the functions in $\mathscr{F}$ are *uniformly bounded*, i.e. there is an M such that $|f(x)| \le M$ for all $x \in U$ and $f \in \mathscr{F}$.

Hint: Choose $A_N = \{x \in X : |f(x)| \le N \text{ for all } f \in \mathscr{F}\}$.

3. Suppose now that X is a Banach space and the functions in $\mathscr{F}$ are linear. Show that there is an M such that for every $x \in X$ and $f \in \mathscr{F}$, $|f(x)| \le M\|x\|$.

4. Let X be a normed vector space. Show that if X is not a Baire space, then it is meager as a subset of itself.
Hint: Suppose that $A \subseteq X$ is meager. Show that αA is meager for every $\alpha \in (0, \infty)$, and $x + A = \{x + y : y \in A\}$ is meager for every $x \in X$.

Part II

Homotopy Theory and Fundamental Groups