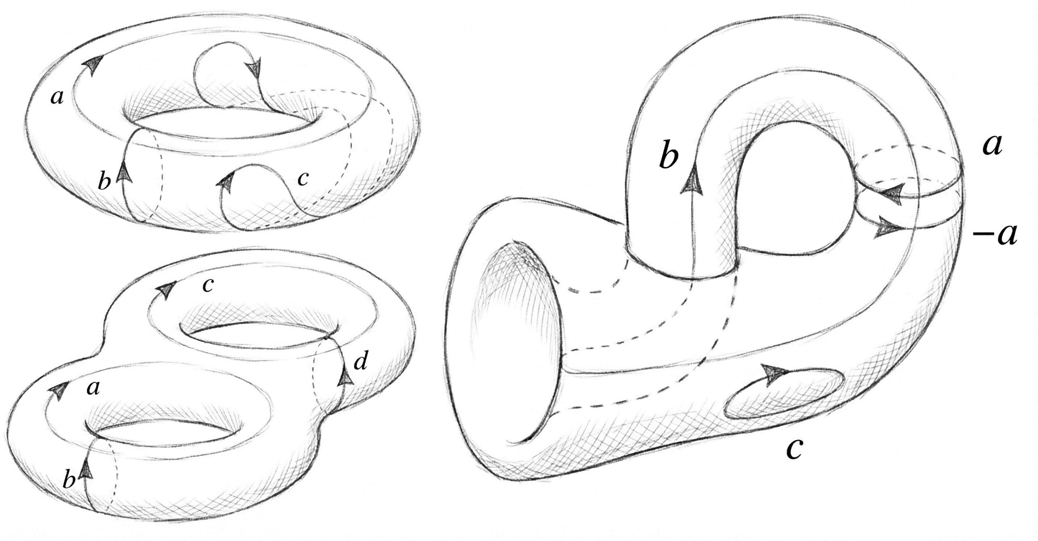
TOPICS ON SMOOTH MANIFOLDS AND DIFFERENTIAL TOPOLOGY

WRITTEN BY

TIANYI WANG



WITH PROBLEMS COMPOSED BY

HERMAN GLUCK

A remark on notation:

In this book we use the so called ***Einstein summation convention*** implicitly without mention. This convention states that when an index variable appears twice in a single term, once upstair and once downstair, and is not otherwise defined, it implies summation of that term over all the values of the index. For example, we write

$$a^\mu b_\mu := a^1 b_1 + a^2 b_2 + \cdots + a^n b_n.$$

This is the same thing as $\sum_{\mu=1}^{n} a^\mu b_\mu$. Note that the dummy index μ does not matter and we can rename it. That is, we can use $a^k b_k$ or $a^\nu b_\nu$ or $a^\alpha b_\alpha$ to express the same quantity.

One important remark is that an upper index in the denominator is to be taken as a lower index. For example,

$$\frac{\partial f}{\partial x^\mu} dx^\mu := \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n.$$

Sometimes another way of expressing the same quantity above is $\partial_\mu f dx^\mu$.

Preface

This is a set of notes that I have written on basics of differentiable and smooth manifolds, and some other topics that I liked a lot while learning, specifically de Rham cohomology theory and its applications in differential topology. I was deeply amazed by the beauty of these elegant results in differential topology, and the concrete geometric (in contrast to homological algebraic) approach to cohomology is what I believe to be the most pedagogical approach to a first exposure in algebraic topology.

The content in the majority of the text come from some of my old notes on differential manifolds that I wrote while self-studying and while taking graduate courses on relevant topics. So technically it does not follow any textbooks or standard references. However, the style of my writing was greatly influenced by [1], a series of wonderful lectures in the WE-Heraeus International Winter School on Gravity and Light by Frederic P. Schuller, and [2]. The readers are encouraged to read these references if needed, since they certainly do a better job than what I did in these notes.

The reader hoping to use these notes should have a solid foundation in multivariable calculus and linear algebra, and a certain level of mathematical maturity. Ideally, they should also have a good background in point set topology, as I will only review the relevant concepts very quickly in the beginning. Readers who does not have relevant background mentioned above may find themselves having difficulties to comprehend the text, and may want to do some extra reading on the background material to catch up along the way.

I am deeply indebted to my professors: Xianzhe Dai, Guofang Wei, Rugang Ye, Fedor Manin, and Daryl Cooper, for teaching me the theory of differentiable manifolds, Riemannian geometry, and differential (algebraic) topology.

Thanks to Merrick Hua and Benjamin Chung, for taking graduate topology classes with me and doing a group reading on Bott and Tu's book. Our collective notes inspired the majority of Chapter 6.

Thanks to Liviu Nicolaescu for answering some of my questions through email.

I also want to thank Sicheng Zhong for carefully reading over the notes, offering suggestions, and correcting several typos. Sicheng trusted me enough to be my lab rat by trying to learn manifold theory from my notes. And his positive feedback gave me a lot of inspiration.

Finally, I want to express my gratitude to Herman Gluck. He read an earlier version of this notes, and liked them (I was flattered). He invited me to TA a first year graduate course in geometry topology with him at UPenn (MATH 6000), and shared his own slides with me, which contain a huge collection of problems. Our hope is that we can compile his slides and my notes together into one document, and make it available to future instructors and students at UPenn taking the same course. Most of the problems at the end of each chapter are composed by him.

Speaking of problems, I encourage readers to work through as many of these as possible. It goes without saying that doing lots of problems is an essential part of mathematical learning. Readers will find that some of these problems are quite elementary, and in fact a large portion of them are in the Euclidean space setting, even people with solid grasp on multivariable calculus can do them. However, I don't think that this is a bad thing, and even deliberately incorperated them into this notes. Because these exercises in Euclidean spaces provide some of the best illustration and examples of the theory developed in the text. Readers might also notice that the problems focuses more about concrete examples rather than proving theorems. I have chosen this style deliberately because I think working with examples is the best way to understand any mathematics. There are many good proofs in the literature but few good examples!

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Part I

Basic Theory

Chapter 0

Review of Basic Topology

In this chapter we review the basic idea of topology and continuity defined on general topological spaces. Here is the definition of a topology.

0.1 Basic Concepts, Continuity

Definition 0.1. Given a set M, a set $\mathcal{O} \in \mathcal{P}(M)$, where $\mathcal{P}(M)$ is the power set of M, is called a ***topology*** of M if

1. $\emptyset, M \in \mathcal{O}$;
2. $U, V \in \mathcal{O}$ implies $U \cap V \in \mathcal{O}$;
3. If $U_i \in \mathcal{O}$ for some $i \in A$, then $\bigcup_{i \in A} U_i \in \mathcal{O}$.

One example. Let $M = \{1, 2, 3\}$. Let $\mathcal{O}_M = \{\emptyset, \{1, 2, 3\}\}$. You can immediately verify that this is indeed a topology on M. We have two trivial topologies: the smallest possible topology $\emptyset$, and the biggest possible topology $\mathcal{P}(M)$, which is the set of all subsets of M.

We introduce the standard topology on $\mathbb{R}^n$.

Definition 0.2. A set $U \in \mathcal{O}_S$, where $U \subset \mathbb{R}^n$, implies that for all $p \in U$, there exists a positive real r such that $B(p, r) \in U$.

You should verify that this topology $\mathcal{O}_S$ on $\mathbb{R}^n$ satisfies the axioms above, thus is indeed a topology. This topology is the the set of open sets in Euclidean space.

Definition 0.3. Suppose some set M is given some topology $\mathcal{O}$. Together, $(M, \mathcal{O})$ is called a ***topological space***.

Now we defined the continuity using topology. Given a map $f : M \to N$, suppose V is in the range of f. Recall the preimage of V, denoted by $f^{-1}(V)$, or sometimes $\mathrm{Pre}_f(V)$, is defined by

$$f^{-1}(V) := \{m : f(m) \in V\}.$$

Definition 0.4. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. Then a map $f : M \to N$ is ***continuous*** if for all $V \in \mathcal{O}_N$, we have $f^{-1}(V) \in \mathcal{O}_M$. A continuous bijection with a continuous inverse is called a ***homeomorphism***.

Now comes the general definition of open and closed sets.

Definition 0.5. Given a topological space $(M, \mathcal{O})$, a set $U \subset M$ is ***open*** in M if $U \in \mathcal{O}$. A set U is ***closed*** in M if $M \setminus U \in \mathcal{O}$.

If we are in Euclidean space, then the definition of open sets and closed sets can be equivalently given by the familiar δ-ϵ language: However, we wish to generalize this statement into abstract topological spaces. For general functions, we use the above definition of continuity rather than the δ-ϵ statement.

Theorem 0.6. *Let $(M, \mathcal{O}_M)$, $(N, \mathcal{O}_N)$, $(P, \mathcal{O}_P)$ be topological spaces. Let $f : M \to N$ and $g : N \to P$ be continuous. Then $g \circ f : M \to P$ is also continuous.*

Proof. Take some $V \in \mathcal{O}_P$. We have

$$\begin{aligned}(g \circ f)^{-1}(V) &= \{m \in M : g \circ f(m) \in V\} \\ &= \{m \in M : f(m) \in g^{-1}(V)\} \\ &= f^{-1}(g^{-1}(V)).\end{aligned}$$

Since g is continuous, we must have $g^{-1}(V) \in \mathcal{O}_N$. But then $f^{-1}(g^{-1}(V)) \in \mathcal{O}_M$ by the continuity of f. This completes the proof. $\square$

Definition 0.7. Let $f : M \to N$ be continuous mapping and M, N be topological spaces with given topology $\mathcal{O}_M, \mathcal{O}_N$, respectively. Given $S \subset M$, defined the restricted map $f|_S : S \to N$. We say $f|_S$ as f restricted to S.

Defined the ***subspace topology*** $\mathcal{O}|_S$ to be

$$\mathcal{O}|_S := \{U \cap S : U \in \mathcal{O}_M\}.$$

The definition above says that every element in $\mathcal{O}|_S$ must be in the form $U \cap S$, where $U \in \mathcal{O}_M$. The good thing about this definition is shown below.

Theorem 0.8. *Given topological spaces $(S, \mathcal{O}|_S)$ and $(N, \mathcal{O}_N)$, the restricted map $f|_S : S \to N$ is continuous.*

Proof. Let $V \in \mathcal{O}_N$. Since $f : M \to N$ is continuous, we have $\mathrm{Pre}_f(V) \in \mathcal{O}_M$. Consider

$$\begin{aligned}f^{-1}(V) \cap S &= \{m \in M : f(m) \in V\} \cap \{m \in S : S \subset M\} \\ &= \{m \in S : f(m) \in V\} = (f|_S)^{-1}(V).\end{aligned}$$

Therefore, we conclude that $(f|_S)^{-1}(V) \in \mathcal{O}|_S$. This completes the proof. $\square$

Theorem 0.9. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous in the δ-ϵ sense iff it is continuous in the sense that we defined above.*

Proof. We prove the case where $n = m = 1$, the general case follows exactly the same.

Let $U \subset \mathbb{R}$ be open. First we show that if f is continuous in the δ-ϵ sense, then $f^{-1}(U)$ is open. By definition, $U = \bigcup_{p \in U} B(p, \epsilon_p)$ where ϵ_p is chosen such that $B(p, \epsilon_p) \subset U$. Since

$$f^{-1}(U) = \bigcup_{p \in U} f^{-1}(B(p, \epsilon_p))$$

it suffices to show that $f^{-1}(B(p, \epsilon))$ is open for any $p \in \mathbb{R}$ and $\epsilon > 0$.

Let $x \in f^{-1}(B(p, \epsilon))$. Then $f(x) \in B(p, \epsilon)$, or equivalently $|f(x) - p| < \epsilon$. Set

$$\eta = \epsilon - |f(x) - p| > 0.$$

Since f is continous at x, we may choose $\delta > 0$ such that $|x_0 - x| < \delta$ implies $|f(x_0) - f(x)| < \eta$. In other words $f(B(x, \delta)) \subset B(f(x), \eta)$. Then by triangle inequality:

$$|f(x_0) - p| \leq |f(x_0) - f(x)| + |f(x) - p| < \eta + |f(x) - p| < \epsilon.$$

Thus $f(x_0) \in B(p, \epsilon)$ for all $x_0 \in B(x, \delta)$. Hence we have shown that there exists $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(p, \epsilon))$. This shows that $f^{-1}(B(p, \epsilon))$ is open.

Conversely, we show that a continuous function $f : \mathbb{R} \to \mathbb{R}$ between topological spaces is continuous in the δ-ϵ sense. Fix $\epsilon > 0, p \in \mathbb{R}$. Since $f^{-1}(B(f(p), \epsilon))$ is an open set (by assumption) that contains p, we can choose $\delta > 0$ such that $B(p, \delta) \subset f^{-1}(B(f(p), \epsilon))$. Thus $f(B(p, \delta)) \subset B(f(p), \epsilon)$. This is exactly the definition of δ-ϵ continuity. □

0.2 Compactness

Definition 0.10. A topological space X is called ***compact*** if for any open cover $\{U_\alpha\}_{\alpha \in A}$ of X (i.e., $X = \bigcup_{\alpha \in A} U_\alpha$), there exists a finite subcover. That is, there exists a finite subcollection $\{U_1, ..., U_n\} \subset \{U_\alpha\}_{\alpha \in A}$ such that $X = U_1 \cup \cdots \cup U_n$.

In $\mathbb{R}^n$, we know exactly what compact sets look like:

Theorem 0.11 (Heine-Borel)**.** *A set $K \subset \mathbb{R}^n$ is compact iff it is closed and bounded, i.e., $K \subset B(\mathbf{0}, r)$ for some sufficiently large $r > 0$.*

Example 0.12. Here are some examples of compact spaces:

- By Heine-Borel theorem, any closed interval $[a, b] \subset \mathbb{R}$ is compact. So is finite union of them.

- Let $X = \{p_1, ..., p_n\}$ be a finite set. Let $\mathcal{T}$ be the collection of all subsets of X. Then with respect to this topology, X is compact.

- Let X be a compact space and $f : X \to Y$ a continuous map between topological spaces. Then $f(X)$ is a compact subspace of Y (subspace topology).

  Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of $f(X)$. Then

  $$X = f^{-1}(f(X)) = f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha).$$

  Since f is continous, $f^{-1}(U_\alpha)$ is open for each α. Since X is compact, there exists a finite collection such that

  $$X = f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n).$$

  Therefore $f(X) = U_1 \cup \cdots \cup U_n$. □

- A curve in a topological space X is the image of a continuous map $\gamma : [0, 1] \to X$. By the last example, a curve is compact in X.

- A closed subspace K of a compact space X is compact.

Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of K. Then $\{U_\alpha\}_\alpha \cup \{X \setminus K\}$ is an open cover of X. Since X is compact, we have can choose a finite subcollection

$$X = (X \setminus K) \cup U_1 \cup \cdots \cup U_n.$$

Then $U_1 \cup \cdots \cup U_n = K$. This is a finite subcollection of $\{U_\alpha\}_\alpha$ that covers K. $\square$

- The most important example is still compact subsets in $\mathbb{R}^n$, which is equivalent to closed and bounded subsets in $\mathbb{R}^n$ by the HeineBorel theorem. This is probably one of the only few examples you need to remember as manifolds are locally Euclidean.

As compactness is some sort of boundedness on a space, functions on that space are also bounded as a result:

Theorem 0.13. *Let $f : X \to \mathbb{R}$ be a continuous real-valued function. Then f is bounded. Moreover, f obtains its maximal and minimal values.*

Proof. Apply Heinel-Borel: Since $f(X)$ is compact in $\mathbb{R}$, clearly f is bounded. Since $f(X)$ is bounded, $\inf_{x\in X} f(x)$ and $\sup_{x\in X} f(x)$ exists. Since $f(X) \subset \mathbb{R}$ is closed, the inf and sup are contained in $f(X)$. Suppose $f(x_0) = \inf_{x\in X} f(X)$ and $f(x_1) = \sup_{x\in X} f(x)$. Then f obtains minimum at x_0 and maximum at x_1. $\square$

0.3 Product Topology, Connectedness

Definition 0.14. Let $(X, \mathcal{O})$ be a topological space. A ***base*** (or ***basis***) for the topology $\mathcal{O}$ is a subcollection $\mathcal{B} \subset \mathcal{O}$ of open sets in X, such that every open sets U in X is a union of open sets in $\mathcal{B}$.

Definition 0.15. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces. Let $X \times Y$ be the set theoretic product of X and Y. We define a topology $\mathcal{O}_{X\times Y}$ on $X \times Y$ as follows: The basis $\mathcal{B}$ of $\mathcal{O}_{X\times Y}$ consists of all open sets of the form

$$\mathcal{B} = \{U \times V : U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}.$$

Example 0.16. $\mathbb{R}^n$ is the set theoretic product of n copies of $\mathbb{R}$. Convince yourself that the basis topology of $\mathbb{R}^n$ coincides with the standard topology on $\mathbb{R}^n$. It is not hard to show that any open sets U in $\mathbb{R}^n$ (with respect to the standard topology) can be written as a union of open boxes, and every open boxes in $\mathbb{R}^n$ can be written as a union of open balls.

Definition 0.17. A topological space X is said to be ***path-connected*** if for every two points p, q, there exists a continuous curve $\gamma : [0, 1] \to X$, such that $\gamma(0) = p, \gamma(1) = q$.

Definition 0.18. A topological space X is said to be ***disconnected***, if there exists *nonempty disjoint open sets* U, V such that $U \cup V = X$. Otherwise, X is said to be ***connected***. Note that the definition of X being connected is equivalent to saying that the only closed and open subset in X is either $\emptyset$ or X.

Lemma 0.19. *Let X be a connected topological space, and $f : X \to Y$ is a continuous map between topological spaces that is locally constant, i.e., for every $p \in X$, there exists an open set $U_p \subset X$ such that f is constant on U_p. Then f is globally constant.*

Proof. Let $p \in X$ be any point and let $s = f(p) \in Y$. Then consider the set

$$S := \{q \in X : f(q) = s\}.$$

Since $S = f^{-1}(q)$ and f is continuous, we see that S is closed. On the other hand, for every $q \in S$, by locally constant hypothesis, there exists an open set $U_q \subset X$ such that $f(q') = s$ for all $q' \in U_q$. That is, f is constant on U_q with value s. Thus $S = \bigcup_{q \in S} U_q$. Hence S is open. Since S is closed and open at the same time, and by assumption $p \in S$, so $S \neq \emptyset$. Then since X is connected, we conclude that $S = X$, i.e., f is globally constant. $\square$

Theorem 0.20. *The interval* $[0, 1]$ *with subspace topology of* $\mathbb{R}$*, is connected. Similarly, any interval* $[a, b]$ *is connected.*

Proof. Suppose $[0, 1]$ is disconnected and U, V be two disjoint nonempty open subsets such that $U \cup V = [0, 1]$. Define a function $f : [0, 1] \to \{\pm 1\}$ via

$$f(x) = \begin{cases} +1 & x \in U \\ -1 & x \in V. \end{cases}$$

Then f is well-defined, since U, V are disjoint, and is nonconstant, since U, V are both nonempty. Note that f is continuous since preimages of open sets in $\{\pm 1\}$ can only be one of $\emptyset, U, V, [0, 1]$, which are both open. But f is clearly locally constant by construction. This contradicts Lemma 0.19. $\square$

Theorem 0.21. *A path connected space is connected. (The converse fails, but in the case of smooth manifolds, it is true)*

Proof. Suppose X is path connected but disconnected. Then we can find nonempty open sets U, V such that $U \cup V = X$. Since both are nonempty, take $p \in U, q \in V$. Since X is path connected, we may find a path $\gamma : [0, 1] \to X$ with $\gamma(0) = p, \gamma(1) = q$. Since U, V are disjoint, $\gamma^{-1}(U), \gamma^{-1}(V)$ are disjoint open sets that covers $[0, 1]$. But $[0, 1]$ is connected, contradiction. $\square$

Chapter 1

Manifolds

1.1 Topological Manifolds

Definition 1.1. A topological space $(M, \mathcal{O})$ is called a ***topological manifold of dimension*** n, if it is:

- ***Hausdorff:*** For every pairs of distinct points $p, q \in M$, there are disjoint open sets $U, V \subset M$ such that $p \in U$ and $q \in V$.
- ***Second-countable:*** There exists a countable basis for the topology of M.
- ***Locally Euclidean:*** For every $p \in M$, there exists an open set U containing p, and a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ from U onto an open subset of $\mathbb{R}^n$.

Often we simply write M^n, to denote a manifold M of dimension n.

Example 1.2. An open disk $S^1 \subset \mathbb{R}^2$ is a 1-dimensional topological manifold. For each $p \in U \subset S^1$, we can project it smoothly onto $\mathbb{R}^1$ by letting φ be the projection map $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

Similarly, a sphere $S^2 \subset \mathbb{R}^3$ is a 2-dimensional topological manifold. For each $p \in S^2$, rotate the hemisphere it is on to the top and make it the upper hemisphere. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map onto $\mathbb{R}^2$ plane, provided that U is contained completely in the hemisphere. Or, just define the open set containing p be the hemisphere it is on without the equator. Then this is clearly an open set containing p. Its projection onto the $\mathbb{R}^2$ plane is just a disk without boundary, which is also an open set.

This generalizes to $S^n \subset \mathbb{R}^{n+1}$.

Definition 1.3. An open subset $U \subset M$ and a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$, together, (U, φ) is called a ***chart*** of $(M, \mathcal{O})$.

Definition 1.4. A set $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of charts is called an ***atlas*** of $(M, \mathcal{O})$ if $M = \bigcup_{\alpha \in A} U_\alpha$. That is, if the union of elements in $\mathcal{A}$ covers M. The maps φ_α are called chart maps.

Consider two overlapping charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ on a manifold M. By overlapping charts we mean $U_\alpha \cap U_\beta \neq \emptyset$. Then one can define a map between subsets of $\mathbb{R}^n$

$$\tau_{\alpha,\beta} \equiv \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$

This is called a ***chart transition map***, for the obvious reason that it serves as a transition map that transform one set of coordinates of points in $U_\alpha \cap U_\beta$ to another set. Note that $\tau_{\alpha,\beta}$ is continuous. See Figure 1.1 for illustration.

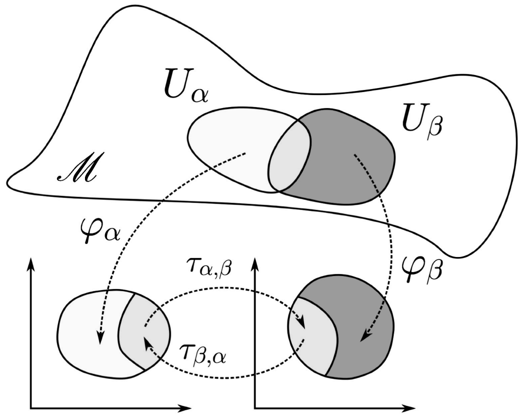


Figure 1.1: Chart transition maps $\tau_{\alpha,\beta}, \tau_{\beta,\alpha}$.

Often, there exists a curve in real world M, defined by $\gamma \subset U \subset M$. As an example, for a physicist, γ can be a particle trajectory changing with respect to time, i.e. $\gamma : \mathbb{R} \to U$. But since there exists a map $\varphi : U \to \mathbb{R}^n$, the composition $\varphi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$ is the trajectory of particle in Euclidean space that we know very well.

But there is one potential danger: the chart φ is arbitrary. This depends on what coordinate system a physicist chooses to use (Cartesian, polar, etc). But γ is independent of the chart we choose. Since physics should not depend on the choice of coordinate, we want to make sure that the defined property in one chart, φ, does not change when we switch to another chart, say ψ. Specifically, we have a commutative diagram

$$\begin{tikzcd}
 & \psi(U) \\
\mathbb{R} \arrow[ur, "\psi\circ\gamma"] \arrow[r, "\gamma"] \arrow[dr, "\varphi\circ\gamma"'] & U \arrow[u, "\psi"'] \arrow[d, "\varphi"] \\
 & \varphi(U)
\end{tikzcd}$$

For example, say if we have $\varphi \circ \gamma$ be a continuous map, then we better hope $\psi \circ \gamma$ to be also continuous. Now,

$$\psi \circ \gamma = \psi \circ (\varphi^{-1} \circ \varphi) \circ \gamma = (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma).$$

Since $\varphi \circ \gamma$ is continuous, and $\psi \circ \varphi^{-1}$ is continuous (since both ψ and φ^{-1} are continuous), we have $\psi \circ \gamma$ be a continuous map.

However, if $\varphi \circ \gamma$ is differentiable, it need not be true that $\psi \circ \gamma$ is also differentiable. This is due to the fact that $\psi \circ \varphi^{-1}$ only needs to be continuous. The composition of a continuous map and a differentiable map is not necessarily differentiable. Thus we need stronger condition for differentiability on manifolds, which we will come back in the next section.

1.2 Smooth Manifolds and Smooth Maps

In the previous section, we tried to appeal to intuition that physicists use by viewing a manifold as the world we live in, and charts as coordinate systems in our maps. We have seen that for a continuous curve $\gamma : \mathbb{R} \to M$, what physicists have in their map $\varphi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$ is also continuous. But we cannot ensure that the differentiability of the curve that physicists describe is independent of the chart we choose without adding further assumptions. In this section we try to solve this issue.

Let $\mathcal{A}_\infty$ be the maximum atlas, the set of all possible atlas we can find, where each atlas covers our manifold M. Let our desired property be $\heartsuit$. Then define two maps to be $\heartsuit$-compatible as follows:

Definition 1.5. Let $(U, \varphi), (V, \psi) \in \mathcal{A} \subset \mathcal{A}_\infty$ be two charts of $(M, \mathcal{O})$. We say these two charts are ***$\heartsuit$-compatible*** if either $U \cap V = \emptyset$, or the chart translation maps $\psi \circ \varphi^{-1}$ and $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$ have property $\heartsuit$.

For example, if the two chart translation maps are at least once differentiable (C^1), then we say the two charts $(U, \varphi), (V, \psi)$ are C^1-compatible.

Definition 1.6. An atlas $\mathcal{A}_\heartsuit$ is ***$\heartsuit$-compatible*** if any two charts $(U, \varphi), (V, \psi) \in \mathcal{A}_\heartsuit$ are $\heartsuit$-compatible.

Definition 1.7. We say that $\mathcal{A}_\heartsuit$ is a ***maximal $\heartsuit$ atlas***, if it is not properly contained in any larger $\heartsuit$-compatible atlas. In other words, $\mathcal{A}_\heartsuit$ is maximal iff any chart that is $\heartsuit$-compatible with every chart in $\mathcal{A}_\heartsuit$ is already in $\mathcal{A}_\heartsuit$.

Definition 1.8. A ***$\heartsuit$-manifold*** is a triple $(M, \mathcal{O}, \mathcal{A}_\heartsuit)$, where $\mathcal{A}_\heartsuit$ is a maximal atlas. The atlas $\mathcal{A}_\heartsuit$ is then called a $\heartsuit$ ***structure.***

Now, we can solve the problem mentioned above. Let $\heartsuit$ be the property that the maps are infinitely differentiable, or C^∞. Then we restrict our choice of charts by only choosing charts in some $\mathcal{A}_{C^\infty}$. Then the chart transition map $\psi \circ \varphi^{-1}$ is differentiable. Thus $\psi \circ \gamma = (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma)$ is differentiable given that $\varphi \circ \gamma$ is differentiable.

For the rest of this notes, we will always focus on smooth manifolds $(M, \mathcal{O}, \mathcal{A}_{C^\infty})$. Smooth manifolds are sometimes also called C^∞-manifolds. Usually, we will just write M, assuming the information $\mathcal{O}, \mathcal{A}_{C^\infty}$ is clear. Usually we just refer to smooth manifolds as manifolds to be concise.

It is generally not easy to construct a maximal smooth atlas on a topological manifold. However, the next theorem saves us and suggests that constructing a smoothly compatible atlas is sufficient to determine a smooth structure:

Theorem 1.9. *Every smooth atlas $\mathcal{A}$ is for a topological manifold M is contained in a unique maximal smooth atlas $\mathcal{A}^*$.*

Proof. Let $\mathcal{A}$ be a smooth atlas for M, and let $\mathcal{A}^*$ be the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. We need to show:

- $\mathcal{A}^*$ is a smooth atlas: Let $(U, \varphi), (V, \psi) \in \mathcal{A}^*$ be intersecting charts, we would like to show $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth. Let $x = \varphi(p) \in \varphi(U \cap V)$ be any point. Because open sets in $\mathcal{A}$ covers M, there exists $(W, \theta) \in \mathcal{A}$ with $p \in W$. Since charts in $\mathcal{A}^*$ are smoothly compatible with every chart in $\mathcal{A}$, we know $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$$

  is smooth in a neighborhood of x. This proves the claim.

- $\mathcal{A}^*$ is maximal: Any chart that is smoothly compatible with every chart in $\mathcal{A}^*$ must be in particular smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\mathcal{A}^*$.

- $\mathcal{A}^*$ is the unique maximal smooth atlas containing $\mathcal{A}$: If $\mathcal{B}$ is another maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subset \mathcal{A}^*$. By maximality of $\mathcal{B}$, we have $\mathcal{B} = \mathcal{A}^*$.

This completes the proof. $\square$

Now we define a smooth map between (smooth) manifolds.

Definition 1.10. A map $f : M^m \to N^n$ between manifolds is said to be ***smooth***, if for every $p \in M$, there exists a chart (U, φ) of M containing p, a chart (V, ψ) of N containing $f(U)$, such that the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is a smooth map (i.e., all partial derivative exists) from an open subset $\varphi(U)$ of $\mathbb{R}^m$ to an open subset $\psi(V)$ of $\mathbb{R}^n$.

Definition 1.11. A smooth map $f : M \to N$ with a smooth inverse $f^{-1} : N \to M$ is called a ***diffeomorphism***.

Note that one can view a chart $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ as a map between manifolds: Here U is given the subspace topology and charts on U are given by charts on M restricted to open subsets of U; we may view $\mathbb{R}^n$ as a manifold with chart just being identity; then $\varphi(U)$ is also a manifold in the same way that U is a manifold. Thus, $\varphi : U \to \varphi(U)$ is a diffeomorphism, since the induced maps are just identity maps between open subsets of Euclidean spaces, which is clearly smooth.

Finally, we remark that the notion of smoothness of $f : M \to N$ is independent of choices of parameterizations φ, ψ in the chart $(U, \varphi), (V, \psi)$ if the open sets U, V are the same. Suppose $(U, \tilde{\varphi})$ and $(V, \tilde{\psi})$ are some other choice of charts, then from the commutative diagram below

$$\begin{tikzcd}
\tilde{\varphi}(U) \arrow[r, "\tilde{\psi}\circ f\circ\tilde{\varphi}^{-1}"] & \tilde{\psi}(V) \\
U \arrow[u, "\tilde{\varphi}"'] \arrow[d, "\varphi"] \arrow[r, "f"] & V \arrow[u, "\tilde{\psi}"] \arrow[d, "\psi"'] \\
\varphi(U) \arrow[uu, bend left=60, "\tilde{\varphi}\circ\varphi^{-1}"] \arrow[r, "\psi\circ f\circ\varphi^{-1}"'] & \psi(V) \arrow[uu, bend right=60, "\tilde{\psi}\circ\psi^{-1}"']
\end{tikzcd}$$

we see that

$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\tilde{\varphi} \circ \varphi^{-1})^{-1}$$

is a composition of smooth maps in Euclidean spaces, hence is also smooth.

1.3 Examples of Smooth Manifolds

In this section, we study some examples of smooth manifolds. Our strategy to show something is a smooth manifold can be summarized as follows: First, we start with a topological space M and check that it is a topological manifold (by constructing a chart on M); Then, we show that the charts are smoothly compatible; Finally, we use Theorem 1.9 and show that such a smooth atlas defines a smooth structure on M, showing that M is indeed a smooth manifold.

Example 1.12 (0-dimensional Manifolds)**.** A zero dimensional topological manifold M is just a discrete sets of points. For each point $p \in M$, the only neighborhood of p homeomorphic to an open set of $\mathbb{R}^0$ is $\{p\}$. There is exactly one coordinate map $\varphi : \{p\} \to \mathbb{R}^0$. Thus the set of all charts on M is trivially smoothly compatible. Hence M is a smooth manifold.

Example 1.13 ($\mathbb{R}^n$)**.** $\mathbb{R}^n$ is an n-dimensional smooth manifold with the smooth atlas consisting of a single chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$. This is called the ***standard smooth structure on*** $\mathbb{R}^n$.

Example 1.14 (S^n)**.** For each $n \geq 0$, the n-sphere defined by

$$S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

is an n-dimensional smooth manifold: We construct an atlas on S^n as follows:

$$\mathcal{A}_{S^n} = \{(U_i^{\pm}, \varphi_i^{\pm}) : i = 1, ..., n+1\}$$

where

$$U_i^+ = \{(x^1, ..., x^{n+1}) \in S^n : x^i > 0\} \qquad U_i^- = \{(x^1, ..., x^{n+1}) \in S^n : x^i < 0\}$$

and the homeomorphisms $\varphi_i^{\pm} : U_i^{\pm} \to B(0,1) \subset \mathbb{R}^n$ is given by

$$\varphi_i^{\pm}(x^1, ..., x^{n+1}) = (x^1, ..., \hat{x^i}, ..., x^{n+1}).$$

It is easy to check that the defined maps are indeed homeomorphisms, with inverse

$$(\varphi_i^{\pm})^{-1}(a^1, ..., a^n) = (a^1, ..., \pm\sqrt{1 - |a|^2}, a^i, ..., a^n).$$

Hence we have a chart on S^n, making it a topological manifold. Now to check that $\mathcal{A}_{S^n}$ defines a smooth structure, we need to check that the transition maps are smooth. In the case where $i < j$, we have

$$\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1}(u^1, ..., u^n) = (u^1, ..., \hat{u^i}, ..., \pm\sqrt{1 - |u|^2}, ..., u)$$

which is clearly smooth on $B(0,1)$. A similar formula holds for $i > j$. When $i = j$, the transition map is just identity on $B(0,1)$. Hence charts in $\mathcal{A}_{S^n}$ are smoothly compatible, hence defining a smooth structure on S^n. Thus S^n is a smooth manifold of dimension n.

Example 1.15 (Vector Spaces)**.** Let V^n be an n-dimensional real vector space. We claim that V is an n-dimensional smooth manifold. Let $e_1, ..., e_n$ be a basis of V. We first show that

Lemma 1.16. *Any two norms on a finite-dimensional real vector space is equivalent, so in particular they all give the same topology to V.*

Proof. Let $\|\cdot\|$ be the norm on V given by $\|\sum_i a_i e_i\| = (\sum_i |a_i|^2)^{1/2}$. Let $F : V \to \mathbb{R}_{\geq 0}$ be any norm. It suffices to prove that F is equivalent to $\|\cdot\|$. Define $\mu := \max_{1 \leq i \leq n} F(e_i) < \infty$. Then by Cauchy-Schwarz inequality one has

$$F(x) = F(\sum_i a_i e_i) \leq \sum_i |a_i| F(e_i) \leq \mu \sum_i |a_i| \leq \mu\sqrt{n}\|x\|. \tag{1.1}$$

Since by the reverse triangle inequality one has $F(x-y) \geq F(x) - F(y)$ and similarly $F(x-y) = F(y-x) \geq F(y) - F(x)$, one has

$$|F(x) - F(y)| \leq |F(x-y)| \leq \mu\sqrt{n}\|x-y\|.$$

Hence F is continuous on $(V, \|\cdot\|)$. Then F attains a minimum on the compact set

$$K = \{x \in V : \|x\| = 1\}$$

by Theorem 0.13. Say the minimum is at $p \in K$. Note that since $p \neq 0 \in V$, one has $F(p) > 0$. Thus

$$F(x) = \|x\|F(x/\|x\|) \geq F(p) \cdot \|x\|. \tag{1.2}$$

Combining (1.1) and (1.2), we conclude that F and $\|\cdot\|$ are equivalent. □

Now we consider V to be a topological space with topology given by any norm. Then we get a homeomorphism $\varphi : \mathbb{R}^n \to V$ by

$$\varphi(x^1, ..., x^n) = x^1 e_1 + \cdots + x^n e_n.$$

Hence we see that (V, φ^{-1}) is a chart. That is, the coordinates on V are given by components of a vector with respect to the basis e_α's. If $e'_1, ..., e'_n$ is another basis for V and $\varphi' : \mathbb{R}^n \to V$ is another chart that is defined by the same recipe but with respect to the basis $e'_1, ..., e'_n$, then there is an invertible matrix $T = (T^\alpha_\beta)$ such that

$$e_\beta = \sum_\beta T^\alpha_\beta e'_\alpha.$$

Thus we see that the chart transition map $\varphi' \circ \varphi$ is given by the linear map $\mathbb{R}^n \to \mathbb{R}^n$ which is exactly (T^α_β). Since this is an invertible linear map, it is a diffeomorphism from $\mathbb{R}^n$ to itself. Thus φ, φ' are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the ***standard smooth structure on V***.

Example 1.17 (Open subsets of a manifold)**.** Let U be an open subset of a smooth manifold M. Then a smooth atlas is defined by

$$\mathcal{A}_U = \{(V, \varphi) \in \mathcal{A}_M : V \subset U\}.$$

This clearly is a smoothly compatible atlas and gives U a smooth structure. Thus, we call any open subset of a smnooth manifold M an ***open submanifold of M.***

Example 1.18 (Matrix groups)**.** Let $M(n, \mathbb{R})$ denote the space of $n \times n$ matrices with real coefficients. This is a real vector space of dimension n^2, hence a smooth manifold of the same dimension by our example above. Let $GL(n, \mathbb{R})$ denote the set of invertible $n \times n$ matrices with real entries. Then $GL(n, \mathbb{R})$ is an open subset of $M(n, \mathbb{R})$ because $GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R}^*)$ where $\det : M(n, \mathbb{R}) \to \mathbb{R}$ is the determinant function (which is continuous because it only involves polynomial operations). Hence $GL(n, \mathbb{R})$ is a smooth manifold of dimension n^2 as well.

Example 1.19 (Product Manifolds)**.** Let M, N be two smooth manifolds. Then $M \times N$ is a topological space with the product topology. We can define a smoothly compatible atlas on $M \times N$ via

$$\mathcal{A}_{M \times N} = \{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}.$$

Hence $M \times N$ is again a smooth manifold of dimension $\dim M + \dim N$.

Example 1.20 ($\mathbb{RP}^n$)**.** We define the ***real projective space*** of dimension n, as a topological space, to be the quotient of $\mathbb{R}^{n+1}$ (or equivalently S^n) under the action of $x \sim \lambda x$ for any $\lambda \in \mathbb{R}, x \in \mathbb{R}^{n+1}$, with quotient topology. Therefore, we have made $\mathbb{RP}^n$ into a topological space. You will show that $\mathbb{RP}^n$ is a smooth manifold of dimension n in one of the exercises at the end of this chapter.

Example 1.21 (Graph of a function)**.** The ***graph*** $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}\}$ of any continuous function $f : \mathbb{R} \to \mathbb{R}$ is a topological manifold, since it is homeomorphic to $\mathbb{R}$. Moreover, any such graph has a smooth structure since $\mathbb{R}$ is a smooth manifold. However, graphs of continuous non-smooth functions are, in general, not smooth submanifolds of $\mathbb{R}^2$ (See one of the exercises at the end of Chapter 2).

1.4 Partition of Unity and Applications

Partition of unity is a tool for us to patch smooth functions defined locally to a global smooth function. This is probably one of the most widely used tools in the theory of smooth manifolds. Despite its importance, proof of the existence of partition of unity is very hard and technical. Readers who do not want to get into such technical intricacies can just learn the definition of a partition of unity, believe in its existence, and move on.

Definition 1.22. Let M be a smooth manifold, and $f : M \to \mathbb{R}$ a smooth function. The ***support*** of f, denoted by $\operatorname{Supp} f$, is the ***closure*** of the set $\{p \in M : f(p) \neq 0\}$.

Definition 1.23. Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. A ***partition of unity subordinate to the cover*** $\{U_\alpha\}$ is a collection of smooth functions $\{\rho_\alpha\}$ defined on the whole manifold M so that

1. $0 \leq \rho_\alpha \leq 1$ for all α;
2. $\operatorname{Supp}(\rho_\alpha) \subset U_\alpha$ for all α;
3. Each $p \in M$ has a neighborhood which intersects only finitely many $\operatorname{Supp}(\rho_\alpha)$'s;
4. $\sum_\alpha \rho_\alpha(p) = 1$ for all $p \in M$. By the previous condition, at every point this is only a finite sum, hence is well-defined.

Our goal is to prove that for any open cover $\mathfrak{U}$ of M, there exists a partition of unity subordinate to $\mathfrak{U}$. However, the proof requires several very technical lemmas in advance. Since the existence of partition of unity subordinate to any open cover of a manifold is such an important result, we will present the full proof for future references. However, this is a good point for impatient readers to stop for now, and come back to it later when needed. We now begin the technical preparation required for the proof.

Construction 1.24. Consider the function $f_1 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

It is a standard exercise in calculus to check that f_1 is smooth (check all derivative exists by L'Hospital rule). One sees that $0 \leq f_1(x) \leq 1$ when $x > 0$ and $f_1(x) = 0$ when $x \leq 0$. See Figure 1.2a.

Next, we consider $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

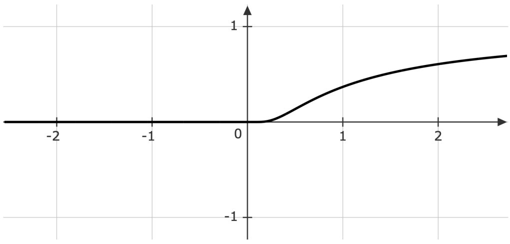
$$f_2(x) = \frac{f_1(x)}{f_1(x) + f_1(1-x)}.$$

Then one sees that f_2 is a nonnegative smooth function, such that $f_2(x) = 0$ when $x \leq 0$, $0 < f_2(x) < 1$ when $0 < x < 1$, and $f_2(x) = 1$ when $x \geq 1$. See Figure 1.2b.

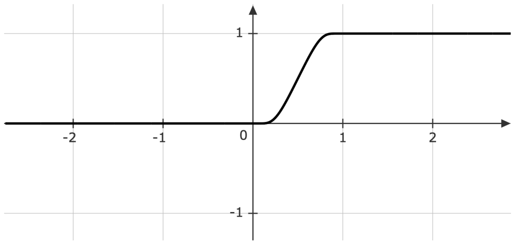
Finally, consider the function $f_3 : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_3(x) = f_2(2 - |x|).$$

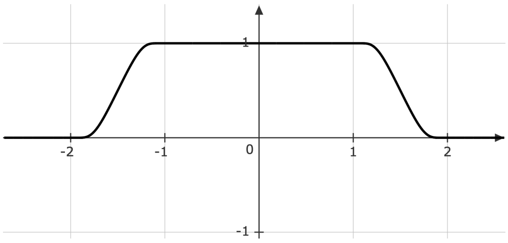
Then one sees that f_3 is a bump nonnegative function that is 1 on $B(0,1)$, and decays smoothly until it vanishes outside of $B(0,2)$. See Figure 1.2c.



(a) Plot of f_1.



(b) Plot of f_2.



(c) Plot of f_3

Figure 1.2: The plot of the three functions defined in Construction 1.24.

Lemma 1.25. *Let M^m be a smooth manifold, $A \subset M$ a compact subset, and $U \subset M$ an open subset that contains A. Then there is a bump function $\phi \in C^\infty(M)$ such that $0 \leq \phi \leq 1$, $\phi|_A \equiv 1$, and $\mathrm{Supp}(\phi) \subset U$.*

Proof. For each $q \in A$, one can find[1] a chart (U_q, φ_q) near q such that $U_q \subset U$ and $B(0,3) \subset \varphi_q(U_q)$. Let $U_q' = \varphi_q^{-1}(B(0,1))$, and let

$$f_q(p) = \begin{cases} f_3(\varphi_q(p)) & p \in U_q \\ 0 & p \notin U_q. \end{cases}$$

Here, f_3 is defined in Construction 1.24. Then clearly $f_q \in C^\infty(M), \mathrm{Supp}(f_q) \subset U_q$, and $f|_{U_q'} \equiv 1$. Now the family $\{U_q'\}_{q \in A}$ is an open cover of A. Since A is compact, we can choose

[1]We can find such a chart because we may take any chart (U, φ) containing q, and a smooth dialation $\lambda : \mathbb{R}^m \to \mathbb{R}^m$ that scales $\varphi(U)$ so that it contains $B(0,3)$. Then denote $U_q = U, \varphi_q = \lambda \circ \varphi$. Clearly (U_q, φ_q) is smoothly compatible with any chart in the atlas of M, hence already contained in the atlas because a smooth structure is maximal by definition.

a finite subcover $U'_{q_1}, ..., U'_{q_n}$. Let $\psi := \sum_{i=1}^n f_{q_i}$. Then one can see that $\psi \in C^\infty(M)$ such that $\psi \geq 1$ on A and $\mathrm{Supp}(\psi) \subset U$. It follows that

$$\phi(p) := f_2(\psi(p))$$

satisfies all the conditions we want, where f_2 is defined in Construction 1.24. □

Lemma 1.26. *For any topological manifold M, there exists a countable collection of open sets $\{X_i\}$, which is called an **exhaustion of** M, so that*

1. *For each j, the closure $\overline{X}_j$ is compact;*
2. *For each j, we have $\overline{X}_j \subset X_{j+1}$;*
3. *$M = \bigcup_j X_j$.*

Proof. Since M is second countable, there is a countable basis $\mathcal{B}$ of the topology of M. Out of this countable collection of open sets, we pick those that have compact closures, and denote them by $Y = \{Y_1, Y_2, ...\}$.

We claim that Y is in fact an open cover of M. To see this, let $p \in M$ be an arbitrary point. We can take a chart (U, φ) such that $p \in U$, and choose $\epsilon > 0$ small enough, and denote $B := B(\varphi(p), \epsilon)$, such that

$$\varphi(p) \in \overline{B} \subset \varphi(U)$$

and an element S in the basis $\mathcal{B}$ such that $p \in S \subset \varphi^{-1}(B)$. Then we see that

$$\overline{S} \subset \overline{\varphi^{-1}(B)} = \varphi^{-1}(\overline{B})$$

is a closed subset of a compact set, hence is compact. Therefore $S \in Y$. Hence Y is indeed an open cover of M.

Now we let $X_1 = Y_1$. Since Y is an open cover of $\overline{X}_1$ which is compact, there exists finitely many open sets $Y_{i_1}, ..., Y_{i_k}$ such that

$$\overline{X}_1 \subset Y_{i_1} \cup \cdots \cup Y_{i_k}.$$

Now define

$$X_2 = Y_2 \cup Y_{i_1} \cup \cdots \cup Y_{i_k}.$$

Clearly, $\overline{X}_2$ is compact and $\overline{X}_1 \subset X_2$. By repeating this procedure we get a sequence of open sets $X_1, X_2, ...$ satisfying the requirements in the statement. Note that since $X_k \supset \bigcup_{j=1}^k Y_j$, we have that $M = \bigcup_k X_k$. □

Lemma 1.27. *For any open covering $\mathcal{U} = \{U_\alpha\}$ of a topological manifold M, one can find two countable family of open covers $\mathcal{V} = \{V_j\}$ and $\mathcal{W} = \{W_j\}$ such that*

(i) *For each j, $\overline{V}_j$ is compact and $\overline{V}_j \subset W_j$;*

(ii) *$\mathcal{W}$ is a refinement of $\mathcal{U}$: For each j, there is an $\alpha = \alpha(j)$ so that $W_j \subset U_\alpha$;*

(iii) *$\mathcal{W}$ is locally finite: Any $p \in M$ has a neighborhood W such that $W \cap W_j \neq \emptyset$ for only finitely many W_j's.*

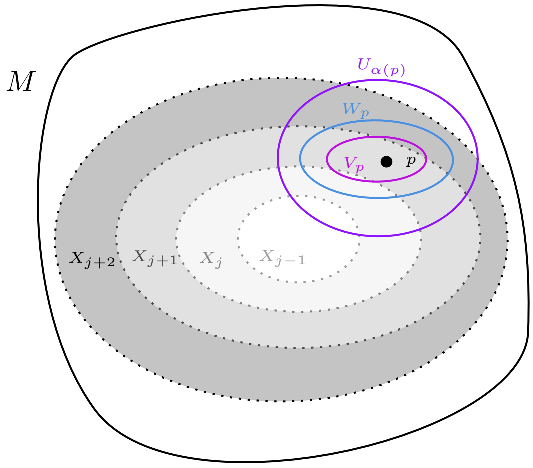


Figure 1.3: Since $U_{\alpha(p)} \cap (X_{j+2} - \overline{X}_{j-1})$ is the intersection of two ooen sets containing p, it is open. Intersecting this open set with a chart near p and map it down to $\mathbb{R}^m$ under the homeomorphism φ, we get an open set $U' \subset \mathbb{R}^m$ under φ containing $\varphi(p)$. Then choose $V', W' \subset U'$ such that $\varphi(p) \in \overline{V'} \subset W'$. Then set $V_p = \varphi^{-1}(V')$ and $W_p = \varphi^{-1}(W')$.

Proof. The proof is in fact very geometric and ingenious: Let $\{X_j\}$ be a countable exhaustion of M as in Lemma 1.26. For each $p \in M$, there is an j and an $\alpha(p)$ so that $p \in \overline{X}_{j+1} - X_j$ and $p \in U_{\alpha(p)}$. Since M is locally Euclidean, one can choose open neighborhoods V_p, W_p of p such that $\overline{V}_p$ is compact and (See Figure 1.3):

$$\overline{V}_p \subset W_p \subset U_{\alpha(p)} \cap (X_{j+2} - \overline{X}_{j-1}).$$

Now for each j, since the strip $\overline{X}_{j+1} - X_j$ is compact, one can choose finitely many points $p_{1,j}, ..., p_{k_j,j}$ so that $V_{p_{1,j}}, ..., V_{p_{k_j,j}}$ is an open cover of $\overline{X}_{j+1} - X_j$. Denote all these $V_{p_{k,j}}$'s by $V_1, V_2, V_3, ...$, and the corresponding $W_{p_{k,j}}$'s by $W_1, W_2, W_3, ...$ Then $\mathcal{V} = \{V_j\}$ and $\mathcal{W} = \{W_j\}$ clearly satisfies (i) and (ii) as in the statement of the lemma. For (iii), the local finiteness property of $\mathcal{W}$ follows from the fact that there are only finitely many W_k's intersecting $X_{j+1} - \overline{X}_{j-1}$. □

Lemma 1.28. *Closure of unions of locally finite sets in a topological space is the union of closure of these sets. In other words, let $\{S_i : i \in I\}$ be a collection of locally finite sets on M, then we have $\overline{\bigcup_{i\in I} S_i} = \bigcup_{i\in I} \overline{S}_i$.*

Proof. Since $S_i \subset \bigcup_{i\in I} S_i$, we have $\overline{S}_i \subset \overline{\bigcup_{i\in I} S_i}$, hence $\bigcup_{i\in I} \overline{S}_i \subset \overline{\bigcup_{i\in I} S_i}$. So the nontrivial part is the inclusion in the other direction.

To prove the other inclusion, by the definition of closure it suffices to prove that $\bigcup_{i\in I} \overline{S}_i$ is closed. To this end, we show its complement is open: Take $x \notin \bigcup_{i\in I} \overline{S}_i$. Since we have a locally finite collection, we may find an open neighborhood U of x that only intersects finitely many S_i's. Say U intersects $S_{i_1}, ..., S_{i_n}$. Now consider the new open set

$$U' := U \setminus (\overline{S}_{i_1} \cup \cdots \cup \overline{S}_{i_n}).$$

Note that U' is still an open set containing x, and it does not intersect any of the S_i's. We claim that this implies U' does not intersect any of the $\overline{S_i}$'s either: Indeed, since the complement $(U')^c$ of U' is a closed set containing all the S_i's, by minimality of the closure, this means it also contains all the $\overline{S}_i$'s. Hence $x \in U' \subset (\bigcup_{i \in I} \overline{S}_i)^c$. □

Now we are finally ready to prove the big theorem:

Theorem 1.29. *Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. Then there exists a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$.*

Proof. We choose $\{V_j\}, \{W_j\}$ as in Lemma 1.27. Since $\overline{V}_j$ is compact and $\overline{V_j} \subset W_j$, by Lemma 1.25 we can find nonnegative functions $\phi_j \in C^\infty(M)$ such that

$$0 \leq \phi_j \leq 1 \qquad \phi_j|_{\overline{V}_j} \equiv 1 \qquad \mathrm{Supp}(\phi_j) \subset W_j.$$

Since $\mathcal{W}$ is a locally finite covering, the function $\phi = \sum_j \phi_j$ is a well-defined smooth function on M. Since each ϕ_j is nonnegative, and $\mathcal{V}$ is a covering of M, ϕ is strictly positive on M. It follows that the functions $\psi_j = \phi_j/\phi$ are smooth and satisfy $0 \leq \psi_j \leq 1$ and $\sum_j \psi_j = 1$.

Next, we need to re-index the family $\{\psi_j\}$ to get the required partition of unity subordinate to $\{U_\alpha\}$. For each j, we fix an index $\alpha(j)$ so that $W_j \subset U_{\alpha(j)}$, and define

$$\rho_\alpha = \sum_{\{j:\alpha(j)=\alpha\}} \psi_j.$$

Note that the right hand side is a finite sum near each point, so it does define a smooth function. Moreover, By Lemma 1.28, we have

$$\mathrm{Supp}(\rho_\alpha) = \overline{\bigcup_{\{j:\alpha(j)=\alpha\}} \mathrm{Supp}(\psi_j)} = \bigcup_{\{j:\alpha(j)=\alpha\}} \overline{\mathrm{Supp}(\psi_j)} = \bigcup_{\{j:\alpha(j)=\alpha\}} \mathrm{Supp}(\psi_j) \subset U_\alpha.$$

This concludes the construction. □

Partition of unity has countless applications in the theory of smooth manifolds. Here is one example: we can approximate continuous functions on a manifold by smooth functions, and the proof uses Theorem 1.29. We first start with a definition:

Definition 1.30. Let M be a smooth manifold, $g : M \to \mathbb{R}$ a function on M, $A \subset M$. We say that g is ***smooth on*** A if there exists open sets $U \supset A$ and smooth function g_0 on U, such that $g_0 = g$ on A.

Theorem 1.31 (Whitney Approximation Theorem)**.** *Let M be a smooth manifold, let A be a closed subset of M. For any continuous function $g : M \to \mathbb{R}$ that is smooth on A, and for any continuous $\delta : M \to \mathbb{R}_{>0}$ that is everywhere positive, there exists a smooth function $f : M \to \mathbb{R}$ such that*

- *For any $p \in A$, we have $f(p) = g(p)$;*
- *For any $p \in M$, we have $|f(p) - g(p)| < \delta(p)$.*

In particular, take $A = \emptyset$ and $\delta(p) = \epsilon$ everywhere for a positive constant $\epsilon > 0$, we see that any continuous function $g : M \to \mathbb{R}$ can be approximated by a smooth function $f : M \to \mathbb{R}$, such that $|f(p) - g(p)| < \epsilon$.

Proof. By definition, there exists open set $U \supset A$ and smooth function g_0 on U such that $g_0 = g$ on A. Let

$$U_0 = \{p \in U : |g_0(p) - g(p)| < \delta(p)\}.$$

Then U_0 is open in M and $U_0 \supset A$. Next we construct a collection of open cover of $M - A$. For any $q \in M - A$, define

$$U_q = \{p \in M - A : |g(p) - g(q)| < \delta(p)\}.$$

Then $\{U_q : q \in M - A\}$ is an open cover of $M - A$. Since M is second countable, we can find countable U_{q_i}, where $i = 1, 2, ...$, such that $\bigcup_i U_{q_i} = M - A$.

Now let $\{\rho_0, \rho_i\}$ be a partition of unity subordinate to the open cover $\{U_0, U_{q_i}\}$ of M. We define

$$f(p) = \rho_0(p)g_0(p) + \sum_i \rho_i(p)g(q_i).$$

Since the sum is locally finite, $f \in C^\infty(M)$. Moreover, by definition we have that on A, $f = g_0 = g$. Moreover, for any $p \in M$, we have

$$\begin{aligned}
|f(p) - g(p)| &= \left|\rho_0(p)g_0(p) + \sum_i \rho_i(p)g(q_i) - \sum_i \rho_i(p)g(p)\right| \\
&\leq \rho_0(p)|g_0(p) - g(p)| + \sum_i \rho_i(p)|g(q_i) - g(p)| \\
&< \rho_0(p)\delta(p) + \sum_i \rho_i(p)\delta(p) \\
&< \delta(p),
\end{aligned}$$

where the last inequality is obtained by the following argument: If $p \in U_0$, then by definition $\rho_0(p)|g_0(p) - g(p)| < \rho_0(p)\delta(p)$, and if $p \notin U_0$, we have $\rho_0(p) = 0$; similarly for every i, if $p \in U_{q_i}$, then we get the inequality $\rho_i(p)|g_i(q) - g_i(p)| < \rho_i(p)\delta(p)$, and if $p \notin U_{q_i}$, we get zero. But at least for one i, we get strict inequality. Hence the last inequality follows, and the theorem is proven. □

By considering each components, obviously we get the generalization for $\mathbb{R}^k$-valued continuous functions:

Theorem 1.32. *Let M be a smooth manifold, let A be a closed subset of M. For any continuous function $g : M \to \mathbb{R}^k$ that is smooth on A, and for any continuous $\delta : M \to \mathbb{R}_{>0}$ that is everywhere positive, there exists a smooth function $f : M \to \mathbb{R}^k$ such that*

- *For any $p \in A$, we have $f(p) = g(p)$;*
- *For any $p \in M$, we have $|f(p) - g(p)| < \delta(p)$.*

In particular, take $A = \emptyset$ and $\delta(p) = \epsilon$ everywhere for a positive constant $\epsilon > 0$, we see that any continuous function $g : M \to \mathbb{R}^k$ can be approximated by a smooth function $f : M \to \mathbb{R}^k$, such that $|f(p) - g(p)| < \epsilon$.

1.5 Problems

1. (Visualization exercise) Figure out good coordinates on the unit 3-sphere $S^3 \subset \mathbb{R}^4$. Show that the volume of S^3 is $2\pi^2$.

2. Show that:

   (a) Any composition of smooth maps between smooth manifolds is itself smooth.

   (b) A map $F : M \to N_1 \times N_2$ is smooth if and only if the component maps $F_i = \pi_i \circ F : M \to N_i$ are smooth, where $\pi_i : N_1 \times N_2 \to N_i$ are the projection maps.

   (c) The family of smooth maps $f : M \to \mathbb{R}$ forms an algebra under pointwise addition and multiplication of values.

3. In Example 1.20, we defined the real projective space $\mathbb{RP}^n$ as a topological space.

   (a) Show that $\mathbb{RP}^n$ is a topological manifold. Hausdorff and second countability should not be hard. To show it is locally Euclidean, explicitly exhibit an Euclidean open cover of $\mathbb{RP}^n$.

   Hint: Show that $\mathbb{RP}^n$ can be covered by $n+1$ open sets U_i, for $i = 1, ..., n+1$, constructed as follows: Points in $\mathbb{RP}^n$ are equivalent classes $[x_0 : x_1 : \cdots : x_n]$ of points in $\mathbb{R}^{n+1}$, and each equivalent classes consists of scalar multiples of a representative of it in $\mathbb{R}^{n+1}$. Define the open sets U_i to be the collection of points in $\mathbb{RP}^n$, such that $x_i \neq 0$.

   (b) Using those U_i's in the previous part as charts on $\mathbb{RP}^n$, explicitly compute the chart transition maps, showing that they are smooth. Conclude that $\mathbb{RP}^n$ is a smooth manifold.

4. Define $F : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$F(x, y, z) = (x^2 - y^2,\, xy,\, xz,\, yz),$$

   and let f denote the restriction of F to the unit sphere $S^2 \subset \mathbb{R}^3$. Show that $f(S^2)$ is a 2-dimensional differentiable manifold in $\mathbb{R}^4$ and that

$$f : S^2 \to f(S^2)$$

   is a *double covering*. Define this! Conclude that the image $f(S^2) \subset \mathbb{R}^4$ is a copy of the real projective plane $\mathbb{RP}^2$.

5. Let $\mathbb{CP}^n$ denote the set of complex lines through the origin in $\mathbb{C}^{n+1}$.

   (a) Show how to put a smooth structure on $\mathbb{CP}^n$ to make it a smooth $2n$-manifold.

   (b) Show that $\mathbb{CP}^1$ is diffeomorphic to S^2.

   (c) What does $\mathbb{CP}^2$ look like?

   (d) There is a well-defined map

$$f : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$$

   which sends each nonzero point of $\mathbb{C}^{n+1}$ to the unique complex line containing it. Restricting this map to the unit $2n+1$-sphere in $\mathbb{C}^{n+1}$ gives the map

$$f : S^{2n+1} \to \mathbb{CP}^n,$$

known as a *Hopf fibration*. When $n = 1$, we can write

$$f : S^3 \to S^2,$$

according to (b) above. Try hard to picture this map.

6. The *orthogonal group* $\mathrm{O}(n)$ consists of all $n \times n$ matrices A such that $AA^{\mathrm{tr}} = I$.

   (a) Show that $\mathrm{O}(n)$ is a group under matrix multiplication.

   (b) Show that $\mathrm{O}(n)$ is the group of linear transformations of $\mathbb{R}^n$ which preserve the ordinary Euclidean inner product:

   $$\langle A(v), A(w)\rangle = \langle v, w\rangle$$

   for all vectors v and w in $\mathbb{R}^n$.

   (c) Identify the set $M(n)$ of all $n \times n$ matrices with Euclidean space of dimension n^2, and then show that the orthogonal group $\mathrm{O}(n)$ is a differentiable submanifold. What is its dimension?

7. The *special linear group* $\mathrm{SL}(n)$ consists of all $n \times n$ matrices with determinant $+1$.

   (a) Show that $\mathrm{SL}(n)$ is a differentiable submanifold of $M(n)$ of dimension $n^2 - 1$.

   (b) Try to draw a good picture of $\mathrm{SL}(2)$.

8. (a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} \exp\bigl(-1/x^2\bigr) & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

   Prove that f is of class C^∞.

   (b) Show that if $a < b$, then $g(x) = f(x-a)f(b-x)$ is a C^∞ function which is positive on the open interval (a, b) and zero elsewhere.

   (c) Show that

   $$h(x) = \frac{\int_{-\infty}^{x} g(x)\, dx}{\int_{-\infty}^{\infty} g(x)\, dx}$$

   is a C^∞ function which satisfies

   $$\begin{aligned} h(x) &= 0 && \text{for } x \le a \\ h(x) &\in (0,1) && \text{for } a < x < b \\ h(x) &= 1 && \text{for } x \ge b. \end{aligned}$$

   (d) Now, given $a < b$, construct a C^∞ function $k : \mathbb{R}^n \to \mathbb{R}$ such that

   $$\begin{aligned} k(x) &= 1 && \text{for } |x| \le a \\ k(x) &\in (0,1) && \text{for } a < |x| < b \\ k(x) &= 0 && \text{for } |x| \ge b. \end{aligned}$$

Chapter 2

Tangent and Cotangent Spaces

2.1 Tangent Space of a Manifold

Usually we have a curve, i.e. a smooth map $\gamma : \mathbb{R} \to M$ (or $\gamma : (a, b) \to M$ where the domain of the curve is an interval. But an interval is clearly homeomorphic to $\mathbb{R}$, so we can use these definitions interchangably), we want to know what is the velocity of the curve γ at some point p.

Definition 2.1. Let $(M, \mathcal{O}, \mathcal{A})$ be smooth manifold. The curve $\gamma : \mathbb{R} \to M$ is at least C^1 with respect to charts of the smooth atlas $\mathcal{A}$. Suppose γ hits a point $p \in M$ at $\lambda_0 \in \mathbb{R}$, i.e. $\gamma(\lambda_0) = p$. Then the ***velocity*** of γ at p is the linear map

$$v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R},$$

where $C^\infty(M)$ is the set of all smooth functions $f : M \to \mathbb{R}$ equipped with addition and multiplication:

$$(f \oplus g)(p) = f(p) + g(p);$$
$$(\lambda \odot g)(p) = \lambda \cdot g(p).$$

More specifically, the velocity is defined by the rule

$$v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0),$$

that is, the derivative of $f \circ \gamma$ at that point p where γ takes λ_0 as an input.

Example 2.2. Let $\gamma : \mathbb{R} \to M$ be the path you take when you are running in real world. Let the temperature be $f : M \to \mathbb{R}$. Then as you run, you plot $f \circ \gamma$, which is the temperature with respect to time. Then $(f \circ \lambda)'(t)$ is the rate of change of the temperature at that time t, which you are so used to.

Definition 2.3. For each point $p \in M$, define the ***tangent space*** to M at p, denoted by T_pM, as

$$T_pM := \{v_{\gamma,p} : \gamma \text{ is a smooth curve}\}.$$

That is, we take all possible curves that passes p, and find their velocity at point p. These velocity vector spans a imaginary plane that forms the tangent space T_pM.

Now it is natural to ask whether T_pM, if it is like any plane in Euclidean space, forms a vector space. We define the addition and multiplication on T_pM and show that it is indeed a vector space.

Definition 2.4. Define the addition $\oplus$ on T_pM to be

$$\oplus : T_pM \times T_pM \to \mathrm{Hom}(C^\infty(M), \mathbb{R})$$

with the rule

$$(v_{\gamma,p} \oplus v_{\delta,p})(f) := v_{\gamma,p}(f) + v_{\delta,p}(f),$$

where $\oplus$ happens in T_pM and $+$ happens in $\mathbb{R}$. Similarly, define the multiplication structure

$$\odot : \mathbb{R} \times T_pM \to \mathrm{Hom}(C^\infty(M), \mathbb{R})$$

to be

$$(\lambda \odot v_{\gamma,p})(f) := \lambda \cdot v_{\gamma,p}(f),$$

where $\odot$ happens in T_pM and $\cdot$ happens in $\mathbb{R}$.

Theorem 2.5. *$(T_pM, \oplus, \odot)$ is a vector space.*

Proof. Suppose $\gamma(\lambda_0) = p$ and $\delta(\lambda_1) = p$. We prove the theorem by showing

1. There exists a curve σ such that $v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$;
2. There exists a curve τ such that $\alpha \odot v_{\gamma,p} = v_{\tau,p}$ for $\alpha \in \mathbb{R}$.

We prove the second claim first. We claim that the curve

$$\tau(\lambda) := \gamma(\alpha\lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda)$$

works, where $\mu_\alpha(\lambda) = \alpha\lambda + \lambda_0$. Notice that $\tau(0) = \gamma(\lambda_0) = p$. Then by definition,

$$v_{\tau,p}(f) : = (f \circ \tau)'(0) = (\gamma \circ \mu_\alpha)'(0) = (f \circ \gamma)'(\lambda_0) \cdot \alpha = \alpha \cdot v_{\gamma,p}(f),$$

where the third step follows from the chain rule. This finished the second half of the proof. Now, for the first part, make a choice of chart (U, φ) where $p \in U$. Define the curve $\sigma : \mathbb{R} \to M$ to be

$$\sigma(\lambda) = \varphi^{-1}((\varphi \circ \gamma)(\lambda_0 + \lambda) + (\varphi \circ \delta)(\lambda_1 + \lambda) - (\varphi \circ \gamma)(\lambda_0)).$$

Notice that

$$\sigma(0) = \varphi^{-1}((\varphi \circ \gamma)(\lambda_0) + (\varphi \circ \delta)(\lambda_1) - (\varphi \circ \gamma)(\lambda_0)) = \delta(\lambda_1) = p.$$

Now,

$$\begin{aligned} v_{\sigma,p}(f) : &= (f \circ \sigma)'(0) \\ &= (f \circ \varphi^{-1} \circ \varphi \circ \sigma)'(0) \\ &= ((f \circ \varphi^{-1}) \circ (\varphi \circ \sigma))'(0) \end{aligned}$$

Notice that $f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}$, and $\varphi \circ \sigma : \mathbb{R} \to \mathbb{R}^m$ are functions that are very easy to differentiate by using the multidimensional chain rule. We continue the calculation and use chain rule to get

$$
\begin{aligned}
v_{\sigma,p}(f) &= ((f \circ \varphi^{-1}) \circ (\varphi \circ \sigma))'(0) \\
&= \sum_{i=1}^{m} \frac{d(\varphi \circ \sigma)^i}{d\lambda} \cdot \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(\sigma(0))) \\
&= \sum_i \left(\frac{d(\varphi \circ \gamma)^i}{d\lambda}(\lambda_0) + \frac{d(\varphi \circ \delta)^i}{d\lambda}(\lambda_1) \right) \cdot \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) \\
&= \sum_i \frac{d(\varphi \circ \gamma)^i}{d\lambda}(\lambda_0) \cdot \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) + \sum_i \frac{d(\varphi \circ \delta)^i}{d\lambda}(\lambda_1) \cdot \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) \\
&= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_1) \\
&= v_{\gamma,p}(f) + v_{\delta,p}(f).
\end{aligned}
$$

And we are done. $\square$

In the remaining of the text we will not use $\oplus, \odot$ and will switch to ordinary addition and multiplication notation for adding and scalar multiplying vectors in T_pM for convenience.

Now let us consider a point $p \in M$ and a chart (U, φ) containing p. Let $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$. Using the same method of calculation in our proof above, we have that for $f \in C^\infty(M)$,

$$
v_{\gamma,p}(f) = (f \circ \gamma)'(0) = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) \cdot \frac{d(\varphi \circ \gamma)^i}{d\lambda}(0).
$$

Now we make two very important notational convention:

$$
\begin{aligned}
\frac{d\gamma^i}{d\lambda}(0) &:= \frac{d(\varphi \circ \gamma)^i}{d\lambda}(0) \\
\left(\frac{\partial}{\partial x^i} \right)_p f &:= \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)).
\end{aligned}
$$

Then we have that

$$
v_{\gamma,p}(f) = \sum_i \frac{d\gamma^i}{d\lambda}(0) \cdot \left(\frac{\partial}{\partial x^i} \right)_p f.
$$

Usually, we use Einstein's summation convention, which dictates that if a repeated index (like i in the equation above) appears twice, once upstairs and once downstairs, will be automatically summed over, with no summation notation explicitly written out. So the compact form of the equation above is

$$
v_{\gamma,p}(f) = \frac{d\gamma^i}{d\lambda}(0) \cdot \left(\frac{\partial}{\partial x^i} \right)_p f.
$$

From now on, we will use this convention throughout this notes, to be concise.

The notation $(\partial/\partial x^i)_p$ looks deceptively simple and familar (after all it is just like the partial derivative), but it is fundamentally different from the familiar partial derivative that we know. So, from now on, whenever it appears, what we mean is the operation

$$
\left(\frac{\partial}{\partial x^i} \right)_p : C^\infty(M) \xrightarrow{\sim} \mathbb{R} \qquad f \mapsto \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))
$$

which is linear over $C^\infty(M)$. Hence $(\partial/\partial x^i)_p$ may be viewed as an element in T_pM. In fact, since any tangent vector $v \in T_pM$ can be expressed as a linear combination of these partials, a curve with tangent vector $(\partial/\partial x^i)_p$ at p is the one with

$$\frac{d\gamma^j}{d\lambda}(0) = \frac{d(\varphi \circ \gamma^i)}{d\lambda} = 0$$

unless $j = i$. That is, a curve γ such that its image $\varphi \circ \gamma$ in $\mathbb{R}^m$ is constant for every $j \neq i$. The existence and uniqueness theorem of ODEs guarantee that this curve is unique at least in a neighborhood of p. See Figure 2.1.

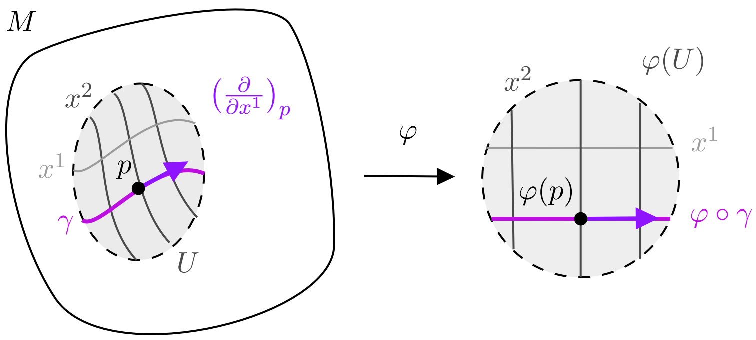


Figure 2.1: A curve γ on M having $(\partial/\partial x^1)_p$ as its tangent vector at p.

Note that the notion of $(\partial/\partial x^i)_p$ only makes sense if a chart (U, φ) around p is specified, as φ is explicitly needed in its definition. But overall we would like to show the following:

Theorem 2.6. *Let $(U, \varphi) \in \mathcal{A}_{C^\infty}$ be a chart containing p. Then the vectors*

$$\left(\frac{\partial}{\partial x^1}\right)_p, \cdots, \left(\frac{\partial}{\partial x^m}\right)_p \in T_pM$$

consitute a basis for T_pM, where m is the dimension of the chart map $\varphi : U \to \varphi(U) \subset \mathbb{R}^m$, or the dimension of the manifold M. We call this basis a chart induced basis for T_pM.

Proof. We have already shown that any vector $v_{\gamma,p} \in T_pM$ can be written as a linear combination of these vectors in the discussion above. So we only need to show that these vectors are linearly independent. We will show that

$$\lambda^i \left(\frac{\partial}{\partial x^i}\right)_p = 0 \quad \text{implies} \quad \lambda^i = 0 \quad \text{for all } i.$$

Now, consider a family $x^1, ..., x^m \in C^\infty(U)$, defined as follows: For every $p \in U$, we define $x^i(p)$ to be the ith component of $\varphi(p) \in \mathbb{R}^m$. Note that $x^j \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}$ is simply the map $(a^1, ..., a^m) \mapsto a^j$, hence we have

$$\lambda^i \left(\frac{\partial}{\partial x^i}\right)_p x^j = \lambda^i \frac{\partial(x^j \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) = \lambda_i \delta_i^j = \lambda^j = 0,$$

which does imply $\lambda^j = 0$ for all j. □

Corollary 2.7. $\dim T_pM = \dim M$.

Proof. This is immediate since given any chart (U, φ) around p, $(\partial/\partial x^1)_p, ..., (\partial/\partial x^m)_p$ is a basis. $\square$

Example 2.8. Let $p = (p^1, ..., p^n) \in \mathbb{R}^n$. The tangent space $T_p\mathbb{R}^n$ consists of all linear maps $C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by C^1-curves $\gamma : \mathbb{R} \to \mathbb{R}^n$. Let $x^1, ..., x^n$ be the standard coordinate on $\mathbb{R}^n$. A basis for $T_p\mathbb{R}^n$ consists of tangent vectors of the constant curves $\eta^i(\lambda) = (p^1, ..., p^i + \lambda, ..., p^n)$ at $\lambda = 0$. It is easy to see that the associated basis $(\partial/\partial x^i)_p$ turns out to be exactly the directional derivatives in the ith direction, evaluated at p. Let $\mathbf{e}_1 = (1, ..., 0), ..., \mathbf{e}_n = (0, ..., 1)$ be the standard basis for $\mathbb{R}^n$. Then the identification

$$v = \sum_{i=1}^{n} v^i \left(\frac{\partial}{\partial x^i}\right)_p \xrightarrow{\cong} \sum_{i=1}^{n} v^i \mathbf{e}_i \in \mathbb{R}^n$$

gives an isomorphism $T_p\mathbb{R}^n \cong \mathbb{R}^n$ of vector spaces.

2.2 Pushforward of a Smooth Map

Definition 2.9. Let $F : M \to N$ be a smooth map between manifolds. For each $p \in M$ we define a map

$$dF_p : T_pM \to T_{F(p)}N$$

as follows: for any given $f \in C^\infty(N)$ and $v \in T_pM$, we define $dF_p(v) \in T_{F(p)}(N)$ via

$$dF_p(v)(f) := v(f \circ F).$$

Note that this definition makes sense, because $f \circ F \in C^\infty(M)$, and v acts linearly on $C^\infty(M)$. This map dF_p is called the ***differential*** or ***pushforward*** of F at p. Equally common notations includes $(F_*)_p$ and $F_{*,p}$. See Figure 2.2 for a more geometric description of the pushforward.

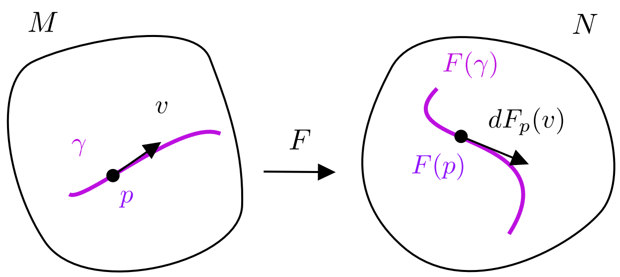


Figure 2.2: If v is the tangent vector of a curve γ at p and $\gamma(0) = p$, then for any $g \in C^\infty(M)$, we have $v(g) = (g \circ \gamma)'(0)$. Then using the definition $dF_p(v)(f) = v(f \circ F) = (f \circ F \circ \gamma)'(0)$. Therefore, the geometric picture of dF_p can be viewed as follows: it maps v, the tangent vector of $\gamma \subset M$ at p, to the tangent vector of $F \circ \gamma \subset N$ at $F(p)$.

Proposition 2.10. Let M, N, P be smooth manifolds, $F : M \to N, G : N \to P$ be smooth maps, and $p \in M$.

1. $dF_p : T_pM \to T_{F(p)}N$ is linear;
2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p$;

3. $d(\mathrm{Id}_M)_p : T_pM \to T_pM$ is exactly id_{T_pM};

4. If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism of vector spaces, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proof. 1. This is clear from definition, since v is linear; 2. This is also clear because for $f \in C^\infty(P)$ and $v \in T_pM$, we have

$$dG_{F(p)} \circ dF_p(v)(f) = dF_p(v)(f \circ G) = v(f \circ G \circ F) = d(G \circ F)_p(v);$$

3. This is clear from definition; 4. This follows from 2 and 3. □

Example 2.11 (Tangent space of a finite-dimensional vector space)**.** Let V be a n-dimensional real vector space, and let $\alpha_1, ..., \alpha_n$ be a basis for V. Any norm on V makes V into a topological space with the same topology (see Example 1.15). There is a map $\phi : V \to \mathbb{R}^n$ which identifies $w = \sum_i w^i \alpha_i \in V$ with $(w^1, ..., w^n) \in \mathbb{R}^n$. This map is a diffeomorphism (in fact a chart) by Example 1.15. Therefore, we can make sense of $d\phi_w : T_wV \to T_{\phi(w)}\mathbb{R}^n$. Then the three isomorphisms in the diagram below defines an identification isomorphism $\theta : V \to T_wV$:

$$\begin{tikzcd} V \arrow[r, "\theta"] \arrow[d, "\phi"'] & T_wV \arrow[d, "d\phi_w"] \\ \mathbb{R}^n \arrow[r, "\cong"'] & T_{\phi(w)}\mathbb{R}^n \end{tikzcd}$$

Hence we have shown that for any finite-dimensional vector space $V \cong T_wV$.

Example 2.12 (Another way of viewing basis for T_pM)**.** Let (U, φ) be a chart around $p \in M$. Then we may view $\varphi^{-1} : \varphi(U) \to U$ a smooth map (in fact a diffeomorphism) between manifolds. Note that because curves on U are curves on M as well, and curves on M restricts to curves on U, this gives an identification $T_pU \cong T_pM$.Since φ^{-1} is a diffeomorphism, the pushforward

$$d(\varphi^{-1})_{\varphi(p)} : T_{\varphi(p)}\varphi(U) \to T_pU \cong T_pM$$

is an isomorphism between vector spaces. We have that, for any $f \in C^\infty(M)$

$$d(\varphi^{-1})_{\varphi(p)}\left(\left(\frac{\partial}{\partial x^i}\right)_{\varphi(p)}\right)(f) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})(\varphi(p)) = \left(\frac{\partial}{\partial x^i}\right)_p f.$$

Hence we see that the basis for T_pM induced by the chart (U, φ) is just the image of the standard basis in $T_{\varphi(p)}\varphi(U) \subset \mathbb{R}^n$ under the isomorphism $d(\varphi^{-1})_{\varphi(p)}$ of vector spaces.

We conclude this section by studying the expression of the differential $dF : T_pM \to T_{F(p)}N$ in coordinates.

Theorem 2.13. *With respect to the basis for T_pM induced by a chart (U, φ) around p and the basis for $T_{F(p)}N$ induced by a chart (V, ψ) of N containing $F(U)$, the matrix representation of the linear map $dF_p : T_pM \to T_{F(p)}N$ is given by the Jacobian of $\hat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ evaluated at $\varphi(p)$.*

Proof. Let $(\partial/\partial x^1)_p, ..., (\partial/\partial x^m)_p$ a basis for T_pU induced by the chart (U,φ), and similarly let $(\partial/\partial y^1)_{F(p)}, ..., (\partial/\partial y^n)_{F(p)}$ a basis for $T_{F(p)}V$ induced by the chart (V,ψ). We would like to study dF_p in matrix form with respect to these base. Now let $f \in C^\infty(N)$ be arbitrary, by chain rule we have

$$\begin{aligned} dF_p\left(\left(\frac{\partial}{\partial x^i}\right)_p\right)f &= \frac{\partial}{\partial x^i}\big(f\circ F\circ\varphi^{-1}\big)(\varphi(p)) \\ &= \frac{\partial}{\partial x^i}\big(f\circ\psi^{-1}\circ\psi\circ F\circ\varphi^{-1}\big)(\varphi(p)) \\ &= \frac{\partial(f\circ\psi^{-1})}{\partial y^j}(\psi(F(p)))\cdot\frac{\partial(\psi\circ F\circ\varphi^{-1})^j}{\partial x^i}(\varphi(p)) \\ &= \frac{\partial\hat{F}^j}{\partial x^i}(\varphi(p))\cdot\left(\frac{\partial}{\partial y^j}\right)_{F(p)} f \end{aligned}$$

where $\hat{F} := \psi\circ F\circ\varphi^{-1} : \varphi(U)\to\psi(V)$. □

The result above also answers the question that how does the component of a vector change under change of coordinates. Let $p \in M$ and let (U,φ) and (V,ψ) be two overlapping charts whose intersection contains p. Then the change of coordinate map is just $\psi\circ\varphi^{-1} : \varphi(U\cap V)\to\psi(U\cap V)$. Note that in this case we have $F = \mathrm{id}_{U\cap V}$. Then by the result above we have

$$\left(\frac{\partial}{\partial x^i}\right)_p = \frac{\partial(\psi\circ\varphi^{-1})}{\partial x^i}(\varphi(p))\cdot\left(\frac{\partial}{\partial y^j}\right)_p.$$

Usually, we use a set of coordinate $(x^1, ..., x^m)$ to reprensent points in $\varphi(U\cap V)$, where $x^i \in \mathbb{R}$. Similarly, we use $(y^1, ..., y^m)$ to represent points in $\psi(U\cap V)$. One can view $y^1, ..., y^m$ as functions of $x^1, ..., x^m$, i.e., for a point $p \in U\cap V$, its y^j coordinate is the j-th component of $\psi\circ\varphi^{-1}(x^1, ..., x^m)$, where $x^1, ..., x^m$ is the coordinate of p in $\varphi(U\cap V)$. Then the formula above can be conveniently written as

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}.$$

This is easy to remember because of its resemblance to the chain rule formula in multivariable calculus. This also generalizes to arbitrary tangent vectors $v \in T_pM$. We first write v as a linear combination of basis vectors, then by above we get

$$v = v^i\frac{\partial}{\partial x^i} = v^i\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = \tilde{v}^j\frac{\partial}{\partial y^j},$$

or simply

$$\tilde{v}^j = v^i\frac{\partial y^j}{\partial x^i}.$$

Example 2.14 (Tangent space of the circle)**.** In this example, we explicitly calculate the tangent space of S^1 at a point (a,b).

Rotating the circle if necessary, we may assume (a,b) lies on the upper half circle, i.e. $b > 0$. Then a local parametrization is given by $\psi : (-1,1)\to S^1$, where

$$\psi(x) = (x, \sqrt{1-x^2}).$$

Then its derivative is given by

$$d\psi_x = \begin{pmatrix} 1 \\ -\frac{x}{\sqrt{1-x^2}} \end{pmatrix}.$$

Note that since $d\psi_x : \mathbb{R} \to \mathbb{R}^2$, this map should take a real number and gives a vector, which is done by scalar multiplication. Now if we let $x = a$, then $\sqrt{1-x^2} = b$ since (a, b) is on the upper half circle. Therefore

$$T_{(a,b)}(S^1) = d\psi_a(\mathbb{R}) = \mathbb{R}\begin{pmatrix} 1 \\ -a/b \end{pmatrix} = \mathbb{R}(-b)\begin{pmatrix} 1 \\ -a/b \end{pmatrix} = \mathbb{R}\begin{pmatrix} -b \\ a \end{pmatrix}.$$

This is what we expect: the tangent space of the circle at the point $p = (a, b)$ should be spanned by the vector $(-b, a)$, which is perpendicular to the radial vector (a, b) at the point p.

2.3 Tensors

We provide a section that briefly reviews the concept of tensors.

Definition 2.15. Let $(V, +, \times)$ be a finite dimensional vector space. An (r, s) ***tensor*** T is a multilinear map

$$T : \underbrace{V \times V \cdots \times V}_{r} \times \underbrace{V^* \times V^* \cdots \times V^*}_{s} \to \mathbb{R}.$$

Multilinear means the tensor is linear in each slots. Say if T is a $(1, 1)$ tensor. Let $\varphi, \psi \in V^*$, let $v, w \in V$, and let $\lambda \in \mathbf{F}$ (some field, e.g. real or complex). Then

$$\begin{aligned} T(v, \varphi + \psi) &= T(v, \varphi) + T(v, \psi) \\ T(v, \lambda\varphi) &= \lambda T(v, \varphi). \end{aligned}$$

Similarly for the first slot.

Example 2.16. Let $\mathcal{P}(\mathbb{R})$ be the vector space of polynomials on $\mathbb{R}$ and let $g : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be defined as

$$g(p, q) = \int_{-1}^{1} p(x)q(x)dx.$$

Then you can verify that g is a $(2, 0)$ tensor. From linear algebra, you know this is an inner product. More generally speaking, inner products are $(2, 0)$ tensors.

Of course, $\varphi : V \to \mathbb{R}$ is a $(1, 0)$ tensor.

Vectors are $(0, 1)$ tensors. This is because, for finite dimensional vector space, $(V^*)^* \cong V$. The identification is given by the correspondence that for any $v \in V$ and $\alpha \in V^*$, we may define $v(\alpha) := \alpha(v) \in \mathbb{R}$, hence viewing v as an element in V^*.

Similar to choosing a basis in vector space V, we can choose a basis for V^*.

Definition 2.17. Let V be a finite dimensional vector space with basis $e_1, \cdots, e_n$. Define $\epsilon^1, \cdots, \epsilon^n \in V^*$ where

$$\epsilon^a(e_b) = \delta^a_b.$$

$\epsilon^1, \cdots, \epsilon^n$ are uniquely determined by the choice of $e_1, \cdots, e_n$. It is easy to verify that $\epsilon^1, ..., \epsilon^n$ form a basis for V^*. We call $\epsilon^1, \cdots, \epsilon^n$ the dual basis for V.

Example 2.18. Consider the vector space $\mathcal{P}_3(\mathbb{R})$, the set of real polynomials with degree no more than 3. Let

$$e_0(x) = 1 \quad e_1(x) = x \quad e_2(x) = x^2 \quad e_3(x) = x^3,$$

Let

$$\epsilon^a = \frac{1}{a!}\partial^a\Big|_{x=0}$$

Then $\epsilon^a(e_b) = \delta^a_b$ is a dual basis.

Definition 2.19. Let T be an (r, s) tensor on a finite dimensional vector space V ($\dim V = n < \infty$). Let $e_1, \cdots, e_n$ be the chosen basis for V. We can define the ***components*** of T by

$$T^{i_1,\cdots,i_r}_{j_1,\cdots,j_s} = T(e_{j_1}, \cdots, e_{j_s}, \epsilon^{i_1}, \cdots, \epsilon^{i_r}),$$

where $i_1, \cdots, i_r, j_1, \cdots, j_s \in \{1, 2, \cdots, \dim V\}$.

Example 2.20. Let T be a $(1, 1)$ tensor. Then $T^i_j = T(e_j, \epsilon^i)$. Therefore,

$$\begin{aligned}
T(v, \varphi) &= T\Big(\sum_{j=1}^{\dim V} v^j e_j, \sum_{i=1}^{\dim V} \varphi_i \epsilon^i\Big) \\
&= \sum_i \sum_j \varphi_i v^j T(e_j, \epsilon^i) \\
&= \sum_i \sum_j \varphi_i v^j T^i_j \\
&= \varphi_i v^j T^i_j,
\end{aligned}$$

where the second step is by the multilinear property of T. In the last step we use Einstein summation convention and contract the indices.

2.4 Cotangent Spaces

Definition 2.21. The dual space of T_pM, or the ***cotangent space***, is defined by

$$T^*_pM = \{\text{linear maps } \alpha : T_pM \to \mathbb{R}\}.$$

Definition 2.22. The ***gradient*** of $f \in C^\infty(M)$ at a point p, denoted by $df_p : T_pM \to \mathbb{R}$, is given by the rule

$$v \mapsto df_p(v) := v(f).$$

We can now compute the component of the gradient. Its jth component is given by gradient acting on the jth basis for T_pM. That is,

$$(df)_p\left(\frac{\partial}{\partial x^j}\right) = \left(\frac{\partial}{\partial x^j}\right)_p f = \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(p)).$$

Now this looks like the gradient that we are familiar with in multivariable calculus!

Another note on our choice of notation: Recall from the previous section that given smooth function $f : M \to \mathbb{R}$, we can define a differential $df_p : T_pM \to T_{f(p)}\mathbb{R}$. If we use the identification $T_{f(p)}\mathbb{R} \cong \mathbb{R}$ by identifying $\left(\frac{\partial}{\partial x}\right)_{f(p)} \in T_{f(p)}\mathbb{R}$ associated to the chart induced by the identity map on $\mathbb{R}$ with $1 \in \mathbb{R}$, then the two definitions coincide, as the formula above is exactly Theorem 2.13.

Theorem 2.23. *Let (U, φ) be a chart and $x^i : U \to \mathbb{R}$ be the smooth function defined for any $p \in U$ by giving the ith component of $\varphi(p) \in \mathbb{R}^m$. Then the gradient vectors*

$$(dx^1)_p, (dx^2)_p, \cdots, (dx^m)_p \in T_p^*M$$

form a basis for T_p^*M.

Proof. By the calculation above, it is easy to see that

$$(dx^i)_p\left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = \frac{\partial(x^i \circ \varphi^{-1})}{\partial x^j}(\varphi(p)) = \frac{\partial x^i}{\partial x^j}(x^1, ..., x^m) = \delta^i_j.$$

That is, $(dx^i)_p$'s are the dual basis to the basis $\left(\frac{\partial}{\partial x^j}\right)_p$'s in T_pM. Hence it is a basis for T_p^*M. □

Now we can talk about the change of components of a covector under a change of charts. Similar to our discussion on change of components of vectors, let $(U, \varphi), (V, \psi)$ be two overlapping charts containing p, and let x^i, y^j be the respective coordinates. Then we have

$$\begin{aligned}
(dx^i)_p v_{\gamma,p} = v_{\gamma,p}(x^i) = (x^i \circ \gamma)'(\lambda_0) &= (x^i \circ \psi^{-1} \circ \psi \circ \gamma)'(\lambda_0) \\
&= (y^j \circ \gamma)'(\lambda_0) \cdot \frac{\partial(x^i \circ \psi^{-1})}{\partial y^j}(\psi(p)) \\
&= v_{\gamma,p}(y^j) \cdot \frac{\partial(x^i \circ \psi^{-1})}{\partial y^j}(\psi(p)) \\
&= \frac{\partial(x^i \circ \psi^{-1})}{\partial y^j}(\psi(p)) \cdot (dy^j)_p v_{\gamma,p}.
\end{aligned}$$

Note that $x^i \circ \psi^{-1} : \psi(U \cap V) \to \mathbb{R}$ is the function that gives the ith component of the map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$. Similar to our previous argument on change of components of tangent vectors, we may view x^i as a function of $y^1, ..., y^m$'s, so that $x^i(y^1, ..., y^m)$ is the ith component of $\varphi \circ \psi^{-1}(y^1, ..., y^m)$. So the equation above can be conveniently written as

$$(dx^i)_p = \left(\frac{\partial x^i}{\partial y^j}\right)_p (dy^j)_p.$$

This formula is very easy to remember as it resembles the differentials we learned in multivariable calculus.

This also works for general covectors, i.e., general elements in T_p^*M, other than basis vectors. Given $\alpha \in T_p^*M$, write

$$\alpha = \alpha_i (dx^i)_p.$$

Then under a change of coordinate from x^i's to y^j's on the level of manifolds, this induces a change of basis in T_pM, which then gives a new representation of α in terms of $(dy^j)'s$ by

$$\alpha = \alpha_i (dx^i)_p = \alpha_i \left(\frac{\partial x^i}{\partial y^j}\right)_p (dy^j)_p = \tilde{\alpha}_j (dy^j)_p,$$

or simply

$$\tilde{\alpha}_j = \frac{\partial x^i}{\partial y^j} \alpha_i.$$

Remark 2.24. Coming back to gradient df_p of a smooth function $f \in C^\infty(M)$. At a local chart around p we may write df_p as a linear combination $df_p = \beta_i (dx^i)_p$. To figure out what the coefficient β_i is, we note that

$$df_p\left(\frac{\partial}{\partial x^j}\right) = \beta_i dx^i\left(\frac{\partial}{\partial x^j}\right) = \beta_i \delta^i_j = \beta_j.$$

Then by the previous calculation we know

$$\beta_j = \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(p)).$$

Note that in the case where $M = \mathbb{R}^n$, and φ is the identity map, we have exactly

$$df_p = \frac{\partial f}{\partial x^j}(dx^j)_p.$$

Therefore, differentials should really be viewed as elements in the cotangent space $T^*_p\mathbb{R}^n$. There are higher dimensional generalizations of them called differential forms, which we will get to later.

2.5 Problems

1. (a) Find a basis for the tangent space to the unit 2-sphere S^2 in $\mathbb{R}^3$ at the point $p = (a, b, c)$.

   (b) What is the tangent space to the hyperboloid in $\mathbb{R}^3$ defined by the equation

   $$x^2 + y^2 - z^2 = a^2$$

   at the point $(a, 0, 0)$?

2. Identify the set of real 2×2 matrices with $\mathbb{R}^4$, as in an earlier problem, and let M^3 denote the 3-dimensional submanifold of matrices of rank 1. Find the tangent space to M^3 at the matrix

   $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

3. (a) Show that the tangent space to the orthogonal group $\mathrm{O}(n)$ at the identity I is the vector space of skew-symmetric matrices, that is, matrices B satisfying $\mathrm{Tr}\, B = -B$.

   (b) What is the dimension of $\mathrm{O}(n)$?

   (c) Show that the tangent space to the special linear group $\mathrm{SL}(n)$ at the identity I consists of all matrices with trace zero.

   (d) What is the dimension of $\mathrm{SL}(n)$?

4. Let $S^1 \subset \mathbb{R}^2$ and K be the boundary of the square of side length 2 centered at the origin. Show that there exists no diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(S^1) = K$.

5. Recall from Example 1.21 that the graph Γ of the function $f(x) = |x|$, $x \in \mathbb{R}$, is a smooth manifold.

   (a) Show that it is diffeomorphic to $\mathbb{R}$.

   (b) Show that, however, there is no diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ taking $\mathbb{R}$ to Γ (or vice versa). Here you can take $\mathbb{R}$ to be the real axis of $\mathbb{R}^2$.

   Hint: By contradiction, assume that there is such an F. Without loss of generality, we can assume that $F(0, 0) = (0, 0)$ (why?). Write $F(x, y) = (g(x, y), h(x, y))$. What does F taking $\mathbb{R}$ to Γ tell you about the form of $F(x, 0)$? This will depend on which side you are approaching the origin. Now what does this information translate to $F_* \left(\frac{\partial}{\partial x}\right)$?

6. Let X be a smooth manifold.

   (a) Let $f : X \to X \times X$ be the mapping $f(x) = (x, x)$. Check that $df_x(v) = (v, v)$.

   (b) If Δ is the diagonal of $X \times X$, show that its tangent space $T_{(x,x)}(\Delta)$ is the diagonal of $T_x(X) \times T_x(X)$.

7. If U is an open subset of the manifold X, check that $T_x U = T_x X$ for $x \in U$.

Chapter 3

Bundles and Vector Fields

3.1 Basic Definitions and Examples

Definition 3.1. A ***bundle*** is a triple (E, M, π), where E is a smooth manifold (called the total space), so is M (called the base space). And $\pi : E \to M$ a smooth and *surjective* map.

- Let $\pi : E \to M$ be a bundle. Let $p \in M$. The ***fiber*** over p is defined to be the preimage of p with respect to π, i.e. $\pi^{-1}(\{p\})$, frequently we simply write $\pi^{-1}(p)$.

- A ***section*** of a bundle σ is a map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. If σ is smooth (in the sense of smooth maps between manifolds), we call σ a smooth section.

There is also the notion of a topological bundle, where the definition is the same as above, except that we only require E, M to be topological spaces, and $\pi : E \to M$ be a continuous and surjective map.

Note that from the definition above, in particular the surjectivity of π, we see that as a set,

$$E = \bigcup_{p \in M} \pi^{-1}(p)$$

is a union of its fibers.

Example 3.2. Let $M = S^1$ be the circle, and $E = S^1 \times \mathbb{R}$ be the cylinder (See Figure 3.1). The projection $\pi : E \to M$ is defined by $\pi(p, \lambda) = p$, where $p \in S^1$, and $\lambda \in \mathbb{R}$. A section $\sigma : M \to E$ can be thought of a map $\sigma(p) = (p, \lambda(p))$, for all $p \in S^1$, and $\lambda : S^1 \to \mathbb{R}$ is some function. If λ varies smoothly with respect to p, then we expect to see a smooth section.

Definition 3.3. Let M be a smooth manifold. A ***(real) smooth vector bundle of rank k over M*** is a bundle with the following two extra conditions:

(i) For each $p \in M$, the fiber $E_p = \pi^{-1}(\{p\})$ over p is endowed with the structure of a k-dimensional real vector space.

(ii) For each $p \in M$, there exists a neighborhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called the ***local trivialization of E over U***), satisfying the following three conditions:

  - $\pi_U \circ \Phi = \pi$ where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection.

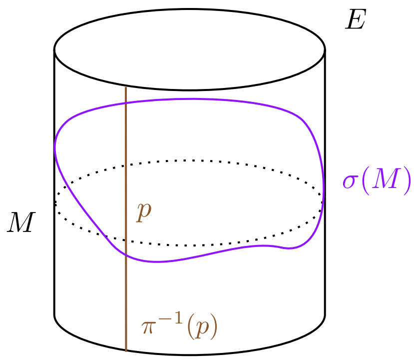


Figure 3.1: An illustration of $E = S^1 \times \mathbb{R}$. For a concrete example, imagine a particle is moving along a circle M with varying speed, and at every point $p \in M$, we associate its speed at that point $v(p)$. Since v should depend smoothly on p, we get a smooth section defined by $\sigma(p) = (p, v(p))$.

- For each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.
- Let $U \cap V \neq \emptyset$ on M and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ be a trivialization. The map

$$\Psi \circ \Phi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \times V) \times \mathbb{R}^k$$

are vector-space automorphisms of $\mathbb{R}^k$ in each fiber and hence give rise to maps $g_{UV} : U \cap V \to GL(n, \mathbb{R})$ defined by $g_{UV}(p) = \Psi \circ \Phi^{-1}|_{\{p\} \times \mathbb{R}^k}$.

3.2 Tangent Bundle and Cotangent Bundle

We will now construct the ***tangent bundle*** of a smooth manifold. This is the most important example of a vector bundle. The construction is rather complicated when one first encounter it, but it is worth going through the whole process, because it will be the guiding examples of how one can construct vector bundles of various ranks over a smooth manifold M (as you will see, there is basically only one way, which is achieved by mimicking our construction of the tangent bundle). Now that I have convinced you that this construction is important, let us start the work. First, let $(M^m, \mathcal{O}, \mathcal{A})$ be a smooth manifold.

Step 1

Define the tangent bundle (on the level of set) as

$$TM := \coprod_{p \in M} T_pM,$$

which is the disjoint union of all the tangent space T_pM. Here, the disjoint union is the set defined by

$$\coprod_{p \in M} T_pM = \{(p, v) : p \in M, v \in T_pM\}.$$

Step 2

Define our map $\pi : TM \to M$ such that

$$\pi : (p, v) \mapsto p,$$

where $p \in M, v \in T_pM$. By the definition of TM, since we unite over all $p \in M$, we know this map π will hit all points on M. Hence π is a surjective map. Now we have $\pi : TM \to M$ where TM only has a set structure and we don't know whether it is smooth or not. π is only a surjective map for now. We don't know if it is smooth either. M is a smooth manifold by assumption. If we want to make this triple a bundle, then we have to make TM a smooth manifold.

Step 3

Now, TM is just a set. We first make TM a topological manifold first by giving it a proper topology. Let us require $\mathcal{O}_{TM}$ to be the coarsest topology to make π continuous. We define

$$\mathcal{O}_{TM} := \{\pi^{-1}(U) : U \in \mathcal{O}\}.$$

We claim that this is indeed a topology.

Proof. We verify that this axiom satisfies the definition of a topology, i.e., Definition 0.1:

1. Note that $\emptyset \in \mathcal{O}$, and $\pi^{-1}(\emptyset) = \emptyset$ since π is surjective, we have $\emptyset \in \mathcal{O}_{TM}$. Also, $M \in \mathcal{O}$ and $\pi^{-1}(M) = TM$ since π surjective, we have $TM \in \mathcal{O}$.

2. Let $U', V' \in \mathcal{O}_{TM}$. Then there exists some $U, V \in \mathcal{O}$ such that $\pi(U') = U$ and $\pi(V') = V$. Then one can easily check that $U' \cap V' = \pi^{-1}(U \cap V)$. Since $U \cap V \in \mathcal{O}$, we have $U' \cap V' \in \mathcal{O}_{TM}$ by definition.

3. Let $U_i' \in \mathcal{O}_{TM}$ for some $i \in I$. Then there exists $U_i \in \mathcal{O}$ such that we have $U_i' = \pi^{-1}(U_i)$ for the same $i \in I$. Then

$$\bigcup_{i\in I} U_i' = \bigcup_{j\in I} \pi^{-1}(U_j).$$

   But since $U_j \in \mathcal{O}$, we have $\bigcup_{i\in I} U_i' \in \mathcal{O}_{TM}$.

This concludes the proof. □

Step 4

Now with a given topology, we construct a C^∞ atlas from $\mathcal{A}$ on M. Define

$$\mathcal{A}_{TM} := \{(\pi^{-1}(U), \xi_\varphi) : (U, \varphi) \in \mathcal{A}\},$$

where $\xi_\varphi : \pi^{-1}(U) \to \mathbb{R}^{2m}$ defined by the rule

$$\xi_\varphi : (p, v) \mapsto (x^1(p), ..., x^m(p), (dx^1)_p(v), ..., (dx^m)_p(v)).$$

Recall that $x^i(p)$ is the ith coordinate of $\varphi(p) \in \mathbb{R}^m$. All this is saying is that given a chart (U, φ) containing p, we take the first m coordinate of $(p, v) \in TM$ under ξ_φ to be the coordinates of p under φ; then for the second m coordinates, express v as a linear combination of basis tangent vectors in T_pM induced by the chart φ (see the previous chapter),

$$v = v^j \left(\frac{\partial}{\partial x^j}\right)_p.$$

Then our second m coordinates will be $v^1, ..., v^m$. It is easy to see that the inverse of ξ_φ is given by

$$\xi_\varphi^{-1} : (\alpha^1, ..., \alpha^m, \beta^1, ..., \beta^m) \mapsto \left(\varphi^{-1}(\alpha^1, ..., \alpha^m), \beta^j \left(\frac{\partial}{\partial x^j}\right)_p\right) \in TM.$$

Step 5

To show that we have constructed a smooth atlas, we still need to show that the chart transition maps are smooth. Hence, let $(U, \varphi), (V, \psi)$ be two overlapping charts on M, and x^i, y^j be the respective coordinates. Then

$$\begin{aligned}
\xi_\psi \circ \xi_\varphi^{-1}(\alpha^1, ..., \alpha^m, \beta^1, ..., \beta^m) &= \xi_\psi \left(\varphi^{-1}(\alpha^1, ..., \alpha^m), \beta^j \left(\frac{\partial}{\partial x^j}\right)_p\right) \\
&= \xi_\psi \left(\varphi^{-1}(\alpha^1, ..., \alpha^m), \beta^j \frac{\partial y^k}{\partial x^j} \left(\frac{\partial}{\partial y^k}\right)_p\right) \\
&= \left(\psi \circ \varphi^{-1}(\alpha^1, ...\alpha^m), \beta^j \frac{\partial y^1}{\partial x^j}, ..., \beta^j \frac{\partial y^m}{\partial x^j}\right).
\end{aligned}$$

This is clearly smooth with respect to $\alpha^1, ..., \alpha^m, \beta^1, ..., \beta^m$, because recall that $\partial y^k / \partial x^j = \partial(\psi \circ \varphi)/\partial x^j$ by our definition, and since $\mathcal{A}$ is a smooth atlas, we know that $\psi \circ \varphi^{-1} \in C^\infty$. By Theorem 1.9, we see that $\mathcal{A}_{TM}$ determines a smooth structure on TM, making TM a smooth manifold.

Conclusion

This completes the construction. See what have we got: we defined a smooth manifold TM, and a smooth and surjective map $\pi : TM \to M$. And finally, we get a bundle TM, which is called the ***tangent bundle*** of M, as it is constructed by gluing pieces of tangent spaces of M. Concretely, we have constructed a space that contains the information of all points of M along with all possible tangent spaces of M. To a physicist, M can be thought of as a space where the particle of interests move along, and the particle, when at a point $p \in M$, has a velocity $v \in T_pM$. As the particle moves along M, physicists may want to write down the pair (p, v), for $p \in M$ all points the particle was at, and v the corresponding velocity at p. What physicist will get eventually is a curve in TM.

Remark 3.4 (The Cotangent Bundle)**.** One can construct ***cotangent bundle*** of a smooth manifold M^m in basically the same way: In step 1, we define the cotangent bundle $T^*M = \coprod_{p \in M} T_p^*M$; The projection map π and the topology $\mathcal{O}_{T^*M}$ is defined exactly as in Step 2 and Step 3 above; The chart construction $(\pi^{-1}(U), \xi_\varphi)$ for a chart (U, φ) is similar to that in step 4: the first m coordinates of $\xi_\varphi(p, \alpha)$ for $p \in U, \alpha \in T_p^*M$ is still given by the coordinates $\varphi(p) \in \mathbb{R}^m$, and then express $\alpha = \alpha_i(dx^i)_p$, then the last m coordinates will be $\alpha_1, ..., \alpha_m$; Similar to Step 5, one can show that this indeed defines a smooth atlas.

Example 3.5 (Tangent Bundle of S^1). The tangent bundle TS^1 of the circle S^1 is diffeomorphic to $S^1 \times \mathbb{R}$, which is shown in Figure 3.1. To see this, note that for any $p = (a, b) \in S^1 \subset \mathbb{R}^2$, we have $\dim T_p S^1 = 1$, as S^1 is one dimensional. In fact, the calculation in Example 2.14 shows that $T_{(a,b)} S^1$ is spanned by a single tangent vector $\partial/\partial\theta|_{(a,b)} = (-b, a)$. Hence we can define a map $\psi : TS^1 \to S^1 \times \mathbb{R}$ via $\psi(p, \lambda \cdot \partial/\partial\theta|_p) = (p, \lambda)$. It is easy to check that ψ is a diffeomorphism.

3.3 Vector, Covector, and Tensor Fields

Definition 3.6. A smooth ***vector field*** X on M is a smooth section $X : M \to TM$ of the tangent bundle, i.e., $\pi \circ X = \mathrm{id}_M$.

Definition 3.7. We collect all smooth sections (vector fields) on a manifold M and denote it by

$$\Gamma(TM) := \{X : M \to TM \text{ smooth sections}\}$$

with addition and multiplication structure

$$(X \oplus \tilde{X})f = X(f) + \tilde{X}(f),$$

where $X(f) : p \mapsto X(p)f$. The second addition $+$ takes place in $C^\infty(M)$. And for multiplication $\odot : C^\infty(M) \times \Gamma(TM) \to \Gamma(TM)$ is defined by

$$(g \odot X)f = g \cdot X(f),$$

where the second multiplication $\cdot$ takes place in $C^\infty(M)$.

This set with addition and multiplication defined as above $(\Gamma(TM), \oplus, \otimes)$ is a C^∞ vector space (verify) over a ring. This is because $\Gamma(TM)$ does not satisfy the inverse of multiplication axiom and is, therefore, not a field, but a ring. We call such structure a module. Thus $(\Gamma(TM), \oplus, \otimes)$ is a $C^\infty(M)$-module.

A vector space over a ring, or a module, may not have all the properties a vector space have. For example, we **cannot** find vector fields, for any manifold M, $X_1, \cdots, X_d \in \Gamma(TM)$, such that for every vector field $X \in \Gamma(TM)$, $X = a^i X_i$. Although this can always be done locally in a chart $(\pi^{-1}(U), \xi_\varphi)$ using the smooth vector fields $\partial/\partial x^i$, where $\partial/\partial x^i|_p$ is the ith basis vector induced by the chart φ on T_pM.

Definition 3.8. Similarly, we could define the covector fields as

$$\Gamma(T^*M) := \{\alpha : M \to T^*M \text{ smooth sections}\}.$$

This is also a $C^\infty(M)$-module.

Then we can introduce the definition of a tensor field.

Definition 3.9. An (r, s) ***tensor field*** T is a multilinear map

$$T : \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{r} \times \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{s} \to C^\infty(M)$$

where we used $C^\infty(M)$ rather than $\mathbb{R}$ because the tensor field need not to be a constant. Instead, it varies from point to point.

Remark 3.10. Notice that, for every $p \in M$, and $X \in \Gamma(TM)$, we have $X(p) \in T_pM$. Hence, at a point, a vector field is just a vector in the tangent space. Similarly for $\alpha \in \Gamma(T^*M)$, we have $\alpha(p) \in T_p^*M$, hence at a point a coverctor field is just a covector. Similarly, at a point p, a tensor field is just a tensor in the sense of Section 2.3, where $V = T_pM$.

3.4 Integral Curves and Flows

Let M be a smooth manifold, and $\phi : I \to M$ a smooth map of an open interval $I \subset \mathbb{R}$ into M. Then for each $t \in I$ we have the associated linear map of tangent spaces,

$$d\phi_t = (\phi_*)_t : T_t I \to T_{\phi(t)} M.$$

For simplicity, we write

$$(\phi_*)_t \left(\frac{\partial}{\partial t} \right) = \frac{d\phi}{dt}$$

and speak of it as the *tangent vector* (or *velocity vector*) to our curve at the point $\phi(t) \in M$.

Now suppose that, in addition to a smooth curve $\phi : I \to M$, we have a smooth vector field V on M. Suppose, in addition, that for each $t \in I$, we have

$$\frac{d\phi}{dt} = V(\phi(t)).$$

In other words, the tangent vector to our curve ϕ at the point $\phi(t) \in M$ coincides with the value of the vector field $V(\phi(t))$ there. Then we say that $\phi : I \to M$ is an ***integral curve*** for the vector field V.

Theorem 3.11. *Let V be a C^∞ vector field on the C^∞ differentiable manifold M. Then there exists an open neighborhood U of $0 \times M$ in $\mathbb{R} \times M$ and a C^∞ map*

$$\Phi : U \to M$$

where we write $\phi_t(x) = \Phi(t, x)$, such that

1. *$\phi_0(x) = x$,*
2. *$V(x)$ is the tangent vector to the curve $t \mapsto \phi_t(x)$ at $t = 0$,*
3. *$\phi_t(\phi_s(x)) = \phi_{t+s}(x)$ whenever both sides are defined.*

Proof. The construction of Φ is an obvious application of integral curves and the existence and uniqueness theorem from ordinary differential equations. Only the last statement is interesting, which we prove: Fix $x \in M$, and consider the curves

$$\alpha(t) = \phi_t \phi_s(x) \qquad \beta(t) = \phi_{t+s}$$

with $\alpha(0) = \beta(0) = \phi_s(x)$. Then $\alpha'(t) = V(\alpha(t))$ and $\beta'(t) = V(\beta(t))$. Hence both $\alpha(t)$ and $\beta(t)$ are integral curves of the smooth vector field V with $\alpha(0) = \beta(0)$. By uniqueness of solutions, we have $\alpha(t) = \beta(t)$. Hence $\phi_t \phi_s = \phi_{t+s}$, as desired. Note that this shows that $\phi : t \mapsto \phi_t$ is a group homomorphism, whenever t is defined. □

Remark 3.12. If the vector field V has compact support (in particular, if M is compact), then $U = \mathbb{R} \times M$ and each $\phi_t : M \to M$ is a diffeomorphism.

In that case, the collection $\{\phi_t : t \in \mathbb{R}\}$ is called a ***one-parameter group*** of diffeomorphisms of M, equivalent to a ***flow*** on M, and the vector field V is said to be its ***infinitesimal generator***.

Example 3.13. Let $M = \mathbb{R}$ and $V = \partial/\partial t$. Then we can take $U = \mathbb{R} \times M$ and the flow is given by

$$\phi_t(x) = x + t.$$

3.5 Problems

1. Write down the induced atlas for $T(TM)$, the tangent bundle of the tangent bundle of M, and compute the corresponding coordinate change.

2. Let M be a nonempty positive-dimensional smooth manifold with or without boundary. Show that $\Gamma(TM)$, the set of smooth vector fields on M, is an infinite-dimensional vector space.

3. (a) Find two "really different" smooth vector fields on the two-sphere S^2 which vanish (i.e., are zero) at just two points.
   (b) Find a smooth vector field on S^2 which vanishes at just one point.
   (c) It is impossible to find a smooth (or even just continuous) vector field on S^2 which never vanishes. The phrase "you can't comb the hair on a billiard ball" refers to this fact. Try to get an intuitive feeling for its validity.
   (d) Find a smooth vector field on the 3-sphere S^3 which never vanishes.

4. Consider the vector field
$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$
in the plane $\mathbb{R}^2$, and consider the curves
$$\phi(t) = (a\cos t,\, a\sin t)$$
for each real value of $a \geq 0$.

   Show that these are all integral curves for the vector field V, and that they fill out the plane.

   Note that when $a = 0$, the integral curve is "stationary", that is, it consists of a single point. This is understandable, since the vector field V is zero at the origin.

5. Interpret the differential equation
$$\frac{dy}{dx} = -y^2$$
as a vector field on the plane, and show that it has integral curves which are not defined for all time.

6. Consider the maps $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$\phi_t(x, y) = (xe^{2t},\, ye^{-3t}).$$
   (a) Show that $\{\phi_t : t \in \mathbb{R}\}$ is a one-parameter group of linear maps of $\mathbb{R}^2$ to itself.
   (b) Find the corresponding vector field V on $\mathbb{R}^2$.
   (c) Sketch V and its integral curves.

7. On $\mathbb{R}^3$, consider the vector fields
$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \qquad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \qquad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$
Show that the flow of $aX + bY + cZ$ is a rotation of $\mathbb{R}^3$ about the line through the origin and the point (a, b, c).

8. What is the tangent bundle of S^n? Answer: $TS^n \cong (S^n \times S^n) - \Delta$.

9. (Lie derivative) We continue to work in the smooth manifold M, but this time with two smooth vector fields V and W.

   We want to measure the rate of change of W with respect to V.

   Again let $\{\phi_t\}$ be the local flow generated by V.

   We define the *Lie derivative* of W with respect to V at the point $x \in M$ to be the vector
$$(\mathcal{L}_V W)(x) = \lim_{t \to 0} \frac{(\phi_{-t})_* W_{\phi_t(x)} - W_x}{t},$$
   lying in the tangent space $T_x M$.

   (a) Show that
$$\mathcal{L}_V(W_1 + W_2) = \mathcal{L}_V(W_1) + \mathcal{L}_V(W_2).$$

   (b) Show that
$$\mathcal{L}_V(fW) = (\mathcal{L}_V f)W + f(\mathcal{L}_V W) = (Vf)W + f(\mathcal{L}_V W).$$

   Hint: Remember that the Leibniz Rule in Freshman Calculus was proved by adding and subtracting a convenient middle term.

10. Let V be a smooth vector field on the smooth manifold M, and let x be a point of M at which $V(x) \neq 0$. Show how to find local coordinates $(x^1, x^2, \ldots, x^n)$ about x, in terms of which $V = \partial/\partial x^1$.

11. Show that for two vector fields V, W on M with the local coordinate representation $V = v^i \partial/\partial x^i$ and $W = w^j \partial/\partial x^j$, then we have the local formula
$$\mathcal{L}_V W = \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

12. (Lie bracket) Given two vector fields V, W on M, their *Lie bracket* $[V, W]$ is another vector field, defined by
$$[V, W]f = (VW - WV)f$$
   for all $f \in C^\infty(M)$. Derive an expression of $[V, W]$ in local coordinates, check that it agree with the local coordinate expression of $\mathcal{L}_V W$, and conclude that
$$\mathcal{L}_V W = [V, W].$$

13. Using the equivalence of Lie derivatives and Lie brackets, $\mathcal{L}_V W = [V, W]$, whenever convenient, prove the following.

   (a) $\mathcal{L}_V W = -\mathcal{L}_W V$, and hence $\mathcal{L}_V V = 0$.
   (b) $\mathcal{L}_{V_1 + V_2} W = \mathcal{L}_{V_1} W + \mathcal{L}_{V_2} W$.
   (c) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$.
   (d) $\mathcal{L}_U [V, W] = [\mathcal{L}_U V, W] + [V, \mathcal{L}_U W]$.

   Thus Lie derivatives act as derivations with respect to the bracket "product" of vector fields.

(e) $\mathcal{L}_{[V,W]} = \mathcal{L}_V\mathcal{L}_W - \mathcal{L}_W\mathcal{L}_V$. A more convinient notation for the right hand side is $[\mathcal{L}_V, \mathcal{L}_W]$. Hence the statement becomes $\mathcal{L}_{[V,W]} = [\mathcal{L}_V, \mathcal{L}_W]$.

(f) $[fV, gW] = fg[V, W] + f(Vg)W - g(Wf)V$.

14. A vector field W is said to be *invariant* under a diffeomorphism $\Phi : M \to M$ if $\Phi_* W = W$.

    (a) Show that if ϕ_t is the local flow of V, then V is invariant under the local diffeomorphisms ϕ_t.

    (b) Show that a vector field W is invariant under a vector field V iff $\mathcal{L}_V W = 0$.

15. Let $h : M \to N$ be a diffeomorphism of smooth manifolds. Let V and W be vector fields on M. Show that
$$h_*[V, W] = [h_*V, h_*W].$$

16. (a) Let $h\colon M \to N$ be a diffeomorphism of smooth manifolds. Let V be a vector field on M with corresponding local flow $\{\phi_t\}$. Show that the vector field h_*V on N generates the local flow $\{h \circ \phi_t \circ h^{-1}\}$.

    (b) Let $h : M \to M$ be a diffeomorphism, and let V be a vector field on M. Show that $h_*V = V$ if and only if
$$h \circ \phi_t = \phi_t \circ h$$
    for all t.

17. Let V and W be vector fields on M, with local flows $\{\phi_s\}$ and $\{\psi_t\}$. Show that $[V, W] \equiv 0$ if and only if
$$\phi_s \circ \psi_t = \psi_t \circ \phi_s$$
for all s and t.

18. Let $V_1, V_2, \ldots, V_k$ be smooth vector fields on M which are linearly independent at every point of some neighborhood of $x \in M$, and such that $[V_i, V_j] = 0$ at every point of that neighborhood.

    Prove that there are local coordinates $x^1, x^2, \ldots, x^n$ on some (generally smaller) neighborhood of x such that
$$V_1 = \frac{\partial}{\partial x^1}, \quad V_2 = \frac{\partial}{\partial x^2}, \quad \ldots, \quad V_k = \frac{\partial}{\partial x^k}$$
    throughout that neighborhood.

19. Show that in Euclidean space $\mathbb{R}^n$, if the vector fields V and W are both divergence-free, then their Lie bracket $[V, W]$ is also divergence-free.

20. View S^3 as a Lie subgroup of unit quaternions. Name a tangent vector to S^3 at the point u by the name of a quaternion orthogonal to u. For example, at the identity 1, any imaginary quaternion $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ will denote a tangent vector there.

    (a) Consider the left-invariant vector fields X, Y and Z on S^3 given by
$$X_u = u\mathbf{i}, \quad Y_u = u\mathbf{j}, \quad Z_u = u\mathbf{k}.$$
    Show that the corresponding flows are given by
$$\varphi_t(u) = u(\cos t + \mathbf{i}\sin t), \quad \psi_t(u) = u(\cos t + \mathbf{j}\sin t), \quad \zeta_t(u) = u(\cos t + \mathbf{k}\sin t).$$

(b) Compute the Lie bracket $[X, Y]$ directly from the definition,

$$[X, Y]_1 = (\mathcal{L}_X Y)_1 = \lim_{t \to 0} \left((\varphi_{-t})_* Y_1 - Y_1 \right) / t,$$

and show that $[X, Y] = 2Z$. Conclude by symmetry that $[Y, Z] = 2X$ and $[Z, X] = 2Y$.

Chapter 4

Differential Forms

4.1 Symmetric and Alternating Tensors, Wedge Product

Denote
$$T^{r,s}(V) = \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}.$$

Then $T^{0,k}(V) = \otimes^k V^*$ is the set of multilinear maps $\prod^k V \to \mathbb{R}$, defined via $\alpha_1 \otimes \cdots \alpha_k : (v_1, ..., v_k) \mapsto \alpha_1(v_1)\alpha_2(v_2)\cdots\alpha_k(v_k)$. Then we have two types of important tensors: If $\alpha \in T^{0,k}(V)$, then α is said to be ***symmetric*** if

$$\alpha(v_1, ..., v_k) = \alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

for any $(v_1, ..., v_k) \in \prod^k V$ and $\sigma \in S_k$. Equivalently, one can define α to be the unchanged by interchanging any pair of arguments. We say α is ***alternating*** if

$$\alpha(v_1, ..., v_k) = \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

for any $(v_1, ..., v_k) \in \prod^k V$ and $\sigma \in S_k$. Equivalently, one can define α to be the tensor that takes the negative of the original value by interchanging any pair of arguments.

Now, given any $\alpha \in T^{0,k}(V)$, we can define its ***symmetrization*** $\mathscr{S}(\alpha)$ to be

$$\mathscr{S}(\alpha)(v_1, ..., v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

It is clear that $\mathscr{S}(\alpha)$ is a symmetric tensor. One can also define the ***antisymmetrization*** $\mathscr{A}(\alpha)$ to be

$$\mathscr{A}(\alpha)(v_1, ..., v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}).$$

Proposition 4.1. For any $\alpha \in T^{0,k}(V)$, its antisymmetrization $\mathscr{A}(\alpha)$ is alternating.

Proof. Let $\tau \in S_k$ and $(v_1, ..., v_k) \in \prod^k V$. Then we calculate

$$\begin{aligned}
\mathscr{A}(\alpha)(v_{\tau(1)}, ..., v_{\tau(k)}) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\alpha(v_{\sigma\circ\tau(1)}, ..., v_{\sigma\circ\tau(k)}) \\
&= \frac{1}{k!}\sum_{\eta \in S_k} \frac{\operatorname{sgn}(\eta)}{\operatorname{sgn}(\tau)}\alpha(v_{\eta(1)}, ..., v_{\eta(k)}) \\
&= \operatorname{sgn}(\tau)\mathscr{A}(\alpha)(v_{\tau(1)}, ..., v_{\tau(k)})
\end{aligned}$$

where we made the substitution $\eta = \sigma \circ \tau$ and we used $\operatorname{sgn}(\eta) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$. But since $\operatorname{sgn}(\tau) = \pm 1$, so dividing by $\operatorname{sgn}(\tau)$ is the same as multiplying by it. $\square$

Now we may define

$$\Lambda^k V^* = \mathscr{A}(\otimes^k V^*) = \{\sigma \in \otimes^k V^* : \sigma \text{ alternating}\}.$$

If $\alpha \in \Lambda^k V^*$, then we define $\deg \alpha := k$ to be the ***degree*** of α.

Definition 4.2. Let V^n be a vector space, and $\epsilon^1, ..., \epsilon^n$ any basis for V^*. Let $I = (i_1, ..., i_k)$ be a multi-index of length k, where $1 \leq i_s \leq n$. We define the so called ***elementary alternating tensor*** ϵ^I to be the tensor such that for any $v_1, ..., v_k \in V$,

$$\epsilon^I(v_1, ..., v_k) := \begin{vmatrix} \epsilon^{i_1}(v_1) & \cdots & \epsilon^{i_1}(v_k) \\ \vdots & \cdots & \vdots \\ \epsilon^{i_k}(v_1) & \cdots & \epsilon^{i_k}(v_k) \end{vmatrix}.$$

By definition, ϵ^I is alternating because the determinant is, hence $\epsilon^I \in \Lambda^k V^*$.

Lemma 4.3. *Let $e_1, ..., e_n$ be a basis for V, and $\epsilon^1, ..., \epsilon^n$ be the dual basis for V^*. Let ϵ^I be defined as above where $I = (i_1, ..., i_k)$.*

(1) If I has an repeated index, then $\epsilon^I = 0$.

(2) If $J = I_\sigma := (i_{\sigma(1)}, ..., i_{\sigma(k)})$ where $\sigma \in S_k$, then $\epsilon^J = (\operatorname{sgn} \sigma)\epsilon^I$.

(3 We have, for a multi-index $J = (j_1, ..., j_k)$

$$\epsilon^I(e_{j_1}, ..., e_{j_k}) = \begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & \cdots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{vmatrix} =: \delta^I_J.$$

Proof. For (1), if I has an repeated index, then the determinant in the definition of ϵ^I will have two identical rows, hence is equal to zero; Next, (2) is immediate because interchanging I by σ amounts to interchanging columns of the determinant in the definition of ϵ^I by σ; Finally, (3) is just obvious from the definition of ϵ^I. $\square$

Theorem 4.4. *Let V^n be a vector space with basis $e_1, ..., e_n$ and $\epsilon^1, ..., \epsilon^n$ the dual basis for V^*. Then, for each positive $k \leq n$, the collection of ϵ^I with increasing multi-index of length k, i.e.,*

$$\mathcal{E} := \{\epsilon^I : I = (i_1, ..., i_k), i_1 < i_2 < \cdots < i_k\}$$

is a basis for $\Lambda^k V^*$. Therefore, $\dim \Lambda^k V^* = \binom{n}{k}$. Specifically, $\dim \Lambda^k V^* = 0$ if $k > n$.

Proof. To show that $\mathcal{E}$ spans $\Lambda^k V^*$: For each multi-index $I = (i_1, ..., i_k)$, define $\alpha_I := \alpha(e_{i_1}, ..., e_{i_k})$. Since α is alternating, $\alpha_I = 0$ if I has an repeated index and $\alpha_J = (\operatorname{sgn} \sigma)\alpha_I$ if $J = I_\sigma$. Thus the collection of real numbers α_I with I increasing determines α completely. We claim that

$$\alpha = \sum_{I \text{ increasing}} \alpha_I \epsilon^I.$$

To see this, let $J = (j_1, ..., j_k)$ be any multi-index set. Note that by the definition of δ^I_J above, we can see that $\delta^I_J = 0$ if I or J has an repeated index or if J is not a permutation of I; and $\delta^I_J = \operatorname{sgn}\sigma$ if neither I nor J has a repeated index and $J = I_\sigma$ for some $\sigma \in S_k$. Then by Lemma 4.3,

$$\sum_{I \text{ increasing}} \alpha_I \epsilon^I(e_{j_1}, ..., e_{j_k}) = \sum_{I \text{ increasing}} \alpha_I \delta^I_J = \alpha_J = \alpha(e_{j_1}, ..., e_{j_k}).$$

This proves the claim, showing that $\mathcal{E}$ spans $\Lambda^k V^*$.

To show that $\mathcal{E}$ is linearly independent: Suppose that for some collections of coefficients α_I we have

$$\sigma := \sum_{I \text{ increasing}} \alpha_I \epsilon^I = 0.$$

Let J be any increasing multi-index. Then again we have

$$0 = \sum_{I \text{ increasing}} \alpha_I \epsilon^I(e_{j_1}, ..., e_{j_k}) = \sum_{I \text{ increasing}} \alpha_I \delta^I_J = \alpha_J = 0.$$

Hence $\alpha_J = 0$ for all coefficients α_J. Thus $\sigma \equiv 0$. This proves linear independence. $\square$

Definition 4.5. The ***wedge product*** $\wedge$ is a bilinear map $\wedge : \wedge^r V^* \times \wedge^s V^* \to \wedge^{r+s} V^*$ by

$$\alpha \wedge \beta = \frac{(r+s)!}{r!s!} \mathscr{A}(\alpha \otimes \beta).$$

Theorem 4.6. *Let V^n be a vector space with basis $e_1, ..., e_n$ and $\epsilon^1, ..., \epsilon^n$ the dual basis for V^*. For any multi-index $I = (i_1, ..., i_k), J = (j_1, ..., j_l)$, define $IJ := (i_1, ..., i_k, j_1, ..., j_l)$. Then we have*

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}.$$

Proof. By linearity, it suffices to show that $\epsilon^I \wedge \epsilon^J$ and ϵ^{IJ} give the same result for any sequence $(e_{p_1}, ..., e_{p_{k+l}})$ of basis vectors. Denote $P = (p_1, ..., p_{k+l})$.

We first reduce the problem by restricting our attention to nontrivial multi-index P in which the equation is not obviously true. Note that:

- If P contains an index that does not appear in either I or J, then by Lemma 4.3, both sides are zero.

- If P contains a repeated index, then since both sides are clearly alternating tensors, both sides are again zero.

- We will discuss the case where $P = IJ$ and has no repeated index later.

- If P is a permutation of IJ and has no repeated indices, and if $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$ is true on e_P where $P = IJ$, then say $P = (IJ)_\sigma$ for some $\sigma \in S_{k+l}$. Again because both sides are alternating, we see that the previous case implies $(\operatorname{sgn}\sigma)(\epsilon^I \wedge \epsilon^J) = (\operatorname{sgn}\sigma)\epsilon^{IJ}$, which proves what we want.

Therefore, we only need to check the case where $P = IJ$ and has no repeated indices. Direct computation using the definition shows that

$$\begin{aligned}
\epsilon^I \wedge \epsilon^J(e_{p_1}, ..., e_{p_{k+l}}) &= \frac{(k+l)!}{k!l!} \mathscr{A}(\epsilon^I \otimes \epsilon^J)(e_{p_1}, ..., e_{p_{k+l}}) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma) \epsilon^I(e_{p_{\sigma(1)}}, ..., e_{p_{\sigma(k)}}) \epsilon^J(e_{p_{\sigma(k+1)}}, ..., e_{p_{\sigma(k+l)}}).
\end{aligned}$$

By Lemma 4.3 again, the only terms in the sum above that is nonzero are the ones in which σ permutes the first k indices and the last l indices of P separately. In other words, $\sigma = \tau\eta$ where $\tau \in S_k$ permutes the index $1, ..., k$, and $\eta \in S_l$ permutes the index $k+1, ..., k+l$. Since $\operatorname{sgn}(\tau\eta) = (\operatorname{sgn}\tau)(\operatorname{sgn}\eta)$, we have

$$\begin{aligned}
\epsilon^I \wedge \epsilon^J(e_{p_1}, ..., e_{p_{k+l}}) &= \frac{1}{k!l!} \sum_{\tau \in S_k, \eta \in S_l} (\operatorname{sgn}\tau)(\operatorname{sgn}\eta)\epsilon^I(e_{p_{\tau(1)}}, ..., e_{p_{\tau(k)}})\epsilon^J(e_{p_{k+\eta(1)}}, ..., e_{p_{k+\eta(l)}}) \\
&= \frac{1}{k!} \sum_{\tau \in S_k} (\operatorname{sgn}\tau)\epsilon^I(e_{p_{\tau(1)}}, ..., e_{p_{\tau(k)}}) \cdot \frac{1}{l!} \sum_{\eta \in S_l} (\operatorname{sgn}\eta)\epsilon^J(e_{p_{k+\eta(1)}}, ..., e_{p_{k+\eta(l)}}) \\
&= \mathscr{A}(\epsilon^I)(e_{p_1}, ..., e_{p_k}) \cdot \mathscr{A}(\epsilon^J)(e_{p_{k+1}}, ..., e_{p_{k+l}}) \\
&= \epsilon^I(e_{p_1}, ..., e_{p_k}) \cdot \epsilon^J(e_{p_{k+1}}, ..., e_{p_{k+l}}) \\
&= 1,
\end{aligned}$$

where the second to last inequality is because ϵ^I, ϵ^J are already alternating, and the last inequality is because of the assumption $P = IJ$. Note that since $P = IJ$, clearly $\epsilon^{IJ}(e_P) = 1$ by Lemma 4.3, so indeed $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$. $\square$

Proposition 4.7. The wedge product is bilinear, associative, and anticommutative, i.e., $\omega \wedge \eta = (-1)^{\deg\omega \deg\eta}\eta \wedge \omega$.

Proof. Bilinearity is immediate from definition of wedge product, because both the tensor product and alternation are bilinear. To prove associativity, note on basis elements associativity holds:

$$(\epsilon^I \wedge \epsilon^J) \wedge \epsilon^K = \epsilon^{IJ} \wedge \epsilon^K = \epsilon^{IJK} = \epsilon^I \wedge \epsilon^{JK} = \epsilon^I \wedge (\epsilon^J \wedge \epsilon^K).$$

The general case follows from bilinearity. To show anticommutativity, note that again this holds on basis elements:

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ} = (-1)^{|I|\cdot|J|}\epsilon^{JI} = (-1)^{|I|\cdot|J|}\epsilon^J \wedge \epsilon^I.$$

The general case follows again from bilinearity. $\square$

Corollary 4.8. *Let $I = (i_1, ..., i_k)$ be an multi-index, then $\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} = \epsilon^I$.*

Proof. This is immediate from Theorem 4.6 and induction. $\square$

Definition 4.9. Let V^n be a vector space and $L : V \to V$ be a linear map. Then this induces a map $L(v_1 \wedge \cdots \wedge v_n) = (Lv_1) \wedge \cdots \wedge (Lv_n)$.

Now, suppose that $Lv_i = \sum_j a_i^j v_j$. Then we see that

$$L(v_1 \wedge \cdots \wedge v_n) = \sum_J (a_1^{j_1} \cdots a_n^{j_n})(v_{j_1} \wedge \cdots v_{j_n})$$

where $J = (j_1, ..., j_n)$ runs over all n-tuples where each entry takes value between $1, ..., n$. However, the previous theorem tells us that if J contains any repeated indices, then $v_{j_1} \wedge \cdots \wedge v_{j_n} = 0$. Therefore we can in fact take J to run over all permutations of $(1, ..., n)$. Therefore the equation above actually becomes

$$\begin{aligned}
L(v_1 \wedge \cdots \wedge v_n) &= \sum_{\sigma \in S_n} (a_1^{\sigma(1)} \cdots a_n^{\sigma(n)})(v_{\sigma(1)} \wedge \cdots v_{\sigma(n)}) && (4.1) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(a_1^{\sigma(1)} \cdots a_n^{\sigma(n)})(v_1 \wedge \cdots \wedge v_n) && (4.2) \\
&= (\det L)(v_1 \wedge \cdots \wedge v_n) && (4.3)
\end{aligned}$$

Definition 4.10. Let V^n be a vector space and let $v_1, ..., v_n$ and $w_1, ..., w_n$ be two basis for V. Then these two basis are said to be ***equivalent*** if the linear map $A : V \to V$ such that $w_j = Av_j$ has positive determinant, i.e., $\det A > 0$.

Corollary 4.11. *The two basis $v_1, ..., v_n$ and $w_1, ..., w_n$ of V are equivalent iff*

$$w_1 \wedge \cdots \wedge w_n = \lambda(v_1 \wedge \cdots \wedge v_n)$$

for some $\lambda > 0$.

4.2 Differential Forms

Now we generalize the notion of alternating k-tensors to alternating k-tensor fields on manifolds, these are known as differential k-forms. Pointwise speaking, at every point $p \in M$ on a manifold, we take $V = T_pM$ in the previous section. But we need to define differential forms globally instead of just at every point. Therefore, we first need to construct a bundle $\Lambda^k T^*M$. This will be almost identical to the construction of the tangent bundle and the cotangent bundle in section 3.2.

First, we define, on the level of sets, that

$$\Lambda^k T^*M = \coprod_{p \in M} \Lambda^k T_p^*M.$$

There is an obvious projection map $\pi : (p, \alpha) \mapsto p$ where $p \in M, \alpha \in \Lambda^k T_p^*M$. We make this set into a topological space by defining $\mathcal{O}_{\Lambda^k T^*M} = \{\pi^{-1}(U) : U \in \mathcal{O}_M\}$. It remains to define an atlas on $\Lambda^k T^*M$ and check that it is smooth. We define

$$\mathcal{A}_{\Lambda^k T^*M} = \{(\pi^{-1}(U), \xi_\varphi) : (U, \varphi) \in \mathcal{A}_M\}$$

where ξ_φ is defined as follows: Let $p \in U$ and $dx^1, ..., dx^m$ be the induced basis for T_p^*M via the chart map φ. Then any $\alpha \in \Lambda^k T_p^*M$ can be written as a summand

$$\alpha = \sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1,...,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where the summand contains $\binom{m}{k}$ terms (some could be zero) by Theorem 4.4 and Corollary 4.8. Then define

$$\xi_\varphi : (p, \alpha) \mapsto (\varphi(p), a_I) \in \mathbb{R}^m \times \mathbb{R}^{\binom{m}{k}}$$

where the first m coordinates are the coordinates of $\varphi(p)$, and the last $\binom{m}{k}$ coordinates are the coefficients $a_{i_1,...,i_k}$. Clearly this map is invertible by definition. To check that this is a smooth atlas, i.e., the transition maps are smooth, let (V, ψ) be an overlapping chart with (U, φ) and let $dy^1, ..., dy^m$ be the induced basis for T_p^*M via the chart map ψ. Then

$$\begin{aligned}
\alpha &= \sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1,...,i_k} \left(\frac{\partial x^{i_1}}{\partial y^{j_1}} dy^{j_1}\right) \wedge \cdots \wedge \left(\frac{\partial x^{i_k}}{\partial y^{j_k}} dy^{j_k}\right) \\
&= \sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1,...,i_k} \left(\frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}}\right) dy^{j_1} \wedge \cdots \wedge dy^{j_k},
\end{aligned}$$

where the indices $j_1, ..., j_k$ are summed over using the summation convention. Since $\partial x^{i_s}/\partial y^{j_s}$ are smooth functions because φ, ψ are smoothly compatible charts, we see that from definition $\xi_\psi \circ \xi_\varphi^{-1}$ is smooth. This concludes the construction.

Definition 4.12. A section of $\Lambda^k T^*M$ is called a ***differential*** ***k-form***. The integer k is called the ***degree*** of the form. We say α is a smooth k-form is as a section $\alpha : M \to \Lambda^k T^*M$ is a smooth map between manifolds. We denote the space of smooth k-forms by $\Omega^k(M)$.

Proposition 4.13. A k-form α can be written, locally in a chart (U, φ), as a summand

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_k \leq m} a_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Then α is smooth iff the functions $a_{i_1,\dots,i_k} \in C^\infty(U)$, for every open set U in the atlas $\mathcal{A}_M$.

Proof. First, assume $\alpha : M \to \Lambda^k T^*M$ is a smooth section, then for every chart (U, φ) in the atlas, the component functions are just the components of the composition of smooth maps $\xi_\varphi \circ \alpha \circ \varphi^{-1}$, which are of course smooth. The converse implies the smoothness of $\alpha : M \to \Lambda^k T^*M$ as a section follows immediately from Definition 1.10. $\square$

Corollary 4.14. *If (U, φ, x^i) and (V, ψ, y^i) are two overlapping smooth coordinate charts on M^n, then the following identity holds on $U \cap V$:*

$$dy^1 \wedge \cdots \wedge dy^n = \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n = [J(\psi \circ \varphi^{-1}) \circ \varphi] dx^1 \wedge \cdots \wedge dx^n.$$

Proof. This follows immediately from (4.3). $\square$

4.3 Orientation on Manifolds

Definition 4.15. A smooth atlas $\mathcal{A}$ of a smooth manifold M is called an ***orientation atlas*** if the following holds true: For all (U, φ, x^i) and (V, ψ, y^j) in $\mathcal{A}$ such that $U \cap V \neq \emptyset$, we have

$$J(\psi \circ \varphi^{-1}) = \det\left(\frac{\partial y^i}{\partial x^j}\right) > 0$$

for all $x \in U \cap V$. A maximal orientation atlas is called an ***orientation***. A manifold M is said to be ***orientable*** if an orientation exists. An ***oriented manifold*** is a manifold with a chosen orientation.

Theorem 4.16. *An n-dimensional manifold M is orientable iff M admits a nowhere vanishing n-form ω.*

Proof. Let ω be a nowhere vanishing n-form on M. Then for each chart (U, φ, x^i) there exists a nowhere vanishing function $f \neq 0$ such that $\omega = f dx^1 \wedge \cdots \wedge dx^n$. We can always take such a chart near each point so that $f > 0$, otherwise we can replace x^1 by $-x^1$. Now suppose $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two such charts. Then suppose $\omega = f dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha = g dx^1_\beta \wedge \cdots \wedge dx^n_\beta$ where $f, g > 0$. Then

$$0 < g = \omega(\partial^\beta_1, ..., \partial^\beta_n) = J(\varphi_\beta \circ \varphi_\alpha^{-1}) \omega(\partial^\alpha_1, ...\partial^n_\alpha) = J(\varphi_\beta \circ \varphi_\alpha^{-1}) f.$$

So $J(\varphi_\beta \circ \varphi_\alpha^{-1}) = g/f > 0$.

Conversely, let $\mathcal{A}$ be an orientation. For each local chart U_α, we consider the form $\omega_\alpha = dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$. Pick a partition of unity ρ_α suboridnate to the open cover U_α. Then consider the n-form

$$\omega = \sum_\alpha \rho_\alpha \omega_\alpha.$$

We claim that ω is nowhere vanishing. For each $p \in M$, there is a neighborhood U of p such that the sum $\sum_\alpha \rho_\alpha \omega_\alpha$ reduces to a finite sum $\sum_{i=1}^k \rho_i \omega_i$. It follows that near p, one has

$$\omega(\partial_1^1, ..., \partial_n^1) = \sum_i J(\varphi_i \circ \varphi_1^{-1}) \rho_i > 0.$$

This shows that ω is nowhere vanishing. □

With this theorem in mind, we can formulate the following (equivalent) definition:

Definition 4.17. We shall say M is ***orientable*** if it is possible to define a smooth n-form ω on M which is nonvanishing, in which case M is said to be ***oriented*** by the choice of ω in the following sense: Two nonvanishing n-forms ω and η on M induces two orientations as shown by the previous theorem. We call those two orientations equivalent iff $\omega = f\eta$ for some smooth function f that is positive everywhere. A nowhere vanishing form on M is usually called an ***orientation form***, for the obvious reason.

Theorem 4.18. *Every connected orientable smooth manifold has exactly two orientations.*

Proof. Let M be a connected smooth orientable manifold. By the previous theorem, there exists a nonvanishing n-form ω on M. Any other choice of orientation will induce a nonvanishing n-form η on M. Since $\omega = f\eta$ for some smooth function $f : M \to \mathbb{R}$ such that $f(p) \neq 0$ for all $p \in M$, then $f(M) \subset \mathbb{R}$ is a connected subset which does not contain 0, hence f is either always positive or always negative. If $f > 0$, then ω and η induces the same orientation at T_pM for each $p \in M$, hence they induce the same orientation. Therefore assume $f < 0$ and ω, η induces different orientations on M. Then any other orientation will induce a non-vanishing n-form τ such that $\tau = g\omega$ for some smooth function $g \in C^\infty(M)$. So $g > 0$ or $g < 0$, in which case τ induces the same orientation as ω or η does. □

4.4 Pullback and Exterior Differentiation

Let M^n be a smooth manifold, and let $\Omega^k(M)$ denote the set of smooth k-form on M, or equivalently $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$, which is the set of all smooth sections of $\Lambda^k T^*M$.

Definition 4.19. Let $F : M \to N$ be a smooth map. Recall that $dF_p : T_pM \to T_pN$ is given by $dF_p(v)(f) = v(f \circ F)$ for $v \in T_pM, f \in C^\infty(N)$. Then its dual map $F_p^* : T^*_{F(p)}N \to T_p^*M$ is called the ***pullback*** of F at p. If $\omega \in \Omega^k(N)$, then we can also define the pullback $F^*\omega \in \Omega^k(M)$ given by

$$(F^*\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(dF_p(v_1), ..., dF_p(v_k)).$$

By definition, if $\omega \in \Omega^0(N)$ is a smooth function, then $F^*\omega := \omega \circ F$.

Using this definition, we have the following easy observation:

Lemma 4.20. *Suppose $F : M \to N$ is smooth. Then*

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge F^*(\eta)$.

(c) In any smooth chart, we have that

$$F^*\left(\sum_I \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

The ***exterior derivative*** d is a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for each k, constructed as follows: Let $\omega \in \Omega^k(M)$. Then in a local chart (U, φ), we may write

$$\omega = \sum_{I:i_1<\ldots<i_k} \omega_I dx^I$$

where ω_I's are smooth functions $U \to \mathbb{R}$. Then we define

$$d\omega = \sum_{I:i_1<\ldots<i_k} d\omega_I \wedge dx^I = \sum_{I:i_1<\ldots<i_k} \left(\sum_j \frac{\partial \omega_I}{\partial x^j} dx^j \right) \wedge dx^I \tag{4.4}$$

$$= \sum_{i_1<\ldots<i_k} \left(\sum_j \frac{\partial \omega_{i_1\cdots i_k} \circ \varphi^{-1}}{\partial x^j} dx^j \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{4.5}$$

Since partial derivatives commutes, using the summation convention we see that

$$\begin{aligned} d(d\omega) &= d\left(\sum_{I,j} \frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I \right) \\ &= \sum_{I,j,k} \frac{\partial^2 \omega_I}{\partial x^j \partial x^k} dx^k \wedge dx^j \wedge dx^I \\ &= \sum_{k<j} \frac{\partial^2 \omega_I}{\partial x^j \partial x^k} dx^k \wedge dx^j \wedge dx^I + \sum_{k>j} \frac{\partial^2 \omega_I}{\partial x^j \partial x^k} dx^k \wedge dx^j \wedge dx^I \\ &= \sum_{k<j} \frac{\partial^2 \omega_I}{\partial x^j \partial x^k} dx^k \wedge dx^j \wedge dx^I - \sum_{k<j} \frac{\partial^2 \omega_I}{\partial x^j \partial x^k} dx^k \wedge dx^j \wedge dx^I = 0. \end{aligned}$$

Therefore, we see that $d \circ d : \Omega^k(M) \to \Omega^{k+2}(M)$ is identically the zero map, i.e., $d \circ d = 0$. More frequently this is compactly written as $d^2 = 0$.

Note that this definition (potentially) depends on the chart we choose. But it turns out that this definition is independent of charts.

Proposition 4.21. Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be an operator that satisfies the following properties:

(i) d is linear over $\mathbb{R}$.

(ii) For any $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

(iii) $d^2 = 0$.

(iv) For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.

Then d exists and is unique. Moreover, in any smooth coordinate chart, d is given by (4.4).

Proof. First, suppose $M = \mathbb{R}^n$. Then for $\omega \in \Omega^k(\mathbb{R}^n)$ we define its exterior derivative to be

$$d\left(\sum_J \omega_J dx^J \right) = \sum_J d\omega_J \wedge dx^J = \sum_J \left(\sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \right) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Then one can check that d defined in $\mathbb{R}^n$ satisfies the properties (i)-(iv). Moreover, one can check that if $F : U \to V$ is a smooth map in Euclidean space, then we have

$$F^*(d\omega) = d(F^*\omega) \tag{4.6}$$

the detailed calculation is in [1, Proposition 14.23]. Now for a general manifold M, we first show the existence of d. Let $\omega \in \Omega^k(M)$, and we define d on chart (U, φ) by

$$d\omega = \varphi^* d(\varphi^{-1*}\omega).$$

This is well-defined, since for any other chart (V, ψ), the map $\varphi \circ \psi^{-1}$ is a diffeomorphism. Hence by (4.6) we have

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) = d((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega) = d(\psi^{-1*}\omega).$$

Therefore, multiplying both sides by ψ^*, we get that $\varphi^* d(\varphi^{-1*}\omega) = \psi^* d(\psi^{-1*}\omega)$. It follows quite easily from the calculation in $\mathbb{R}^n$ that this definition satisfies (i)-(iv).

Next, we prove uniqueness. Suppose d is any operator satisfying (i)-(iv). We claim that if $\omega_1, \omega_2 \in \Omega^k(M)$ such that $\omega_1 = \omega_2$ on an open set $U \subset M$, then $d\omega_1 = d\omega_2$ on U. To see this, let $p \in U$ and $\eta = \omega_1 - \omega_2$. Let $\psi \in C^\infty(M)$ be a bump function that is identically 1 on some neighborhood of p and supported in U. Then $\psi\eta$ is identically zero. So (i)-(iv) implies that

$$0 = d(\psi\eta) = d\psi \wedge \eta + \psi d\eta.$$

Evaluating this at p using $\psi(p) = 1$ and $d\psi_p = 0$, we have that $d\omega_1|_p - d\omega_2|_p = d\eta_p = 0$.

Now let $\omega \in \Omega^k(M)$ be arbitrary, and (U, φ) be a smooth chart on M. Write $\omega = \sum_I \omega_I dx^I$ on U. For any $p \in U$, by means of a bump function we can construct global smooth functions $\hat{\omega}_I, \hat{x}^i$ that agrees with ω_I, x^i in a neighborhood of p. By virtual of (i)-(iv), we see that $d\omega$ has to have the expression of (4.4) at p: More detailed explanation can be found in [6, p.221]. Since p is arbitrary, this d must be everywhere equal to the one we defined locally by (4.4). □

Proposition 4.22. If $F : M \to N$ is a smooth map, then for each k, the pullback map $F^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d: $F^*(d\omega) = d(F^*\omega)$. In other words, the diagram below commutes:

$$\begin{tikzcd}
\Omega^k(N) \arrow[r, "F^*"] \arrow[d, "d"'] & \Omega^k(M) \arrow[d, "d"] \\
\Omega^{k+1}(N) \arrow[r, "F^*"'] & \Omega^{k+1}(M)
\end{tikzcd}$$

Proof. We first claim that for any $f \in C^\infty(N)$, we have $F^*(df) = d(f \circ F)$. This is a simple calculation:

$$F_p^*(df_{F(p)})(v) = df_{F(p)}(dF_p(v)) = (dF_p(v))(f) = v(f \circ F) = d(f \circ F)_p(v).$$

Next, since d is local, for any $\omega \in \Omega^k(N)$, we can restrict to local charts (V, ψ, x^i) of M and (U, φ, y^j) of N such that $F(V) \subset U$. Let

$$\omega = \sum_J \omega_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Then

$$d\omega = \sum_J d\omega_{j_1 \cdots j_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Hence

$$\begin{aligned}
F^*(d\omega) &= \sum_J F^*(d\omega_{j_1\cdots j_k}) \wedge F^*(dx^{j_1}) \wedge \cdots \wedge F^*(dx^{j_k}) \\
&= \sum_J d(\omega_{j_1\cdots j_k} \circ F) \wedge d(x^{j_1} \circ F) \wedge \cdots \wedge d(x^{j_k} \circ F).
\end{aligned}$$

Next,

$$\begin{aligned}
F^*\omega &= \sum_J (\omega_{j_1\cdots j_k} \circ F) \wedge F^* dx^{j_1} \wedge \cdots \wedge F^* dx^{j_k} \\
&= \sum_J (\omega_{j_1\cdots j_k} \circ F) \wedge d(x^{j_1} \circ F) \wedge \cdots \wedge d(x^{j_k} \circ F).
\end{aligned}$$

Therefore,

$$d(F^*\omega) = \sum_J d(\omega_{j_1\cdots j_k} \circ F) \wedge d(x^{j_1} \circ F) \wedge \cdots \wedge d(x^{j_k} \circ F) = F^*(d\omega).$$

This completes the proof. □

Theorem 4.23. *Let $F : N^n \to M^n$ be a smooth map, and let $\omega \in \Omega^n(M)$. Let (U, φ, x^i) and (V, ψ, y^j) be two charts. Suppose $\omega = h dy^1 \wedge \cdots \wedge dy^n$ on (V, y^j). Then the following holds on $U \cap F^{-1}(V)$:*

$$F^*\omega = F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det dF) dx^1 \wedge \cdots \wedge dx^n.$$

Proof. We have shown in the previous proof that

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F).$$

Now how does the one form $d(y^j \circ F)$ relate to dx^i's? By the way how 1-forms transform (p.281 of Lee's Smooth manifold), we have that

$$d(y^j \circ F) = d(y^j \circ \psi \circ F) = \frac{\partial(y^j \circ \psi \circ F \circ \varphi^{-1})}{\partial x^i} dx^i = \frac{\partial \hat{F}^j}{\partial x^i} dx^i.$$

Therefore, we see that

$$\begin{aligned}
F^*(u dy^1 \wedge \cdots \wedge dy^n) &= (u \circ F) \left(\frac{\partial \hat{F}^1}{\partial x^{j_1}} \cdots \frac{\partial \hat{F}^n}{\partial x^{j_n}} \right) dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\
&= (u \circ F) \det \left(\frac{\partial \hat{F}^j}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n \\
&= (u \circ F)(\det dF) dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

This completes the proof. □

Definition 4.24. A differential form $\omega \in \Omega^k(M)$ is said to be ***closed*** if $d\omega = 0$. It is said to be ***exact*** if there is some $\alpha \in \Omega^{k-1}(M)$ such that $d\alpha = \omega$.

Corollary 4.25. *Every exact form is closed.*

Proof. If ω is exact, then $\omega = d\alpha$. Hence $d\omega = d^2\alpha = 0$ since $d^2 = 0$. □

One naturally asks is the converse of this statement true, namely, is closed form exact? Answer: it need not be true in general. Discussions on when the converse is true is deferred to the second part of this book. We conclude with a coordinate invariant formulation of exterior differentiation:

Proposition 4.26. Let $\omega \in \Omega^1(M)$. Then for smooth vector fields $X, Y \in \Gamma(TM)$,

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{4.7}$$

Proof. Since ω is local and linear and the commutator is multilinear, it suffices to prove the statement for $X = \partial/\partial x^i, Y = \partial/\partial x^j$ on a chart. Write $\omega = f_k dx^k$, then

$$d\omega = df_k \wedge dx^k = \frac{\partial f_k}{\partial x^l} dx^l \wedge dx^k.$$

Therefore,

$$\begin{aligned} d\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= \frac{\partial f_k}{\partial x^l} dx^l \wedge dx^k \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &= \frac{\partial f_k}{\partial x^l}(\delta^l_i \delta^k_j - \delta^k_i \delta^l_j) \\ &= \frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j}. \end{aligned}$$

One can check that this is equal to the right hand side of 4.7 applies to the vector fields $X = \partial_i, Y = \partial_j$. This concludes the proof. $\square$

4.5 Problems

1. Let ω be a k-form and η an r-form. Show that

$$(\omega \wedge \eta)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+r}) = \sum_{\sigma \in S'} (\operatorname{sgn} \sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+r)}),$$

where S' is the *subset* of the symmetric group S_{k+r} consisting of all permutations σ such that

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+r).$$

2. Show that the 1-forms $\varphi_1, \dots, \varphi_k$ are linearly independent if and only if

$$\varphi_1 \wedge \cdots \wedge \varphi_k \neq 0.$$

3. If V is an n-dimensional vector space, define

$$\Lambda^*(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V),$$

where $\Lambda^0(V)$ is defined to be the real numbers. Then $\Lambda^*(V)$ is of course a vector space.

   (a) Show that $\dim \Lambda^*(V) = 2^n$.

   (b) Extend the wedge product to $\Lambda^*(V)$ by linearity. Show that now $\Lambda^*(V)$ is an algebra. It is called the *exterior algebra* of forms on V.

   (c) If $f : V \to W$ is a linear mapping, explain why $f^* : \Lambda^*(W) \to \Lambda^*(V)$ is an algebra homomorphism.

4. A k-form ω on V is said to be *decomposable* if there are 1-forms $\varphi_1, \dots, \varphi_k$ on V such that

$$\omega = \varphi_1 \wedge \cdots \wedge \varphi_k.$$

   (a) Let $v_1, \dots, v_4$ be a basis for $\mathbb{R}^4$ and let $\varphi_1, \dots, \varphi_4$ be the dual basis for $\Lambda^1\mathbb{R}^4 = (\mathbb{R}^4)^*$. Show that the 2-form $\omega = \varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4$ is *not* decomposable.

   (b) Show that every 2-form on $\mathbb{R}^3$ is decomposable.

   (c) Show that every $n-1$ form on $\mathbb{R}^n$ is decomposable.

5. Let ω be a 2-form on $\mathbb{R}^4$. Show that either there is a basis $v_1, \dots, v_4$ for $\mathbb{R}^4$ so that

$$\omega = \varphi_1 \wedge \varphi_2$$

in terms of the dual basis, or else a basis $v_1, \dots, v_4$ so that

$$\omega = \varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4$$

in terms of the dual basis, and that these two cases are mutually exclusive.

6. Likewise show that every 2-form ω on $\mathbb{R}^n$ can be written as

$$\omega = \varphi_1 \wedge \varphi_2 + \cdots + \varphi_{2k-1} \wedge \varphi_{2k}$$

in terms of a suitably chosen dual basis, where $2k \leq n$, and that these "normal forms" for a 2-form are mutually exclusive. Given a 2-form ω, explain how to compute k.

7. Let $\alpha_1, \ldots, \alpha_n$ be independent 1-forms. Let $\{\varphi_{ij} : 1 \leq i, j \leq n\}$ be a set of 1-forms with $\varphi_{ij} = -\varphi_{ji}$. Show that if $\sum_i \alpha_i \wedge \varphi_{ij} = 0$ for all j, then each $\varphi_{ij} = 0$.

8. Let V be an n-dimensional vector space. Let $\alpha_1, \ldots, \alpha_k$ be independent 1-forms on V. Suppose $\beta_1, \ldots, \beta_k$ are 1-forms on V such that

$$\alpha_1 \wedge \beta_1 + \cdots + \alpha_k \wedge \beta_k = 0.$$

Show that $\beta_i = \sum a_{ji}\alpha_j$, with $a_{ji} = a_{ij}$. This is known as *Cartan's Lemma.*

9. (a) Show that a Möbius band, a projective plane, and a Klein bottle are all non-orientable surfaces.
   (b) Explain, intuitively, why a closed surface in $\mathbb{R}^3$ must be orientable. Recall that "closed" in this context means "compact and no boundary".

10. Let $M^{n-1} \subset \mathbb{R}^n$ be a smooth manifold. Let $M(\epsilon)$ be the set of endpoints of normal vectors (in both directions) of length ϵ.
    (a) Show that if ϵ is small enough, then $M(\epsilon)$ is a smooth manifold in $\mathbb{R}^n$.
    (b) Show that $M(\epsilon)$ is orientable, even if M is not.
    (c) Describe $M(\epsilon)$ if M is a Möbius band in $\mathbb{R}^3$.

11. (Lie derivative of differential forms) Let M be a smooth manifold, V a smooth vector field of M with flow $\{h_t\}$, φ a differential form on M, p a point of M.

    Then the *Lie derivative of φ with respect to V* is the differential form $\mathcal{L}_V\varphi$ defined by

$$(\mathcal{L}_V\varphi)_p = \lim_{t\to 0} \frac{h_t^*(\varphi_{h_t(p)}) - \varphi_p}{t} = \left.\frac{d}{dt}\right|_{t=0} h_t^*(\varphi_{h_t(p)}).$$

    (a) Suppose that V and W are smooth vector fields on M, and that φ is a 1-form on M. Show that

$$\mathcal{L}_V(\varphi(W)) = (\mathcal{L}_V\varphi)(W) + \varphi(\mathcal{L}_V W).$$

    (b) If φ and η are differential forms on M, show that

$$\mathcal{L}_V(\varphi \wedge \eta) = (\mathcal{L}_V\varphi) \wedge \eta + \varphi \wedge (\mathcal{L}_V\eta).$$

    (c) Let V be a smooth vector field and φ a differential form on M. Show that

$$\mathcal{L}_V(d\varphi) = d(\mathcal{L}_V\varphi).$$

Chapter 5

Integration on Manifolds

5.1 Basic Definitions and Properties

Let M^n be an oriented smooth manifold (i.e. orientable and has a chosen orientation). We want to define integral of an n-form on M. First, assume ω is a compactly supported n-form, i.e., $\omega \in \Omega_c^n(M)$. Further assume that the support of ω is contained in a single chart (U, φ) of M. Then we define the ***integral of*** ω ***over*** M as

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega \tag{5.1}$$

where the expression on the right hand side is in $\mathbb{R}^n$. And if φ is ***positively oriented***, i.e., the coordinate frame $(\partial/\partial x^i)$ is a positively oriented basis for each tangent space, then we choose the plus sign in the expression above. If φ is negatively oriented, then we choose the minus sign in the expression above.

We should be more explicit about the definition above. Let $\eta \in \Omega^n(D)$ where $D \subset \mathbb{R}^n$. Then using the standard coordinate $(x^1, ..., x^n)$ of $\mathbb{R}^n$, one can write

$$\eta = g(x^1, ..., x^n) dx^1 \wedge \cdots \wedge dx^n$$

for some continuous function $g : D \to \mathbb{R}$. Then we define the ***integral of*** η ***over*** D to be

$$\int_D \eta = \int_D g(x^1, ..., x^n) dx^1 \wedge \cdots \wedge dx^n := \int_D g(x^1, ..., x^n) dx^1 \cdots dx^n.$$

Then, we can now write (5.1) more explicitly. Suppose that on the chart (U, φ, x^i), we have that

$$\omega = f dx^1 \wedge \cdots \wedge dx^n$$

for some smooth function $f : \operatorname{Supp} \omega \subset M \to \mathbb{R}$. Then the pullback formula gives (recall that on manifolds, the coordinate function x^i is really $x^i \circ \varphi$):

$$\begin{aligned}
(\varphi^{-1})^* \omega &= (\varphi^{-1})^* (f dx^1 \wedge \cdots \wedge dx^n) \\
&= (f \circ \varphi^{-1}) d(x^1 \circ \varphi \circ \varphi^{-1}) \wedge \cdots \wedge d(x^n \circ \varphi \circ \varphi^{-1}) \\
&= (f \circ \varphi) dx^1 \wedge \cdots \wedge dx^n
\end{aligned}$$

where the last expression is a differential form in a subset of $\mathbb{R}^n$, and x^i's are coordinate functions in $\mathbb{R}^n$. Therefore, using the definition of integral in $\mathbb{R}^n$ above, we can rewrite (5.1) as

$$\int_M \omega = \pm \int_M (\varphi^{-1})^* (f dx^1 \wedge \cdots \wedge dx^n) := \pm \int_{\varphi(U)} (f \circ \varphi^{-1}) dx^1 \cdots dx^n.$$

One needs to check that this definition is indepent of charts chosen. First, recall the change of variable formula in $\mathbb{R}^n$:

Lemma 5.1. *Let $F : U \to F(U)$ be a diffeomorphism between subsets of $\mathbb{R}^n$. Let $J(F) = (\partial F^i/\partial x^j)$ denote the Jacobian of F. Let (y^j) denote the coordinate systems in $F(U)$ and let (x^i) denote that of U. Then we have, for an integrable function f,*

$$\int_{F(U)} f dy^1 \cdots dy^n = \int_U (f \circ F)|J(F)| dx^1 \cdots dx^n.$$

Proposition 5.2. The definition of integral above does not depend on which chart we choose.

Proof. We only need to check the case for positively oriented charts, since the other cases will follow exact same calculation, except we will get a minus sign when computing Jacobian determinants. So suppose (U, φ, x^i) and (V, ψ, y^j) are two intersecting charts such that $\operatorname{Supp}\omega \subset U \cap V$. And suppose

$$\omega = f dx^1 \wedge \cdots \wedge dx^n = h dy^1 \wedge \cdots \wedge dy^n.$$

Then the transformation for dx^i into dy^j gives the relation

$$\begin{aligned}
\omega &= h dy^1 \wedge \cdots \wedge dy^n \\
&= h \left(\frac{\partial y^1}{\partial x^{i_1}} \cdots \frac{\partial y^n}{\partial x^{i_n}} \right) \circ \varphi dx^{i_1} \wedge \cdots \wedge dx^{i_n} \\
&= h \det \left(\frac{\partial y^i}{\partial x^j} \right) \circ \varphi dx^1 \wedge \cdots \wedge dx^n
\end{aligned}$$

where in the last step, we reorder every $dx^{i_1} \wedge \cdots dx^{i_n}$ into $dx^1 \wedge \cdots \wedge dx^n$, at a cost of a $\operatorname{sgn}(\sigma)$ where $\sigma(i_1, ..., i_n) = (1, ..., n)$. But that with the partial derivatives combined is exactly the determinant of the matrix $\left(\partial y^i/\partial x^j\right)$. Therefore, the expression above gives

$$f = \det \left(\frac{\partial y^i}{\partial x^j} \right) \circ \varphi.$$

Now, since $\psi \circ \varphi^{-1}$ is a positively oriented chart, we have that $J(\psi \circ \varphi^{-1}) > 0$. Therefore, we have that

$$\begin{aligned}
\int_{\psi(V)} (h \circ \psi^{-1}) dy^1 \cdots dy^n &= \int_{\varphi(U)} h \circ \psi^{-1} \circ (\psi \circ \varphi^{-1}) |J(\psi \circ \varphi^{-1})| dx^1 \cdots dx^n \\
&= \int_{\varphi(U)} h \circ \psi^{-1} \circ (\psi \circ \varphi^{-1}) J(\psi \circ \varphi^{-1}) dx^1 \cdots dx^n \\
&= \int_{\varphi(U)} \left(h \cdot \det \left(\frac{\partial y^i}{\partial x^j} \right) \circ \varphi \right) \circ \varphi^{-1} dx^1 \cdots dx^n \\
&= \int_{\varphi(U)} (f \circ \varphi^{-1}) dx^1 \cdots dx^n.
\end{aligned}$$

Therefore, this definition is indeed independent of coordinate charts. Note that we may assume $(\psi \circ \varphi^{-1})(\varphi(U \cap V)) = \psi(V)$ since $\operatorname{Supp}\omega$ is contained in $U \cap V$, so outside of $\psi(U \cap V)$ the integration gives zero. $\square$

Now suppose that $\omega \in \Omega_c^n(M)$. Let (U_i, φ_i) be positive charts and let (ρ_i) be a partition of unity suboridnate to the open cover (U_i) of M. Then we may write

$$\omega = \sum_i \rho_i \omega =: \sum_i \omega_i.$$

Since $\operatorname{Supp} \omega$ is compact, it intersects finitely many open cover (U_i)'s. Hence we really have a finite sum

$$\omega = \sum_{i=1}^m \omega_i.$$

Then we define the ***integral of*** ω ***over*** M to be

$$\int_M \omega = \sum_{i=1}^m \int_M \omega_i.$$

Proposition 5.3. This definition is independent of which partition of unity we choose.

Proof. Let (η_j) be a partition of unity subordinate to another set of charts (V_j, ψ_j). So $\omega = \sum \eta_j \omega$. Note that we can write $\omega \rho_i = \sum_j (\omega \rho_i) \eta_j$. Thus we can swap the sums since they are finite. Therefore, we have that

$$\sum_i \int_M \omega \rho_i = \sum_i \sum_j \int_M \omega \rho_i \eta_j = \sum_j \int_M \left(\sum_i \omega \eta_j \right) \rho_i = \sum_j \int_M \omega \eta_j.$$

Hence this definition is indeed independent of the choice of partition of unity. □

Some properties are immediate:

Lemma 5.4. $\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta$. *Also, if* ω *is a positively oriented orientation form, then* $\int_M \omega > 0$.

Proof. See [1], Proposition 16.6. □

Theorem 5.5 (Integration and Orientations)**.** *Let* M^n *be a smooth manifold and* $\omega \in \Omega_c^n(M)$.

(a) If $-M$ *denotes* M *with the opposite orientation, then*

$$\int_{-M} \omega = -\int_M \omega.$$

(b) If $F : N \to M$ *is a diffeomorphism, then*

$$\int_M \omega = \pm \int_N F^* \omega$$

where the $\pm$ *depends on whether* F *is orientation preserving or reversing. We say* F *is* ***orientation preserving*** *if for each* $p \in N$, *the isomorphism* $F_{*p} : T_pN \to T_pM$ *takes positively oriented bases of* T_pN *to positively oriented bases of* T_pM, *and say* F *is* ***orientation reversing*** *if it is not orientation preserving.*

Proof. (a) Let (U, φ, x^i) be a positively oriented chart of M containing $\operatorname{Supp}\omega$. Then since $-M$ has opposite orientation of M, this would be a negatively oriented chart. A positively oriented chart on $-M$ containing $\operatorname{Supp}\omega$ would be (U, ψ, y^j), where $\psi = T \circ \varphi$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $T(x^1, x^2, ..., x^n) = (x^2, x^1, ..., x^n)$. Then suppose $\omega = f dx^1 \wedge \cdots \wedge dx^n = h dy^1 \wedge \cdots dy^n$. By the previous calculation, we have that

$$f = h \cdot \det\left(\frac{\partial y^i}{\partial x^j}\right) \circ \varphi = h \cdot \det\left(\frac{\partial(\psi \circ \varphi^{-1})^i}{\partial x^j}\right) \circ \varphi = h \cdot \det\left(\frac{\partial T^i}{\partial x^j}\right) \circ \varphi = -h.$$

Then by the change of variable formula, one sees that

$$\begin{aligned}
\int_{-M} \omega &= \int_{\psi(U)} (\psi^{-1})^* \omega \\
&= \int_{\psi(U)} (h dy^1 \wedge \cdots dy^n) \\
&= \int_{\psi(U)} (h \circ \psi^{-1}) dy^1 \cdots dy^n \\
&= \int_{\varphi(U)} (h \circ \psi^{-1} \circ \psi \circ \varphi^{-1}) |J(\psi \circ \varphi^{-1})| dx^1 \cdots dx^n \\
&= \int_{\varphi(U)} (h \circ \varphi^{-1}) dx^1 \cdots dx^n \\
&= -\int_{\varphi(U)} (f \circ \varphi^{-1}) dx^1 \cdots dx^n \\
&= -\int_{\varphi(U)} (\varphi^{-1})^* \omega \\
&= -\int_M \omega.
\end{aligned}$$

Note that we have used the fact that $|J(\psi \circ \varphi^{-1})| = |J(T)| = 1$. This proves (a).

(b) Suppose F is orientation preserving, and (U, φ) is a positive chart of M such that $\operatorname{Supp}\omega \subset U$. Then we see that $(F^*\omega)_x(v_1, ..., v_n) \neq 0$ implies $\omega_{F(x)}(F_*(v_1), ..., F_*(v_n)) \neq 0$, which implies that $\omega \neq 0$ at $F(x)$. Hence $F(\operatorname{Supp} F^*\omega) \subset \operatorname{Supp}\omega \subset U$. Therefore $\operatorname{Supp} F^*\omega \subset F^{-1}(U)$. Since F is orientation preserving, then $(F^{-1}(U), \varphi \circ F)$ is a positively oriented chart of N containing $\operatorname{Supp} F^*\omega$. Thus

$$\begin{aligned}
\int_N F^*\omega &= \int_{\varphi(U)} [(\varphi \circ F)^{-1}]^* F^* \omega \\
&= \int_{\varphi(U)} (F \circ F^{-1} \circ \varphi^{-1})^* \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega = \int_M \omega.
\end{aligned}$$

If F is orientation reversing, then $(F^{-1}(U), \varphi \circ F)$ is a negatively oriented chart of N containing $\operatorname{Supp} F^*\omega$. Hence by the definition of the integral, we have

$$\int_N F^*\omega = -\int_{\varphi(U)} [(\varphi \circ F)^{-1}]^* F^* \omega = -\int_{\varphi(U)} (\varphi^{-1})^* \omega = -\int_M \omega$$

from the previous calculation. This proves (b).

□

Remark 5.6. The previous theorem says that if $F : N \to M$ is a diffeomorphism, then we have $\int_N F^*\omega = \pm \int_M \omega$. Using the pullback formula we obtained from Theorem 4.23, we see that when N and M are Euclidean spaces, then this statement is just a reformulation of the change of variable formula Lemma 5.1. This formulation is more concise and elegant. However, we should not think that Lemma 5.1 as a consequence of this statement. Remember that the whole theory of integration on manifolds is built on integration on $\mathbb{R}^n$ at the first place. Also, recall that in the proof of the previous theorem, we used the change of variable formula. It would be useful, however, to use this more concise statement as an aid to remember the change of variable formula in $\mathbb{R}^n$.

5.2 Manifolds with Boundary and Boundary Orientation

We denote the n-dimensional upper half-space $H^n = \{(x^1, ..., x^n) \in \mathbb{R}^n : x^n \geq 0\}$. An n-dimensional ***manifold with boundary*** is a second-countable Hausdorff space M in which every point has a neighborhood U homeomorphic to an open subset in H^n via a chart map $\varphi : U \to \varphi(U) \subset H^n$. A point $p \in M$ is called an ***interior point*** of M if it is in the domain of some chart (U, φ) such that $\varphi(U) \cap \partial H^n = \emptyset$. We call such a chart an ***interior chart***. We say $p \in M$ is a ***boundary point*** if it is in the domain of some ***boundary chart*** (V, ψ), i.e., a chart with $\psi(V) \cap \partial H^n \neq \emptyset$. The boundary ∂M of M consists of all boundary points of M. A smooth structure on a manifold M with boundary is defined similarly to

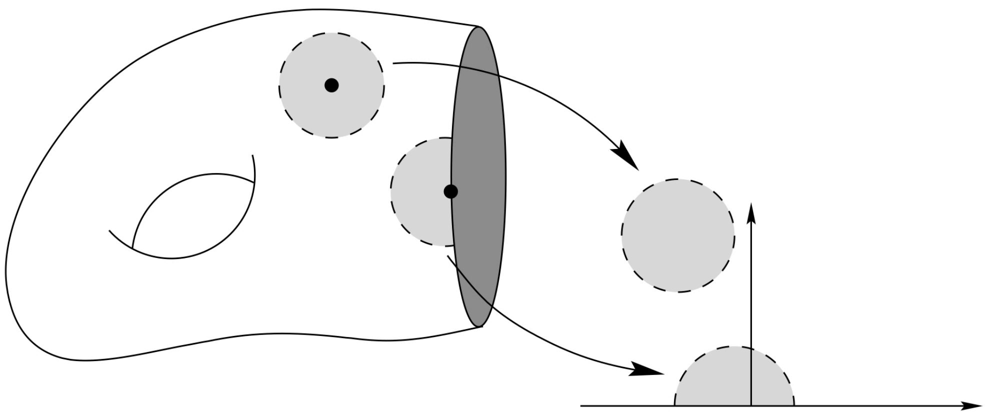


Figure 5.1: A manifold with boundary. An interior point and a boundary point are shown.

our previous construction of smooth structure on a manifold. A map $f : U \to \mathbb{R}^k$ where U is a subset of H^n is said to be ***smooth*** if it admits a smooth extension to some open subset $V \subset \mathbb{R}^n$ that contains U. We define a smooth structure on M to be a maximal smooth atlas, smooth in the sense that the chart transition maps are smooth as defined above.

The following theorem is Theorem 5.11 of [1], whose proof is omitted here:

Theorem 5.7. *If M is a smooth manifold with boundary, then with subspace topology, ∂M is a topological $(n-1)$-manifold without boundary, and has a smooth structure such that it is a properly embedded submanifold of M.*

Definition 5.8. Let M be a manifold with boundary and let $p \in \partial M$. A vector $v \in T_pM$ is said to be ***inward-pointing*** if $v \notin T_p(\partial M)$, and there exists some $\epsilon > 0$, a smooth curve $\gamma : [0, \epsilon] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. A vector $v \in T_pM$ is said to be ***outward-pointing*** if $-v$ is inward pointing.

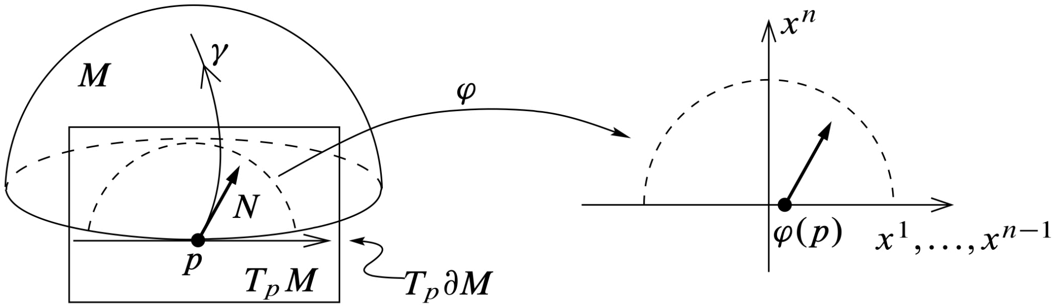


Figure 5.2: An inward-pointing vector $N \in T_pM$.

Lemma 5.9. *Suppose M is a smooth manifold with boundary,$p \in \partial M$, and (U, φ, x^i) is a chart containing p. Then the inward-pointing vectors $v = v^i\partial/\partial x^i$ in T_pM are precisely those with $v^n > 0$, the outward-pointing ones are those with $v^n < 0$.*

Proof. Let $p \in \partial M$. Suppose $v \in T_pM$ is an inward-pointing vector. Then $v \notin T_p(\partial M)$ and there exists a curve $\gamma : [0, \epsilon] \to M$ such that $\gamma(0) = p, \gamma'(0) = v$. Then under the coordinate basis, we have that

$$v = v^i \frac{\partial}{\partial x^i} = \gamma'(0) = \frac{d(\varphi \circ \gamma)^i}{dt}(0)\frac{\partial}{\partial x^i}.$$

We claim that since $v \notin T_p(\partial M)$, then $v^n \neq 0$. In fact, we will show that $v \in T_p(\partial M)$ iff $v^n = 0$: Suppose $v \in T_p(\partial M)$. Then there exists a curve $\eta : [0, \epsilon] \to M$ such that $\eta \in \partial M$ and $\eta'(0) = v$. Then under the chart (U, φ), we see that a portion of $\eta([0, \epsilon])$ near p is mapped to ∂H^n. Hence $(\varphi \circ \eta)^n(t) = 0$ for sufficiently small t by definition. Therefore we see that $v^n = d(\varphi \circ \eta)^n/dt|_{t=0} = 0$. Conversely, if $v^n = 0$, then consider $\hat{\eta} : [0, \epsilon] \to H^n$ defined by $\hat{\eta}^i(t) = v^i(t) + \varphi^n(p)$. Then it is easy to check that the curve $\eta = \varphi^{-1} \circ \hat{\eta} : [0, \epsilon] \to M$ satisfies $\eta(0) = p, \eta'(0) = v$. Since $\hat{\eta} \subset \partial H^n$, we have $\eta \subset \partial M$. Therefore $v \in T_p(\partial M)$.

Now that we argued since $v \notin T_p(\partial M)$, we have $v^n \neq 0$, then we must have $(\varphi \circ \gamma)^n(t) > 0$ for all $t > 0$, otherwise $d(\varphi \circ \gamma)^n(t)/dt|_{t=0} = v^n$ will be zero, which is a contradiction. But then this shows that $v^n > 0$ since

$$\frac{d(\varphi \circ \gamma)^n(t)}{dt}|_{t=0} = \lim_{t \to 0^+} \frac{(\varphi \circ \gamma)^n(t) - \varphi^n(p)}{t} = \lim_{t \to 0^+} \frac{(\varphi \circ \gamma)^n(t)}{t} > 0.$$

Conversely, suppose that $v = v^i\partial/\partial x^i$ with $v^n > 0$. Then by our argument above $v \notin T_p(\partial M)$. Consider the curve $\hat{\gamma} : [0, \epsilon] \to H^n$ defined by $\hat{\gamma}^i(t) = v^i t + \varphi^i(p)$. Then it is easy to check that $\gamma = \varphi^{-1} \circ \hat{\gamma} : [0, \epsilon] \to M$ satisfies $\gamma(0) = p, \gamma'(0) = v$. Hence v is inward-pointing. The outward-pointing statement follows immediately from definition. $\square$

Theorem 5.10. *If M is any smooth manifold with boundary, there is a smooth outward pointing vector field along ∂M.*

Proof. Cover a neighborhood of ∂M by smooth boundary charts $(U_\alpha, \varphi_\alpha)$. In each such chart,

$$\nu_\alpha = -\frac{\partial}{\partial x^n}\bigg|_{\partial M \cap U_\alpha}$$

is a smooth vector field along $\partial M \cap U_\alpha$, and is outward-pointing by the previous lemma. Let (ρ_α) be a partition of unity subordinate to the cover $(\partial M \cap U_\alpha)_\alpha$ of ∂M, and define a global vector field ν along ∂M by

$$\nu = \sum_\alpha \rho_\alpha \nu_\alpha.$$

Clearly ν is a smooth vector field along ∂M. To show that ν is outward-pointing, let (y^j) be any smooth boundary coordinate in a neighborhood of $p \in \partial M$. Because each ν_α is outward pointing, it satisfies $dy^n(v_\alpha) < 0$. Hence

$$\nu^n = dy^n(\nu) = \sum_\alpha \rho_\alpha dy^n(\nu_\alpha) < 0.$$

This concludes the proof. $\square$

Now we define, for an oriented manifold M, its ***boundary orientation*** on ∂M. In non-rigorous terms, an orientation on ∂M is a consistent choice of orientation at each $T_p(\partial M)$. Let $N \in T_pM$ be an outward pointing vector at ∂M. Let $(E_1, ..., E_{n-1})$ be an ordered basis for $T_p(\partial M)$. We define this basis to have positive orientation iff $(N, E_1, ..., E_{n-1})$ is a positively oriented basis for T_pM.

More rigorously, let $\omega \in \Omega^n(M)$, and let N be an outward-pointing vector field on ∂M. Then define $i^*_{\partial M}(N \lrcorner\, \omega) \in \Omega^{n-1}(\partial M)$ by

$$i^*_{\partial M}(N \lrcorner\, \omega)_p(v_1, ..., v_{n-1}) = \omega(di_{\partial M}(N_p), di_{\partial M}(v_1), ..., di_{\partial M}(v_{n-1}))$$

where $i_{\partial M} : \partial M \to M$ is the inclusion map, and $v_1, ..., v_{n-1} \in T_p(\partial M)$.

Proposition 5.11. If ω is an orientation form on M, then $i^*_{\partial M}(N \lrcorner\, \omega)$ is an orientation form on ∂M.

Proof. Denote $\sigma = i^*_{\partial M}(N \lrcorner\, \omega)$. Then $\sigma \in \Omega^{n-1}(\partial M)$. It will follow that σ is an orientation form on ∂M if we can show it never vanishes. Given a basis $(E_1, ..., E_{n-1})$ of $T_p(\partial M)$, the fact that $N_p \notin T_p(\partial M)$ implies $(N_p, E_1, ..., E_{n-1})$ is a basis for T_pM. Therefore

$$\sigma_p(E_1, ..., E_{n-1}) = \omega_p(N_p, E_1, ..., E_{n-1}) \neq 0$$

since ω_p is non-vanishing at every p, and if ω_p is zero on a basis, then a multilinear-argument would show that $\omega_p \equiv 0$. We also used the fact that $i^*_{\partial M}$ is just restriction to vectors tangent to ∂M. Since σ_p does not vanish and p is arbitrary, σ is an orientation form. Moreover, since $\sigma(E_1, ...E_{n-1}) > 0$ iff $\omega_p(N_p, E_1, ..., E_{n-1}) > 0$, the orientation determined by σ is exactly as described in our informal statement.

It remains to show that this orientation is independent of the choice of N. Let (x^i) be coordinate of a boundary chart near $p \in \partial M$. If N and $\tilde{N}$ are two different outward-pointing vector field on ∂M, then both $(N_p, \partial_1, ..., \partial_{n-1})$ and $(\tilde{N}_p, \partial_1, ..., \partial_{n-1})$ are bases for T_pM, and the transition matrix between them has determinant given by $N^n(p)/\tilde{N}^n(p) > 0$. Thus both bases determine the same orientation for T_pM. Hence both $N, \tilde{N}$ determines the same orientation for $T_p(\partial M)$. $\square$

Corollary 5.12. *Let ∂H^{n-1} be equipped with the induced orientation when H^n itself has the standard orientation inherited from $\mathbb{R}^n$. Then we can identify ∂H^n with $\mathbb{R}^{n-1}$. But the induced orientation agrees with the standard orientation on $\mathbb{R}^{n-1}$ only when n is even. That is,*

$$\partial H^n = (-1)^n \mathbb{R}^{n-1}.$$

Proof. The standard orientation on $\mathbb{R}^{n-1}$ is positively oriented for ∂H^n iff the ordered basis $[-\partial_n, \partial_1, ..., \partial_{n-1}]$ is a positively oriented basis for $\mathbb{R}^n$, where $-\partial_n$ is an outward-pointing vector field of H^n. Note that

$$[-\partial_n, \partial_1, ..., \partial_{n-1}] = -[\partial_n, \partial_1, ..., \partial_{n-1}] = (-1)^n[\partial_1, ..., \partial_n].$$

This concludes the proof. □

5.3 Stokes' Theorem

Theorem 5.13 (Stokes' Theorem)**.** *Let M^n be an oriented manifold with boundary, and $\omega \in \Omega_c^{n-1}(M)$. Then let ∂M to have the induced boundary orientation. We then have that*

$$\int_M d\omega = \int_{\partial M} i_{\partial M}^* \omega,$$

where $i_{\partial M} : \partial M \to M$ is the inclusion map. More concisely, we can write

$$\int_M d\omega = \int_{\partial M} \omega. \tag{5.2}$$

Proof. First suppose $M = H^n$. Since ω is of compact support, choose $R > 0$ such that $\operatorname{Supp}\omega \subset [-R, R]^{n-1} \times [0, R] =: A$. Write ω in standard coordinate as

$$\omega = \sum_{i=1}^n \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where the hat means that dx^i is omitted. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^n d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

We can change the order of integration in each term so as to do the x^i integration first. By the fundamental theorem of calculus, the terms for which $i \neq n$ reduce to

$$\begin{aligned} &\sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^1 \cdots dx^n \\ &= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R [\omega_i(x)]_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0, \end{aligned}$$

because we have chosen R large enough such that $\omega = 0$ when $x^i = \pm R$. The only term that might not be zero is the one for which $i = n$. For that term we have

$$\begin{aligned}
\int_{H^n} d\omega &= (-1)^{n-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \int_{0}^{R} \frac{\partial \omega_n}{\partial x^n}(x) dx^n dx^1 \cdots dx^{n-1} \\
&= (-1)^{n-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \left[\omega_n(x)\right]_{x^n=0}^{x^n=R} dx^1 \cdots dx^{n-1} \\
&= (-1)^{n} \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n\left(x^1, \ldots, x^{n-1}, 0\right) dx^1 \cdots dx^{n-1}
\end{aligned}$$

because $\omega_n = 0$ when $x^n = R$. To compare this to the other side of (5.2), we compute as follows:

$$\int_{\partial H^n} \omega = \sum_i \int_{A \cap \partial H^n} \omega_i\left(x^1, \ldots, x^{n-1}, 0\right) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Because x^n vanishes on ∂H^n, the pullback of dx^n to the boundary is identically zero. Thus, the only nonzero term is the one for which $i = n$, which becomes

$$\begin{aligned}
\int_{\partial H^n} \omega &= \int_{A \cap \partial H^n} \omega_n\left(x^1, \ldots, x^{n-1}, 0\right) dx^1 \wedge \cdots \wedge dx^{n-1} \\
&= (-1)^n \int_{A \cap \partial H^n} \omega_n(x^1, ..., x^{n-1}, 0) dx^1 \cdots dx^{n-1}
\end{aligned}$$

which is equal to $\int_{H^n} d\omega$. Note that in the last step we used Corollary 5.12.

Next we consider another special case: $M = \mathbb{R}^n$. In this case, the support of ω is contained in a cube of the form $A = [-R, R]^n$. Exactly the same computation goes through, except that in this case the $i = n$ term vanishes like all the others, so the left-hand side of (5.2) is zero. Since M has empty boundary in this case, the right-hand side is zero as well.

Now let M be a smooth manifold with boundary, $\omega \in \Omega_c^{n-1}(M)$ whose support is contained in a single chart (U, φ). WLOG assume this chart is positively oriented. The definition yields

$$\int_M d\omega = \int_{H^n} (\varphi^{-1})^* d\omega = \int_{H^n} d(\varphi^{-1*}\omega) = \int_{\partial H^n} (\varphi^{-1})^* \omega$$

from the calculation above, where ∂H^n has the induced orientation. Since φ_* takes outward-pointing vectors on ∂M to outward-pointing vectors on H^n, it follows that $\varphi|_{U \cap \partial M}$ is an orientation preserving diffeomorphism on $\varphi(U) \cap \partial H^n$, and the expression above is equal to $\int_{\partial M} \omega$. For an interior chart, we get the same computations with H^n replaced by $\mathbb{R}^n$. This proves the theorem in this case.

Finally, let $\omega \in \Omega_c^{n-1}(M)$ be arbitrary. Choose a cover of $\operatorname{Supp} \omega$ by finitely many domains of positively oriented smooth charts $(U_i)_{i \in I}$, and choose a subordinate smooth partition of unity $(\rho_i)_{i \in I}$. We can apply the argument to $\rho_i \omega$ for each i and obtain

$$\begin{aligned}
\int_{\partial M} \omega = \sum_i \int_{\partial M} \rho_i \omega = \sum_i \int_M d\left(\rho_i \omega\right) = \sum_i \int_M d\rho_i \wedge \omega + \rho_i d\omega \\
= \int_M d\left(\sum_i \rho_i\right) \wedge \omega + \int_M \left(\sum_i \rho_i\right) d\omega = 0 + \int_M d\omega
\end{aligned}$$

because $\sum_i \psi_1 \equiv 1$. This completes the proof of Stokes' theorem. □

5.4 Applications of Stokes' Theorem

In this section, we use Stokes' theorem to deduce all the major theorems in elementary single and multivariable calculus (with probably more restrictive condition on singularities). In fact, as we will see, these theorems are all just special cases of the Stokes' theorem (Theorem 5.2) that we obtained in the previous section, which is what makes Stokes' theorem so powerful and elegant. The computations in these examples are very straightforward, and the more challenging part (which serves as good practice) is to figure out the orientation and the boundary orientation.

Corollary 5.14 (The fundamental theorem of calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, then we have*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof. Take $M = [a, b]$ in Stokes' theorem. We define on orientation form on $[a, b]$ to be dx and a basis v of $T_p[a, b]$ is positively oriented if $dx(v) > 0$ and negatively oriented otherwise. Notice that $\partial[a, b]$ has only two points a, b. Hence the integral degenerates into a sum. Since the outward-pointing normal at a is a negatively oriented basis for $T_a[a, b]$, we assign a minus sign to a. Since the outward-pointing normal at b is a positively oriented basis for $T_b[a, b]$, we assign a plus sign to b, i.e., $\partial[a, b] = b - a$.

Now take $\omega = f(x)dx$. Then by Stokes' theorem,

$$f(b) - f(a) = \int_{\partial[a,b]} \omega = \int_{[a,b]} d\omega = \int_a^b f'(x)dx.$$

□

Corollary 5.15 (The fundamental theorem of line integral)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth nonsingular curve, i.e., γ is a diffeomorphism onto its image. Then*

$$\int_a^b \nabla f \cdot \gamma' d\lambda = f(\gamma(b)) - f(\gamma(a)).$$

This version is slightly more restrictive since we do not allow singularity on γ.

Proof. We define the orientation form on $\gamma([a, b])$ to be $(\gamma^{-1})^*(dx)$ where dx is the orientation form on $[a, b]$. Therefore, similar to the previous argument, $\partial\gamma([a, b]) = \gamma(b) - \gamma(a)$. Now take the zero form $\omega = f \in \Omega(\gamma([a, b]))$. We have

$$\begin{aligned}\int_{\gamma([a,b])} d\omega = \int_{[a,b]} \gamma^* d\omega = \int_{[a,b]} d(\gamma^*\omega) = \int_{[a,b]} d(f \circ \gamma) &= \int_{[a,b]} \frac{\partial f}{\partial x^i}\frac{d\gamma^i}{d\lambda} d\lambda \\ &= \int_a^b \nabla f \cdot \gamma' d\lambda.\end{aligned}$$

while

$$\int_{\partial\gamma([a,b])} \omega = f(\gamma(b)) - f(\gamma(a))$$

as discussed. □

Corollary 5.16 (Green's theorem)**.** *Let $D \subset \mathbb{R}^2$ be a region and ∂D is smooth. If P, Q are functions defined on an open region containing D and have continuous partial derivatives there, then*

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Proof. Choose $\omega = Pdx + Qdy \in \Omega^1(D)$, then the claim immediately follows from Stokes' theorem. $\square$

Corollary 5.17 (Divergence theorem)**.** *Let $S \subset \mathbb{R}^3$ be a region with smooth boundary ∂S. Let $\mathbf{F} = (F_x, F_y, F_z) : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable vector field. Then*

$$\int_S (\nabla \cdot \mathbf{F}) dV = \int_{\partial S} \mathbf{F} \cdot \mathbf{dS}$$

where $dV = dx \wedge dy \wedge dz$ is the volume form on S, and $\mathbf{dS} = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$ is the area element. One can check that $\mathbf{dS}(v, w) = v \times w$ for two vectors v, w in $\mathbb{R}^3$.

Proof. This is just a direct application of Stokes theorem with

$$\omega = \mathbf{F} \cdot \mathbf{dS} = F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy.$$

Simple calculation gives

$$d\omega = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx \wedge dy \wedge dz = (\nabla \cdot \mathbf{F}) dV.$$

$\square$

Corollary 5.18 (Classical Stokes' theorem)**.** *Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface with smooth boundary $\partial\Sigma$ (which is a curve). Let $\mathbf{F} = (F_x, F_y, F_z)$ be a vector field that is continously differentiable in a region containing S. Then*

$$\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = \int_{\partial\Sigma} \mathbf{F} \cdot \mathbf{dl}$$

where $\mathbf{dS} = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$ is again the area element, and $\mathbf{dl} = (dx, dy, dz)$ is the curve element.

Proof. This again follows from straightforward calculation with suitable choice of form $\omega = F_x dx + F_y dy + F_z dz$. $\square$

5.5 Problems

1. Let V be a smooth vector field and φ a k-form on M. Show that

$$\mathcal{L}_V \varphi = V \lrcorner\, d\varphi + d(V \lrcorner\, \varphi).$$

2. Show that the integral of the 1-form $\omega = xdy$ around the boundary of a region in the plane is equal to the area of the region. Check this out by explicit computation for the unit disk.

3. Let $U = \mathbb{R}^2 - \{(0,0)\}$, and consider the curve $c : [0,1] \to U$ defined by

$$c(t) = (\cos 2\pi t, \sin 2\pi t).$$

   Let ω be a closed 1-form defined in U. Show that ω is exact if and only if

$$\int_c \omega = 0.$$

4. Show that *locally* we can always rescale a nowhere vanishing 1-form α to make it closed. That is, show that each point $p \in U$ has a neighborhood U_p on which there is a nowhere vanishing function $g_p : U \to \mathbb{R}$ so that $d(g_p\alpha) = 0$.

5. Find a nowhere vanishing vector field V defined on an open set $U \subset \mathbb{R}^2$ such that V *cannot* be rescaled to be divergence-free. Conclude that the corresponding 1-form α cannot be rescaled to be closed.

6. Let α be a nowhere vanishing 1-form defined on an open set $U \subset \mathbb{R}^3$. When can we find a nowhere vanishing function $g : U \to \mathbb{R}$ such that $d(g\alpha) = 0$?

   Show that a necessary condition for the existence of such a function g is that $\alpha \wedge d\alpha = 0$. Note that this condition is automatically satisfied in $\mathbb{R}^2$.

7. Let α be a nowhere vanishing 1-form defined on an open set $U \subset \mathbb{R}^3$, and suppose that $\alpha \wedge d\alpha = 0$ throughout U. Show that each point $p \in U$ has a neighborhood U_p on which there is a nowhere vanishing function $g_p : U \to \mathbb{R}$ so that $d(g_p\alpha) = 0$.

8. Consider the 1-form $\alpha = x\,dy + dz$ in $\mathbb{R}^3$. Note that

$$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$$

   is never zero. Such a 1-form in $\mathbb{R}^3$ is called a *contact form.*

   At each point p of $\mathbb{R}^3$, the kernel of α is a 2-plane in the tangent space $(\mathbb{R}^3)_p$, and the collection of these is a *2-plane field* on $\mathbb{R}^3$.

   (a) At each point $p = (x, y, z)$ of $\mathbb{R}^3$, find a basis for the 2-plane $\ker \alpha(p)$.

   (b) Draw a good picture of the 2-plane field $\ker \alpha$.

   (c) Show that it is impossible to find a piece of surface Σ in $\mathbb{R}^3$, no matter how small, whose tangent plane at each point $p \in \Sigma$ is $\ker \alpha(p)$.

Part II

Cohomology and Its Applications to Differential Topology

Chapter 6

Cohomology

6.1 Category Theory, de Rham Cohomology

In this chapter, we introduce a powerful tool that is used to study topology of manifolds, called cohomology. The idea of differential and algebraic topology is to associate, to each topological space X, some algebraic objects $A(X)$. These objects are usually groups or vector spaces. But we do not just want any algebraic objects, we want some algebraic objects that are topological invariants in the following sense:

> Suppose X, Y are topological spaces that are homeomorphic, then we want $A(X), A(Y)$ to be isomorphic algebraic objects.

The motivation behind this request is simple: Since we are studying topology, and the existence of a homeomorphism between X, Y means that these two objects are indistinguishable from a topological point of view, then any information we obtain from them, including algebraic information $A(X), A(Y)$, should be indistinguishable in their respective sense. So now the goal of the game is clear: Given topological spaces, we want to produce algebraic objects that are invariant under homeomorphisms. But this can be a difficult task. To accomplish this goal, we first need to introduce our playgrounds for the game, categories:

Definition 6.1. A ***category*** $\mathcal{C}$ consists of the following data:

- A class of objects, which we denote by $\mathrm{Ob}(\mathcal{C})$;
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$;
- For each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a composition map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, denoted $(f, g) \mapsto f \circ g$.

These data must satisfy the following axioms:

(C1) For every $X \in \mathrm{Ob}(\mathcal{C})$, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ whenever these compositions make sense.

(C2) Composition is associative. That is, $(f \circ g) \circ h = f \circ (g \circ h)$ whenever these compositions make sense.

Example 6.2. There are many examples of categories:

- Consider the category **Set**, where the objects are sets, and for $A, B \in \mathrm{Ob}(\mathbf{Set})$, let $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ be the set of all maps from A to B.

- Consider the category **Top**, where the objects are topological spaces, and for $X, Y \in \mathrm{Ob}(\mathbf{Top})$, let $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ be the set of all continuous maps from X to Y.
- Consider the category **Vec**, where the objects are vector spaces, and for $V, W \in \mathrm{Ob}(\mathbf{Vec})$, let $\mathrm{Hom}_{\mathbf{Vec}}(V, W)$ be the set of all linear maps from V to W.
- Consider the category **Grp**, where the objects are groups, and for $G, H \in \mathrm{Ob}(\mathbf{Grp})$, let $\mathrm{Hom}_{\mathbf{Grp}}(G, H)$ be the set of all group homomorphisms from G to H.
- Consider the category **Mfd**, where the objects are smooth manifolds, and for $M, N \in \mathrm{Ob}(\mathbf{Mfd})$, let $\mathrm{Hom}_{\mathbf{Mfd}}(M, N)$ be the set of all smooth maps from M to N.

Recall that when we first learned what a set is, we then went on to study functions between sets; similarly, now we study functions between categories, called functors:

Definition 6.3. Let $\mathcal{A}, \mathcal{B}$ be two categories. A ***(covariant) functor*** $F : \mathcal{A} \to \mathcal{B}$ contains the following data:

- A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$, denoted by $A \mapsto F(A)$.
- For every pair $X, Y \in \mathrm{Ob}(\mathcal{A})$, a map $F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$, denoted by $\phi \mapsto F(\phi)$.

These data should be compatible with identity and composition in the following sense:

1. We must have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
2. For every composable pair (f, g), we have $F(f \circ g) = F(f) \circ F(g)$. Note that the composition $f \circ g$ is composition of morphisms in $\mathcal{A}$, while $F(f) \circ F(g)$ is composition of morphisms in $\mathcal{B}$.

One can visualize a functor $F : \mathcal{A} \to \mathcal{B}$ with the following picture: we have a sequence of objects and morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{A}$. Then we imagine having a machine F that takes objects in $\mathcal{A}$ to objects in $\mathcal{B}$, and morphisms in $\mathcal{A}$ to morphisms in $\mathcal{B}$. Applying this machine to the sequence above gives a new sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

of objects and morphisms in $\mathcal{B}$.

Example 6.4 (the tangent functor)**.** The description above might seem very abstract. But really this is just a fancier language that can be used to describe many examples we have already seen. One might think of an example where, for any smooth manifold M and a point $p \in M$, its tangent space T_pM is a vector space, and for $F : M \to N$ smooth maps, we have a linear map $dF_p : T_pM \to T_{F(p)}N$. And what's lucky is that associativity is satisfied because $d(G \circ F)_p = dG_{F(p)} \circ dF_p$. Does this mean we have a functor $\mathbf{Mfd} \to \mathbf{Vec}$? Almost, but not quite: This is base-point dependent. However, we can introduce the ***category of pointed manifolds***, denoted by $\mathbf{Mfd}_*$, where the objects are ordered pair (M, p), where M is a manifold and $p \in M$ is a point, and the morphisms are ***pointed maps*** $F : (M, p) \to (N, p')$, i.e., maps $F : M \to N$ with $p' = F(p)$. Then we get a functor $\mathbf{Mfd}_* \to \mathbf{Vec}$ because we can define the functor to take the sequence of pointed manifolds and morphisms

$$(M, p) \xrightarrow{F} (N, p') \xrightarrow{G} (Q, p'')$$

to a sequence of vector spaces and linear maps

$$T_pM \xrightarrow{dF_p} T_{p'}N \xrightarrow{dG_{p'}} T_{p''}Q.$$

But note that given a smooth map $F : M \to N$, we can define its global differential $dF : TM \to TN$, via $(p, X) \mapsto (F(p), dF_p(X))$. This is a smooth map between tangent bundles. So we get a functor **Mfd** $\to$ **Mfd**, by sending the sequence

$$M \xrightarrow{F} N \xrightarrow{G} Q$$

to the sequence

$$TM \xrightarrow{dF} TN \xrightarrow{dG} TQ.$$

Now we define what is called a contravariant functor. It almost just behaves like a functor, but it reverses the direction of the morphism:

Definition 6.5. Let $\mathcal{A}, \mathcal{B}$ be two categories. A ***contravariant functor*** $F : \mathcal{A} \to \mathcal{B}$ contains the following data:

- A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$, denoted by $A \mapsto F(A)$.
- For every pair $X, Y \in \mathrm{Ob}(\mathcal{A})$, a map $F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(Y), F(X))$, denoted by $\phi \mapsto F(\phi)$.

These data should be compatible with identity and composition in the following sense:

1. We must have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
2. For every composable pair (f, g), we have $F(f \circ g) = F(g) \circ F(f)$.

Again, one should have the following picture in mind to visualize a contravariant functor F: Suppose we have a sequence of objects and morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{A}$. Then we imagine having a machine F that takes objects in $\mathcal{A}$ to objects in $\mathcal{B}$, and morphisms in $\mathcal{A}$ to morphisms in $\mathcal{B}$, but with the direction of the arrow reversed. Applying this machine to the sequence above gives a new sequence

$$F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X).$$

Example 6.6 (pullback functor)**.** Note that for any smooth map $F : M \to N$, the pullback $F^* : C^\infty(N) \to C^\infty(M)$, $f \mapsto F^*(f) = f \circ F$ defines a contravariant functor **Mfd** $\to$ **Vec** by sending the sequence

$$M \xrightarrow{F} N \xrightarrow{G} Q$$

to the sequence

$$C^\infty(Q) \xrightarrow{G^*} C^\infty(N) \xrightarrow{F^*} C^\infty(M).$$

But note that we can also define pullbacks of differential forms. Hence we get another contravariant functor **Mfd** $\to$ **Vec** by sending the same sequence to

$$\Omega^*(Q) \xrightarrow{G^*} \Omega^*(N) \xrightarrow{F^*} \Omega^*(M).$$

But how can this abstract category theory nonsense help us achieve our goal of constructing topological invariants? The insight is that instead of just thinking about getting an algebraic object from each topological space, we should really be thinking about constructing functors **Top** $\to$ **Grp**.

Proposition 6.7. Let $A : \mathbf{Top} \to \mathbf{Grp}$ be a functor between categories, either covariant or contravariant. Then if X, Y are homeomorphic topological spaces, then $A(X), A(Y)$ are isomorphic groups.

Proof. This is almost immediate from all definitions. Homeomorphic assumption means that there exist continuous maps $f : X \to Y, g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Then since A is a (covariant) functor, it respects composition, meaning we get group homomorphisms $A(f) : A(X) \to A(Y)$ and $A(g) : A(Y) \to A(X)$, such that $A(f) \circ A(g) = \mathrm{id}_{A(Y)}$ and $A(g) \circ A(f) = \mathrm{id}_{A(X)}$. This is exactly the definition of groups $A(X), A(Y)$ being isomorphic. The argument works similarly for contravariant functors. $\square$

Remark 6.8. The argument above also works if we have a functor **Top** $\to$ **Vec**. Then homeomorphic spaces gives isomorphic vector spaces (existence of linear isomorphisms).

From the example above, one might immediately think we have already accomplished our task: we already have a contravariant functor $\Omega^* : \mathbf{Mfd} \to \mathbf{Vec}$. But as it turns out, the vector space $\Omega^*(M)$ is too large to be useful: Just imagine zero forms on $\mathbb{R}$, i.e., $\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$. This is an infinite-dimensional real vector space. But we have the right idea, we just need to reduce $\Omega^*(M)$ to something smaller. This motivates the definition of cohomology groups.

In the previous chapter, we have defined exterior differentiation operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$, we recall that $d^2 = 0$. This means that

$$\operatorname{im} d|_{\Omega^{k+1}(M)} \subset \ker d|_{\Omega^k(M)}$$

for all k. Therefore, we have a sequence of ***cochain complex***, i.e. a sequence of abelian groups with ascending index with a sequence of maps d connecting them such that $d^2 = 0$:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots$$

Definition 6.9. We define the kth ***(de Rham) cohomology group with real coefficients*** of M by the quotient of vector spaces (in particular it is a group under vector addition)

$$H^k(M) = \frac{\ker d|_{\Omega^k(M)}}{\operatorname{im} d|_{\Omega^{k-1}(M)}} = \frac{Z^k(M)}{B^k(M)}.$$

Therefore, this group is a quotient of ***closed*** k-forms $Z^k(M)$, i.e., $d\omega = 0$, by the ***exact*** k-forms $B^k(M)$, i.e., $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. Then

$$H(M) := \bigoplus_{0 \le k \le \dim M} H^k(M).$$

is a ring with wedge product as multiplication operation.

Definition 6.10. Let M be a manifold and $\omega \in \Omega^k(M)$. We define the ***support*** $\operatorname{Supp}(\omega)$ to be the closure of the set $\{p \in M : \omega(p) \neq 0\}$. Denote $\Omega^k_c(M)$ to be the space of ***smooth k-forms with compact support***. Note that

$$\Omega^k_c(M) \cong C^\infty_c(M) \otimes \Omega^k(M),$$

where $C_c^\infty(M) = \Omega^0(M)$ is the set of smooth functions on M with compact support. Define the kth ***cohomology group with compact support*** to be the quotient vector spaces

$$H_c^k(M) = \frac{\ker d|_{\Omega_c^k(M)}}{\operatorname{im} d|_{\Omega_c^{k-1}(M)}} = \frac{Z_c^k(M)}{B_c^k(M)}.$$

and

$$H_c(M) := \bigoplus_{0 \le k \le \dim M} H_c^k(M).$$

In the remaining part of this chapter, we will first develop the basic theory of de Rham cohomology. In the next chapter, we will see some beautiful applications of cohomology and then we can appreciate how powerful it is: we can actually deduce topological information about manifolds from cohomology.

Theorem 6.11. *Given a smooth map $F : M \to N$, the induced pullback map on forms $F^* : \Omega^k(N) \to \Omega^k(M)$ induces a well-defined linear map*

$$F^* : H^k(N) \to H^k(M).$$

This makes H^k into a contravariant functor **Top** $\to$ **Grp** *(in fact it is also a contravariant functor* **Top** $\to$ **Vec***).*

Proof. By Proposition 4.22, we know that the pullback commutes with d. This means that if $\alpha \in \Omega^k(N)$ is a closed form, i.e., $d\alpha = 0$, then $F^*\alpha \in \Omega^k(M)$ is also closed, because $d(F^*\alpha) = F^*(d\alpha) = 0$. Therefore $F^*(Z^k(N)) \subset Z^k(M)$. Similarly, if $\beta \in \Omega^k(N)$ is an exact form, i.e., $\beta = d\eta$, then $F^*\beta = F^*(d\eta) = d(F^*\eta) \in \Omega^k(M)$ is also exact, hence $F^*(B^k(N)) \subset B^k(M)$.

Now, we define $F^* : H^k(N) \to H^k(M)$ as follows: For every $[\alpha] \in H^k(N)$, it is represented by some closed form $\alpha \in \Omega^k(N)$. We define

$$F^*([\alpha]) = [F^*\alpha] \in H^k(M).$$

We must check that F^* is well-defined. To this end, let $\alpha' \in \Omega^k(N)$ be another (closed) representative of $[\alpha]$. Then by definition, there exists some $\beta \in \Omega^{k-1}(N)$ such that $\alpha - \alpha' = d\beta$. Then we see that

$$F^*[\alpha] - F^*[\alpha'] = [F^*\alpha] - [F^*\alpha'] = [F^*(\alpha - \alpha')] = [F^*(d\beta)] = [d(F^*\beta)] = 0$$

because exact form are quotient to zero in $H^k(M)$. Hence $F^* : H^k(N) \to H^k(M)$ is indeed well-defined. Clearly F^* is a linear map, because $[\alpha] + [\beta] := [\alpha + \beta]$ by definition and $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear, which means it is also a group homomorphism. □

We now introduce some important concepts in topology.

Definition 6.12. Let M, N be smooth manifolds and $F_0, F_1 : M \to N$ be two smooth maps.

- A ***homotopy from*** F_0 ***to*** F_1 is a smooth map $H : M \times [0,1] \to N$. Equivalently, it is a family of smooth maps $H_t : M \to N$, with $t \in [0,1]$, such that $H_0 = F_0$ and $H_1 = F_1$. If there exists a homotopy from F_0 to F_1, we say that F_0, F_1 are ***homotopic***, written $F_0 \simeq F_1$.

- Let $A \subset M$ be a subset. Then F_0, F_1 are said to be ***homotopic relative to*** A (or simply homotopic rel A), if there exists a homotopy H from F_0 to F_1, with the property that $H(p,t) = F_0(p) = F_1(p)$ for all $p \in A$. That is, H_t fixes every point in A for all t.
- Two paths $\gamma_0, \gamma_1 : [0,1] \to M$ are said to be ***path-homotopic***, if they are homotopic rel $\{0,1\}$. The notation is $\gamma_0 \simeq \gamma_1$.
- A ***retraction*** from M onto a subspace A is a map $r : M \to M$ such that $r(M) = A$ and $r|_A = \mathrm{id}_A$.
- A ***deformation retraction*** of M onto a subspace A is a homotopy from id_M to a retraction of M onto A. If there exists a deformation retraction of M onto A, we say M ***deformation retracts*** onto A, or A is a ***deformation retract*** of M.
- A smooth map $F : M \to N$ is called a ***homotopy equivalence*** if there is a smooth map $G : N \to M$ such that $F \circ G \simeq \mathrm{id}_N$ and $G \circ F \simeq \mathrm{id}_M$. In this case we say that M, N are ***homotopy equivalent***.
- A space X is said to be ***contractible*** if it is homotopy equivalent to a point. In particular, if X deformation retracts to a point $p \in X$, then X is contractible.

Remark 6.13. Equivalently, a homotopy from F_0 to F_1 can be think of a map $H : M \times \mathbb{R} \to N$ such that $H(x,t) = F_1(x)$ for $t \geq 1$ and $F(x,t) = F_0(x)$ for $t \leq 0$, and for all $x \in M$. Sometimes one is more convenient than the other when it comes to solving certain problems. We will use these two definitions interchangeably throughout the text.

We have already seen that cohomology group $H^*(M)$ is a topological invariant (i.e., invariant under homeomorphisms) simply because it is functorial. What is surprising is that it is invariant under homotopy equivalence (note that homotopic equivalence is weaker than being homeomorphic). We will prove this claim later, but for now, let us conclude this section by proving homotopy invariance of integral of a closed form along curves.

Theorem 6.14. *Let $\gamma_0, \gamma_1 : [0,1] \to M$ be two (piecewise) differentiable paths that are path-homotopic. Let $\omega \in \Omega^1(M)$ be a closed 1-form (i.e., $d\omega = 0$). Then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. Here we will only prove the claim for differentiable paths. Piecewise differentiable path can be handled similarly by doing each pieces at a time and adding them up.

Let $H : [0,1] \times [0,1] \to M$ be a path-homotopy from γ_0 to γ_1. Denote $I^2 := [0,1] \times [0,1]$. Since exterior derivative commutes with pullback by Theorem 4.22, and the two vertical edges Γ_2, Γ_3 of ∂I^2 are crushed into two points $\gamma_0(0), \gamma_0(1)$ via H (See Figure 6.1), Stokes' theorem gives

$$\int_{\partial I^2} H^*\omega = \int_{I^2} d(H^*\omega) = \int_{I^2} H^*(d\omega) = 0$$

since ω is closed. But on the other hand (See Figure 6.1 for the orientation of integration),

$$0 = \int_{\partial I^2} H^*\omega = \sum_{i=0}^{3} \int_{\Gamma_i} H^*\omega = \int_{\Gamma_1} H^*\omega - \int_{\Gamma_3} H^*\omega = \int_{\gamma_0} \omega - \int_{\gamma_1} \omega.$$

This is what we want to prove. $\square$

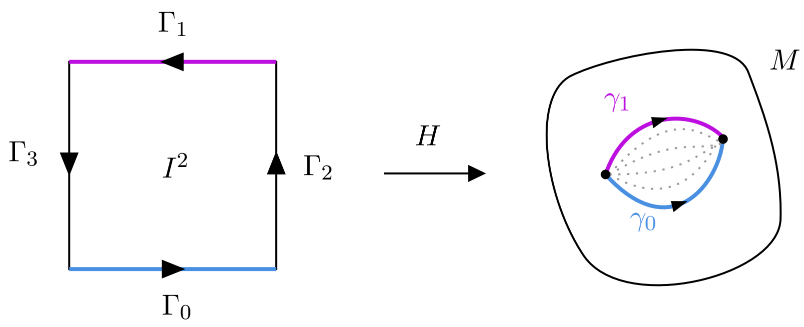


Figure 6.1: The homotopy H maps the square I^2 to the loops enclosed by γ_0, γ_1, maps Γ_0 to γ_0 and Γ_1 to γ_1, collapses Γ_2 to $\gamma_0(1) = \gamma_1(1)$, and collapses Γ_3 to $\gamma_0(0) = \gamma_1(0)$.

Example 6.15 (Winding number)**.** Consider $\omega \in \Omega^1(\mathbb{R}^2 - 0)$ defined by

$$\omega := \frac{1}{2\pi}\frac{xdy - ydx}{x^2 + y^2}.$$

Simple calculation suggests that $d\omega = 0$. Hence ω is a closed form. For each integer k, let $C_k : [0,1] \to \mathbb{R}^2 - 0$ be the curve defined by $C_k(t) = (\cos 2\pi kt, \sin 2\pi kt)$, i.e., a circle that warps around the origin k times. One can calculate and verify that $\int_{C_k} \omega = k$. Hence, the form ω, integrated along C_k, detects the number of times the curve C_k goes around the origin.

Now, consider an arbitrary piecewise differentiable loop $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = (1,0)$. What is $\int_\gamma \omega$? It is a result in algebraic topology that $\gamma \simeq C_k$ for some unique integer k (which is exactly the number of times γ warps around the origin). Hence by Theorem 6.14, we have $\int_\gamma \omega = k$ for that k. Therefore, ω is called the ***winding number form***, because it detects the number of times any γ winds around the origin.

This example is also very instructive, because of the following reason: We know that $d^2 = 0$. If ω is exact, i.e., $\omega = d\eta$ for some $\eta \in \Omega^0(\mathbb{R}^2 - 0)$, then $d\omega = d^2\eta = 0$, as expected. So there are two related questions one can ask:

- Does there exists $\eta \in \Omega^0(\mathbb{R}^2 - 0)$ such that $\omega = d\eta$?
- Is it true that on $\mathbb{R}^2 - 0$, every closed 1-form is exact? This is equivalent to asking whether $H^1(\mathbb{R}^2 - 0) = 0$ is true.

The answers are no to both the questions above. For the first question, suppose $\omega = d\eta$ for some 0-form η. Then just take k a nonzero integer and consider the loop C_k. Since $\partial C_k = 0$ (because it is a loop), Stokes' theorem gives a contradiction

$$k = \int_{C_k} \omega = \int_{C_k} d\eta = \int_{\partial C_k} \eta = 0.$$

Note that the use of Stokes' theorem here is legit because η, although not necessarily having compact support on $\mathbb{R}^2 - 0$, does have compact support on C_k because C_k is compact. In fact, here we are really talking about $i_{C_k}^*\eta$, where $i_{C_k} : C_k \to \mathbb{R}^2 - 0$ is the inclusion map.

Therefore, immediately we know the answer to the second question is no, with ω being a counterexample. In fact, we have $H^1(\mathbb{R}^2 - 0) \cong \mathbb{R}$, because $\mathbb{R}^2 - 0$ deformation retracts onto S^1 and $H^1(S^1) \cong \mathbb{R}$, as we will see later.

Definition 6.16. A path-connected manifold M is said to be ***simply connected*** if every path $\gamma : [0,1] \to M$ is path-homotopic to the constant loop based at $\gamma(0)$.

Proposition 6.17. Let M be a simply connected manifold. Then $H^1(M) = 0$.

Proof. This is immediate from the definition and Theorem 6.14. $\square$

Remark 6.18. From the previous proposition and Example 6.15, we can conclude that $\mathbb{R}^2 - 0$ is not simply connected. Indeed, the definition of being simply connected means that one can continuously (even smoothly) shrink any loop γ to its basepoint at $\gamma(0)$. However, this clearly cannot be done for a loop in $\mathbb{R}^2 - 0$ containing the missing origin, since there is no continuous shrinking possible because of the missing hole. This is the reason why $\mathbb{R}^2 - 0$ is not simply connected, and the reason why $H^1(\mathbb{R}^2 - 0) \neq 0$. We have come to a very important observation which shows the geometric meaning of cohomology $H^k(M)$: the kth cohomology detects k-dimensional holes in spaces. We will not go on and discuss why a missing point is a 1-dimensional hole since it would be too much of a detour.

6.2 Computations: First Installments

Our next main goal is to develop results and tools to eventually calculate basic cohomology groups, for example, $H^k(S^n), H^k(\mathbb{R}^n), H^k_c(\mathbb{R}^n)$, and so on. The process, although worth our effort, will be technical. For now, we dedicate this section to computing some examples directly from definitions.

Example 6.19. If M is an m-dimensional manifold, by definition $H^k(M) = 0$ for all $k > m$.

Example 6.20 (Simply connected manifold)**.** We already learned in the previous section that if M is simply connected, then $H^1(M) = 0$.

Example 6.21 (Cohomology of disjoint unions)**.** Let $M = \coprod_\alpha M_\alpha$ is a countable unions of disjoint manifolds M_α. Then we claim that $H^k(M) \cong \prod_\alpha H^k(M_\alpha)$. This is because on the level of forms, we already have an isomorphism $\Omega^k(M) \to \prod_\alpha \Omega^k(M)$ via $\omega \mapsto (i_\alpha^*\omega)_\alpha$.. Hence, the corresponding cochain complex are isomorphic, which gives isomorphic cohomology groups.

This shows that we only need to focus on computing cohomology groups of connected manifolds.

Example 6.22 (Cohomology of degree 0)**.** Let M be a connected manifold. Since there are no (-1)-forms, hence $B^0(M) = 0$. Thus $H^0(M) = Z^0(M)$. A closed 0-form is the one such that $df = (\partial f/\partial x^i)dx^i = 0$ on a local chart. Hence f must be a locally constant function. But on a connected topological space, locally constant functions are globally constant by Lemma 0.19. Hence $Z^0(M) \cong \mathbb{R}$. Thus $H^0(M) \cong \mathbb{R}$.

Example 6.23 (Cohomology of zero-manifolds)**.** Let M be a manifold of dimension zero, i.e., a discrete set of n points. Then $H^0(M) \cong \mathbb{R}^n$ by the previous two examples. Since $\dim T_p^*M = 0$ for every $p \in M$, we have $\Omega^k(M) = 0$ for $k > 0$. Hence $H^k(M) = 0$ for all $k > 0$. Similarly, we have $H^0_c(\mathbb{R}^0) = \mathbb{R}$ and $H^k_c(\mathbb{R}^0) = 0$ for $k > 0$.

Example 6.24. In this example we compute $H^*_c(\mathbb{R})$. Since there are no k-forms for $k > 1$ on $\mathbb{R}$, we have $H^k_c(\mathbb{R}) = 0$ for $k > 2$. So we only need to compute $H^0_c(\mathbb{R})$ and $H^1_c(\mathbb{R})$.

Since there is no constant function on $\mathbb{R}$ with compact support, $H^0_c(\mathbb{R}) = 0$. We show that $H^1_c(\mathbb{R}) = \mathbb{R}$ via first isomorphism theorem: Let $I : \Omega^1_c(\mathbb{R}) \to \mathbb{R}$ denote the integration map, $I(\omega) = \int_{\mathbb{R}} \omega$.

Clearly I is linear.

I is surjective since if we pick a positive function $g \in C_c^\infty(\mathbb{R})$, which amounts to picking $\omega = g(x)dx \in \Omega_c^1(\mathbb{R})$, then $I(\omega) = \int_{\mathbb{R}} \omega < \infty$. Thus, we can hit any real number by considering $I(\lambda\omega)$, for arbitrary $\lambda \in \mathbb{R}$.

We claim that $\ker I = B_c^1(\mathbb{R})$, the set of exact 1-forms with compact support on $\mathbb{R}$. If $df \in B_c^1(\mathbb{R})$ for some $f \in C_c^\infty(\mathbb{R})$, then say $\operatorname{Supp} f \subset [a,b]$, then we see that

$$\int_{\mathbb{R}} df = \int_{[a,b]} df = f(b) - f(a) = 0 - 0 = 0.$$

Thus $B_c^1(\mathbb{R}) \subset \ker I$. Conversely, if $g(x)dx \in \ker I \subset \Omega_c^1(\mathbb{R})$, then it is easy to check that

$$f(x) = \int_{-\infty}^{x} g(u)du \in C_c^\infty(\mathbb{R})$$

and $df = g(x)dx$. Thus $\ker I \subset B_c^1(\mathbb{R})$. Hence we conclude that $B_c^1(\mathbb{R}) = \ker I$.

We also note that $Z_c^1(\mathbb{R}) = \Omega_c^1(\mathbb{R})$, i.e., every 1-form on $\mathbb{R}$ is closed. One inclusion is obvious, and to see $\Omega_c^1(\mathbb{R}) \subset Z_c^1(\mathbb{R})$, let $g(x)dx \in \Omega_c^1(\mathbb{R})$. Then

$$d(g(x)dx) = g'(x)dx \wedge dx = 0.$$

Finally, by the first isomorphism theorem, we know that

$$\mathbb{R} \cong \frac{\Omega_c^1(\mathbb{R})}{\ker I} = \frac{Z_c^1(\mathbb{R})}{B_c^1(\mathbb{R})} = H_c^1(\mathbb{R}).$$

As a final remark, note that $I : \Omega_c^1(\mathbb{R}) \to \mathbb{R}$ descends to a map $I : H_c^1(\mathbb{R}) \to \mathbb{R}$, still defined by literal integration. The result $\ker I = B_c^1(\mathbb{R})$ suggests that the descended integration map $I : H_c^1(\mathbb{R}) \to \mathbb{R}$ is an isomorphism.

Example 6.25. As a generalization to the example above, we have $H_c^k(\mathbb{R}^n) = \mathbb{R}$ if $k = n$ and $H^k(\mathbb{R}^n) = 0$ otherwise. This is called the Poincaré lemma for cohomology with compact support, which we will prove later.

6.3 Some Homological Algebra

Although it is possible to compute the cohomology groups of many spaces directly from definition, as we did in the previous section, it should be apparent now that this is no easy task. Fortunately, there is an algebraic tool, called the exact sequence, that makes computations of unknown cohomology of spaces using known spaces significantly easier, and in fact almost like a dull procedure. However, such a tool does not come for free. We have to develop some technical algebraic machinery to be able to use it, and this is the goal of this and the next section.

Definition 6.26. A ***(cochain) complex*** is a sequence of vector spaces (or abelian groups) and linear maps (group homomorphisms):

$$\cdots \to A^{q-1} \xrightarrow{d^{q-1}} A^q \xrightarrow{d^q} A^{q+1} \to \cdots$$

such that $d^q \circ d^{q-1} = 0$ for all q. We denote a chain complex like this by A^*.

We have already seen an example of a chain complex, that is $\Omega^*(M)$ with the exterior derivative operator. The requirement that $d^q \circ d^{q-1} = 0$, or simple $d^2 = 0$, is to ensure that $\operatorname{im} d^{q-1} \subset \ker d^q$ for every q, hence the quotient space $\ker d^q / \operatorname{im} d^{q-1}$ is well-defined.

Definition 6.27. We define the kth ***cohomology*** of the cochain complex A^* by

$$H^k(A) := \ker d^q / \operatorname{im} d^{q-1}.$$

Definition 6.28. Let A^*, B^* be cochain complexes. A ***cochain map*** $F : A^* \to B^*$ is a collection of linear maps $F^q : A^q \to B^q$ such that the following diagram commutes for each q:

$$\begin{tikzcd}
\cdots \arrow[r] & A^q \arrow[r, "d_A^q"] \arrow[d, "F^q"'] & A^{q+1} \arrow[r] \arrow[d, "F^{q+1}"] & \cdots \\
\cdots \arrow[r] & B^q \arrow[r, "d_B^q"'] & B^{q+1} \arrow[r] & \cdots
\end{tikzcd}$$

Remark 6.29 (Cochain maps induces maps on cohomology groups)**.** An example of cochain map is the pullback: Let $F : M \to N$ be a smooth map, then we get a chain map defined by pullback $F^* : \Omega^q(N) \to \Omega^q(M)$. We know that $dF^* = F^*d$ from Proposition 4.22. Hence F^* is a cochain map. In exactly the same way we prove Theorem 6.11, one can show that a chain map $F : A^* \to B^*$ induces a morphism on cohomology groups $F^q : H^q(A) \to H^q(B)$ for each q.

Definition 6.30. A sequence

$$\cdots \to A^{q-1} \xrightarrow{d^{q-1}} A^q \xrightarrow{d^q} A^{q+1} \to \cdots$$

is said to be ***exact at*** A^q if one has $\ker d^q = \operatorname{im} d^{q-1}$. A sequence is said to be ***exact*** if it is exact at every A^q. Note that for a cochain complex, we only require $d^q \circ d^{q-1} = 0$, i.e., $\operatorname{im} d^{q-1} \subset \ker d^q$. So an exact sequence is a more restrictive type of cochain complex.

Remark 6.31. It should be clear from definition that for an exact sequence A^*, one has $H^k(A^*) = 0$ for all k.

The following lemma is immediate from definition:

Lemma 6.32. *Let A, B, C be vector spaces (or abelian groups).*

1. *The sequence $0 \to A \xrightarrow{f} B$ is exact iff f is injective.*
2. *The sequence $B \xrightarrow{g} C \to 0$ is exact iff g is surjective.*
3. *The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff f is injective, g is surjective, and $\operatorname{im} f = \ker g$. Such a sequence is called a **short exact sequence**.*

Theorem 6.33 (Snake lemma)**.** *Let $f : A^* \to B^*, g : B^* \to C^*$ are cochain maps such that for each q, the maps*

$$0 \to A^q \xrightarrow{f^q} B^q \xrightarrow{g^q} C^q \to 0$$

is a short exact sequence. That is, we have the following big commutative diagrams with *exact rows:*

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^{q+1} \arrow[u, "d_A^{q+1}"] \arrow[r, "f^{q+1}"] & B^{q+1} \arrow[u, "d_B^{q+1}"] \arrow[r, "g^{q+1}"] & C^{q+1} \arrow[u, "d_C^{q+1}"] \arrow[r] & 0 \\
0 \arrow[r] & A^q \arrow[u, "d_A^q"] \arrow[r, "f^q"] & B^q \arrow[u, "d_B^q"] \arrow[r, "g^q"] & C^q \arrow[u, "d_C^q"] \arrow[r] & 0 \\
0 \arrow[r] & A^{q-1} \arrow[u, "d_A^{q-1}"] \arrow[r, "f^{q-1}"] & B^{q-1} \arrow[u, "d_B^{q-1}"] \arrow[r, "g^{q-1}"] & C^{q-1} \arrow[u, "d_C^{q-1}"] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

then this induces a (long) exact sequence of cohomology groups:

$$\cdots \to H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \xrightarrow{d^*} H^{q+1}(A) \to \cdots$$

Proof Sketch. Detailed proof (e.g., check exactness) can be found in [3], though quite unreadable. We only define the maps f^*, g^*, d^* here and leave the rest to the readers who are interested. Here $f^* : H^q(A) \to H^q(B)$ is induced by the map $f^q : A^q \to B^q$ and similarly $g^* : H^q(B) \to H^q(C)$ is induced by $g^q : B^q \to C^q$, according to Remark 6.29. We briefly review the construction of the map $d^* : H^q(C) \to H^{q+1}(A)$: Let $c \in C^q$ with $d_C^q(c) = 0$, i.e., c represents a class in $H^q(C)$, then since g^q is surjective, there exists $b \in B^q$ such that $g^q(b) = c$. Now

$$g^{q+1} d_B^q(b) = d_C^q g^q(b) = d_C^q(c) = 0.$$

Therefore $d_B^q(b) \in \ker g^{q+1} = \operatorname{im} f^{q+1}$. Thus there exists $a \in A^{q+1}$ such that $d_B^q(b) = f^{q+1}(a)$. Thus we define

$$d^*[c] = [a].$$

□

Theorem 6.34 (The Five Lemma). *Given a commutative diagram of Abelian groups and group homomorphisms*

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "f_2"] \arrow[d, "\beta"'] & C \arrow[r, "f_3"] \arrow[d, "\gamma"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"'] & E \arrow[r] \arrow[d, "\varepsilon"'] & \cdots \\
\cdots \arrow[r] & A' \arrow[r, "f_1'"'] & B' \arrow[r, "f_2'"'] & C' \arrow[r, "f_3'"'] & D' \arrow[r, "f_4'"'] & E' \arrow[r] & \cdots
\end{tikzcd}$$

in which the rows are exact, if the maps α, β, δ and ε are isomorphisms, then so is the middle one γ.

Proof. First, we prove that γ is injective. Suppose for $c \in C$ we have $\gamma(c) = 0$. Then $f_3'(\gamma(c)) = 0$, and by commutativity of the diagram $\delta(f_3(c)) = 0$. Since δ is an isomorphism, $f_3(c) = 0$, and thus $c \in \ker f_3 = \operatorname{im} f_2$. It follows that there exists $b \in B$ such that $f_2(b) = c$. Now, $\gamma(f_2(b)) = \gamma(c) = 0$, and by commutativity of the diagram $f_2'(\beta(b)) = 0$. So, $\beta(b) \in \ker f_2' = \operatorname{im} f_1'$, and thus there exists $a' \in A'$ such that $\beta(b) = f_1'(a')$. Since α is an isomorphism, there exists $a \in A$ such that $\alpha(a) = a'$. We then have $\beta(b) = f_1'(\alpha(a)) = \beta(f_1(a))$ by commutativity, and since β is an isomorphism we have $b = f_1(a)$. Now, we see that $c = f_2(f_1(a)) = 0$ by exactness, so γ is indeed surjective.

Now, we prove that γ is surjective. Let $c' \in C'$ be arbitrary. By exactness $f_4'(f_3'(c')) = 0$. Since δ, ε are isomorphisms, $f_4' = \varepsilon \circ f_4 \circ \delta^{-1}$, so we have $\varepsilon(f_4(\delta^{-1}(f_3'(c')))) = 0$. Since ε is an isomorphism, $f_4(\delta^{-1}(f_3'(c'))) = 0$, and thus $\delta^{-1}(f_3'(c')) \in \ker f_4 = \operatorname{im} f_3$, and thus there exists $c \in C$ such that $\delta^{-1}(f_3'(c')) = f_3(c)$. Now, consider $\gamma(c)$. We have $f_3'(\gamma(c)) = \delta(f_3(c)) = f_3'(c')$, and thus $c' - \gamma(c) \in \ker f_3' = \operatorname{im} f_2'$. Therefore, there exists $b' \in B'$ such that $c' - \gamma(c) = f_2'(b')$. Since β is an isomorphism, there exists $b \in B$ so that $b' = \beta(b)$. It follows by commutativity of the diagram that $c' - \gamma(c) = f_2'(\beta(b)) = \gamma(f_2(b))$, and thus $c' = \gamma(c + f_2(b))$. This proves surjectivity, and we are done. □

6.4 the Mayer-Vietoris Sequence

Suppose $M = U \cup V$ where U, V are two open sets, then we have a sequence of inclusions

$$U \cap V \underset{i_V}{\overset{i_U}{\rightrightarrows}} U \coprod V \xrightarrow{i} M.$$

Here, the map $U \coprod V \to M$ is just the obvious map, sending $(u, U) \in U \subset U \coprod V$ to $u \in M$ and $(v, V) \in V \subset U \coprod V$ to $v \in M$. Applying the contravariant functor Ω^* gives a sequence

$$\Omega^*(M) \xrightarrow{i^*} \Omega^*(U) \oplus \Omega^*(V) \underset{i_V^*}{\overset{i_U^*}{\rightrightarrows}} \Omega^*(U \cap V).$$

Here, the first map is given by $i^*(\omega) = (i_U^*\omega, i_V^*\omega)$. Then we obtain the ***Mayer-Vietoris sequence***

$$0 \to \Omega^*(M) \xrightarrow{i^*} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\varphi} \Omega^*(U \cap V) \to 0$$

where the map $\varphi : \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V)$ is given by

$$\varphi(\omega, \tau) = i_{U\cap V}^*\tau - i_{U\cap V}^*\omega = \tau - \omega.$$

Proposition 6.35. The Mayer-Vietoris sequence is exact.

Proof. We first show i^* is injective: If $i^*(\omega) = 0$, then it means $i_U^*\omega = i_V^*\omega = 0$. Thus $\omega \equiv 0$ on U and on V, so it is zero on the entire $M = U \cup V$.

Now we show $\ker \varphi = \operatorname{im} i^*$. Suppose $(\omega, \tau) \in \ker \varphi$, then $\tau - \omega \equiv 0$ on $U \cap V$. Thus $\tau = \omega$ on $U \cap V$. Therefore we may define $\eta \in \Omega^*(M)$ by setting $\eta = \omega$ on U and $\eta = \tau$ on V. Then $i^*(\eta) = (\omega, \tau)$.

Finally we show φ is surjective. Let $\omega \in \Omega^*(U \cap V)$. Let ρ_U, ρ_V be a partition of unity subordinate to U, V. Then we claim that $\varphi(-\rho_V\omega, \rho_U\omega) = \omega$. Indeed,

$$\varphi(-\rho_V\omega, \rho_U\omega) = \rho_U\omega - (-\rho_V\omega) = (\rho_U + \rho_V)\omega = \omega.$$

Lastly, notice that $(-\rho_V\omega, \rho_U\omega) \in \Omega^*(U) \oplus \Omega^*(V)$. □

By snake lemma (Theorem 6.33), the Mayer-Vietoris sequence above induces the following long exact sequence (which is also called ***Mayer-Vietoris sequence***):

$$\cdots H^q(M) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(M) \to H^{q+1}(U) \oplus H^{q+1}(V) \to \cdots$$

Remark 6.36 (Explicit description of d^*)**.** Recall how we diagram chase the snake lemma in the proof sketch of Theorem 6.33: Let $\omega \in \Omega^q(U \cap V)$ be such that $d\omega = 0$. Then by exactness of the row there is $\xi \in \Omega^q(U) \oplus \Omega^q(V)$ such that $\varphi(\xi) = \omega$. One can see that

$$\xi = (-\rho_V \omega, \rho_U \omega).$$

Since the snake diagram commutes and $d\omega = 0$, we know $d\xi = 0$ on $\Omega^{q+1}(U \cap V)$. That is, $-d(\rho_V \omega) \equiv d(\rho_U \omega)$ on $U \cap V$. Hence $d\xi$ is the image of an element in $\Omega^{q+1}(M)$ which is closed and well-defined. Explicitly one sees that

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U \\ [d(\rho_U \omega)] & \text{on } V. \end{cases}$$

6.4.1 Mayer-Vietoris Sequence for Compact Supports

If $j : U \to M$ is the inclusion of an open subset, then we define the induced map $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ to be the map which extends a form on U by zero to a form on M. Then Ω_c^* is a **covariant** functor under inclusion of open sets.

The sequence of inclusions

$$U \cap V \underset{i_V}{\overset{i_U}{\rightrightarrows}} U \coprod V \xrightarrow{\;i\;} M.$$

again gives rise to a sequence of forms with compact support

$$0 \to \Omega_c^*(U \cap V) \xrightarrow{\delta} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{+} \Omega_c^*(M) \to 0$$

where

$$\delta(\omega) = (-(j_U)_* \omega, (j_V)_* \omega).$$

Proposition 6.37. The Mayer-Vietoris sequence for compact support above is exact.

Proof. The map δ is injective because if $\delta(\omega) = (-(j_U)_*\omega, (j_V)_*\omega) = (0, 0)$, then $(j_U)_*\omega = 0$, so $\omega \equiv 0$ on U, in particular on $U \cap V$.

Clearly, we have $+(-(j_U)_*\omega, (j_V)_*\omega) = -(j_U)_*\omega + (j_V)_*\omega = 0$. Thus $\operatorname{im}\delta \subset \ker +$. Conversely, let $(\omega, \tau) \in \Omega_c^*(U) \oplus \Omega_c^*(V)$ such that $\omega + \tau = 0$ on M. Then $\omega = -\tau$ on $U \cap V$. Thus if we define $\eta = \omega$ on $U \cap V$, then we have $\delta(\eta) = (\omega, \tau) = (-\tau, \tau) = (-j_*\omega, j_*\omega)$. So $\ker + \subset \operatorname{im}\delta$.

Finally, let $\omega \in \Omega_c^*(M)$. Then $\omega = +(\rho_U \omega, \rho_V \omega)$. Moreover, since

$$\operatorname{Supp}(\rho_U \omega) = \operatorname{Supp} \rho_U \cap \operatorname{Supp} \omega$$

which is a closed set ($\operatorname{Supp} \rho_U$) intersected with a compact set ($\operatorname{Supp} \omega$), which is compact in a Hausdorff space, we have $\rho_U \omega \in \Omega_c^*(U)$, and similarly $\rho_V \omega \in \Omega_c^*(V)$. So $+$ is surjective. This concludes the proof. ☐

Again by snake lemma (Theorem 6.33), we get a long exact sequence, also called the Mayer-Vietoris sequence for compact supports, from the short exact sequence of cochain complexes above:

$$\cdots \to H_c^q(U \cap V) \to H_c^q(U) \oplus H_c^q(V) \to H_c^q(M) \xrightarrow{d_*} H_c^{q+1}(U \cap V) \to \cdots$$

Remark 6.38 (Explicit description of d_*)**.** Just like what we did in the ordinary MV sequence, we give an explicit description of d_* here. Let $\tau \in \Omega_c^q(M)$ be a closed form representing $[\tau] \in H_c^q(M)$. By exactness of the MV sequence for Ω_c^q's, there exists an element in $\Omega_c^q(U) \oplus \Omega_c^q(V)$ that sums to τ, namely $(\rho_U\tau, \rho_V\tau)$. Note that $\rho_U\tau, \rho_V\tau$ indeed have compact supports (why?). Then under exterior differentiation it goes to $(d(\rho_U\tau), d(\rho_V\tau)) \in \Omega_c^{q+1}(U) \oplus \Omega_c^{q+1}(V)$. Since $d\tau = 0$ in $\Omega_c^{q+1}(U \cap V)$ by assumption, exactness of the MV sequence for Ω_c^{q+1}'s implies there is an element $d_*\tau \in \Omega_c^{q+1}(U \cap V)$, such that

$$(-(j_U)_*(d_*\tau), (j_V)_*(d_*\tau)) = (d(\rho_U\tau), d(\rho_V\tau)).$$

6.4.2 Cohomology of S^1

We now calculate $H^*(S^1), H^*(S^1)$ using the Mayer-Vietoris sequence. We do the regular cohomology first.

By Example 6.22, and the fact that S^1 is connected, we see that $H^0(S^1) = \mathbb{R}$. By Example 6.19, we know that $H^k(S^1) = 0$ for $k > 1$. Hence it remains to calculate $H^1(S^1)$. We cover S^1 by two open sets U, V as suggested by Figure 6.2:

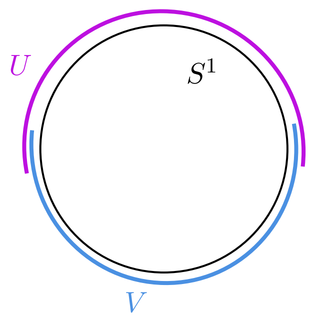


Figure 6.2: Choose U to be an open subset of S^1 that contains the upper hemisphere, and V an open subset that contains the lower hemisphere, such that $U \cap V$ is homeomorphic to two disjoint copies of intervals.

Now the Mayer-Vietoris sequence reads

$$H^0(S^1) \to H^0(U) \oplus H^0(V) \xrightarrow{\delta} H^0(U \cap V) \xrightarrow{d^*} H^1(S^1) \to H^1(U) \oplus H^1(V) \to \cdots$$

Since $U, V \cong \mathbb{R}, U \cap V \cong \mathbb{R} \coprod \mathbb{R}$, using $H^0(\mathbb{R}) = \mathbb{R}, H^1(\mathbb{R}) = 0$, we get that the MV sequence reads:

$$\mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d^*} H^1(S^1) \to 0.$$

Here, note that δ is induced by the map $\varphi : \Omega^0(U) \oplus \Omega^0(V) \to \Omega^0(U \cap V)$, where $\varphi(\omega, \tau) = \tau - \omega$. Thus δ is the map $\delta : (\omega, \tau) \mapsto (\tau - \omega, \tau - \omega)$. Hence $\dim \operatorname{im} \delta = 1$. By exactness of the sequence, we know d^* is surjective. Hence

$$H^1(S^1) = \frac{\mathbb{R} \oplus \mathbb{R}}{\ker d^*} = \frac{\mathbb{R} \oplus \mathbb{R}}{\operatorname{im} \delta} \cong \mathbb{R}.$$

Remark 6.39 (A generator for $H^1(S^1)$)**.** We now find an explicit generator of the group $H^1(S^1) \cong \mathbb{R}$. Since $H^1(S^1) = \mathbb{R}^2 / \ker d^*$, anything that is not killed by d^* in $H^0(U \cap V)$ will represent a generator. Hence, let $\alpha \in \Omega^0(U \cap V)$ be a closed 0-form (i.e., locally constant function on $U \cap V$) that is not in $\operatorname{im} \delta = \ker d^*$, then $d^*\alpha$ will represent a generator of $H^1(S^1)$. For example, if we take α to be a locally constant function that is 1 on one connected piece of $U \cap V$, and 0 on the other, then it will represent the element $(1,0) \in H^0(U \cap V)$, which is clearly not in $\operatorname{im} \delta$.

Now by Remark 6.36, we have

$$d^*\alpha = \begin{cases} -d(\rho_V \alpha) & \text{on } U \\ d(\rho_U \alpha) & \text{on } V \end{cases}$$

where ρ_U, ρ_V is a partition of unity subordinate to U, V. Note that this form $d^*\alpha$ is well-defined, since on the intersection

$$-d(\rho_V \alpha) - d(\rho_U \alpha) = -d((\rho_U + \rho_V)\alpha) = -d\alpha = 0$$

since α is closed by assumption.

Hence we see that

$$H^k(S^1) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = 1 \\ 0 & k \geq 2. \end{cases}$$

Next, we compute $H_c^*(S^1)$. Obviously, since S^1 is compact, $H_c^*(S^1) = H^*(S^1)$, so we already have the answer above. But let us compute it using the MV sequence as an illustration. Again $H_c^k(S^1) = 0$ for $k \geq 2$, so we only worry about the zeroth and first cohomology groups with compact support. We still choose U, V as in Figure 6.2. Then the MV sequence reads

$$\begin{aligned} H_c^0(U \cap V) \to H_c^0(U) \oplus H_c^0(V) \to H_c^0(S^1) \to \\ \to H_c^1(U \cap V) \xrightarrow{\delta} H_c^1(U) \oplus H_c^1(V) \to H^1(S^1) \to 0. \end{aligned}$$

By Example 6.24 and Since $U, V \cong \mathbb{R}, U \cap V \cong \mathbb{R} \coprod \mathbb{R}$, we see that the first two terms of the sequence are zero, and $H_c^1(U \cap V) \cong \mathbb{R} \oplus \mathbb{R}$. Recall that the map δ is induced by the extension map $\Omega_c^1(U \cap V) \to \Omega_c^1(U) \oplus \Omega_c^1(V)$, where $\alpha \mapsto (-(j_U)_*\alpha, (j_V)_*\alpha)$. Hence the map δ is given by

$$\begin{aligned} \delta : H_c^1(U \cap V) &\to H_c^1(U) \oplus H_c^1(V) \\ \omega &\mapsto (-(j_U)_*\omega, (j_V)_*\omega). \end{aligned}$$

Proposition 6.40. $\dim \operatorname{im} \delta = 1$.

Proof. By Example 6.24, for any open set W homeomorphic to $\mathbb{R}$, we have $H_c^1(W) \cong \mathbb{R}$, with the isomorphism defined by $\alpha \mapsto \int_W \alpha$. In other words, while making the identification $H^1(U \cap V) \cong \mathbb{R}^2$, we are really using the isomorphism $\omega \mapsto (\int_A \omega, \int_B \omega)$, where A, B are the two connected components of $U \cap V$. Now the image $\delta(\omega) = (-(j_U)_*\omega, (j_V)_*\omega)$ is identified with $(\int_U -(j_U)_*\omega, \int_V (j_V)_*\omega) = (-\int_{U \cap V} \omega, \int_{U \cap V} \omega)$. in $\mathbb{R}^2$. Now putting all the identification together, and viewing δ as a map $\mathbb{R}^2 \to \mathbb{R}^2$, the map is actually

$$\delta : (\lambda, \mu) \mapsto (-\lambda - \mu, \lambda + \mu)$$

which makes it clear that $\dim \operatorname{im} \delta = 1$. $\square$

By exactness, we see that the map $\phi : H_c^1(S^1) \to H_c^1(U \cap V)$ in the MV sequence is an injection, since the previous term in the sequence is 0. Hence

$$H_c^0(S^1) \cong \operatorname{im} \phi = \ker \delta \cong \mathbb{R}$$

by the previous proposition and the rank-nullity theorem. And since $H_c^1(U) \oplus H_c^1(V) \to H^1(S^1)$ is a surjection by exactness,

$$H_c^1(S^1) = \frac{\mathbb{R} \oplus \mathbb{R}}{\ker \delta} \cong \operatorname{im} \delta \cong \mathbb{R}$$

by the 1st isomorphism theorem and the previous proposition.

6.5 Homotopy Invariance and Applications

We now prove that de Rham cohomology is a homotopy invariant, and use this result to compute $H^*(\mathbb{R}^n), H^*(S^n)$, and some other examples.

Theorem 6.41. *Homotopic maps induce the same map in cohomology.*

Proof. Let $f, g : M \to N$ be homotopic. By Remark 6.13, there exists $H : M \times \mathbb{R} \to N$ such that $H(x,t) = f(x)$ for $t \geq 1$ and $H(x,t) = g(t)$ for $t \leq 0$. Equivalently let $s_0, s_1 : M \to M \times \mathbb{R}$ be the 0-section and 1-section, then $f = H \circ s_1$ and $g = H \circ s_0$. Then $f^* = s_1^* \circ H^*$ and $g^* = s_0^* \circ H^*$.

Now, let $x^i : M \times \mathbb{R} \to \mathbb{R}$ with $i = 1, ..., n+1$, and $x^{n+1} = t$. Then we see that $x^i \circ s : M \to M \times \mathbb{R} \to \mathbb{R}$ is given in local coordinate by

$$x^i \circ s_0(x^1, ..., x^n) = x^i(x^1, ..., x^n, 0) = x^i$$

for $i \neq n+1$ and 0 when $i = n+1$. Thus $s_0^*(dx^i) = d(x^i \circ s_0) = dx^i$ when $i \neq n+1$ and $s_0^*(dt) = 0$. Similarly we see s_1^* is given by the same expression. Thus $s_0^* = s_1^*$. Thus $f^* = g^*$. □

Corollary 6.42. *If $M \simeq N$, then $H^*(M) \cong H^*(N)$.*

Corollary 6.43. *If A is a deformation retract of M, then $H^*(M) \cong H^*(A)$.*

Proof. A deformation retraction is a homotopy H from id_M to a retraction $r : M \to A$ of M onto A, where $r|_A = \mathrm{id}_A$. Let $i : A \to M$ be the inclusion map, then $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_M$ via H. Thus i^*, r^* are inverses of each other by homotopy invariance, which establishes an isomorphism from $H^*(M)$ to $H^*(A)$. □

Example 6.44 (Cohomology of $\mathbb{R}^n$)**.** Since $\mathbb{R}^n$ deformation retract onto a point via $H(x,t) = tx$, for $t \in [0,1]$, we see that $H^*(\mathbb{R}^n) \cong H^*(\mathrm{pt})$, hence

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0. \end{cases}$$

Example 6.45 ($S^1 \simeq \mathbb{R}^2 - 0$)**.** Viewing $S^1 \subset \mathbb{R}^2$, we can define a retraction $r : \mathbb{R}^2 - 0 \to S^1$, where $r(x) := x/\|x\|$. We can also find a deformation retraction from $\mathbb{R}^2 - 0$ onto S^1: This is simply a straight line homotopy $H(x,t) := tx/\|x\|$. In particular, we see that $r \circ i = \mathrm{id}_{S^1}$ and $i \circ r \simeq \mathrm{id}_{\mathbb{R}^2 - 0}$ via H. Hence S^1 and $\mathbb{R}^2 - 0$ is homotopy equivalent. Hence by the homotopy invariance of cohomology, we immediately get that

$$H^*(\mathbb{R}^2 - 0) \cong H^*(S^1)$$

and we know $H^*(S^1)$ from the Mayer-Vietoris calculation. Thus $H^k(\mathbb{R}^2 - 0) \cong \mathbb{R}$ if $k = 1$ and is zero otherwise.

We will briefly sketch the construction of an explicit isomorphism

$$\psi : H^1(\mathbb{R}^2 - 0) \xrightarrow{\cong} \mathbb{R}.$$

Let $[\alpha] \in H^1(\mathbb{R}^2 - 0)$, which is represented by a closed form $\alpha \in \Omega^1(\mathbb{R}^2 - 0)$. Take any smooth closed loop γ in $\mathbb{R}^2 - 0$ and define

$$\psi([\alpha]) := \int_\gamma \alpha.$$

We note that ψ is well-defined by Stokes' theorem, and ψ is surjective because of the existence of the winding number form defined in Example 6.15. To check that ψ is injective, it is equivalent to showing that if $\alpha \in \Omega^1(\mathbb{R}^2 - 0)$ such that $d\alpha = 0$, and $\int_\gamma \alpha = 0$, then there exists $F \in \Omega^0(\mathbb{R}^2 - 0)$ such that $dF = \alpha$. We define F as follows: Fix a base point $p \in \mathbb{R}^2 - 0$. For any $x \in \mathbb{R}^2 - 0$, pick a smooth path η such that $\eta(0) = p$ and $\eta(1) = x$. Then define $F(x) := \int_\eta \alpha$. One can check that F is well-defined by Theorem 6.14 and the assumption that $\int_\gamma \alpha = 0$. Finally, one can check that $dF = \alpha$ by a simple $\delta - \epsilon$ limit argument (see the proof of Morera's theorem, for example).

Example 6.46. By the example above, we may consider another interesting application of the Mayer-Vietoris sequence. The problem is to calculate $H^*(P_s)$, where

$$P_s := \mathbb{R}^2 - \{p_1, ..., p_s\}$$

is the plane with s distinct points removed. Clearly $H^k(P_s) = 0$ for $k > 2$ because of the dimension, and $H^0(P_s) = \mathbb{R}$ because P_s is connected. Hence the problem is really about the calculation of $H^1(P_s)$. We choose U, V in the Mayer-Vietoris sequence as Figure 6.3 suggests:

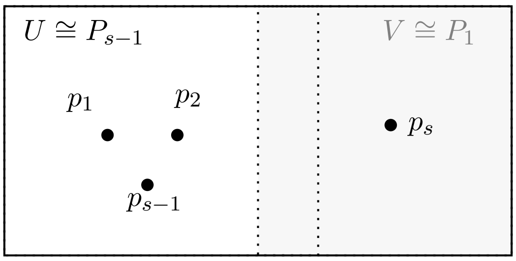


Figure 6.3: The relative position of $p_1, ..., p_s$ does not matter since cohomology is invariant under homeomorphisms. So we may choose U, V as suggested, with $U \cong P_{s-1}, V \cong P_1, U \cap V \cong \mathbb{R}^2$.

We claim that $H^1(P_s) \cong \mathbb{R}^s$, for any integer $s > 0$. We argue by induction. By the previous remark we already know this is true for $s = 1$. Now given $s > 1$, we may assume $H^1(P_{s-1}) \cong \mathbb{R}^{s-1}$ inductively. Now the Mayer-Vietoris sequence reads

$$\cdots H^0(U) \oplus H^0(V) \xrightarrow{\delta} H^0(U \cap V) \xrightarrow{d^*} H^1(P_s) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \cdots$$

which by what we discussed and Poincaré lemma, becomes

$$\cdots \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} \mathbb{R} \xrightarrow{d^*} H^1(P_s) \xrightarrow{\alpha} \mathbb{R}^{s-1} \oplus \mathbb{R} \to 0 \to \cdots$$

where $\delta(a, b) = a - b$, which implies $\dim \operatorname{im} \delta = 1$. Hence $\dim \ker d^* = \dim \operatorname{im} \delta = 1$. By exactness, we know α is a surjection, hence

$$H^1(P_s)/\ker \alpha \cong \mathbb{R}^s$$

by the first isomorphism theorem. Again by exactness, $\ker \alpha = \operatorname{im} d^*$. And since

$$\dim \ker d^* + \dim \operatorname{im} d^* = 1$$

we conclude that $\dim \operatorname{im} d^* = 0$. Hence $\dim \ker \alpha = 0$. This implies that $\ker \alpha = 0$ and

$$H^1(P_s) \cong \mathbb{R}^s$$

which proves the claim. Similar to the previous remark, we can construct an explicit isomorphism $\psi : H^1(P_s) \to \mathbb{R}^s$, by choosing s smooth closed loops $\gamma_1, ..., \gamma_s$, each only contains $p_1, ..., p_s$ in their interior (ref. Jordan curve theorem), respectively. Then define $\psi([\alpha]) = (\int_{\gamma_1} \alpha, ..., \int_{\gamma_s} \alpha)$.

Example 6.47 (Cohomology of S^n)**.** Cover S^n by two open sets U and V where U is slightly larger than the northern hemisphere and V slightly larger than the southern hemisphere (See Figure 6.2 for the case where $n = 1$). Then $U \cap V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}^1$, which is homotopic to S^{n-1} via a deformation retraction. We claim that

$$H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

for $n > 0$ by induction. We already proved the statement for $n = 1$ in Section 6.4.2. Assuming the statement holds for S^{n-1}. Then the Mayer-Vietoris sequence reads

$$\begin{tikzcd}
 & S^n & U \coprod V & U \cap V \\
H^{n+1} & 0 \arrow[r] & 0 \arrow[r] & 0 \\
H^n & ? \arrow[r] & 0 \arrow[r] & 0 \\
H^{n-1} & ? \arrow[r] & 0 \arrow[r] & \mathbb{R} \\
H^{n-2} & ? \arrow[r] & 0 \arrow[r] & 0 \\
\vdots & \vdots & \vdots & \vdots \\
H^1 & ? \arrow[r] & 0 \arrow[r] & 0 \\
H^0 & ? \arrow[r, "i"'] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta"'] & \mathbb{R}
\end{tikzcd}$$

Exactness alone gives

$$\begin{aligned} &H^n(S^n) = \mathbb{R}, \\ &H^k(S^n) = 0 \text{ for } k = 2, \cdots, n-1, \end{aligned}$$

And $H^0(S^n) = \mathbb{R}$ by connectedness, so all that remains is to compute $H^1(S^n)$. To do this, note that that $\ker \delta = \operatorname{im} i$ is one-dimensional by injectivity of i. Thus, $\operatorname{im} \delta$ is also one-dimensional, so δ is surjective, and $H^1(S^n) = 0$, which completes the induction.

Example 6.48. Let M be a manifold, we claim that $H^*(M\times\mathbb{R}) \cong H^*(M)$. Clearly $M\times\mathbb{R}$ deformation retracts onto its zero section $M \times \{0\}$ via the homotopy $H_t : (p, x) \mapsto (p, tx)$ for $p \in M, x \in \mathbb{R}, t \in [0, 1]$. By the same construction, we have $H^*(M \times I) \cong H^*(M)$.

6.6 Poincaré Lemmas

6.6.1 Poincaré Lemma and Calculation of $H^*(\mathbb{R}^n)$

In the previous section, we already used homotopy invariance to deduce $H^*(\mathbb{R}^n)$ and $H^*(M\times\mathbb{R})$. This section will present an alternative way of arriving at the same conclusion, via Poincaré Lemma. The proofs are more involved, but some readers might find it more concrete. More importantly, it introduces the tool of a ***chain homotopy operator***. Since the important calculation are already done in the previous section, we encourage the reader to skip this section first and return to it later if they are not interested at the moment (We will need this result in the proof of Poincaré duality later).

Theorem 6.49 (Poincaré Lemma)**.** *Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first factor, and let $s : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the zero section. That is,*

$$\pi(x^1, ..., x^n, t) = (x^1, ..., x^n) \qquad s(x^1, ..., x^n) = (x^1, ..., x^n, 0).$$

Then the induced map on cohomology $\pi^* : H^*(\mathbb{R}^n) \to H^*(\mathbb{R}^{n+1})$ and $s^* : H^*(\mathbb{R}^{n+1}) \to H^*(\mathbb{R}^n)$ are mutual inverses. Therefore, $H^*(\mathbb{R}^{n+1}) \cong H^*(\mathbb{R}^n)$.

Therefore, by induction, and the computation of $H^*(\mathbb{R})$, one immediately conclude that $H^k(\mathbb{R}^n) = 0$ for all $k \neq 0$, and $H^0(\mathbb{R}^n) = \mathbb{R}$, for any n.

Proof of Poincaré Lemma. By Theorem 6.11, we see that π^*, s^* are well-defined morphisms on cohomology. Since $\pi \circ s = \mathrm{id}_{\mathbb{R}^n}$, by functoriality, we see that $s^*\pi^* = \mathrm{id}_{H^*(\mathbb{R}^n)}$. Therefore, all it remains is to show that $\pi^*s^* = \mathrm{id}_{H^*(\mathbb{R}^{n+1})}$. This is more difficult: note that even on the level of forms, $\pi^*s^* \neq \mathrm{id}$.

To show the desired result, we shall construct, for each q, an operator $K^q : \Omega^q(\mathbb{R}^{n+1}) \to \Omega^{q-1}(\mathbb{R}^{n+1})$, such that

$$\mathrm{id} -\pi^*s^* = (-1)^{q-1}(dK^q - K^{q+1}d) \quad \text{on} \quad \Omega^q(\mathbb{R}^{n+1}). \tag{*}$$

Such a family of operators is called a ***chain homotopy operator.*** Note that (*) implies that π^*s^* is *equal* to identity on $H^q(\mathbb{R}^{n+1})$, although it is not identity on $\Omega^*(\mathbb{R}^{n+1})$. Because any element $[\alpha] \in H^q(\mathbb{R}^{n+1})$ is represented by a closed form α, and exact forms are set to be zero in $H^q(\mathbb{R}^{n+1})$. Hence the two terms on the right hand side of (*) will both be zero when we descend to $H^q(\mathbb{R}^{n+1})$. This will finish the proof.

It remains to construct the family of operators $K^q : \Omega^q(\mathbb{R}^{n+1}) \to \Omega^{q-1}(\mathbb{R}^{n+1})$ described above. If we let $x^1, ..., x^n, t$ be the coordinate on $\mathbb{R}^{n+1}$, the we note that for any form in $\Omega^q(\mathbb{R}^{n+1})$, it either contains the differential of the last coordinate dt in the wedge, or it does not. We define K^q linearly as follows: If the input does not contain dt, i.e., the input is of the form

$$\alpha = f(x, t)dx^{i_1} \wedge \cdots \wedge dx^{i_q},$$

then define

$$K^q(\alpha) := 0.$$

If the input contains dt, i.e., the input is of the form

$$\beta = f(x, t)dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \wedge dt,$$

then define

$$K^q(\beta) := dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \cdot \int_0^t f(x,\lambda)d\lambda.$$

We now check that (*) holds for forms α, β respectively. Since elements in $\Omega^q(\mathbb{R}^{n+1})$ are linear combinations of elements of the form α, β, we are done since K^q's are defined linearly. We first check (*) holds for α. By definition of π, we have $s^*(f) = f \circ s$, and $\pi^*(dx^a) = dx^a, s^*(dx^a) = dx^a$, and so on. Therefore,

$$\begin{aligned}(\mathrm{id} - \pi^* s^*)\alpha &= \alpha - \pi^*(f(x,0)dx^{i_1} \wedge \cdots \wedge dx^{i_q}) \\ &= \alpha - f(x,0)dx^{i_1} \wedge \cdots \wedge dx^{i_q} \\ &= [f(x,t) - f(x,0)]dx^{i_1} \wedge \cdots \wedge dx^{i_q}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(dK^q - K^{q+1}d)\alpha &= -K^{q+1}\alpha \\ &= -K^{q+1}\left(\frac{\partial f}{\partial t}dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}\right) \\ &= -K^{q+1}\left((-1)^q \frac{\partial f}{\partial t}dx^{i_1} \wedge \cdots \wedge dx^{i_q} \wedge dt\right) \\ &= (-1)^{q+1}dx^{i_1} \wedge \cdots \wedge dx^{i_q} \cdot \int_0^t \frac{\partial f}{\partial t}(x,\lambda)d\lambda \\ &= (-1)^{q-1}[f(x,t) - f(x,0)]dx^{i_1} \wedge \cdots \wedge dx^{i_q}\end{aligned}$$

by the fundamental theorem of calculus. This proves that (*) holds on forms of type α. Now we check that (*) holds on forms of type β. Note that since $s^*dt = d(t \circ s) = d(0) = 0$, we have

$$(\mathrm{id} - \pi^* s^*)\beta = \beta.$$

Now, we have

$$\begin{aligned}K^{q+1}d\beta &= K^{q+1}\left(\frac{\partial f}{\partial x^a}dx^a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \wedge dt\right) \\ &= (-1)^{q-1}dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \wedge dx^a \cdot \int_0^t \frac{\partial f}{\partial x^a}(x,\lambda)d\lambda\end{aligned}$$

and

$$\begin{aligned}dK^q\beta &= d\left(dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \cdot \int_0^t f(x,\lambda)d\lambda\right) \\ &= \frac{\partial}{\partial x^a}\int_0^t f(x,\lambda)d\lambda \cdot dx^a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \\ &\quad + \frac{\partial}{\partial t}\int_0^t f(x,\lambda)d\lambda \cdot dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \\ &= \int_0^t \frac{\partial f}{\partial x^a}dt \cdot (-1)^{q-1}dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \wedge dx^a + f(x,t)dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \\ &= \int_0^t \frac{\partial f}{\partial x^a}dt \cdot (-1)^{q-1}dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}} \wedge dx^a + (-1)^{q-1}\beta.\end{aligned}$$

Therefore,

$$dK^q\beta - K^{q+1}d\beta = (-1)^q\beta$$

which again proves (*). □

Therefore, as we discussed, the Poincaré lemma implies that

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0. \end{cases}$$

Remark 6.50 (General Poincaré Lemma)**.** In fact, let M be any smooth manifold, and $\pi : M \times \mathbb{R} \to M$ be the projection, and $s : M \to M \times \mathbb{R}$ be the zero section. The proof of the Poincaré lemma carries over and shows that π^*, s^* gives an isomorphism

$$H^*(M \times \mathbb{R}) \cong H^*(M).$$

6.6.2 Poincaré Lemma for Compactly Supported Cohomology and Calculation of $H_c^*(\mathbb{R}^n)$

One last important computational task we are yet to finish is computing $H_c^*(\mathbb{R}^n)$. This is a consequence of a more general statement:

Theorem 6.51 (Poincaré Lemma for Compactly Supported Cohomology)**.** *Let M be a smooth manifold. Then there is an isomorphism $H_c^{*+1}(M \times \mathbb{R}) \to H_c^*(M)$.*

The construction of the isomorphisms are more difficult, hence we will talk through the proof for the remainder of this section.

To prove this theorem, again consider $\pi : M \times \mathbb{R} \to M$, the projection map. But $\pi^*\omega \in \Omega^*(M \times \mathbb{R})$ clearly has noncompact support, for example for a compactly supported $f \in \Omega_c^0(M)$, we have $\pi^* f(x,t) = f(\pi(x,t)) = f(x)$, so $\mathrm{Supp}(\pi^* f) = \mathrm{Supp}(f) \times \mathbb{R}$ is noncompact. Therefore we cannot hope π^* to send $\Omega_c^*(M)$ to $\Omega_c^*(M \times \mathbb{R})$. However, we may define an ***integration along fiber*** map

$$\pi_* : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M)$$

by again only caring about forms with dt in it: Give $M \times \mathbb{R}$ the product charts of M and $\mathbb{R}$, with t being the coordinate on $\mathbb{R}$. Every $\omega \in \Omega_c^q(M \times \mathbb{R})$ is of the following two types: Either

$$\omega = (\pi^*\phi) f(x,t) \qquad \phi \in \Omega_c^q(M),\ f \in C_c^\infty(M \times \mathbb{R})$$

or

$$\omega = (\pi^*\phi) \wedge f(x,t)dt \qquad \phi \in \Omega_c^{q-1}(M),\ f \in C_c^\infty(M \times \mathbb{R}).$$

Then we define π_* as follows, for two types of forms respectively:

$$\pi_* : (\pi^*\phi) f(x,t) \mapsto 0$$

$$\pi_* : (\pi^*\phi) \wedge f(x,t)dt \mapsto \phi \int_{-\infty}^{\infty} f(x,t)dt.$$

We will need to show that $\pi_* d = d\pi_*$. So that this gives a well-defined map

$$\pi_* : H_c^*(M \times \mathbb{R}) \to H_c^{*-1}(M).$$

Lemma 6.52. $\pi_* d = d\pi_*$.

Proof. To prove the equality, we just need to show that it holds for both types of forms. First let $\omega \in \Omega_c^k(M \times \mathbb{R})$ be of "type 1", i.e. $\omega = (\pi^*\phi) f$ for some $\phi \in \Omega^k(M)$ and some compactly supported smooth $f : M \times \mathbb{R} \to \mathbb{R}$. Then

$$d(\pi_*\omega) = d(0) = 0,$$

and

$$\begin{aligned}\pi_*(d\omega) &= \pi_*(d(\pi^*\phi)\wedge f + (-1)^k(\pi^*\phi)\wedge df)\\ &= \pi_*(d(\pi^*\phi)\wedge f) + (-1)^k\pi_*((\pi^*\phi)\wedge df)\\ &= \pi_*(\pi^*(d\phi)\wedge f) + (-1)^k\pi_*((\pi^*\phi)\wedge df)\\ &= (-1)^k\pi_*((\pi^*\phi)\wedge df)\end{aligned}$$

where the last equality holds by definition of π_*. Therefore, we just need to show that $\pi_*((\pi^*\phi)\wedge df) = 0$. In local coordinates $(x^1, \cdots, x^n, t)$, we have

$$\begin{aligned}\pi^*\phi\wedge df &= \pi^*\phi\wedge\left(\frac{\partial f}{\partial x^1}dx^1 + \cdots + \frac{\partial f}{\partial x^n}dx^n + \frac{\partial f}{\partial t}dt\right)\\ &= \pi^*\phi\wedge\left(\pi^*\eta + \frac{\partial f}{\partial t}dt\right)\\ &= \pi^*\phi\wedge\pi^*(\eta) + \pi^*\phi\wedge\frac{\partial f}{\partial t}dt\\ &= \pi^*(\phi\wedge\eta) + \pi^*\phi\wedge\frac{\partial f}{\partial t}dt,\end{aligned}$$

where $\eta\in\Omega^1(M)$ is given by $\eta = d(f\circ s)$, with $s : M\to M\times\mathbb{R}$ being the zero section. Clearly η is globally defined, and $(\partial f/\partial t)dt$ is globally defined when a product atlas is used. Noting that f is compactly supported, we then have

$$\pi_*(\pi^*\phi\wedge df) = \phi\int_{-\infty}^{\infty}\frac{\partial f}{\partial t}dt = \phi[f(x,\infty) - f(x,-\infty)] = 0$$

as desired. This establishes the equality $d\pi_*\omega = \pi_*d\omega$ in the case that ω is of type 1.

Now assume ω is of type 2, i.e. $\omega = (\pi^*\phi)f dt$ for some $\phi\in\Omega^{k-1}(M)$ and some compactly supported smooth $f : M\times\mathbb{R}\to\mathbb{R}$. Then, in local coordinates $(x^1, ..., x^n, t)$, we have

$$\begin{aligned}d(\pi_*\omega) &= d\left(\phi\int_{-\infty}^{\infty} f dt\right)\\ &= d\phi\wedge\int_{-\infty}^{\infty} f dt + (-1)^{k-1}\phi d\left(\int_{-\infty}^{\infty} f dt\right)\\ &= d\phi\wedge\int_{-\infty}^{\infty} f dt + (-1)^{k-1}\phi\wedge\left(dx^a\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^a}dt\right)\end{aligned}$$

while

$$\begin{aligned}\pi_*(d\omega) &= \pi_*\left(d(\pi^*\phi)\wedge f dt + (-1)^{k-1}(\pi^*\phi)\wedge\left(\frac{\partial f}{\partial x^a}dx^a\wedge dt\right)\right)\\ &= d\phi\wedge\int_{-\infty}^{\infty} f dt + \pi_*\left((-1)^{k-1}\pi^*\left(\phi\wedge\frac{\partial f}{\partial x^a}dx^a\right)\wedge dt\right)\\ &= d\phi\wedge\int_{-\infty}^{\infty} f dt + \pi_*\left((-1)^{k-1}\pi^*\left(\phi\wedge dx^a\right)\frac{\partial f}{\partial x^a}dt\right)\\ &= d\phi\wedge\int_{-\infty}^{\infty} f dt + (-1)^{k-1}(\phi\wedge dx^a)\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^a}dt\\ &= d\phi\wedge\int_{-\infty}^{\infty} f dt + (-1)^{k-1}\phi\wedge\left(dx^a\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^a}dt\right)\\ &= d(\pi_*\omega).\end{aligned}$$

This concludes the proof of the lemma. □

Now we need to define another map e_* in the reverse direction, and show that they are mutual inverses in the cohomology level. We define, on forms,

$$e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R})$$

as follows: Let $e = e(t)dt \in \Omega_c^1(\mathbb{R})$ such that $\int_{\mathbb{R}} e = 1$. Then define $e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R})$ by

$$e_* : \phi \mapsto (\pi^*\phi) \wedge e.$$

It is easy to see that $de_* = e_*d$. Thus e_* descends to a well-defined map on the cohomology groups. It follows directly from definition that

$$\pi_* e_* = \mathrm{id}_{\Omega_c^*(M)}.$$

But again, $e_*\pi_* \neq \mathrm{id}_{\Omega_c^*(M\times\mathbb{R})}$. So we need to construct a chain homotopy $K : \Omega_c^*(M\times\mathbb{R}) \to \Omega_c^{*-1}(M \times \mathbb{R})$ again. This time, we define

$$K : (\pi^*\phi)f(x,t) \mapsto 0$$

and

$$K : (\pi^*\phi) \wedge f(x,t)dt \mapsto \pi^*\phi\left(\int_{-\infty}^{t} f(x,\lambda)d\lambda - A(t)\int_{-\infty}^{\infty} f(x,\lambda)d\lambda\right)$$

where

$$A(t) = \int_{(-\infty,t)} e = \int_{-\infty}^{t} e(\lambda)d\lambda.$$

Finally, we conclude the proof of Theorem 6.51 because this shows that e_*, π_* are isomorphisms that are inverses of each other on the level of cohomology groups, proving the Poincaré lemma for compactly supported cohomology.

Theorem 6.53. *We have*

$$\mathrm{id} - e_*\pi_* = (-1)^{q-1}(dK - Kd) \quad on \quad \Omega_c^q(M \times \mathbb{R}).$$

Proof. Again we check the statement on the two types of forms. First, let

$$\omega = \pi^*\phi \cdot f(x,t) \qquad \phi \in \Omega_c^q(M), f \in C_c^\infty(M \times \mathbb{R})$$

be of type 1. Then $(\mathrm{id} - e_*\pi_*)\omega = \omega - 0 = \omega$. On the other hand, we have

$$\begin{aligned}
(dK - Kd)\omega &= -Kd\omega \\
&= -K\pi^*\left(d\phi \cdot f(x,t) + (-1)^q\phi \wedge \frac{\partial f}{\partial x^a}dx^a + (-1)^q\phi \wedge \frac{\partial f}{\partial t}dt\right) \\
&= (-1)^{q-1}K\left(\pi^*\phi \wedge \frac{\partial f}{\partial t}dt\right) \\
&= (-1)^{q-1}\pi^*\phi\left(\int_{-\infty}^{t} \frac{\partial f}{\partial t}(x,\lambda)d\lambda - A(t)\int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(x,\lambda)d\lambda\right) \\
&= (-1)^{q-1}\pi^*\phi f(x,t) = (-1)^{q-1}\omega
\end{aligned}$$

because $\partial f/\partial t(x, \pm\infty) = 0$ by the assumption that $f \in C_c^\infty(M \times \mathbb{R})$. Therefore, the claim holds for forms of type 1. Now let

$$\omega = \pi^*\phi \wedge f(x,t)dt \qquad \phi \in \Omega_c^{q-1}(M), f \in C_c^\infty(M \times \mathbb{R})$$

be a form of type 2. Now

$$(\mathrm{id} - e_* \pi_*)\omega = \omega - e_*\left(\phi \cdot \int_{-\infty}^{\infty} f(x,t)dt\right) = \omega - \left(\pi^*\phi \cdot \int_{-\infty}^{\infty} f(x,t)dt\right) \wedge e.$$

Now we calculate $(dK - Kd)\omega$, which is a bit tedious because it involves cancellation of many terms. First, for simplicity, we put

$$\beta(x,t) := \int_{-\infty}^{t} f(x,\lambda)d\lambda - A(t)\int_{-\infty}^{\infty} f(x,\lambda)d\lambda.$$

Then we have

$$\begin{aligned}
dK\omega &= d(\pi^*\phi\beta(x,t)) \\
&= \pi^*\left(d\phi \cdot \beta(x,t) + (-1)^{q-1}\phi \wedge \frac{\partial \beta}{\partial x^a}dx^a + (-1)^{q-1}\phi \wedge \frac{\partial \beta}{\partial t}dt\right) \\
&= \pi^* d\phi \int_{-\infty}^{t} f(x,\lambda)d\lambda - \pi^* d\phi A(t)\int_{-\infty}^{\infty} f(x,\lambda)d\lambda \\
&\quad + (-1)^{q-1}\pi^*\phi \wedge dx^a \frac{\partial}{\partial x^a}\int_{-\infty}^{t} f(x,\lambda)d\lambda - (-1)^{q-1}\pi^*\phi A(t) dx^a \frac{\partial}{\partial x^a}\int_{-\infty}^{\infty} f(x,\lambda)d\lambda \\
&\quad + (-1)^{q-1}\pi^*\phi \wedge dt \frac{\partial}{\partial t}\left(\int_{-\infty}^{t} f(x,\lambda)d\lambda - A(t)\int_{-\infty}^{\infty} f(x,\lambda)d\lambda\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
Kd\omega &= K\left(\pi^* d\phi \wedge f(x,t)dt + (-1)^q \pi^*\phi \wedge \frac{\partial f}{\partial x^a}dx^a \wedge dt\right) \\
&= \pi^* d\phi\left(\int_{-\infty}^{t} f(x,\lambda)d\lambda - A(t)\int_{-\infty}^{\infty} f(x,\lambda)d\lambda\right) \\
&\quad + (-1)^{q-1}\pi^*\phi \wedge dx^a\left(\int_{-\infty}^{t} \frac{\partial f}{\partial x^a}(x,\lambda)d\lambda - A(t)\int_{-\infty}^{\infty} \frac{\partial f}{\partial x^a}(x,\lambda)d\lambda\right) \\
&= \pi^* d\phi \int_{-\infty}^{t} f(x,\lambda)d\lambda - \pi^* d\phi A(t)\int_{-\infty}^{\infty} f(x,\lambda)d\lambda \\
&\quad + (-1)^{q-1}\pi^*\phi \wedge dx^a \frac{\partial}{\partial x^a}\int_{-\infty}^{t} f(x,\lambda)d\lambda - (-1)^{q-1}\pi^*\phi A(t) dx^a \frac{\partial}{\partial x^a}\int_{-\infty}^{\infty} f(x,\lambda)d\lambda.
\end{aligned}$$

Hence we see that

$$\begin{aligned}
(dK - Kd)\omega &= (-1)^{q-1}\pi^*\phi \wedge dt \frac{\partial}{\partial t}\left(\int_{-\infty}^{t} f(x,\lambda)d\lambda - A(t)\int_{-\infty}^{\infty} f(x,\lambda)d\lambda\right) \\
&= (-1)^{q-1}\pi^*\phi \wedge f(x,t)dt - (-1)^{q-1}\pi^*\phi \wedge dt \cdot \int_{-\infty}^{\infty} f(x,\lambda)d\lambda \cdot e(t) \\
&= (-1)^{q-1}\omega - (-1)^{q-1}\left(\pi^*\phi \cdot \int_{-\infty}^{\infty} f(x,t)dt\right) \wedge e = (-1)^{q-1}(\mathrm{id} - e_*\pi_*)\omega.
\end{aligned}$$

□

As a simple consequence of Theorem 6.51, Example 6.24, and induction, we see that

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & \text{else.} \end{cases}$$

Recall in Example 6.24, the isomorphism $H_c^1(\mathbb{R}) \cong \mathbb{R}$ is achieved by integration. By the proof of Theorem 6.51, the isomorphism $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$ is given by iterated applications of π_*. Hence, the isomorphism $H^n(\mathbb{R}^n) \cong \mathbb{R}$ is also given by integration. Therefore, a generator of $H_c^n(\mathbb{R}^n)$ is a bump n-form $\alpha = f(x^1, ..., x^n)dx^1 \wedge \cdots \wedge dx^n$ with $f \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \alpha = 1$. We may take $\mathrm{Supp}(f)$ to be as small as we like.

6.7 The Mayer-Vietoris Argument

Definition 6.54. An open cover $\mathfrak{U} = (U_\alpha)$ of a manifold M^n is called a ***good cover*** if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. A manifold which has a finite good cover is said to be of ***finite type***.

The proof of the following theorem requires knowledge in Riemannian geometry, which we will not cover. Hence, the reader can simply accept the statement of the theorem.

Theorem 6.55. *Every manifold has a good cover. If the manifold is compact, then the cover may be chosen to be finite.*

Proof. Endow a smooth manifold with a Riemannian metric, which is possible for any smooth manifold: see [1], Proposition 13.3. Then every point on the manifold has a geodesically convex neighborhood, and intersection of geodesically convex neighborhoods is again geodesically convex (see [4], p.72, Theorem 3.7 and p.76, Proposition 4.2). □

Let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}, \mathfrak{U}' = (U'_\beta)_{\beta \in J}$ be two open covers of M. If for all $\beta \in J$, $U'_\beta \subset U_\alpha$ for some $\alpha \in I$, then we say $\mathfrak{U}'$ is a ***refinement*** of $\mathfrak{U}$, and we write $\mathfrak{U} < \mathfrak{U}'$.

Corollary 6.56. *Every open cover on a manifold has a refinement which is a good cover.*

Definition 6.57. A ***directed set*** is a set I with a relation $<$ such that

(a) $a < a$ for all $a \in I$;

(b) If $a < b$ and $b < c$ then $a < c$;

(c) For any $a, b \in I$, there exists $c \in I$ such that $a < c$ and $b < c$.

Lemma 6.58. *The set of open covers on a manifold is a directed set.*

Definition 6.59. A subset J of a directed set I is ***cofinal*** in I if for every $i \in I$ there exists $j \in J$ such that $i < j$. (Clearly such J will also be a directed set.)

Corollary 6.60. *The good covers are cofinal in the set of all covers of a manifold M.*

The existence of good covers on manifolds gives a powerful tool of proving many results by using the Mayer-Vietoris sequence and five lemma, inducting on the cardinality of a good cover. This proof method is called the Mayer-Vietoris argument. We will prove several statements using the Mayer-Vietoris argument. But in the proof we will always assume that our manifold is of ***finite type***, i.e., admits a finite good cover (for example, take any compact manifolds). Many statements that we will prove are much more general and holds for all smooth manifolds (and even topological spaces), but then the proof will require much more work. Here, we are willing to sacrifice generality for more concise and elegant proofs.

Finite Dimensionality of de Rham Cohomology

Proposition 6.61. If M has a finite good cover, then $H^*(M)$ are all finite dimensional.

Proof. The Mayer-Vietoris sequence says

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{d^*} H^q(U \cup V) \xrightarrow{r} H^q(U) \oplus H^q(V) \to \cdots$$

And we have an exact sequence $0 \to \ker r \to H^q(U \cup V) \to \operatorname{im} r \to 0$. Since $\operatorname{im} r \cong \mathbb{R}^n$ for some n, it is free, so the exact sequence splits, therefore

$$H^q(U \cup V) \cong \ker r \oplus \operatorname{im} r \cong \operatorname{im} d^* \oplus \operatorname{im} r$$

where the last isomorphism is by the exactness of the MV sequence. So if $H^q(U), H^q(V), H^{q-1}(U \cap V)$ are finite dimensiobal, then so is $H^q(U \cup V)$.

If M is diffeomorphic to $\mathbb{R}^n$, then Poincaré lemma proves that $H^*(M)$ is finite dimensional. Now induct on the cardinality of a good cover. Suppose the cohomology of any manifold having good cover with at most p open sets is finite dimensional. Let M be a manifold with $p+1$ good cover $U_0, ..., U_p$. Now $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$ is a manifold with a good cover consisting of p open sets $U_0 \cap U_p, U_1 \cap U_p, ..., U_{p-1} \cap U_p$. Thus we know that $H^*(U_p), H^*(U_0 \cup \cdots \cup U_{p-1}), H^*((U_0 \cup \cdots \cup U_{p-1}) \cap U_p)$ are all finite dimensional. Then the claim that $H^*(M)$ is finite dimensional follows from the previous paragraph. □

Similarly,

Proposition 6.62. If M has a finite good cover, then $H_c^*(M)$ are all finite dimensional.

Poincaré Duality on an Orientable Manifold

Definition 6.63. A pariting between two finite-dimensional vector spaces $\langle\ ,\ \rangle : V \otimes W \to \mathbb{R}$ is said to be ***nondegenerate*** if $\langle v, w\rangle = 0$ for all $w \in W$ implies $v = 0$ and for all $v \in V$ implies $w = 0$. Equivalently, the map $v \mapsto \langle v, \ \rangle$ defines an injection $V \to W^*$ and the map $w \mapsto \langle\ , w\rangle$ defines an injection $W \to V^*$.

Lemma 6.64. *The pairing between two finite-dimensional vector spaces $\langle\ ,\ \rangle : V \otimes W \to \mathbb{R}$ is nondegenerate iff the map $v \mapsto \langle v, \ \rangle$ defines an isomorphism $V \to W^*$.*

Proof. Suppose the pairing is nondegenerate. Then the map $v \mapsto \langle v, \ \rangle$ defines an injection $V \to W^*$ and the map $w \mapsto \langle\ , w\rangle$ defines an injection $W \to V^*$. So we have

$$\dim V \le \dim W^* = \dim W \le \dim V^* = \dim V.$$

Thus $\dim V = \dim W$ and the injection $v \mapsto \langle v, \ \rangle$ is actually an isomorphism.

Conversely, suppose the map $\varphi : v \mapsto \langle v, \ \rangle$ defines an isomorphism $V \to W^*$. Suppose $\langle v, w\rangle = 0$ for all $w \in W$. Then denote the map $\varphi(v)(w) := \langle v, w\rangle = \varphi_v(w)$. So $\varphi_v(w) = 0$ for all $w \in W$, which means $\varphi_v \equiv 0$, so $v = 0$ since φ is an isomorphism (hence injective). Now suppose $\langle v, w\rangle = 0$ for all $v \in V$. Then pick $\psi \in W^*$ such that $\psi : W \to \mathbb{R}$ is injective. Since φ is an isomorphism, $\psi = \varphi_u$ for some $u \in V$. By assumption, $\psi(w) = \varphi_u(w) = \langle u, w\rangle = 0$. Hence $w = 0$ since ψ is injective. □

Now let M^n be an oriented smooth manifold with no boundary (For manifold with boundary, Poincare duality does not hold, see this post. There is a generalization called

Lefschetz duality for manifold with boundary). We consider the pairing

$$\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$$
$$\alpha \otimes \beta \mapsto \int_M \alpha \wedge \beta.$$

Here, we are slightly abusing notation and use α instead of $[\alpha]$ to denote a cohomology class in $H^q(M)$. This map is well-defined: If $\alpha = \alpha' \in H^q(M)$, i.e., $\alpha - \alpha' = d\eta$ for some $\eta \in \Omega^{q-1}(M)$, then

$$\begin{aligned}\int_M (\alpha - \alpha') \wedge \beta = \int_M d\eta \wedge \beta &= \int_M d(\eta \wedge \beta) - (-1)^{\deg \eta} \int_M \eta \wedge d\beta \\ &= \int_{\partial M} \eta \wedge \beta - (-1)^{\deg \eta} \int_M \eta \wedge d\beta = 0.\end{aligned}$$

In the second to last equality, we used Stokes' theorem in the first term, which we can because $\eta \wedge \beta \in \Omega_c^{n-1}(M)$, and since $\partial M = \emptyset$, the first term is zero. The second term is also zero because β is closed since it represents a cohomology class. Similarly one can show that if $\beta = \beta' \in H_c^{n-q}(M)$, the resulting pairing is also well-defined.

Theorem 6.65 (Poincaré Duality for Oriented Manifold)**.** *Let M^n be an oriented manifold. The paring $\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$ defined by $\alpha \otimes \beta \mapsto \int_M \alpha \wedge \beta$ is nondegenerate. Equivalently, we have an isomorphism*

$$H^q(M) \xrightarrow{\cong} (H_c^{n-q}(M))^*$$

Given by $\alpha \mapsto \int_M \alpha \wedge (\bullet)$.

The proof of the Poincaré duality theorem relies on the following key technical lemma:

Lemma 6.66. *The following diagram obtained by putting the MV exact sequence for U, V on top, and the MV exact sequence for cohomology of U, V with compact support below and dualizing, has exact rows and is sign commutative (i.e., commutative up to a plus or minus sign):*

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U \cup V) \arrow[r, "r"] \arrow[d, "\alpha \mapsto \int_{U \cup V} \alpha \wedge (\bullet)"] & H^q(U) \oplus H^q(V) \arrow[r, "-"] \arrow[d, "(\sigma,\tau) \mapsto \int_U \sigma \wedge (\bullet) + \int_V \tau \wedge (\bullet)"] & H^q(U \cap V) \arrow[r, "d^*"] \arrow[d, "\omega \mapsto \int_{U \cap V} \omega \wedge (\bullet)"] & H^{q+1}(U \cup V) \arrow[r] \arrow[d, "\eta \mapsto \int_{U \cup V} \eta \wedge (\bullet)"] & \cdots \\
\cdots \arrow[r] & H_c^{n-q}(U \cup V)^* \arrow[r, "+^*"] & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* \arrow[r, "\delta^*"] & H_c^{n-q}(U \cap V)^* \arrow[r, "(d_*)^*"] & H_c^{n-q-1}(U \cup V)^* \arrow[r] & \cdots
\end{tikzcd}$$

Here, r and $-$ are restriction and difference maps defined in the MV sequence, and d^* is the boundary map defined in the MV sequence. The map $\delta : H_c^{n-q}(U \cap V) \to H_c^{n-q}(U) \oplus H_c^{n-q}(V)$ is the signed inclusion $\delta(\omega) = (-j_*\omega, j_*\omega)$, and $+ : H_c^{n-q}(U) \oplus H_c^{n-q}(V) \to H_c^{n-q}(U \cup V)$ is the summation map.

Proof. We check the (sign) commutativity of each squares:

- For the first square, let $\alpha \in H^q(U \cup V)$. We first go down and then go left. Denote the linear functional $\int_{U \cup V} \alpha \wedge (\bullet)$ by $\hat{\alpha}$. By definition of dualization of a linear map,

$$+^*(\hat{\alpha})(\beta_1, \beta_2) = \hat{\alpha}(+(\beta_1, \beta_2)) = \int_{U \cup V} \alpha \wedge (\beta_1 + \beta_2)$$

where $\beta_1 \in H_c^{n-q}(U), \beta_2 \in H_c^{n-q}(V)$. Clearly we get the same result by first going left and then go down in the first square. Note that the integral on $U \cup V$ indeed splits into an integral on U plus an integral on V, because β_1 is supported in U, and similarly for β_2.

- For the second square, we again first go down and then go left. Let $(\sigma, \tau) \in H^q(U) \oplus H^q(V)$. The linear functional we get in $H_c^{n-q}(U \cap V)^*$, recalling the definition of dualization, will be defined as follow: for any $\alpha' \in H_c^{n-q}(U \cap V)$, it gets mapped to

$$\int_U \sigma \wedge (-j_* \alpha') + \int_V \tau \wedge (j_* \alpha') = -\int_U \sigma \wedge \alpha' + \int_V \tau \wedge \alpha'.$$

  Since the map $H^q(U) \oplus H^q(V) \to H^q(U \cap V)$ is defined by $(\sigma, \tau) \mapsto \tau - \sigma$, we clearly get the same result when we go down.

- For the third square, let $\omega \in H^q(U \cap V)$. Recall that $d^*\omega \in H^{q+1}(U \cup V)$ is defined by $d^*\omega|_U = -d(\rho_V \omega)$ and $d^*\omega|_V = d(\rho_U \omega)$. Also recall the definition of d_* from Remark 6.38: For $\tau \in H^{n-q-1}(U \cup V)$, we showed that $d_*\tau$ is a form in $H_c^{n-q}(U \cap V)$ such that

$$(-(j_U)_*(d_*\tau), (j_V)_*(d_*\tau)) = (d(\rho_U \tau), d(\rho_V \tau)).$$

  Note that since τ is represented by a closed form, we have $d(\rho_V \tau) = (d\rho_V) \wedge \tau$. Similarly, $d(\rho_V \omega) = (d\rho_V) \wedge \omega$.

  Now we first go down and go left in the third square of the diagram. The result will be

$$\int_{U \cap V} \omega \wedge d_*\tau = \int_{U \cap V} \omega \wedge (d\rho_V) \wedge \tau = (-1)^{\deg \omega} \int_{U \cap V} (d\rho_V) \wedge \omega \wedge \tau.$$

  Now if we first go left and go down in the third square, we would get (note that $\mathrm{Supp}(\omega) \subset U \cap V$):

$$\int_{U \cup V} d^*\omega \wedge \tau = -\int_{U \cap V} (d\rho_V) \wedge \omega \wedge \tau$$

  Hence we see that the result we get from two different paths only differ by a sign, i.e., differ by $(-1)^{\deg \omega + 1}$.

$\square$

Remark 6.67. There are still two quick remarks I want to make about the proof of the previous lemma, and how it helps us to prove Poincaré duality:

- There is one small detail: We know MV sequence for H_c^* is exact, but we do not know yet when we dualize everything the sequence remain exact. Luckily in this case, it is.

  Reason: The functor $\mathrm{Hom}_R(-, M)$ is exact iff M is an injective R-module. Given a field k, every k-vector space Q is an injective k-module. See details in the example section of Wikipedia. In particular, $\mathrm{Hom}_{\mathbb{R}}(-, \mathbb{R})$ is exact.

- So we know from lemma 6.66 that the diagram there is sign commutative. But to prove Poincaré duality we want to use the Five Lemma. But the Five Lemma requires the diagram to be commutative instead of just sign commutative. The trick here is to fix the vertical map a little bit so that everything becomes commutative: Suppose we have a diagram with exact rows and sign commutative squares, with vertical maps being isomorphisms as below, but say the first square is only commutative up to a minus sign:

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, dashed, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & \cdots \\
\cdots \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & \cdots
\end{tikzcd}$$

That is, $\beta f = -f'\alpha$. Then replacing β by $-\beta : B \to B'$ gives again an isomorphism, and this time the square commutes. Now if the next square is again sign commutative only, we replace γ by $-\gamma$. And we did not change the rows, so exactness is preserved. After fixing the diagram, we may apply the Five Lemma.

Finally we present the proof of the Poincaré duality for an oriented manifold with a finite good cover (this is necessary for induction). Although this is a weaker version of the Poincaré duality theorem, but its proof is extremely concise and elegant.

Proof of Poincaré Duality. When M is diffeomorphic to $\mathbb{R}^n$, then Poincaré duality follows from the Poincaré lemma, i.e., the calculation of $H^*(\mathbb{R}^m)$ and $H_c^*(\mathbb{R}^m)$. We need to show that the integration $\int : H^q(\mathbb{R}^n) \to H_c^{n-q}(\mathbb{R}^n)^*$ is actually an isomorphism. When $q \neq 0$, by Poincaré lemma $H^q(\mathbb{R}^n) = 0 = H_c^{n-q}(\mathbb{R}^n)^*$. So any map between the trivial groups is an isomorphism. When $q = 0$, by Poincaré lemma we have $H^0(\mathbb{R}^n) = H_c^n(\mathbb{R}^n) \cong \mathbb{R}$. Therefore $H_c^n(\mathbb{R}^n)^* \cong \mathbb{R}$ as well, so $\int : H^0(\mathbb{R}^n) \to H_c^n(\mathbb{R}^n)^*$ can be viewed as a linear map between $\mathbb{R}$. We claim that this map is surjective: Take any $\varphi \in H_c^n(\mathbb{R}^n)^*$ and $\omega \in H_c^n(\mathbb{R}^n)$ a generator of the group, i.e., $\int_{\mathbb{R}^n} \omega = 1$. Then say $\varphi(\omega) = \lambda \in \mathbb{R}$. Then the map $\int(\lambda) : \alpha \mapsto \lambda \int_{\mathbb{R}^n} \alpha$ is also a linear functional on $H_c^n(\mathbb{R}^n)$ that agrees with φ on the generator ω, hence they are equal. This shows the integration map is a surjective linear map between vector spaces of same dimension, hence is an isomorphism.

Now induct on the cardinality of a good cover. Suppose Poincaré duality holds for any manifold having a good cover with at most p. Let M be a manifold with $p+1$ good cover $U_0, ..., U_p$. Now $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$ is a manifold with a good cover consisting of p open sets $U_0 \cap U_p, U_1 \cap U_p, ..., U_{p-1} \cap U_p$. Use MV sequence on the open sets $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$ and U_p and Lemma 6.66. This completes the inductive step. ☐

Corollary 6.68. *If M is a connected oriented manifold of dimension n, then $H_c^n(M) \cong \mathbb{R}$. In prticular if M is compact, oriented, and connected, then $H^n(M) \cong \mathbb{R}$.*

Künneth Formula

Theorem 6.69 (Künneth Formula)**.** *Let M, F be two manifolds, then for every nonnegative integer n, one has*

$$H^n(M \times F) \cong \bigoplus_{p=0}^{n} H^p(M) \otimes H^{n-p}(F).$$

Or written more compactly, we have

$$H^*(M \times F) \cong H^*(M) \otimes H^*(F),$$

where $H^*(M) = \bigoplus_{p=1}^{\infty} H^p(M)$ denote the cohomology ring, and similarly for $H^*(F)$ and $H^*(M \times F)$.

Proof. We again prove this theorem for manifolds with finite good cover, using the beautiful MV argument. There are two obvious projection maps

$$\begin{aligned}\pi &: M \times F \to M \\ \rho &: M \times F \to F.\end{aligned}$$

These two maps induces a map in cohomology rings

$$\begin{aligned}\psi : H^*(M) \otimes H^*(F) &\to H^*(M \times F) \\ \omega \otimes \phi &\mapsto \pi^*\omega \wedge \rho^*\phi\end{aligned}$$

for $\omega \in H^p(M), \phi \in H^{n-p}(N)$, and for any $0 \le p \le n$. We shall prove Künneth formula by showing that ψ defined above is an isomorphism.

For $M = \mathbb{R}^m$ this is simply the Poincaré lemma. But we still need to show that $\psi : \bigoplus_{p=0}^n H^p(\mathbb{R}^m) \otimes H^{n-p}(F) \to H^n(\mathbb{R}^m \times F)$ induces an isomorphism using the Poincaré lemma. If $p \neq 0$ then $H^p(\mathbb{R}^m) = 0$, so ψ is really just a map $\psi : \mathbb{R} \otimes H^n(F) \to H^n(\mathbb{R}^m \times F)$. Recall in the proof of the Poincaré lemma, we constructed an isomorphism $\pi^* : H^n(\mathbb{R}^{m-1} \times F) \to H^n(\mathbb{R}^m \times F)$ from the projection $\pi : (\mathbb{R}^{m-1} \times F) \times \mathbb{R}^1 \to \mathbb{R}^{m-1} \times F$. Therefore by inductively applying projection map and eliminating one coordinate of $\mathbb{R}^m$, we get an isomorphism $(\pi^*)^n : H^n(F) \to H^n(\mathbb{R}^m \times F)$, where $\pi^n : \mathbb{R}^m \times F \to F$. But then π^n is just $\rho : \mathbb{R}^m \times F \to F$. Thus ψ can actually be written as a composition of two isomorphisms

$$\begin{tikzcd}
\mathbb{R} \otimes H^n(F) \arrow[r, "\psi"] \arrow[d, "\epsilon : a\otimes\omega \mapsto a\omega"'] & H^n(\mathbb{R}^m \times F) \\
H^n(F) \arrow[ur, "\rho^*"'] &
\end{tikzcd}$$

Here, ϵ is an isomorphism just by the property of tensor product, and $\rho^* = (\pi^*)^n$ is an isomorphism because it is a composition of π^*'s, each is an isomorphism by Poincaré lemma. The diagram obviously commutes, so we have $\psi = \rho^* \circ \epsilon$ which is an isomorphism.

For general M, let U, V be two open sets in M. Then $U \times F, V \times F$ are two open sets in $M \times F$. Since tensoring with a vector space preserves exactness, we tensor the MV sequence of the pair (U, V) with $H^{n-p}(F)$ and get an exact sequence

$$\begin{aligned}\cdots \to H^p(U \cup V) \otimes H^{n-p}(F) \to (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \\ \to H^p(U \cap V) \otimes H^{n-p}(F) \to \cdots\end{aligned}$$

And direct summing also preserves exactness, so we get an exact sequence

$$\begin{aligned}\cdots \to \bigoplus_{p\in\mathbb{Z}} H^p(U \cup V) \otimes H^{n-p}(F) \to \bigoplus_{p\in\mathbb{Z}} (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \\ \to \bigoplus_{p\in\mathbb{Z}} H^p(U \cap V) \otimes H^{n-p}(F) \to \cdots\end{aligned}$$

The trick is to put this exact sequence on top, and the MV exact sequence of the pair $(U \times F, V \times F)$ on the bottom, match them up using ψ componentwise, and show that we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{p=0}^n H^p(U \cup V) \otimes H^{n-p}(F) \arrow[r] \arrow[d, "\psi"] & \bigoplus_{p=0}^n (H^p(U) \oplus H^p(V)) \otimes H^{n-p}(F) \arrow[r] \arrow[d, "\psi"] & \bigoplus_{p=0}^n H^p(U \cap V) \otimes H^{n-p}(F) \arrow[d, "\psi"] \\
H^n((U \cup V) \times F) \arrow[r] & H^n(U \times F) \oplus H^n(V \times F) \arrow[r] & H^n((U \cap V) \times F)
\end{tikzcd}$$

Note that here instead of writing $\bigoplus_{p\in\mathbb{Z}}$ we write $\bigoplus_{p=0}^{n}$. This is because when $p<0$ or $p>n$, one of the two groups we are tensoring is the trivial group, so the direct sum becomes a finite direct sum with only nontrivial cohomology groups tensored together.

The commutativity of the first and the second square commutes is left as an exercise. The first one is easy once all the relevant maps are written out and we can just chase the diagram. The second square commutes because we are using the isomorphism (See Remark 6.70)

$$\bigoplus_{p=0}^{n}(H^p(U)\oplus H^p(V))\otimes H^{n-p}(F)\xrightarrow{\cong}\bigoplus_{p=0}^{n}(H^p(U)\otimes H^{n-q}(F))\oplus(H^p(V)\otimes H^{n-q}(F))$$
$$(\alpha^p,\beta^p)\otimes\tau^{n-p}\xrightarrow{\cong}(\alpha^p\otimes\tau^{n-p},\beta^p\otimes\tau^{n-p})$$

for $\alpha^p\in H^p(U),\beta^p\in H^p(V),\tau^{n-p}\in H^{n-p}(F)$.

Accepting this, then the commutativity of the second square can be verified easily. Now we verify commutativity of the third square

$$\begin{tikzcd}
\bigoplus_{p=0}^{n} H^p(U\cap V)\otimes H^{n-p}(F) \arrow[r, "d^*"] \arrow[d, "\psi"'] & \bigoplus_{p=-1}^{n} H^{p+1}(U\cup V)\otimes H^{n-p}(F) \arrow[d, "\psi"] \\
H^n((U\cap V)\times F) \arrow[r, "d^*"] & H^{n+1}((U\cup V)\times F)
\end{tikzcd}$$

Note that in the second term on the right, the sum $\bigoplus_{p\in\mathbb{Z}}$ actually becomes $\bigoplus_{p=-1}^{n}$.

It suffices to verify the commutativity of the diagram for pure tensors (Remark 6.71). Let $\omega\otimes\phi\in H^p(U\cap V)\otimes H^{n-p}(F)$. Then

$$\psi d^*(\omega\otimes\phi)=\pi^*(d^*\omega)\wedge\rho^*\phi$$
$$d^*\psi(\omega\otimes\phi)=d^*(\pi^*\omega\wedge\rho^*\phi).$$

Recall that if ρ_U,ρ_V is a partition of unity subordinate to U,V, then $d^*\omega|_U=-d(\rho_V\omega)$ and $d^*\omega|_V=d(\rho_U\omega)$. Note that $\pi^*\rho_U,\pi^*\rho_V$ form a partition of unity on $(U\cup V)\times F$ subordinate to the cover $U\times F,V\times F$, and since ϕ is closed, on $(U\cap V)\times F$ we have

$$\begin{aligned}
d^*(\pi^*\omega\wedge\rho^*\phi)&=d((\pi^*\rho_U)\pi^*\omega\wedge\rho^*\phi)\\
&=(d\pi^*(\rho_U\omega))\wedge\rho^*\phi\\
&=\pi^*(d^*\omega)\wedge\rho^*\phi.
\end{aligned}$$

Hence the diagram is indeed commutative.

Finally to finish off the proof of Künneth theorem: By the Five Lemma, if the theorem is true for $U,V,U\cap V$, then it is true for $U\cup V$. Use the MV argument and induct on the cardinality of a good cover. □

Remark 6.70. tensor product distributes over direct sum is a standard result in algebra. A detailed proof of this isomorphism can be found in this post. To get an idea, we can do a simple case and prove $(A\oplus B)\otimes C\cong(A\otimes C)\oplus(B\otimes C)$. The isomorphism can be constructed via $\varphi:(A\oplus B)\otimes C\to(A\otimes C)\oplus(B\otimes C)$ where

$$\varphi:(a,b)\otimes c\mapsto(a\otimes c,b\otimes c).$$

The inverse of this map is given by $\psi:(A\otimes C)\oplus(B\otimes C)\to(A\oplus B)\otimes C$ where

$$\psi:(a\otimes c_1,b\otimes c_2)\mapsto(a,0)\otimes c_1+(0,b)\otimes c_2.$$

One can verify that both φ, ψ are well-defined homomorphisms that are inverses of each other. For example

$$\varphi\psi(a \otimes c_1, b \otimes c_2) = \varphi((a, 0) \otimes c_1 + (0, b) \otimes c_2) = (a \otimes c_1, 0) + (0, b \otimes c_2) = (a \otimes c_1, b \otimes c_2).$$

The key here is to realize that $(A \oplus B) \otimes C$ does not only consists of elements of the form $(a, b) \otimes c$, but linear combinations of elements like this.

Remark 6.71. Finally, I shall remark that for our purpose, it suffices to check the diagram above commutes for elements with the same coefficient in $H^{n-q}(F)$, i.e., elements of the form $(\alpha^p \otimes \tau^{n-p}, \beta^p \otimes \tau^{n-p})$. This is because for any random element, it can be decomposed into elements with the same $H^{n-q}(F)$ coefficients:

$$(\alpha^p \otimes \sigma^{n-p}, \beta^p \otimes \tau^{n-p}) = (\alpha^p \otimes \sigma^{n-p}, 0 \otimes \sigma^{n-p}) + (0 \otimes \tau^{n-p}, \beta^p \otimes \tau^{n-p}).$$

Therefore by linearity, it suffices to check the diagram commutes for elements with the same $H^{n-q}(F)$ coefficients.

Example 6.72. To demonstrate the power of Künneth theorem with an example, let us prove that the cohomology of the n-fold torus $T^n := \prod_{i=1}^n S^1 = S^1 \times \cdots S^1$ satisfies

$$H^k(T^n) = \mathbb{R}^{\binom{n}{k}}$$

for all k. We prove this statement by induction on n. First, the claim obviously holds for the base case where $n = 1$, by Section 6.4.2. Next, we assume that

$$H^k(T^{n-1}) = \mathbb{R}^{\binom{n-1}{k}}$$

holds for all k. Now by Theorem 6.69, we have that

$$\begin{aligned} H^k(T^n) = H^k(T^{n-1} \times S^1) &= \bigoplus_{m+n=k} H^m(T^{n-1}) \otimes H^n(S^1) \\ &= (H^k(T^{n-1}) \otimes \mathbb{R}) \oplus (H^{k-1}(T^{n-1}) \otimes \mathbb{R}) \\ &\cong H^k(T^{n-1}) \oplus H^{k-1}(T^{n-1}) \\ &\cong \mathbb{R}^{\binom{n-1}{k}} \oplus \mathbb{R}^{\binom{n-1}{k-1}} \cong \mathbb{R}^{\binom{n}{k}} \end{aligned}$$

where one can check the identity

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

easily with some algebra. This proves the claim. But one can also repeatedly apply Theorem 6.69 n times on individual factors S^1 in the product $T^n = S^1 \times \cdots S^1$ and get

$$H^k(T^n) = \bigoplus_{i_1+\cdots+i_n=k} H^{i_1}(S^1) \otimes \cdots \otimes H^{i_n}(S^1).$$

Now, $H^{i_p}(S^1) = 0$ unless $i_p = 0, 1$ for all $1 \le p \le n$. If we have $H^{i_p}(S^1) = 0$, then the term in the direct sum containing it will be the trivial group, which does not contribute to $H^k(T^n)$. And since any finite tensor products of $\mathbb{R}$ is isomorphic to $\mathbb{R}$, the number of nontrivial factors of $H^k(T^n)$ in the direct sum above corresponds to the number of distinct ways one can write $i_1 + \cdots + i_n = k$, where $i_1, ..., i_n \in \{0, 1\}$. And the answer of course is $\binom{n}{k}$. This is why the binomial coefficient appears in this problem, because it is just a combinatorial problem in disguise.

Poincaré Dual of a Closed Oriented Submanifold

Let M^n be an oriented manifold and S^k a closed (as subspace of M) submanifold of M. Let $\omega \in \Omega_c^k(M)$. Then $i_S^*\omega \in \Omega_c^k(S)$. So the integral $\int_S i_S^*\omega$ is defined. Then integration on S induces a linear functional on $H_c^k(M)$ via $I_S : \omega \mapsto \int_S i_S^*\omega$. By Poincaré duality:

$$(H_c^k(M))^* \cong H^{n-k}(M).$$

Recall that the isomorphism $H^{n-k}(M) \to (H_c^k(M))^*$ is given by $\omega \mapsto \int_M (\bullet) \wedge \omega$ up to a sign. Therefore there exists a unique $\mathrm{PD}(S) \in H^{n-k}(M)$ such that $\int_M (\bullet) \wedge \mathrm{PD}(S) = I_S$. That is, there exists unique $\mathrm{PD}(S) \in H^{n-k}(M)$ such that

$$\int_S i_S^*\omega = \int_M \omega \wedge \mathrm{PD}(S).$$

We call $\mathrm{PD}(S)$ the ***Poincaré dual of*** S**.**

Proposition 6.73. When S is compact, its Poincaré dual may be taken to have compact support.

Proof. Now suppose M^n has a finite good cover, and S^k is compact. Then since S is compact, $\int_S i_S^*\omega < \infty$ for any $\omega \in \Omega^k(M)$. Thus this time $\omega \mapsto \int_S i_S^*\omega$ is a map $I_S : H^k(M) \to \mathbb{R}$. Now since M has a finite good cover, its de Rham cohomology (with compact support) are finite-dimensional vector spaces, so Poincaré duality $H^k(M) \cong (H_c^{n-k}(M))^*$ implies that $H_c^{n-k}(M) \cong (H^k(M))^*$. The isomorphism $H_c^{n-k}(M) \to (H^k(M))^*$ is given by $\eta \mapsto \int_M (\bullet) \wedge \eta$ up to a sign. Therefore, there exists a unique $\mathrm{PD}(S)' \in H_c^{n-k}(M)$ such that $I_S(\omega) = \int_M \omega \wedge \mathrm{PD}(S)'$, for all $\omega \in H^k(M)$. That is,

$$\int_S i_S^*\omega = \int_M \omega \wedge \mathrm{PD}(S)'.$$

□

6.8 The Thom Isomorphism

Let M^m be a smooth manifold, and let E be a rank n vector bundle over M (recall Definition 3.3). Throughout this section, we consider the situation where $(U_\alpha)_\alpha$ is a family of open covers of M, and let $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ be the corresponding family of trivialization maps on E. The maps

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

are vector space automorphisms of $\mathbb{R}^n$ in each fiber and hence give rise to maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}) \qquad g_{\alpha\beta}(x) = \phi_\alpha \circ \phi_\beta|_{\{x\}\times\mathbb{R}^n}.$$

Definition 6.74. A trivialization $(U_\alpha, \phi_\alpha)_{\alpha\in I}$ on E is said to be ***oriented*** if for every α, β, the transition function $g_{\alpha\beta}$ has positive determinant everywhere on $U_\alpha \cap U_\beta$. Two trivializations $(U_\alpha, \phi_\alpha), (V_\beta, \psi_\beta)$ are ***equivalent*** if for every $x \in U_\alpha \cap V_\beta$, the function $\phi_\alpha \circ \psi_\beta^{-1}(x) : \mathbb{R}^n \to \mathbb{R}^n$ has positive determinant, written as $(U_\alpha, \phi_\alpha) \sim (V_\beta, \psi_\beta)$

One can easily check that

Proposition 6.75. $\sim$ is an equivalence relation on the set of oriented trivializations on E.

This immediately gives

Proposition 6.76. On a connected maniflod M the equivalence relation $\sim$ partitions all the oriented trivializations of E into two equivalence classes. Either equivalence class is called an ***orientation*** on the vector bundle E.

Lemma 6.77. *An orientable vector bundle E over an orientable manifold M is an orientable manifold.*

Proof. Let $(U_\alpha, \psi_\alpha)_\alpha$ be an oriented atlas for M with transition functions $h_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$. Let $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ be the family of local trivializations for E with transition functions $g_{\alpha\beta}$. Then the composition

$$\pi^{-1}(U_\alpha) \xrightarrow{\phi_\alpha} U_\alpha \times \mathbb{R}^n \xrightarrow{\psi_\alpha \times \mathrm{id}} \mathbb{R}^m \times \mathbb{R}^n$$

gives an atlas for E. The transition function of this atlas is given by

$$\sigma_{\alpha\beta} : (\psi_\alpha \times \mathrm{id}) \circ g_{\alpha\beta} \circ (\psi_\beta^{-1} \times \mathrm{id}) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$

which is given by

$$\sigma_{\alpha\beta}(x, y) = (h_{\alpha\beta}(x), g_{\alpha\beta}(\psi_\beta^{-1}(x))y).$$

Then its Jacobian is given by

$$J(\sigma_{\alpha\beta}) = \begin{pmatrix} J(h_{\alpha\beta}) & * \\ 0 & g_{\alpha\beta}(\psi_\beta^{-1}(x)) \end{pmatrix}$$

which clearly has positive determinant. $\square$

Remark 6.78 (Co-orientation)**.** Let E be an oriented vector bundle of rank n over an oriented manifold M. The orientation given by Lemma 6.77 induces a natural orientation on the fibers. We may orient each fiber $E_p = \pi^{-1}(p)$ as follows: Near any point $p \in M$, let $x^1, ..., x^m$ be a local coordinate of M determined by the open set U_α, the zero section in E, and $t^1, ..., t^n$ be a fiber coordinate of E. Since near p we have $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^n$ via the local trivialization ϕ_α. This means that at the point q with $\pi(q) = p$, we have $T_qE \cong T_pM \oplus T_qE_p$, where we identify M with its zero section in E. Then we orient the fiber E_p such that the direct sum orientation $T_pM \oplus T_qE_p$ agrees with that of T_qE. This is called the ***co-orientation***.

Here is a result that we will need, but we will not prove it here. For a proof, see [2] Theorem 6.8 and Corollary 6.9 on p.57-59.

Lemma 6.79. *If M is contractible and E is any rank n vector bundle over M, then E is trivial: $E \cong M \times \mathbb{R}^n$.*

Corollary 6.80. *If M has a good cover $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$, then $\pi^{-1}(\mathfrak{U})$ is a good cover on E, where $\pi : E \to M$ is a vector bundle. In particular, if M is of finite type, then so is E.*

Proof. Let $U_\alpha \subset \mathfrak{U}$ be a good cover for M. Then $\pi^{-1}(U_\alpha)$ is a vector bundle over U, which is a contractible manifold. Hence by Lemma 6.79, $\pi^{-1}(U_\alpha)$ is trivial, i.e., there are diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \cong \mathbb{R}^m \times \mathbb{R}^n$. Moreover, $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha \cap U_\beta) \cong (U_\alpha \cap U_\beta) \times \mathbb{R}^n \cong \mathbb{R}^m \times \mathbb{R}^n$. Hence we see that $\pi^{-1}(\mathfrak{U})$ is indeed a good cover for E. $\square$

We now start our computation of $H^*(E), H^*_c(E)$ using the information of cohomologies of M. First of all, we have

Proposition 6.81. Let E be any vector bundle over M. Then we have $H^*(E) \cong H^*(M)$.

Proof. Consider the zero section $s_0 : M \to E$ defined by $s_0(x) = (x, 0) \in E$. Then clearly $\pi \circ s_0 = \mathrm{id}_M$. We claim that $s_0 \circ \pi \simeq \mathrm{id}_E$. The homotopy is given by $H : E \times \mathbb{R} \to E$ where $H(x, v, t) = (x, tv)$. Hence by homotopy invariance, we see that $E \simeq M$ (In fact, the zero section $M \times \{0\} \subset E$ is a deformation retract of E and is diffeomorphic to M). Thus by homotopy invariance we get the desired result. □

However, it is *not* true in general that $H^*_c(E) \cong H^{*-n}_c(M)$, despite it being true for $E = M \times \mathbb{R}$ according to Poincaré Lemma for compactly supported cohomology. For example, the open Möbius strip E satisfies $H^k_c(E) = 0$ for all k, but E is a rank one vector bundle over S^1 and we don't have $H^2(E) \cong H^1(S^1)$. But our prediction is indeed true for the following special cases:

Theorem 6.82 (Thom isomorphism)**.** *Let M be an oriented manifold of finite type, and E an oriented rank n vector bundle over M. Then for all k, we have*

$$H^{k+n}_c(E) \cong H^k_c(M).$$

Proof. By Lemma 6.77, we know that E is also an oriented manifold, and by Corollary 6.80, we know that E is of finite type. Then the statement can be proven by using Poincaré duality twice, and $H^*(E) \cong H^*(M)$

$$H^{k+n}_c(E) \cong (H^{m+n-k}(E))^* \cong (H^{m+n-k}(M))^* \cong H^{k-n}_c(M).$$

This proves the claim. □

This is Thom isomorphism, in its simplest form. However, there is a more general version of Thom isomorphism, which we will now study. The proof is very simliar to the Poincaré Lemmas. We need some definitions first:

Definition 6.83. Let E be a vector bundle of rank n over a smooth manifold M^m. A differential k-form $\omega \in \Omega^k(E)$ is said to have ***compact vertical support*** if for every compact subset $K \subset M$, the subset $\operatorname{Supp}\omega \cap \pi^{-1}(K)$ is compact in E. Define

$$\Omega^k_{\mathrm{cv}}(E) := \{\omega \in \Omega^k(E) : \omega \text{ has compact vertical support}\}.$$

One can easily see that exterior differentiation preserves forms with vertical compact support. Hence we define the kth ***cohomology of E with compact vertical support***

$$H^k_{\mathrm{cv}}(E) := \frac{\ker(d : \Omega^k_{\mathrm{cv}}(E) \to \Omega^{k+1}_{\mathrm{cv}}(E))}{\operatorname{im}(d : \Omega^{k-1}_{\mathrm{cv}}(E) \to \Omega^k_{\mathrm{cv}}(E))}.$$

Now we define a map, called ***integration along fiber***, similar to what we did in order to prove Poincaré Lemma.

Definition 6.84. Define $\pi_* : \Omega^{k+n}_{\mathrm{cv}}(E) \to \Omega^k(M)$ as follows:

- First suppose $E = M \times \mathbb{R}^n$. Let $x^1, ..., x^m$ be a coordinate on the base M and $t^1, ..., t^n$ be a coordinate on the fiber. Then any $\omega \in \Omega^{k+n}(E)$ has the form

$$\omega = (\pi^*\phi) f(x, t) dt^1 \wedge \cdots \wedge dt^n + \zeta$$

where $\phi \in \Omega^k(M)$, f has compact support for each fixed x on M, and ζ is a form with less than n fiber differential in its wedges. We then define

$$(\pi_*\omega)(p) = \phi(p) \cdot \int_{E_p} f(x,t)dt^1 \wedge \cdots \wedge dt^n.$$

Here, we orient E_p by co-orientation as in Remark 6.78. In other words,

$$\pi_*(\omega) = \phi \int_{\mathbb{R}^n} f(x, t^1, ..., t^n)dt^1 \cdots dt^n.$$

- Now suppose E be an arbitrary oriented vector bundle, with oriented trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ of E. Write $\omega_\alpha := \omega|_{\pi^{-1}(U_\alpha)}$. Suppose on $\pi^{-1}(U_\alpha)$ we have coordinates $(x^1, ..., x^m, t^1, ..., t^n)$, and

$$\omega_\alpha = (\pi^*\phi)f(x^1, ..., x^m, t^1, ..., t^n)dt^1 \wedge \cdots \wedge dt^n + \zeta_\alpha$$

  where ζ_α again contains less than n fiber differential in its wedges. Then define $\pi_* : \Omega^{k+n}_{\text{cv}}(\pi^{-1}(U_\alpha) \to \Omega^k(U_\alpha)$ via

$$(\pi_*\omega_\alpha)(p) = \phi(p) \cdot \int_{E_p} f(x,t)dt^1 \wedge \cdots \wedge dt^n$$

  for any $p \in U_\alpha$. Again, we give E_p the co-orientation. In other words,

$$\pi_*(\omega_\alpha) := \phi \int_{\mathbb{R}^n} f(x, t^1, ..., t^n)dt^1 \cdots dt^n.$$

Proposition 6.85. If E is an oriented vector bundle and $\omega \in \Omega^{k+n}_{\text{cv}}(E)$ as in the second case in the definition above, then $\pi_*\omega_\alpha = \pi_*\omega_\beta$ on the intersection. Hence $(\pi_*\omega_\alpha)_\alpha$ pieces together to give a global form $\pi_*\omega$ on M. Moreover, this definition is independent of the choice of oriented trivialization for E. Hence we get a well-defined map

$$\pi_* : \Omega^{k+n}_{\text{cv}}(E) \to \Omega^k(M).$$

Proof. First, suppose

$$\omega_\beta = (\pi^*\tau)g(y^1, ..., y^m, u^1, ..., u^n)du^1 \wedge \cdots \wedge du^n + \zeta_\beta$$

for another set of coordinate. Suppose

$$\omega_\alpha = \pi^*(\rho dx^{i_1} \wedge \cdots \wedge dx^{i_k})f dt^1 \wedge \cdots dt^n + \zeta_\alpha$$

for $\rho \in C^\infty(U_\alpha)$ and f has compact support on every fiber of E. Then on $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$, note that since $\pi^* dx^i = dx^i$, one has

$$\begin{aligned}\omega_\alpha &= (\pi^*\rho)f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dt^1 \wedge \cdots \wedge dt^n \\ &= (\pi^*\rho)f \frac{\partial x^{i_1}}{\partial y^{p_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{p_k}} \frac{\partial t^1}{\partial u^{q_1}} \cdots \frac{\partial t^n}{\partial u^{q_n}} dy^{p_1} \wedge \cdots dy^{p_n} \wedge du^{q_1} \wedge \cdots du^{q_n}.\end{aligned}$$

Hence on $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$, we should have

$$\tau = \rho \frac{\partial x^{i_1}}{\partial y^{p_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{p_k}} dy^{p_1} \wedge \cdots \wedge dy^{p_n}$$

and

$$\begin{aligned} g du^1 \wedge \cdots \wedge du^n &= f \frac{\partial t^1}{\partial u^{q_1}} \cdots \frac{\partial t^n}{\partial u^{q_n}} du^{q_1} \wedge \cdots du^{q_n} \\ &= f \det\left(\frac{\partial t^\mu}{\partial u^\nu}\right) du^1 \wedge \cdots \wedge du^n \end{aligned}$$

where in the second equality, we used the argument in the proof of (4.3). Now for a fixed point p on the intersection $U_\alpha \cap U_\beta$, we have, by the change of variable formula of integration,

$$\begin{aligned} \pi_* \omega_\alpha(p) &= \rho dx^{i_1} \wedge \cdots \wedge dx^{i_k}(p) \cdot \int_{E_p} f dt^1 \wedge \cdots \wedge dt^n \\ &= \rho dx^{i_1} \wedge \cdots \wedge dx^{i_k}(p) \cdot \int_{\mathbb{R}^n} f \circ \phi_\alpha^{-1}(p,t) dt^1 \cdots dt^n \\ &= \rho \frac{\partial x^{i_1}}{\partial y^{p_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{p_k}} dy^{p_1} \wedge \cdots \wedge dy^{p_n}(p) \cdot \int_{\mathbb{R}^n} f \circ \phi_\beta^{-1}(p,u) \left| \det\left(\frac{\partial t^\mu}{\partial u^\nu}\right)\right| du^1 \cdots du^n \\ &= \tau(p) \cdot \int_{\mathbb{R}^n} g \circ \phi_\beta^{-1}(p,u) du^1 \cdots du^n \\ &= \tau(p) \cdot \int_{E_p} g du^1 \wedge \cdots \wedge du^n = \pi_* \omega_\beta(p) \end{aligned}$$

where $(p,u) = \phi_\beta \circ \phi_\alpha^{-1}(p,t)$ and the Jacobian of $(\phi_\beta \circ \phi_\alpha)(p) : \mathbb{R}^n \to \mathbb{R}^n$ is

$$\det\left(\frac{\partial t^\mu}{\partial u^\nu}\right) = \det(g_{\beta\alpha}) > 0$$

by assumption, hence this gives the 4th equality. It remains to prove that π_* is independent of trivializations. Let

$$\phi'_\beta : \pi^{-1}(V_\beta) \to V_\beta \times \mathbb{R}^n$$

be another family of oriented trivializations for E. We would like to show $\pi_*(\phi)\omega = \pi_*(\phi')\omega$ on M. For any $p \in M$, choose α, β such that $p \in U_\alpha \cap V_\beta$. If $\phi'_\beta \circ \phi_\alpha^{-1}$ has positive determinant everywhere on $U_\alpha \cap V_\beta$, then repeat the proof above where we replace ϕ_β by ϕ'_β. Otherwise, we would get a minus sign in the 4th equality above, but the 5th equality still holds because this is how we define integration on an orientation reversing chart. □

Proposition 6.86. Integration along fiber π_* commutes with exterior differentiation d.

Proof. Let (U_α, ϕ_α) be a trivialization for E, and let ρ_α be a partition of unity subordinate to U_α (note that this time U_α's are actually on E instead of M). Since for any $\omega \in \Omega^{k+n}_{\text{cv}}(E)$, we have $\omega = \sum_\alpha \rho_\alpha \omega$, and both π_* and d are linear, it suffices to prove the proposition for $\rho_\alpha \omega \in \Omega^{k+n}_{\text{cv}}(U_\alpha)$. This means that we may as well assume that $E = M \times \mathbb{R}^n$ is trivial. We consider two situations: First, if ω is a top degree form in the fiber coordinate:

$$\omega = (\pi^* \phi) f(x,t) dt^1 \wedge \cdots \wedge dt^n$$

for some $\phi \in \Omega^k(M)$, then we have

$$\begin{aligned} d\pi_* \omega &= d\left(\phi \int_{\mathbb{R}^n} f(x,t) dt^1 \cdots dt^n\right) \\ &= (d\phi) \int_{\mathbb{R}^n} f(x,t) dt^1 \cdots dt^n + (-1)^k \phi \wedge dx^\mu \int_{\mathbb{R}^n} \frac{\partial f}{\partial x^\mu} dt^1 \cdots dt^n \\ &= \pi_*\left((\pi^* d\phi) f dt^1 \wedge \cdots \wedge dt^n + (-1)^k \pi^* \phi \wedge \frac{\partial f}{\partial x^\mu} dx^\mu \wedge dt^1 \wedge \cdots \wedge dt^n\right) \\ &= \pi_* d\omega. \end{aligned}$$

Now suppose ω does not have top degree in the fiber coordinate:

$$\omega = (\pi^*\phi)f(x,t)dt^{i_1} \wedge \cdots \wedge dt^{i_r}$$

where $\phi \in \Omega^{k+n-r}(M)$. Then by definition $d\pi_*\omega = 0$. On the other hand, we have

$$\pi_* d\omega = (-1)^{k+n-r}\pi_*\left(\pi^*\phi \frac{\partial f}{\partial t^\mu} dt^\mu \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_r}\right)$$

which is zero if $dt^\mu \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_r} \neq \pm dt^1 \wedge \cdots \wedge dt^n$. Otherwise, we note that

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial t^\mu} dt^\mu \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_r} = 0$$

because $\omega \in \Omega^{k+n}_{\mathrm{cv}}(M)$, hence for each fixed $x \in M$, we have $\pi^{-1}(x) \cap \operatorname{Supp}\omega$ is compact. Hence ω has compact support on each fiber for a fixed x. Thus

$$\int_{-\infty}^{+\infty} \frac{\partial f}{\partial t^\mu} dt^\mu = f(..., +\infty, ...) - f(..., -\infty, ...) = 0.$$

This concludes the proof. □

Since $\pi_* d = d\pi_*$, it is a cochain map. A cochain map descends to a well-defined map on cohomology, which can be proven using exactly the same argument as in the proof of Theorem 6.11. Hence, we get a well-defined map

$$\pi_* : H^{k+n}_{\mathrm{cv}}(E) \to H^k(M)$$

by $\pi_*[\omega] = [\pi_*\omega]$, where on the right hand side the π_* is the integration along fiber map $\pi_* : \Omega^{k+n}_{\mathrm{cv}}(E) \to \Omega^k(M)$.

Theorem 6.87 (Projection formula). *Let $\pi : E \to M$ be an oriented real rank n vector bundle over a smooth manifold M^m and $\pi_* : \Omega^{n+k}_{cv}(E) \to \Omega^k(M)$ be the integration along fiber map. Then:*

(a) For any $\omega \in \Omega^\ell(M)$ and $\tau \in \Omega^{n+k}_{cv}(E)$, we have

$$\pi_*(\pi^*\omega \wedge \tau) = \omega \wedge \pi_*\tau.$$

(b) Suppose in addition that M is oriented, and $\tau \in \Omega^{k+n}_{cv}(E)$, and $\omega \in \Omega^{m-k}_c(M)$. If we orient E as in Lemma 6.77, then we have

$$\int_E \pi^*\omega \wedge \tau = \int_M \omega \wedge \pi_*\tau.$$

Proof. (a). Since two forms are the same iff they are the same at every point, we may treat this problem locally and assume $E = M \times \mathbb{R}^n$. If

$$\tau = \pi^*\phi f(x,t)dt^{i_1} \wedge \cdots dt^{i_r} \tag{6.1}$$

with $\phi \in \Omega^{n+k-r}(M)$ and $r < n$, then we see that

$$\pi_*(\pi^*\omega \wedge \tau) = \pi_*(\pi^*(\omega \wedge \phi)f(x,t)dt^{i_1} \wedge \cdots \wedge dt^{i_r}) = 0 = \omega \wedge \pi_*\tau.$$

Now suppose

$$\tau = (\pi^*\phi)f(x,t)dt^1 \wedge \cdots \wedge dt^n \tag{6.2}$$

a top degree form in fiber coordinate with $\phi \in \Omega^k(M)$. Then

$$\pi_*(\pi^*\omega \wedge \tau) = \omega \wedge \phi \int_{\mathbb{R}^n} f(x,t)dt^1 \cdots dt^n = \omega \wedge \pi_*\tau.$$

(b). Let (ρ_α, U_α) be a partition of unity for M, with $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ oriented trivializations. Then $(\pi^*\rho_\alpha, \pi^{-1}(U_\alpha))$ is a partition of unity for E. Then $\tau = \sum_\alpha (\pi^*\rho_\alpha)\tau$. Therefore,

$$\int_E \pi^*\omega \wedge \tau = \sum_\alpha \int_{\pi^{-1}(U_\alpha)} \pi^*\omega \wedge (\pi^*\rho_\alpha)\tau$$

and

$$\int_M \omega \wedge \pi_*\tau = \sum_\alpha \int_M \omega \wedge \pi_*((\pi^*\rho_\alpha)\tau) = \sum_\alpha \int_M \omega \wedge \rho_\alpha \pi_*\tau = \sum_\alpha \int_{U_\alpha} \omega \wedge \pi_*((\pi^*\rho_\alpha)\tau)$$

by part (a). Hence again we reduced the problem to a local trivialization. That is, it suffices to assume that $E = M \times \mathbb{R}^n$ again. With this setup, first suppose τ is given by (6.1). Then $\int_M \omega \wedge \pi_*\tau = 0$ because $\pi_*\tau = 0$. On the other hand,

$$\int_E \pi^*\omega \wedge \tau = \int_E \pi^*(\omega \wedge \phi) \wedge f(x,t)dt^{i_1} \wedge \cdots \wedge dt^{i_r} = 0$$

because

$$\deg(\omega \wedge \phi) = \deg(\omega) + \deg(\phi) = (m-k) + (k+n-r) = m+n-r > m$$

because $r < n$ by assumption. Now assume τ is given by (6.2). Assume that $\omega = \sum_{|I|=m-k} \omega_I dx^I$ and $\phi = \sum_{|J|=k} \phi_J dx^J$ with ω_I, ϕ_J functions on M. Then we have

$$\begin{aligned}
\int_M \omega \wedge \pi_*\tau &= \int_M (\omega_I dx^I \wedge \phi) \cdot \int_{\mathbb{R}^n} f(x,t)dt^1 \cdots dt^n \\
&= \int_M \omega_I \phi_J dx^I \wedge dx^J \cdot \int_{\mathbb{R}^n} f(x,t)dt^1 \cdots dt^n \\
&= \int_{M \times \mathbb{R}^n} \omega_I \phi_J dx^I \wedge dx^J f(x,t)dt^1 \wedge \cdots \wedge dt^n \\
&= \int_{M \times \mathbb{R}^n} \pi^*(\omega_I \phi_J dx^I \wedge dx^J) f(x,t)dt^1 \wedge \cdots \wedge dt^n \\
&= \int_{M \times \mathbb{R}^n} \pi^*(\omega_I dx^I) \wedge \pi^*(\phi_J dx^J) f(x,t)dx^1 \wedge \cdots dt^n \\
&= \int_E \pi^*\omega \wedge \tau.
\end{aligned}$$

This completes the proof of the projection formula. □

First, we utilize our previous work in the proof of Poincaré Lemma and get

Proposition 6.88. Integration along the fiber gives an isomorphism

$$\pi_* : H^{k+n}_{\mathrm{cv}}(M \times \mathbb{R}^n) \to H^k(M).$$

Proof. This is exactly like the proof of the Poincaré Lemma for compactly supported cohomology (Theorem 6.51). We have already constructed π_* and showed that it commutes with d. Now we define $e \in \Omega^n_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} e = 1$, and define $e_* : \Omega^k(M) \to \Omega^{k+n}_{\mathrm{cv}}(M \times \mathbb{R}^n)$ via $e_* : \phi \mapsto (\pi^*\phi) \wedge e$. Then we construct the chain homotopy operator just like we did previously, and proof of Theorem 6.53 works identically. □

Now we are ready to prove the Thom isomorphism theorem in its most general form.

Theorem 6.89 (General Thom isomorphism)**.** *Let E be an orientable vector bundle of rank n over an orientable manifold of finite type. Then integration along fiber gives an isomorphism*

$$\pi_* : H^{n+k}_{cv}(E) \to H^k(M).$$

Proof. Note that E is an orientable manifold of finite type by Theorem 6.77 and Corollary 6.80. The proof of Thom isomorphism is another example of the Mayer-Vietoris argument. Let U, V be open cover of M. Using a partition of unity from the base M, we obtain a partition of unity on E, subordinate to $U' = \pi^{-1}(U), V' = \pi^{-1}(V)$. We then see that the sequence

$$0 \to \Omega^*_{\mathrm{cv}}(E|_{U'\cup V'}) \to \Omega^*_{\mathrm{cv}}(E|_{U'}) \oplus \Omega^*_{\mathrm{cv}}(E|_{V'}) \to \Omega^*_{\mathrm{cv}}(E|_{U'\cap V'}) \to 0$$

is exact. So this induces a long exact sequence of compact vertical cohomology groups. We consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k+n}_{\mathrm{cv}}(E|_{U'\cup V'}) \arrow[r] \arrow[d, "\pi_*"] & H^{k+n}_{\mathrm{cv}}(E|_{U'}) \oplus H^{k+n}_{\mathrm{cv}}(E|_{V'}) \arrow[r] \arrow[d, "\pi_*"] & H^{k+n}_{\mathrm{cv}}(E|_{U'\cap V'}) \arrow[r, "d^*"] \arrow[d, "\pi_*"] & H^{k+n+1}_{\mathrm{cv}}(E|_{U'\cup V'}) \arrow[r] \arrow[d, "\pi_*"] & \cdots \\
\cdots \arrow[r] & H^k(U\cup V) \arrow[r] & H^k(U) \oplus H^k(V) \arrow[r] & H^k(U\cap V) \arrow[r, "d^*"'] & H^{k+1}(U\cup V) \arrow[r] & \cdots
\end{tikzcd}$$

The first two squares of this diagram are clearly commutative. We check the commutativity for the third square. Recalling from Remark 6.36, we have that

$$\pi_* d^* \omega = \pi_* d((\pi^* \rho_U)\omega) = \pi_*((\pi^* d\rho_U) \wedge \omega) = d\rho_U \wedge \pi_* \omega = d^* \pi_* \omega$$

where we used the projection formula, i.e., part (a) of Theorem 6.87, and the fact that both ω and $d\omega$ are closed since they represent cohomology classes. So the diagram is commutative.

Note that if U, V are good covers, then $E|_{U'}, E|_{V'}, E|_{U'\cap V'}$ are trivial and U', V' are good covers on E, by Lemma 6.79 and Corollary 6.80. So in these cases, π_* are isomorphisms by Proposition 6.88. By the Five Lemma if the Thom isomorphism holds for $U', V', U' \cap V'$, then it holds for $U' \cup V'$. The proof now proceeds by induction on the cardinality of a good cover for the base (which is also the cardinality of the corresponding good cover for E). □

Definition 6.90 (Thom class)**.** Let E be an orientable rank n vector bundle over an orientable manifold M of finite type. The Thom isomorphism $\pi_* : H^n_{\mathrm{cv}}(E) \to H^0(M)$ determines a cohomology class

$$\tau(E) = (\pi_*)^{-1}(1) \in H^n_{\mathrm{cv}}(E)$$

called the ***Thom class*** of E. It is characterized by the property that $\pi_* \tau(E) = 1$. That is, the Thom class $\tau(E)$, when integrated on each fiber (with co-orientation) gives unit.

Proposition 6.91. The inverse Thom isomorphism $(\pi_*)^{-1} : H^k(M) \to H^{n+k}_{\mathrm{cv}}(E)$ is given by

$$(\pi_*)^{-1}(\alpha) = \pi^* \alpha \wedge \tau(E).$$

Proof. Since $\pi_* \tau(E) = 1$, by the projection formula (Theorem 6.87), one has

$$\pi_*(\pi^* \alpha \wedge \tau(E)) = \alpha \wedge \pi_* \tau(E) = \alpha \wedge 1 = \alpha.$$

This proves the claim. □

6.8.1 Poincaré Duals and the Thom Class

Let S^k be an oriented submanifold of an oriented manifold M^m. Recall that the Poincaré dual $\mathrm{PD}(S)$ is the unique cohomology class in $H^{m-k}(M)$ characterized by

$$\int_S \omega = \int_M \omega \wedge \mathrm{PD}(S)$$

for any $\omega \in H_c^k(M)$. It turns out that the Poincaré dual of a submanifold relates to the Thom class of a bundle (normal bundle of S). We conclude this section by explaining this connection.

Definition 6.92. Let E, E', E'' be a sequence of vector bundles over M. A sequence of vector bundles $0 \to E \to E' \to E''$ is said to be ***exact*** if at every $p \in M$, the sequence of vector spaces $0 \to E_p \to E'_p \to E''_p \to 0$ is exact.

Definition 6.93 (Normal bundle)**.** Let $S^k \subset M^m$ be defined as above. Then the ***normal bundle of*** S is the vector bundle NS over S defined by the exact sequence

$$0 \to TS \to TM|_S \to NS \to 0$$

where $TM|_S$ is the restriction of the tangent bundle of M to S.

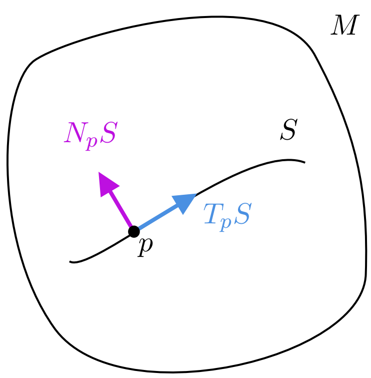


Figure 6.4: One should think of the normal bundle NS of S literally as being the bundle whose fiber at each point $p \in S$ is tangent to S, i.e., orthogonal to T_pS. At every $p \in S$ we have an exact sequence of vector spaces $0 \to T_pS \to T_pM \to N_pS \to 0$, which means that $N_pS \cong T_pM/T_pS$, i.e., N_pS is isomorphic to the vector space that is the complement of T_pS in T_pM, which is isomorphic to the orthogonal complement of T_pS in T_pM, if we have a Riemannian metric on M.

By the tubular neighborhood theorem, which we will not prove, every submanifold S has a tubular neighbohood of $\mathcal{T}(S)$, and in fact $\mathcal{T}(S)$ is diffeomorphic to an open neighborhood the normal bundle NS. The reader is encourage to refer to [1, Theorem 6.24] and [7, p.67] for a proof, but we will not dive into this rabbit hole.

The orientation of $\mathcal{T}(S)$ and NS, when viewed as subsets of M, is given by the induced orientation of M. However, we may orient the fiber of these bundles again using ***co-orientation***. For example, the normal bundle is oriented in such a way that $TS \oplus NS$ has the orientation of $TM|_S$.

Now, we apply the Thom isomorphism theorem to the normal bundle NS. Then we get a sequence of maps

$$H^0(S) \xrightarrow{(\pi_*)^{-1}} H^{m-k}_{\mathrm{cv}}(NS) \xrightarrow{j_*} H^{m-k}(M)$$

where j_* is extension by zero. Here, j_* is defined because every form $\tau \in H^{m-k}_{\mathrm{cv}}(NS)$ has compact support on each fiber since $\pi^{-1}(p) \cap \operatorname{Supp} \tau$ is compact for every $p \in S$. Hence τ would vanish near the boundary of NS. Under the above sequence, we get an element $j_*\tau(NS)$.

Theorem 6.94. *Let $S^k \subset M^m$ be a compact oriented submanifold of an oriented manifold M. The Poincaré dual $\mathrm{PD}(S) \in H^{m-k}(M)$, which may be taken to have compact support by Proposition 6.73, can be chosen to be the Thom class of the normal bundle NS, i.e.,*

$$\mathrm{PD}(S) = j_*\tau(NS).$$

Proof. We only need to show that $j_*\tau(NS)$ satisfies the defining property of Poincaré dual. Let $\omega \in \Omega^k_c(M)$ be any closed form with compact support on M. Now,

$$\int_M \omega \wedge j_*\tau(NS) = \int_{NS} \omega \wedge \tau(NS) \tag{6.3}$$

because $j_*\tau(NS)$ has support in NS. Let $i : S \to NS$ be the inclusion, regarded as the zero section of the bundle $\pi : NS \to S$. Since π is a deformation retraction of NS onto S, we know that π^*, i^* are inverse isomorphisms in cohomology $H^*(NS) \cong H^*(S)$. Hence, if we view ω (or rather its restriction) as a form in NS, then $\omega = \pi^*i^*\omega + d\alpha$, i.e., ω differ from $\pi^*i^*\omega$ by an exact form $\alpha \in \Omega^{k-1}(NS)$. Hence

$$\int_{NS} \omega \wedge \tau(NS) = \int_{NS} (\pi^*i^*\omega + d\alpha) \wedge \tau(NS). \tag{6.4}$$

Note that since S is compact, $H^{m-k}_{\mathrm{cv}}(NS) = H^{m-k}_c(NS)$. So in fact $\tau(NS)$ has compact support in NS. Applying Stokes' theorem we see that

$$\int_{NS} d\alpha \wedge \tau(NS) = \int_{NS} d(\alpha \wedge \tau(NS)) = \int_{\partial(NS)} \alpha \wedge \tau(NS) = 0$$

because $\partial(NS) = \emptyset$. Note that the dimension in the equation above works out: $\dim \partial(NS) = \dim(NS) - 1 = m - 1$, and $\deg(\alpha \wedge \tau(NS)) = (k-1) + (m-k) = m - 1$. Hence (6.4) reads

$$\int_{NS} \omega \wedge \tau(NS) = \int_{NS} \pi^*i^*\omega \wedge \tau(NS) = \int_S i^*\omega \wedge \pi_*\tau(NS) = \int_S i^*\omega \tag{6.5}$$

by part (b) of the projection formula (Theorem 6.87), and by the definition of Thom class that $\pi_*\tau(NS) = 1$. Putting (6.3), (6.4), and (6.5) together, we see that

$$\int_M \omega \wedge j_*\tau(NS) = \int_S i^*\omega.$$

This is exactly the characterizing property of the Poincaré dual. Hence we conclude that $\mathrm{PD}(S) = j_*\tau(NS)$. □

Corollary 6.95. *Suppose $\pi : E \to M$ is an oriented vector bundle over an oriented compact manifold. Then the Poincaré dual η of the zero section of E can be chosen to be the Thom class $\tau(E)$ of E.*

Proof. Note that M embeds diffeomorphically as the zero section of E, hence there is an exact sequence of vector bundles

$$0 \to TM \to TE|_M \to E \to 0.$$

In other words, the normal bundle of M in E is E itself. Then the claim follows immediatly from Theorem 6.94. $\square$

Corollary 6.96. *The support of the Poincaré dual* $\mathrm{PD}(S)$ *of a compact submanifold* S *can be shrunk into any tubular neighborhood of* S.

Proof. Let $\mathcal{N}$ be any tubular neighborhood of S. We identify S with the zero section of $\mathcal{N}$. Then by Corollary 6.95, we see that $\mathrm{PD}(S)$ can be chosen to be the Thom class

$$\tau(\mathcal{N}) \in H^{m-k}_{\mathrm{cv}}(\mathcal{N}) = H^{m-k}_{c}(\mathcal{N}),$$

where the last equality is due to the assumption that S is compact. Hence we see that we may assume $\mathrm{PD}(S)$ has (compact) support in $\mathcal{N}$. $\square$

Theorem 6.94, Corollary 6.95, and Corollary 6.96 will play crucial roles in the foundation of intersection theory, as we will see in the next chapter.

6.9 Problems

1. Let U and V be open subsets of $\mathbb{R}^n$ with U star-shaped. Suppose there is a diffeomorphism $f : U \to V$. Show that every closed form on V is exact.

2. Let U and V be two open star-shaped subsets of $\mathbb{R}^n$. Are U and V diffeomorphic?

3. Compute the cohomology of the real projective spaces $\mathbb{RP}^n$ using the Mayer-Vietoris sequence.

4. Compute the cohomology of the complex projective spaces $\mathbb{CP}^n$ using the Mayer-Vietoris sequence.

5. Let M be a compact orientable manifold with $\dim M = 4n+2$. Show that the Betti number $\beta_{2n+1} = \operatorname{rank} H^{2n+1}(M)$ is even.

6. Two sets of transition functions $(g_{\alpha\beta})$ and $(g'_{\alpha\beta})$ of vector bundles of rank n over M with respect to the same open cover (U_α) of M are said to be equivalent if there exists a collection of continuous maps $\lambda_\alpha : U_\alpha \to GL_n(\mathbb{R})$ such that

   $$g_{\alpha\beta} = \lambda_\alpha \cdot g'_{\alpha\beta} \cdot (\lambda_\beta)^{-1}.$$

   Here $\cdot$ is multiplication in $GL_n(\mathbb{R})$.

   Show that two rank n vector bundles E, F over M with local trivializations relative to some open cover $(U_\alpha)_\alpha$ of M are isomorphic iff their transition functions are equivalent.

Chapter 7

Differential Topology

The goal of this chapter is to prove some of the most important results in differential topology using de Rham cohomology theory. The main tools that we will introduce are degree theory and intersection theory.

The readers may recognize that degree and intersection theory are also the main tools used to prove these results in standard differential topology textbooks like [7] and [8]. But we want to stress that these tools arises naturally under the framework of de Rham cohomology, whereas in the texts mentioned above it is not obvious how this is true. Moreover, topological invariants like Euler characteristics and Lefschetz number are defined using intersection numbers directly in these texts without involving (co)homology groups, thus readers who choose to continue their study in algebraic topology may find it difficult to see why these definitions are equivalent when they read a textbook like [3]. Hence the goal of this writing is to bridge the gap between existing literature and hopefully provide a smooth transition between the frameworks of differential and algebraic topology.

Unfortunately, the prerequisites needed to understand the proofs are by no ways trivial. It would be ideal if the reader knows proofs of results in differential topology like regular value theorem, Thom's transversality theorem, Sard's theorem, and tubular neighborhood theorem. This is well introduced in Chapter 6 of [1] or in standard texts like [7] and [8]. However, it is sufficient to just know the results without knowing the proof. Still, readers might find it difficult to understand this chapter because we quickly introduce relevant concept and omit the proofs of many important results. This is completely normal, since our only goal here is to provide a rapid introduction to this vast and deep subject. My goal here is not trying to make the readers understand everything in this chapter, but to show them many interesting and beautiful results. If you are fascinated by some of these results despite not understanding a considerable amount of material, then I have successfully fulfilled my duty. In that case, grab a book on differential topology and start reading! And don't worry: it will certainly be more well-written than anything you read here!

7.1 Degree Theory

Definition 7.1. A continuous map $f : X \to Y$ between topological spaces is said to be ***proper*** if for any $K \subset Y$ compact, $f^{-1}(K)$ is a compact subset of X.

If $f : M \to N$ is a smooth map between oriented smooth manifolds, and $\omega \in \Omega_c^k(N)$, then $f^*\omega$ need not have compact support. For example, consider the pullback of functions under the projection $N \times \mathbb{R} \to N$. Hence, Ω_c^* is not a functor on the category of manifolds and smooth maps. However, if f is proper, then since $\operatorname{Supp} f^*\omega = f^{-1}(\operatorname{Supp}\omega)$, then we see that $f^*\omega \in \Omega_c^k(M)$. Moreover, one can show that

Proposition 7.2. Ω_c^* (and also H_c^*) is a *contravariant* functor in the category of proper smooth manifolds (where the objects are smooth manifolds and the morphisms are *proper smooth maps*) to the category of vector spaces.

Note that we know from Section 6.4.1 that Ω_c^* is a *covariant* functor under inclusions of open sets.

Now suppose that M^m is a connected oriented manifold. It was shown in Corollary 6.68 that $H_c^n(M) \cong \mathbb{R}$.

Definition 7.3. If both M^m, N^m are oriented and connected manifolds of the same degree, and $f : M \to N$ is a proper map, then $f^* : H_c^m(N) \to H_c^m(M)$ will be a linear map between $\mathbb{R}$, i.e., is multiplication by a number. This real number is denoted by $\deg(f)$, called the ***degree of f.***

One can also remove the proper restriction from the definition above, but then we must assume that both M and N are compact manifolds. But of course in this case, $H_c^k(M) = H^k(M)$ since M is compact.

Remark 7.4. Note that the Poincaré duality theorem states that the isomorphism $H_c^n(M) \to \mathbb{R}$ is given by integration. Therefore, we have that for any $\omega \in \Omega_c^n(M)$,

$$\int_M f^*\omega = (\deg f) \cdot \int_N \omega.$$

We first prove a lemma on proper mappings:

Lemma 7.5. *Let $f : X \to Y$ be a proper map between topological spaces. Assume Y is locally compact and Hausdorff (in particular this holds if Y is a manifold). Then f is a closed map.*

Proof. Let $C \subset X$ be closed. We want to show $Y \setminus f(C)$ is open. So we let $y \in Y \setminus f(C)$. Then by assumption y has an open neighborhood V with compact closure. Then $f^{-1}(\overline{V})$ is compact since f is proper. Let $E = C \cap f^{-1}(\overline{V})$. Then E is compact, hence $f(E)$ is compact. Since Y is Hausdorff, $f(E)$ is closed. Now let $U = V \setminus f(E)$. Then we claim that U is an open neighborhood of y disjoint from $f(C)$, which will complete the proof.

To show the claim, we note that because $y \in V \setminus f(C)$, and $f(E) \subset f(C) \cap \overline{V} \subset f(C)$, we conclude that $y \notin f(E)$. Hence $y \in V \setminus f(E) = U$. And clearly U is open.

Finally we show $U \cap f(C) = \emptyset$. Otherwise suppose $y' \in U \cap f(C)$, then there exists $x \in C$ such that $f(x) = y'$, and $f(x) \in U \subset V \subset \overline{V}$, i.e., $x \in E = C \cap f^{-1}(\overline{V})$. But then $y' = f(x) \in f(E)$, which is a contradiction for $y' \in U = V \setminus f(E)$. □

Then we can show:

Corollary 7.6. *If M^m, N^m are orientable smooth manifolds and $f : M \to N$ is a proper map that is not surjective, then* $\deg(f) = 0$.

Proof. Let $q \notin f(M)$ in N. By the lemma above, $f(M)^c$ is open, so we choose an open neighborhood V of q such that $f(M) \cap V = \emptyset$. Choose $\omega \in \Omega_c^m(V)$ such that $\int_N \omega = 1$. But $f^*\omega = 0$. Hence $\deg(f) = 0$. □

Definition 7.7. Let $F : M \to N$ be a smooth map between manifolds. A point $q \in N$ is called a ***regular value of*** f if for every $p \in F^{-1}(q) \subset M$, the differential $dF_p : T_pM \to T_qN$ is surjective. If q is not a regular value of F, it is called a ***singular value*** of F.

It is a deep theorem of Sard, proven in Chapter 6 of [1], that the set of critical values in N has measure zero. Hence, almost every point in N is a regular value. In particular, it is always possible to find a regular value of any smooth map $F : M \to N$ on N.

Now we show how the definition of degree (usually defined in algebraic topology) relates the degree in differential topology:

Theorem 7.8. *Assume $f : M^m \to N^m$ a proper map between orientable manifolds, and $q \in N$ is a regular value of F, then we have*

$$\deg(f) = \sum_{x \in f^{-1}(q)} \operatorname{sign}(df_x).$$

Proof. By [1] Corollary 5.14, $f^{-1}(q)$ is a properly embedded submanifold of M of dimension 0. Since f is proper, $f^{-1}(q)$ is compact. Hence it consists of finitely many points (A discrete compact set cannot be infinite. Otherwise, because each singleton in the set is open, we get an open cover of the set with no finite subcover, which contradicts compactness). Say $f^{-1}(q) = \{p_1, ..., p_k\}$. Then by the inverse function theorem, there exists open neighborhoods V of q and U_i of q_i for $i = 1, ..., k$, such that

- $U_i \cap U_j = \emptyset$ if $i \neq j$;
- $f^{-1}(V) = \coprod_{i=1}^k U_i$;
- $f|_{U_i} : U_i \to V$ is a diffeomorphism.

Then pick $\omega \in \Omega_c^m(V)$ such that $\int_N \omega = 1$. In other words, ω represents a generator in $H_c^m(N)$. Then

$$\int_M f^*\omega = \sum_{i=1}^k \int_{U_i} f^*\omega = \sum_{i=1}^k \operatorname{sign}(df_x) \int_V \omega = \sum_{x \in f^{-1}(q)} \operatorname{sign}(df_x).$$

This proves the claim. □

Example 7.9. Let $i \neq m+1$ and consider the reflection

$$r_i(x_1, ..., x_{m+1}) = (x_1, ..., -x_i, ..., x_{m+1})$$

on S^m. The north pole $x = (0, ..., 0, 1) \in S^m$ is a fixed point of r_i. Since r_i is a diffeomorphism on a neighborhood of the north pole, x is a regular point of r_i. Let $\varphi : B(0,1) \cong \mathbb{R}^m \to S^m$ be the chart

$$\varphi(x_1, ..., x_m) = \Big(x_1, ..., x_m, (1 - x_1^2 - \cdots - x_m^2)^{1/2}\Big).$$

By definition of orientation on S^m, we know φ is orientation preserving. So by definition, $\operatorname{sign}(dr_i)_x = \operatorname{sign}(d\psi_0)$ where $\psi = \varphi^{-1} \circ r_i \circ \varphi : B_1(0) \to B_1(0)$ is the induced map such that the diagram

$$\begin{tikzcd} S^m \arrow[r, "r_i"] & S^m \\ B_1(0) \arrow[u, "\varphi"] \arrow[r, "\psi"'] & B_1(0) \arrow[u, "\varphi"'] \end{tikzcd}$$

commutes. By straightforward computation, $\psi(x_1, ..., x_m) = (x_1, ..., -x_i, ..., x_m)$. Hence $d\psi_0 = \operatorname{diag}(1, ..., -1, ..., 1)$, which has determinant -1. Thus $\deg(r_i) = -1$.

Now as a result of the above example, we prove the famous

Theorem 7.10 (Hairy Ball Theorem)**.** *There does not exist nowhere vanishing vector fields on S^{2n}.*

Proof. We view $S^{2n} \subset \mathbb{R}^{2n+1}$. Let X be a vector field on S^{2n} that is nonvanishing, for the sake of contradiction. Then we may normalize at every point (i.e., by multiplying a smooth function) and assume WLOG that $|X_p| = 1$ for all $p \in S^{2n}$. View both $p \in S^{2n}$ and X_p as vectors in $\mathbb{R}^{2n+1}$. Consider the map

$$\begin{aligned} F : S^{2n} \times [0,1] &\to S^{2n} \\ (p,t) &\mapsto p\cos(\pi t) + X_p \sin(\pi t). \end{aligned}$$

Note that the target is indeed in S^{2n} by straightforward computation using inner product because $|p| = |X_p| = 1$, and $p \perp X_p$. Then $F(\cdot, 0) = \mathrm{id}_{S^{2n}}$, while on the other hand $F(p,1) = -p$ is the antipodal map. Thus we get a homotopy from the identity to the antipodal map. Identity has degree 1, while the antipodal map is a composition of reflection, and by the example above, has degree $(-1)^{2n+1} = -1$. This is a contradiction because degree is a homotopy invariant by definition. □

7.2 The Brouwer Fixed Point Theorem

In this section we use the tool of degree theory to prove the famous Brouwer fixed point theorem. We start with a useful lemma:

Lemma 7.11. *Let M^m be a connected, orientable, compact manifold with boundary, and suppose ∂M is also connected. Let X^{m-1} be a connected, oriented manifold without boundary. If a smooth map $f : \partial M \to X$ can be extended to a smooth map $g : M \to X$, then* $\deg(f) = 0$.

Proof. Let $i : \partial M \to M$ be the inclusion. Then $f = g \circ i$. Let $\omega \in \Omega^{m-1}(X)$ such that $\int_X \omega = 1$. Then $d\omega = 0$ since ω is already a top degree form on X. Then

$$\deg(f) = \deg(f)\int_X \omega = \int_{\partial M} f^*\omega = \int_{\partial M} i^* g^* \omega = \int_M d(g^*\omega) = \int_M g^*(d\omega) = 0$$

by Stokes' theorem. This concludes the proof. □

Corollary 7.12. *Let M^m be a connected, orientable, compact manifold with boundary, and suppose ∂M has k connected component $\partial_j M$, where $j = 1, ..., k$. Let X^{m-1} be a connected, oriented manifold without boundary. If a smooth map $f : \partial M \to X$ can be extended to a smooth map $g : M \to X$, then if we denote $f_j = f|_{\partial_j M}$, then*

$$\sum_{j=1}^k \deg(f_j) = 0.$$

Proof. The proof is completely analogous to the proof of the previous lemma. Let $i_j : \partial_j M \to M$ be the inclusion. Then $f_j = g \circ i_j$. Let $\omega \in \Omega^{m-1}(X)$ such that $\int_X \omega = 1$. Then $d\omega = 0$ since ω is already a top degree form on X. Then

$$\begin{aligned} \sum_{j=1}^k \deg(f_j) = \sum_{j=1}^k \deg(f_j) \int_X \omega &= \sum_{j=1}^k \int_{\partial_j M} f_j^* \omega \\ &= \sum_{j=1}^k \int_{\partial_j M} i_j^* g^* \omega = \int_M d(g^*\omega) = \int_M g^*(d\omega) = 0 \end{aligned}$$

by Stokes' theorem. This concludes the proof. $\square$

Now we prove

Theorem 7.13 (Brouwer Fixed Point Theorem). *Let $D^m \subset \mathbb{R}^m$ be the closed unit disk. Then every continuous map $f : D^m \to D^m$ has a fixed point.*

Proof. We argue by contradiction. Suppose $F_0 : D^m \to D^m$ has no fixed point. Then since F_0 is continuous, we have $\inf_{p\in D^m} |p - F_0(p)|/3 > 0$. Choose a positive number r such that $0 < r < \inf_{p\in D^m} |p - F_0(p)|/3 > 0$. By Whitney approximation theorem (Theorem 1.32), there exists a smooth map $F_1 : D^m \to \mathbb{R}^m$ such that $|F_0(p) - F_1(p)| < r$ for all $p \in D^m$. Now consider the map $F : D^m \to D^m$ given by

$$F(p) = \frac{F_1(p)}{1 + r}.$$

Then F maps D^m to D^m by triangle inequality, and F is a smooth map from D^m to itself. Moreover,

$$\begin{aligned}
|F(p) - F_0(p)| &\leq |F(p) - F_1(p)| + |F_1(p) - F_0(p)| \\
&< |F(p)|r + r \\
&\leq 2r \\
&< |F_0(p) - p|.
\end{aligned}$$

Thus $|F(p) - p| \leq |F(p) - F_0(p)| + |F_0(p) - p| < 2|F_0(p) - p|$. Hence $F : D^m \to D^m$ is smooth and does not have a fixed point.

Now consider the smooth map $G : D^m \to S^{m-1}$ defined by

$$G(p) = \frac{p - F(p)}{|p - F(p)|}.$$

And we put $g = G|_{S^{m-1}} : S^{m-1} \to S^{m-1}$. Clearly G is an extension of g. So by Lemma 7.11, we have $\deg(g) = 0$.

On the other hand, consider the map

$$\begin{aligned}
H : S^{m-1} \times [0, 1] &\to S^{m-1} \\
(p, t) &\mapsto \frac{p - tF(p)}{|p - tF(p)|}
\end{aligned}$$

is well-defined, because $|F(p)| \leq 1$ and $p \neq F(p)$, and moreover H is a homotopy from the identity on S^{m-1} to g. Thus $\deg(g) = \deg(\mathrm{Id}_{S^{m-1}}) = 1$. This is the desired contradiction. $\square$

7.3 The Borsuk-Ulam Theorem

The goal of this section is to prove the Borsuk-Ulam theorem. The proof, however, requires the following technical lemma. In fact, the proof of this lemma will probably be the most technical (also the most clever) proof of this whole section (one can argue that this is even the case for the whole book):

Lemma 7.14. *Suppose $f : S^m \to S^m$ is a smooth odd map, which means for any p we have $f(-p) = -f(p)$, then $\deg(f)$ is an odd number.*

Proof. We induct on the number m.

For the base case $m = 1$, let $f : S^1 \to S^1$ be an odd map. For simplicity, we denote the upper, lower, left, right hemisphere of S^1 by $S^1_\wedge, S^1_\vee, S^1_<, S^1_>$, respectively. WLOG [1] that $N = (0, 1)$ is a regular value of f. Then $f^{-1}(\pm N)$ is closed in S^1, hence compact, thus has finitely many points. Again we may assume WLOG that $f((\pm 1, 0)) \neq N$.[2] Hence $f((\pm 1, 0)) \neq -N$ as well by oddity. WLOG we assume $f((1, 0)) \in S^1_>$.

We claim that it suffices to show $f^{-1}(N)$ has odd number of points: By Theorem 7.8 we have $\deg(f) = \sum_{x \in f^{-1}(N)} \operatorname{sign}(df_x)$. If the sum has more $+1$'s than -1's, cancel them out in pairs and we get a sum with only $+1$'s. If $|f^{-1}(N)|$ is odd, then after canceling points in pairs we get $\deg(f)$ is odd. If the sum has more -1's than $+1$'s, we consider $-\deg(f)$ and use the same argument. Hence our goal now is to show that $|f^{-1}(N)|$ is an odd number.

Since f is an odd map, we have that

$$|f^{-1}(N)| = \frac{1}{2}|f^{-1}(\pm N)| = |f^{-1}(\pm N) \cap S^1_\wedge|,$$

where the first equality is by symmetry of odd maps, and the second equality is again by symmetry and the assumption that $f((\pm 1, 0)) \neq N$.

Now, points in $f^{-1}(\pm N)$ separates $S^1_\wedge$ into finitely many arcs (or segments). We claim that adjacent segments must be sent to $S^1_>$ and $S^1_<$ respectively by f: To see this, note that since N is a regular value of f, so for any $x \in f^{-1}(\pm N) \cap S^1_\wedge$, we have $df_x : T_x S^1 \to T_{f(x)} S^1$ is surjective. Then for any vector $v \in T_x S^1$, there exists a curve γ such that $\gamma'(0) = v$ and $\gamma(0) = x$. If f sends adjacent arcs separated by x to the same hemisphere, then[3] $df_x(v) = (f \circ \gamma)'(0) = 0$, since $f \circ \gamma$ changes direction at $f(x)$, i.e., at $t = 0$. Thus df_x is not surjective.

Therefore, adjacent arcs goes to different hemisphere $S^1_>$ or $S^1_<$. Also note that $f((1, 0))$ and $f(0, 1)$ belong to $S^1_>$ and $S^1_<$ (or the other way around) separately. So we must start with arc mapping into $S^1_>$, and then arc mapped into $S^1_<$, and alternate... and eventually end with arc mapping into $S^1_<$. So there must be an even number of arcs, i.e., odd number of endpoints. So $|f^{-1}(\pm N) \cap S^1_\wedge| = |f^{-1}(N)|$ is odd. This proves the base case.

Now we prove the inductive case. To this end, suppose $f : S^m \to S^m$ is an odd map. WLOG $N = (0, ..., 0, 1)$ is a regular value of f. Denote S^{m-1} the equator of S^m by setting $x_{m+1} = 0$. By the same argument, we have $f^{-1}(\pm N) \cap S^m_\wedge = \{p_1, ..., p_k\}$ and

$$|f^{-1}(N)| = \frac{1}{2}|f^{-1}(\pm N)| = |f^{-1}(\pm N) \cap S^m_\wedge|.$$

By the inverse function theorem and possibly shrinking neighborhoods, we can choose neighborhoods U of N, disjoint U_i of p_i for $i = 1, ..., k$, such that f sends each U_i diffeomorphically to U or $-U$, and such that $U_i \subset S^m_\wedge$ for all i. Now put

$$M = S^m_\wedge - \coprod_{i=1}^{k} U_i$$

and

$$g = r \circ \pi \circ f|_M$$

[1] Otherwise, we may compose f with a rotation which is an orientation preserving diffeomorphism, then the claim follows from $\deg(fg) = \deg(f)\deg(g)$ and Sard's theorem.

[2] Rotation argument again!

[3] Note that since f is continuous, f will first map one arc with endpoint x to a segment on a hemisphere, and then map the adjacent arc back to a portion of the same segment with the opposite orientation.

where $\pi(x_1, ..., x_{m+1}) = (x_1, ..., x_m)$ is the projection and $r(x) = x/|x|$ retract from $B_1(0) \setminus \{0\}$ to S^{m-1}. We set $g_0 = g|_{S^{m-1}}$. Then by straightforward calculation, g_0 is an odd map. Now consider $g_i = g|_{\partial U_i}$. Since we have a sequence of diffeomorphisms

$$\partial U_i \xrightarrow{f} \pm\partial U \xrightarrow{\pi} \pi(\partial U) \xrightarrow{r} S^{m-1}$$

we see that g_i is a diffeomorphism for each i. Now:

- By construction, $g_0 : S^{m-1} \to S^{m-1}$ is odd. So by inductive hypothesis, $\deg(g_0)$ is odd.
- $g_i : \partial U_i \to S^{m-1}$ is a diffeomorphism for each i, so $\deg(g_i) = \pm 1$ for each i.
- By Corollary 7.12, we have $\sum_{i=0}^{k} \deg(g_i) = 0$, since clearly g is a smooth extension of g_i's.

Thus we conclude that k is an odd number. This concludes the proof. $\square$

We conclude this section by proving

Theorem 7.15 (Borsuk-Ulam Theorem)**.** *Let $f : S^m \to \mathbb{R}^m$ be continuous. Then there exists a point $p_0 \in S^m$ such that $f(-p_0) = f(p_0)$.*

Proof. For $m = 1$, define $g(p) = f(-p) - f(p)$. We want to show g has a zero on S^1. Note $g(1, 0) = f(-1, 0) - f(1, 0)$ and $g(-1, 0) = f(1, 0) - f(-1, 0)$. Hence $g(1, 0) = -g(-1, 0)$. If $g(1, 0) = 0$ we are done, otherwise $g(S^1)$ is connected and contains both positive and negative numbers on the real line, so $0 \in g(S^1)$. This proves the base case.

Now for $m \neq 1$, suppose $f : S^m \to \mathbb{R}^m$ is continuous, and suppose $f(-p) \neq f(p)$ for all $p \in S^m$. Let $\epsilon = \inf_{p \in S^m} |f(-p) - f(p)| > 0$. Then take an $\epsilon/4$-smooth approximation f' of f by Whitney approximation theorem (Theorem 1.32). If there exists a p such that $f'(-p) - f'(p) = 0$, then

$$|f(-p) - f(p)| \leq |f(p) - f'(p)| + |f'(p) - f'(-p)| + |f'(-p) - f(-p)| \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon$$

which contradicts the choice of ϵ. Therefore we have found a smooth map $f' : S^m \to \mathbb{R}^m$, such that $f'(-p) \neq f(p)$ for all $p \in S^m$. Now let $g : S^m \to S^{m-1}$ be defined by

$$g(p) = \frac{f'(p) - f'(-p)}{|f'(p) - f'(-p)|}.$$

By definition, $\tilde{g} = g|_{S^{m-1}}$ is a map from S^{m-1} to itself, and is an odd map by construction. Therefore by the lemma above, $\deg(\tilde{g})$ is odd. Hence in particular $\deg(\tilde{g}) \neq 0$. Now

$$g|_{S^m_\wedge} : S^m_\wedge \to S^{m-1}$$

is clearly an extension of $\tilde{g}$, as $\partial S^m_\wedge = S^{m-1}$. Thus by Lemma 7.11, we have $\deg(\tilde{g}) = 0$, which is the desired contradiction. $\square$

7.4 Intersection Theory

Definition 7.16. Let S^s, L^ℓ be oriented submanifolds of an oriented manifold N^n with complementary dimension, i.e., $s + \ell = n$. We say that S and L intersect ***transversally***, written $S \pitchfork L$, if for every $p \in S \cap L$, one has $T_pS \oplus T_pL = T_pM$ as vector spaces (but we are not concerned with whether the orientations on both sides of the equation agrees or not, we just need T_pS and T_pL to span T_pN). See Figure 7.1 for examples.

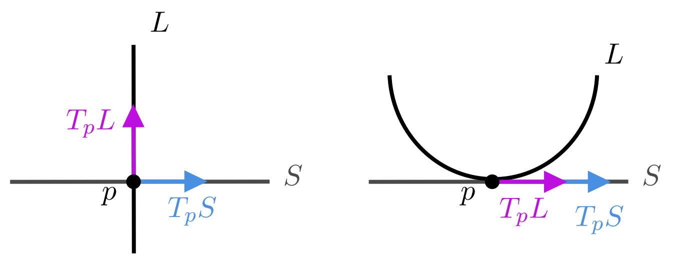


Figure 7.1: We have two examples of intersecting submanifolds S, L in $\mathbb{R}^2$ with complementary dimension. On the left, S and L intersect transversally because their tangent spaces span a two dimensional plane, which is exactly $T_p\mathbb{R}^2$; However, on the right, S and L does not intersect transversally because their tangent spaces are the same one dimensional subspaces, so the tangent spaces do not span the whole $T_p\mathbb{R}^2$. Thus one should think of having transversal intersection as the property of having minimal dimension of intersection, because transversality is equivalent to $\dim T_p(S \cap L) = 0$.

Now we introduce a notion that allows us to count signed intersections of two submanifolds:

Definition 7.17. Let S^s, L^ℓ be oriented submanifolds of an oriented manifold N^n, where $n = s + \ell$ and $S \pitchfork L$.

- If $p \in S \cap L$, we define the ***local intersection number*** $\epsilon(p, S, L)$ to be $+1$ if the direct sum orientation $T_pS \oplus T_pL$ coincides with the orientation of T_pM; and we define $\epsilon(p, S, L) = -1$ if otherwise.
- If S is compact, then $L \cap S$ is a finite set of points ($S \cap L$ has dimension 0, hence a set of points, which is a closed subset of a compact set S). We define the ***intersection number of S and L to be***

$$I(S, L) = \sum_{p \in S \cap L} \epsilon(p, S, L).$$

Note that $\epsilon(p, S, L) = (-1)^{s\ell}\epsilon(p, L, S)$, and similarly $I(S, L) = (-1)^{s\ell} I(L, S)$.

Now we present the key result of this section:

Theorem 7.18. *Suppose that S^s, L^ℓ are closed, oriented submanifolds of an oriented manifold N^n of finite type (i.e., has a finite good cover), such that*

- *S is compact;*
- *$s + \ell = n$;*
- *$L \pitchfork S$.*

Now let $\mathrm{PD}(S)$ be the compactly supported closed form representing the Poincaré dual of S (by Proposition 6.73) in N, and let $\mathrm{PD}(L)$ be a closed form representing the Poincare dual of L in N. Then we have

$$I(S, L) = \int_M \mathrm{PD}(S) \wedge \mathrm{PD}(L).$$

Proof. For simplicity, we write $\epsilon(p) := \epsilon(p, S, L)$. Since $L \pitchfork S$, for all $p \in S \cap L$, we can find[4] a local coordinate $(x^1, ..., x^n)$ defined on an open neighborhood U_p of p such that

$$U_p \cap S = \{x^{s+1} = \cdots = x^{s+\ell} = 0\}$$

and

$$U_p \cap L = \{x^1 = \cdots = x^s = 0\}.$$

The orientation from on $S \cap U_p$ is given by $dx^1 \wedge \cdots dx^s$, and the orientation form on $L \cap U_p$ is given by $dx^{s+1} \wedge \cdots \wedge dx^n$. Then the wedge $dx^1 \wedge \cdots \wedge dx^n$ will correspond to direct sum orientation. Hence the orientation form on U_p is given by

$$\epsilon(p) dx^1 \wedge \cdots \wedge dx^n. \tag{7.1}$$

By shrinking U_p's if necessary, we may assume $U_p \cap U_q = \emptyset$ if $p \neq q$ in $L \cap S$. Now we construct a tubular neighborhood (See Figure 7.2) $\mathcal{N}$ of S in N such that for all $p \in L \cap S$, we have

$$\mathcal{N}_p = (L \cap U_p) \cap \mathcal{N} =: L_p$$

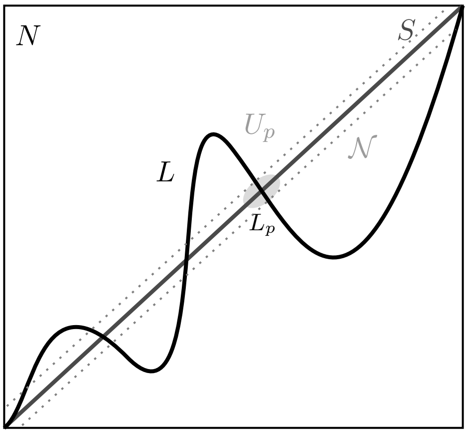


Figure 7.2: Construction of the tubular neighborhood $\mathcal{N}$ of S. We may choose $\mathcal{N}$ thin enough such that $L \cap \mathcal{N} = \coprod_{p \in S \cap L} \mathcal{N}_p$. This can be done because we can construct $\mathcal{N}$ arbitrarily as long as it is diffeomorphic to the normal bundle of S. In this case, $N = M \times M$ and $S = \Delta$ the diagonal of M, where M is a manifold. But the construction works in general as well.

where the equality above is to be understood as equality of sets. But as oriented manifolds, they have different orientations: By definition L_p is a subset of L, hence its orientation is the induced orientation from the manifold L. $\mathcal{N}_p$ is the fiber of $\mathcal{N}$. By convention, the orientation of $\mathcal{N}_p$ (namely the *co-orientation*) is induced by the direct sum $T_pN = T_pS \oplus N_pS$ where NS is the normal bundle of S. By the tubular neighborhood theorem, there is an orientation preserving diffeomorphism from NS to $\mathcal{N}$. That is, $\mathcal{N}_p$ has the orientation such that direct sum orientation of $T_pS \oplus \mathcal{N}_p$ agrees with that of T_pN. Since by (7.1), the orientation of N is given by

$$T_pN = \epsilon(p) T_pS \oplus T_pL$$

[4]Short proof: Since $T_pL \oplus T_pS = T_pN$, we can choose $v_1, ..., v_\ell \in T_pL$ and $v_{\ell+1}, ..., v_{\ell+s} \in T_pS$. Then give N a Riemannian metric, and let the curve $x^i(\lambda) = \exp_p(\lambda v_i)$. Then the coordinates $x^1, ..., x^{s+\ell}$ will do the job. For a more detailed and rigorous proof, see Lemma 18 on p.119 of [12].

we have that

$$L_p = \epsilon(p)\mathcal{N}_p.$$

Now, by Corollary 6.96, we may assume $\mathrm{PD}(S) \in \Omega_c^\ell(\mathcal{N})$. In addition, We may choose $\mathcal{N}$ thin enough such that

$$L \cap \mathcal{N} = \coprod_{p \in S \cap L} \mathcal{N}_p.$$

Then we have, by the definition of Poincaré dual and discussion above, that

$$\int_M \mathrm{PD}(S) \wedge \mathrm{PD}(L) = \int_L \mathrm{PD}(S) = \sum_{p\in S\cap L} \int_{L_p} \mathrm{PD}(S) = \sum_{p\in S\cap L} \epsilon(p) \int_{\mathcal{N}_p} \mathrm{PD}(S) = \sum_{p \in S\cap L} \epsilon(p)$$

where in the last equality we used Corollary 6.95. $\square$

Example 7.19. Let $N = T^2 = S^1 \times S^1$ be a torus, and $S = S^1 \times b, L = a \times S^1$ where $p = (a, b) \in M$ is a fixed point. See Figure 7.3.

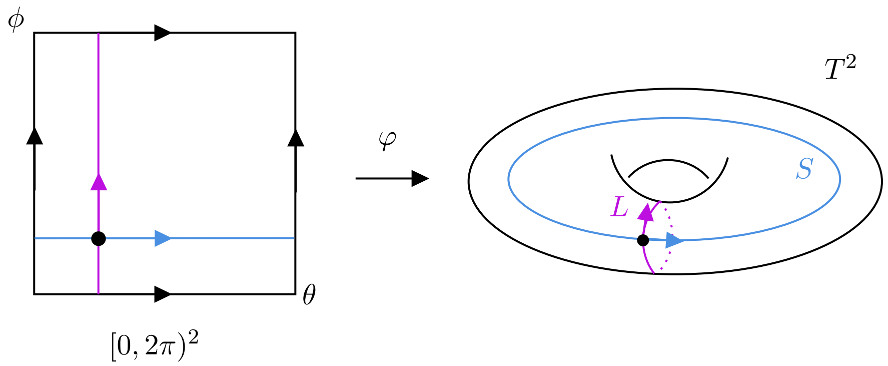


Figure 7.3: Let S, L be two submanifolds of T^2 with orientations indicated as in the figure, and let $\varphi : [0, 2\pi)^2 \to T^2$ be the diffeomorphism parameterizing the torus as shown in the figure. It is easy to see that φ is orientation preserving, where we define the orientation of T^2 as follows: View T^2 as a submanifold of $\mathbb{R}^3$, and let $v, w \in T_pT^2$. We declear the ordered basis $v, w \in T_pT^2$ to be positive iff v, w, n is a positive basis for $\mathbb{R}^3$, where n is the normal to T^2 at p.

We see that $I(S, L) = \epsilon(p, S, L) = +1$ from Figure 7.3 where $x = (a, b)$. We know that $\mathrm{PD}(S)$ satisfies

$$\int_{N_pS} \mathrm{PD}(S) = 1,$$

where NS is the fiber of a tubular neighborhood of S at p by Theorem 6.94. Let φ : $[0, 2\pi)^2 \to T^2$ be an orientation preserving diffeomorphism parameterizing the torus, as Figure 7.3 suggests. We take $\rho : [0, 2\pi) \to \mathbb{R}$ to be a smooth compactly supported function with

$$\int_0^{2\pi} \rho(\lambda) d\lambda = 1.$$

Then take

$$\varphi^* \mathrm{PD}(S) = \rho(\phi) d\phi \qquad \varphi^* \mathrm{PD}(L) = -\rho(\theta) d\theta.$$

Then we see that

$$\int_{T^2} \mathrm{PD}(S) \wedge \mathrm{PD}(L) = \int_{[0,2\pi)^2} \varphi^* \mathrm{PD}(S) \wedge \varphi^* \mathrm{PD}(L) = \int_{[0,2\pi)^2} \rho(\theta)\rho(\phi) d\theta \wedge d\phi = 1.$$

Again this is consistent with the result we get from counting $I(S, L)$.

One can check that our choice of $\mathrm{PD}(S), \mathrm{PD}(L)$ are indeed the Poincaré dual of S and L. To see this, let $\alpha \in \Omega^1(T^2)$ be any closed 1-form. Write $\varphi^*\alpha = F(\theta, \phi)d\theta + G(\theta, \phi)d\phi$. Then we see that $\varphi^*(\alpha \wedge \mathrm{PD}(S)) = F(\theta, \phi)\rho(\phi)d\theta \wedge d\phi$. Since $\operatorname{Supp}\mathrm{PD}(S)$ can be chosen to be sufficiently small as long as it is contained in a thin tubular neighborhood of S, and ρ has mass 1, we see that we can make $\mathrm{PD}(S)$ more and more concentrated on S: think of it as a dirac delta function in the limit. If we denote ϕ' the ϕ coordinate of $\varphi^{-1}(S)$ in $[0, 2\pi)^2$, we can make $\rho(\phi) \to \delta(\phi')$, then it is clear that

$$\int_{[0,2\pi)^2} \varphi^*(\alpha \wedge \mathrm{PD}(S)) = \int_0^{2\pi}\int_0^{2\pi} F(\theta, \phi)\rho(\phi)d\theta d\phi = \int_0^{2\pi} F(\theta, \phi')d\theta = \int_{\varphi^{-1}(S)} \varphi^*\alpha.$$

Under the diffeomorphism φ, we see that on T^2 the integral equality above becomes

$$\int_{T^2} \alpha \wedge \mathrm{PD}(S) = \int_S \alpha,$$

which is exactly the defining integral equation of $\mathrm{PD}(S)$, similarly we can check $\mathrm{PD}(L)$ is indeed the Poincaré dual of L.

7.5 Lefschetz Fixed Point Theorem

In this section, our goal is to prove the Lefschetz fixed point theorem for compact manifold. The proof of Lefschetz fixed point theorem is not hard using the tools of intersection theory we developed in the previous section. However, we need a preliminary result to prove the Lefschetz fixed point theorem, which involves some technical calculations.

First, we recall the Künneth formula from Theorem 6.69.Künneth theorem says that if we have M, N two orientable manifolds of finite type, then

$$H^k(M \times N) \cong \bigoplus_{p+q=k} H^p(M) \oplus H^q(N).$$

In particular, if we define $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ be the projection maps onto each factors, then the map

$$\begin{aligned} \psi : H^*(M) \otimes H^*(N) &\to H^*(M \times N) \\ \omega \otimes \phi &\mapsto \pi_M^*\omega \otimes \pi_N^*\phi \end{aligned}$$

is an isomorphism of cohomology rings.

Now the preliminary result we want to study concerns the following situation: Let M^m be an oriented closed (compact and has no boundary) manifold, and let $f : M \to M$ be a smooth map. We define the graph of f to be

$$\Gamma_f = \{(x, f(x)) : x \in M\} \subset M \times M.$$

Then, what is the Poincaré dual $\mathrm{PD}(\Gamma_f) \in H^m(M \times M)$? Note that since Γ_f is diffeomorphic to M via the projection onto M, codimension of Γ_f in M must be m.

Note that by the Mayer-Vietoris argument, if M^m is of finite type, then $H^j(M)$ is finite dimensional for each j by Proposition 6.61.

Proposition 7.20. Let $\beta_j = \dim H^j(M)$ for each j. Let ω_i^j be a basis of $H^j(M)$, where $1 \leq i \leq \beta_j$. Let θ_i^{m-j} be the dual basis of ω_i^j for $1 \leq i \leq \beta_{m-j} = \beta_j$ under Poincaré duality. Let $\pi_i : M \times M \to M$ be the projection onto the 1st and 2nd factor where $i = 1, 2$. $f^* : H^j(M) \to H^j(M)$ be given by

$$f^*(\omega_i^j) = \sum_k A_j^{ki} \omega_k^j.$$

Then

$$\mathrm{PD}(\Gamma_f) = \sum_{i,j,k} (-1)^j A_j^{ik} (\pi_1^* \omega_i^j \wedge \pi_2^* \theta_k^{m-j}). \tag{7.2}$$

Proof. Since all the cohomology groups are finite dimensional vector spaces, under Poincaré duality and dual vector space isomorphism, we have

$$H^j(M) \cong H_c^{m-j}(M)^* \cong H^{m-j}(M)^* \cong H^{m-j}(M).$$

Under these isomorphisms, we see that we have the relation

$$\int_M \omega_i^j \wedge \theta_k^{m-j} = \delta_{ik}$$

for all $1 \leq i, k \leq \beta_j$. By Künneth theorem, the set

$$\{\pi_1^* \omega_i^j \wedge \pi_2^* \theta_k^{m-j} : 0 \leq j \leq m, 1 \leq i, k \leq \beta_j\}$$

is a basis of $H^m(M \times M)$.

Now we begin our calculation:

$$\begin{aligned}
&\int_{M \times M} (\pi_1^* \omega_i^j \wedge \pi_2^* \theta_k^{m-j}) \wedge (\pi_1^* \theta_s^{m-u} \wedge \pi_2^* \omega_t^u) \\
&\quad = (-1)^{m(m-j)} \int_{M \times M} \pi_1^*(\omega_i^j \wedge \theta_s^{m-u}) \wedge \pi_2^*(\omega_t^u \wedge \theta_k^{m-j}) \\
&\quad = (-1)^{m(m-j)} \int_M \omega_i^j \wedge \theta_s^{m-u} \int_M \omega_t^u \wedge \theta_k^{m-j} \\
&\quad = (-1)^{m(m-j)} \delta_{ju} \int_M \omega_i^j \wedge \theta_s^{m-j} \int_M \omega_t^j \wedge \theta_k^{m-j} \\
&\quad = (-1)^{m(m-j)} \delta_{ju} \delta_{is} \delta_{tk}.
\end{aligned}$$

In the calculation above, the first equality can be seen by swapping the 2nd term in the wedge to the last; the 3rd equality can be seen by considering integrating on local coordinate of product $M \times M$; the 4th equality can be seen by the following argument: Since all forms are compactly supported, if one coordinate or more is missing, say x^n, then $\int_{\varphi(U)} (\cdots) dx^n = 0$ where φ is some local chart and $U \subset M$ is some open set containing support of the form in question (if it is not contain in one chart, use a partition of unity argument).

Now we write $\mathrm{PD}(\Gamma_f)$ as a linear combination of basis elements:

$$\mathrm{PD}(\Gamma_f) = \sum_{i,j,k} c_j^{ik} (\pi_1^* \omega_i^j \wedge \pi_2^* \theta_k^{m-j})$$

and denote the expression on the right hand side by ω. For fixed u, s, t, by the definition of Poincaré dual we have

$$\int_{M \times M} (\pi_1^* \theta_s^{m-u} \wedge \pi_2^* \omega_t^u) \wedge \omega = \int_{\Gamma_f} i^*(\pi_1^* \theta_s^{m-u} \wedge \pi_2^* \omega_t^u) \tag{7.3}$$

where $i : \Gamma_f \to M \times M$ denote the inclusion map. We now calculate the left hand side and the right hand side of (7.3) separately. The left hand side is

$$
\begin{aligned}
&(-1)^{m^2} \sum_{i,j,k} c_j^{ik} \int_{M\times M} (\pi_1^*\omega_i^j \wedge \pi_2^*\theta_k^{m-j}) \wedge (\pi_1^*\theta_s^{m-u} \wedge \pi_2^*\omega_t^u) \\
&= (-1)^{m^2} \sum_{i,j,k} c_j^{ik} \delta_{ju}\delta_{is}\delta_{tk}(-1)^{m(m-j)} \\
&= (-1)^{m^2+m(m-u)} c_u^{st} \\
&= (-1)^{-mu} c^{st} \\
&= (-1)^{mu} c_u^{st}.
\end{aligned}
$$

Now recall that we set

$$f^*(\omega_i^j) = \sum_k A_j^{ki} \omega_k^j.$$

Let $\phi : M \to \Gamma_f$ be the orientation preserving diffeomorphism $\phi(x) = (x, f(x))$. Note that $\pi_1 \circ i \circ \phi = \mathrm{Id}$ and $\pi_2 \circ i \circ \phi = f$. Then the right hand side of (7.3) is given by

$$
\begin{aligned}
\int_M \phi^* i^* (\pi_1^*\theta_s^{m-u} \wedge \pi_2^*\omega_t^u) &= \int_M \theta_s^{m-u} \wedge f^*\omega_t^u \\
&= \sum_k A_u^{kt} \int_M \theta_s^{m-u} \wedge \omega_k^u \\
&= (-1)^{u(m-u)} \sum_k A_u^{kt} \delta_{sk} \\
&= (-1)^{u(m-u)} A_u^{st}.
\end{aligned}
$$

Equating the left hand side and right hand side of (7.3) gives

$$(-1)^{mu} c_u^{st} = (-1)^{u(m-u)} A_u^{st}.$$

Hence

$$c_u^{st} = (-1)^{-u^2} A_u^{st} = (-1)^u A_u^{st}.$$

This proves (7.2). □

As an easy corollary, we obtain the Poincare dual of the diagonal Δ in $M \times M$:

Corollary 7.21. *Assuming conditions and notations in the previous proposition. Let $\Delta \subset M \times M$ be the diagonal, i.e., $\Delta = \{(x, x) : x \in M\}$. Then*

$$\mathrm{PD}(\Delta) = \sum_j (-1)^j (\pi_1^*\omega_i^j \wedge \pi_2^*\theta_i^{m-j}). \tag{7.4}$$

Proof. Use (7.2) where $f : M \to M$ is the identity. Then $\Gamma_f = \Delta$. Therefore $f^* = \mathrm{id} : H^j(M) \to H^j(M)$ hence all $A_j^{ki} = \delta_{ik}$. □

Definition 7.22. Let $f : M \to M$ be a smooth map from a compact oriented manifold M to itself. We define the ***Lefschetz number*** $L(f)$ of f to be

$$L(f) = \sum_j (-1)^j \operatorname{Tr}\big(f^* : H^j(M) \to H^j(M)\big).$$

Lemma 7.23. *Let $i : \Delta \to M \times M$ be the inclusion map, and $f : M \to M$ a smooth map where M is an oriented compact manifold. Then*

$$\int_\Delta \mathrm{PD}(\Gamma_f) = L(f).$$

More specifically, we have

$$\int_\Delta i^* \mathrm{PD}(\Gamma_f) = \sum_j (-1)^j \operatorname{Tr}\big(f^*|_{H^j(M)}\big).$$

Proof. Consider the orientation preserving diffeomorphism $\phi : M \to \Delta$ where $\phi(x) = (x, x)$. Then $\pi_i \circ i \circ \phi : M \to M$ are both identity map where $i = 1, 2$. Hence by our previous calculation,

$$\begin{aligned}
\int_\Delta i^* \mathrm{PD}(\Gamma_f) &= \int_M \phi^* i^* \mathrm{PD}(\Gamma_f) \\
&= \sum_j (-1)^j \sum_{i,k} A_j^{ik} \int_M \phi^* i^* \pi_1^* \omega_i^j \wedge \phi^* i^* \pi_2^* \theta_k^{m-j} \\
&= \sum_j (-1)^j \sum_{ik} A_j^{ik} \int_M \omega_i^j \wedge \theta_k^{m-j} \\
&= \sum_j (-1)^j \sum_{i,k} A_k^{ik} \delta_{ik} \\
&= \sum_j (-1)^j \sum_i A_j^{ii} \\
&= \sum_j (-1)^j \operatorname{Tr}\big(f^*|_{H^j(M)}\big).
\end{aligned}$$

This concludes the proof. □

Definition 7.24. Let $f : M \to M$ be a smooth map between compact manifolds. Then f is said to be ***Lefschetz*** or a ***Lefschetz map*** if $\Gamma_f \pitchfork \Delta$ in $M \times M$.

Now we are ready to prove

Theorem 7.25 (Lefschetz Fixed Point Theorem)**.** *Let M^m be a compact, oriented, smooth manifold, and $f : M \to M$ a smooth map. Then*

$$\int_{M\times M} \mathrm{PD}(\Gamma_f) \wedge \mathrm{PD}(\Delta) = L(f).$$

Hence, if $\Gamma_f \pitchfork \Delta$ (i.e., f is a Lefschetz map), then

$$I(\Gamma_f, \Delta) = L(f).$$

In particular, if $L(f) \neq 0$, then f has at least one fixed point.

Proof. By the definition of Poincaré dual, and the previous lemma, we have

$$\int_{M\times M} \mathrm{PD}(\Gamma_f) \wedge \mathrm{PD}(\Delta) = \int_\Delta i_\Delta^* \mathrm{PD}(\Gamma_f) = L(f).$$

The second part of the theorem follows directly from Theorem 7.18. Finally, if f has no fixed point, then $\Gamma_f \cap \Delta = \emptyset$. Hence by the definition of intersection number, $I(\Gamma_f, \Delta) = 0$. Thus $L(f) = 0$. □

Remark 7.26. Therefore, we see an alternative definition of $L(f)$ for a smooth map $f : M \to M$ between a compact oriented manifold, which is $L(f) := I(\Gamma_f, \Delta)$. Note that some books like [7] defined $L(f) = I(\Delta, \Gamma_f)$. But we insist on defining $L(f)$ to be $I(\Gamma_f, \Delta)$ because it agrees with the cohomological definition, i.e., that $L(f)$ is the alternating trace of f^* on cohomology groups, where the other convention is off by a sign when dimension is odd.

We conclude this section with a final result, which relates Lefschetz number $L(f)$ with its local Lefschetz number $L_x(f)$ at every fixed point x.

Lemma 7.27. *If $f : M^m \to M^m$ is a Lefschetz map between compact manifolds, then at every fixed point x of f (guaranteed to exist at least one by Lefschetz fixed point theorem), 1 is not an eigenvalue of df_x. In other words,* $\det(I - df_x) \neq 0$.

Proof. Let $(x, x) = p \in \Gamma_f \cap \Delta \subset M \times M$. Since f is Lefschetz, we have $T_p\Gamma_f + T_p\Delta = T_p(M \times M)$. If 1 is an eigenvalue of df_x, then say $df_x(v) = v$ for some $v \in T_xM$. Then $(v, v) \in T_p\Gamma_f \cap T_p\Delta$. But this contradicts the transersality condition since $\dim(\Gamma_f) = \dim(\Delta) = m$, hence $\dim(T_p\Gamma_f \cap T_p\Delta) = 0$. □

This motivates the following definition:

Definition 7.28. Let $f : M \to M$ be a Lefschetz map from a oriented, compact manifold to itself. Let x be a fixed point of f. Then define the ***local Lefschetz degree*** $L_x(f)$ of f at x by

$$L_x(f) = \operatorname{sign}\det(I - df_x).$$

We conclude this section by proving

Theorem 7.29. *Let $f : M^m \to M^m$ be a Lefschetz map and M a compact, oriented manifold. Then f has finitely many fixed points, and moreover,*

$$L(f) = \sum_{p \in Fix(f)} L_p(f).$$

Proof. We have $\dim(\Gamma_f \cap \Delta) = 0$ by transversality. Since M is compact, $\Gamma_f \cap \Delta$ is a compact 0-dimensional manifold, which must have finitely many points by compactness argument again. This shows that $\text{Fix}(f)$ is finite. To prove the final statement, note that by the Lefschetz fixed point theorem, we have

$$L(f) = \int_\Delta i_\Delta^* \operatorname{PD}(\Gamma_f) = \int_{M \times M} \operatorname{PD}(\Gamma_f) \wedge \operatorname{PD}(\Delta) = I(\Gamma_f, \Delta) = \sum_{p \in \text{Fix}(f)} \epsilon(p, \Gamma_f, \Delta). \quad (7.5)$$

Hence it suffices to prove that $\epsilon(p, \Gamma_f, \Delta) = L_p(f)$ for every fixed point p. Let $v_1, ..., v_m$ be a positively oriented basis of T_pM. Then $v_1 \oplus v_1, ..., v_m \oplus v_m$ is a positively oriented basis of $T_{(p,p)}\Delta$, and $v_1 \oplus df_p(v_1), ..., v_m \oplus df_p(v_m)$ a positively oriented basis of $T_{(p,p)}\Gamma_f$. Then a positively oriented basis of $T_{(p,p)}\Gamma_f \oplus T_{(p,p)}\Delta$ is

$$v_1 \oplus df_p(v_1), ..., v_m \oplus df_p(v_m), v_1 \oplus v_1, ..., v_m \oplus v_m.$$

We now examine the orientation of this basis in $T_{(p,p)}(M \times M)$ Let ω be a (positive) orientation form on M. Then let $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ be the respective projections. Then a (positive) orientation form on $M \times M$ is given by $\alpha = \pi_1^*\omega \wedge \pi_2^*\omega \in \Omega^{2m}(M)$. Since α is alternating, we see that

$$\begin{aligned}
&\alpha(v_1 \oplus df_p(v_1), ..., v_m \oplus df_p(v_m), v_1 \oplus v_1, ..., v_m \oplus v_m)\\
&\quad = \alpha(0 \oplus (df_p - I)(v_1), ..., 0 \oplus (df_p - I)(v_m), v_1 \oplus v_1, ..., v_m \oplus v_m)\\
&\quad = \alpha(0 \oplus (df_p - I)(v_1), ..., 0 \oplus (df_p - I)(v_m), v_1 \oplus 0, ..., v_m \oplus 0)\\
&\quad = (-1)^{m^2}\alpha(v_1 \oplus 0, ..., v_m \oplus 0, 0 \oplus (df_p - I)(v_1), ..., 0 \oplus (df_p - I)(v_m))\\
&\quad = (-1)^{m}\alpha(v_1 \oplus 0, ..., v_m \oplus 0, 0 \oplus (df_p - I)(v_1), ..., 0 \oplus (df_p - I)(v_m))\\
&\quad = \alpha(v_1 \oplus 0, ..., v_m \oplus 0, 0 \oplus (I - df_p)(v_1), ..., 0 \oplus (I - df_p)(v_m)),
\end{aligned}$$

where in the second equality we used the fact that $I - df_p : T_pM \to T_pM$ is an isomorphism by the lemma above, hence v_i's are linear combinations of $(I - df_p)(v_j)$'s. Then the second equality follows from multilinearity and alternating property of α.

Since we have the identification $T_{(p,p)}(M \times M) \cong T_pM \oplus T_pM$ and the first m terms is a positive basis for T_pM, the final result depends on whether $I - df_p$ is orientation preserving or not. That is, we have that

$$\epsilon(p, \Gamma_f, \Delta) = \operatorname{sign}\det(I - df_p) = L_p(f).$$

This is what we wanted to prove. ☐

By the way, this section provides a full solution to Exercise 11.26 in [2], in case you haven't noticed (although the original problem has a typo, as the next remark suggests).

Remark 7.30. One would note that the local Lefschetz degree in our definition $L_x = \operatorname{sign}\det(I - df_x)$ is off by a factor $(-1)^m$ compared to literature like [7], where they define $L_x = \operatorname{sign}\det(df_x - I)$. This is because we want to stick to the definition where $L(f)$ is defined to be $\sum_j (-1)^j \operatorname{Tr}\big(f^*|_{H^j(M)}\big)$, and we would also like to have the relation $L(f) = \sum_{x \in \mathrm{Fix}(f)} L_x(f)$. These two requirements forces $L_x(f) = \operatorname{sign}\det(I - df_x)$, which disagrees with the convention like [7] when the dimension is odd. These different sign conventions can be quite annoying, and I think Bott and Tu missed this in their Exercise 11.26 on p.129 of [2]. In fact, in [12], it is calculated on p.123 that the local Lefschetz degree must be $L_x(f) = \operatorname{sign}\det(I - df_x)$ for Theorem 7.29 to hold. See also the discussion in this MSE post.

7.6 Euler Characteristics and Poincaré-Hopf Theorem

Now we introduce the notion of self-intersection numbers:

Definition 7.31. Let N^{2m} be a smooth, oriented, compact manifold. For any compact submanifold $S^m \subset N^{2m}$ of dimension m, we define the self-intersection number $I(S, S)$ to be

$$I(S, S) = \int_N \mathrm{PD}(S) \wedge \mathrm{PD}(S).$$

Remark 7.32. Since the wedge product is alternating, one has $dx^i \wedge dx^i = 0$ for all i. Therefore one may wonder why don't we immediately get $\mathrm{PD}(S) \wedge \mathrm{PD}(S) = 0$. This is not necessarily true. The reason is that $\mathrm{PD}(S)$ may not be a pure wedge, it can also be linear combination of them. For example, take $\mathrm{PD}(S) = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$, then $\mathrm{PD}(S) \wedge \mathrm{PD}(S) = 2dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$.

By Thom transversality theorem, as proved in [7] or [1], in such a situation, one can always find a smooth homotopy $F : S \times [0,1] \to M$, such that

- For all t, F_t is an embedding;
- $F_0 = i : S \to M$ is the inclusion map;
- $F_t \pitchfork S$, i.e., $F_t(S) \pitchfork S$.

We denote $S_t = F_t(S)$. Then one can show

Proposition 7.33. $\mathrm{PD}(S_t) = \mathrm{PD}(S)$ in $H^m(N)$.

Proof. We prove the following more general statement: Let $S^k, L^k \subset M^m$ be submanifolds such that there exists a smooth map $F : S \times [0,1] \to M$, with the property that $F_0(S) = S$ and $F_1(S) = L$. Then we claim that $\mathrm{PD}(S) = \mathrm{PD}(L)$.

To see this, take any representative ω of $H^k_c(M)$. Then $d\omega = 0$. Now by Stokes' theorem, one has

$$\int_S \omega - \int_L \omega = \pm \int_{\partial F(S \times [0,1])} \omega = \pm \int_{F(S \times [0,1])} d\omega = 0.$$

Thus we have

$$\int_S i_S^* \omega = \int_L i_L^* \omega$$

for any representative ω in $H^k_c(M)$. Then by the definition of Poincaré dual:

$$\int_S i_S^* \omega = \int_L i_L^* \omega = \int_M \omega \wedge \mathrm{PD}(L).$$

Since ω is arbitrary, we have $\mathrm{PD}(S) = \mathrm{PD}(L)$, as desired. □

We can now define

$$I(S,S) = I(S_t, S) = \sum_{p \in S_t \cap S} \epsilon(p, S_t, S)$$

for every $t \in [0,1]$. Hence our most important tool in intersection theory, Theorem 7.18, can be generalized to self-intersection number again.

Definition 7.34. Let M be a smooth manifold of finite type. Then the ***Euler characteristic*** $\chi(M)$ of M is defined to be the number

$$\chi(M) = \sum_j (-1)^j \operatorname{rank} H^j(M) = \sum_j (-1)^j \beta_j.$$

Then we have the following easy corollary:

Corollary 7.35. *Let M be a compact manifold. Then we have $I(\Delta, \Delta) = \chi(M)$.*

Proof. Take $f : M \to M$ to be id_M in the Lefschetz fixed point theorem, and note that

$$\begin{aligned} I(\Delta,\Delta) &:= \int_{M \times M} \mathrm{PD}(\Delta) \wedge \mathrm{PD}(\Delta) \\ &= L(\mathrm{id}_M) \\ &= \sum_j (-1)^j \operatorname{Tr}\big(\mathrm{id}_M : H^j(M) \to H^j(M)\big) \\ &= \sum_j (-1)^j \operatorname{rank} H^j(M). \end{aligned}$$

where the 2nd equality is by the first part of the Lefschetz fixed point theorem. This concludes the proof. ☐

We immediately get

Corollary 7.36. *If M^{2n+1} is a compact manifold of odd dimension, then $\chi(M) = 0$.*

Proof. Just note that $I(\Delta, \Delta) = (-1)^{(2n+1)^2} I(\Delta, \Delta) = -I(\Delta, \Delta)$ if we swap the two inputs by the definition of intersection number. Then $\chi(M) = I(\Delta, \Delta) = 0$ by the above corollary. ☐

Remark 7.37. The compactness assumption in the corollaries above cannot be removed, because in the proof of Corollary 7.35, we used Theorem 7.18, which requires Δ to be compact. As a counterexample, we note that $\chi(\mathbb{R}^1) = 1$ which is nonzero.

Remark 7.38. Textbooks like [7] uses $I(\Delta, \Delta)$ as the definition of the Euler characteristic $\chi(M)$. The discussion above gives the reason why this definition is compatible with the definition using cohomology, which is more common in algebraic topology literature.

We conclude this section by giving a proof of the Poincaré-Hopf theorem, which requires almost no additional work other than what we have already developed.

Let M be a compact manifold, X a smooth vector field on M. For sufficiently small t, let $\varphi^t : M \to M$ be the flow generated by X. Then it is not hard to see that $x \in M$ is a zero of the vector field X iff φ^t fixes x. Then from $d\varphi^t_x : T_xM \to T_xM$ we get a linear map

$$A_x = \frac{d}{dt}(d\varphi^t_x)\Big|_{t=0}.$$

Definition 7.39. Let M, X be defined as above. If $X(x) = 0$, and $\det(A_x) \neq 0$, then we call x a ***non-degenerate zero*** of X. We define the ***index*** $\iota(X, x)$ ***of*** X ***at*** x to be

$$\iota(X, x) = \operatorname{sign}(\det A_x).$$

In Chapter 3 of [7], it is showed that $\iota(X, x)$ is the degree of the associated Gauss map of X near the non-degenerate zero x. However, it is worth remarking that for non-degenerate zeros x of a vector field, $\iota(X, x)$ can only be ± 1, whereas in textbooks like [7], they consider a more general situation where a vector field has *isolated zeros*, in which case $\iota(X, x)$, which is defined to be the degree of the associated Gauss map, can take other values.

Now we prove the celebrated

Theorem 7.40 (Poincaré-Hopf Theorem)**.** *Let M be a compact oriented manifold and X a smooth vector field on M that has only non-degenerate zeros $x_1, ..., x_k$. Then*

$$\sum_{j=1}^{k} \iota(X, x_j) = \chi(M).$$

Proof. For $t > 0$ sufficiently small, we note that

$$\det\big(d\varphi^t_x - I\big) = \det\big(tA_x + O(t^2)\big) = t^m \det(A_x) + O(t^{m+1}).$$

Hence $L_x(\varphi^t) = \operatorname{sign}\det\big(I - d\varphi^t_x\big) = (-1)^m \operatorname{sign}(\det A_x)$. Therefore, we have

$$(-1)^m \sum_{j=1}^{k} \operatorname{sign}(\det A_{x_j}) = \sum_{x \in \operatorname{Fix}(\varphi^t)} L_x(\varphi^t) = L(\varphi^t)$$

by Theorem 7.29. But φ^t is obviously homotopic to id_M. Hence $L(\varphi^t) = L(\mathrm{id}_M) = \chi(M)$. This gives

$$(-1)^m \sum_{j=1}^{k} \mathrm{sign}(\det A_{x_j}) = \chi(M).$$

Now if m is even, then this is exactly the Poincaré-Hopf theorem. If m is odd, then we already know $\chi(M) = 0$ by Corollary 7.36. Hence multiplying both sides of the equation above by $(-1)^m$ we again get Poincaré-Hopf theorem as desired. □

Note that one can use Poincaré-Hopf theorem to give an alternative proof of the hairy ball theorem:

Proof of Hairy Ball Theorem. Since $\chi(S^{2n}) = 2$, we conclude that smooth vector fields on S^{2n} must have a zero. □

7.7 Euler Class

Suppose M is a connected, oriented, compact manifold. Let E be an oriented rank r vector bundle over M. Let $s : M \to E$ be any section of E, i.e., any smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$ where $\pi : E \to M$ is the projection map. Let $\tau(E) \in H^r_c(E) = H^r_{\mathrm{cv}}(E)$ denote the Thom class if E. Then we get an element $s^*(\tau(E)) \in H^r(M)$.

Proposition 7.41. The cohomology class $s^*(\tau(E))$ is independent of the choice of the section $s : M \to E$.

Proof. Let $s_0 : M \to E$ be the zero section. For any section $s : M \to E$ we have a homotopy from s_0 to s defined by

$$\begin{aligned} H : M \times [0,1] &\to E \\ (p,t) &\mapsto (p, ts(p)). \end{aligned}$$

Therefore, $s^*(\tau(E)) = s_0^*(\tau(E))$. □

Definition 7.42. Let M be a connected, oriented, compact manifold, and E an oriented rank r vector bundle over M. Then define the ***Euler class*** $e(E)$ of E to be

$$e(E) := s^*(\tau(E)) \in H^r(M).$$

Lemma 7.43. *Let E_1 an oriented rank r vector bundle over M_1, and E_2 an oriented rank r vector bundle over M_2, where both M_1, M_2 are oriented, compact manifolds. Suppose there exists an orientation preserving bundle isomorphism $\psi : E_1 \to E_2$ that descends to an orientation preserving diffeomorphism $\psi_0 : M_1 \to M_2$, and sections $s_1 : M_1 \to E_1, s_2 : M_2 \to E_2$, such that the following diagram commutes:*

$$\begin{tikzcd} E_1 \arrow[r, "\psi"] & E_2 \\ M_1 \arrow[u, "s_1"] \arrow[r, "\psi_0"'] & M_2 \arrow[u, "s_2"'] \end{tikzcd}$$

Then we have

$$\int_{M_1} e(E_1) = \int_{M_2} e(E_2).$$

Proof. We have already shown that the Euler class is independent of the choice of sections. The commutative diagram implies $s_1^*\psi^* = \psi_0^* s_2^*$. Hence we have

$$\psi_0^* e(E_2) = \psi_0^* s_2^* \tau(E_2) = s_1^* \psi^* \tau(E_2).$$

But we claim that $\psi^*\tau(E_2) = \tau(E_1)$. This is because the Thom class of E_1 is characterized by the property that fiber integration map applied to it gives unit: $\pi_*\tau(E_1) = 1$ by definition. But since ψ is an orientation preserving bundle isomorphism, it preserves fiber integration. Or in other words

$$\int_{(E_1)_p} \psi^*\tau(E_2) = \int_{\psi((E_1)_p)} \tau(E_2) = \int_{(E_2)_{\psi_0(p)}} \tau(E_2) = 1$$

for any $p \in M$. Hence indeed we have $\psi^*\tau(E_2) = \tau(E_1)$. Therefore, we have

$$\psi_0^* e(E_2) = s_1^*\tau(E_1) = e(E_1).$$

This implies

$$\int_{M_1} e(E_1) = \int_{M_1} \psi_0^* e(E_2) = \int_{\psi_0(M_1)} e(E_2) = \int_{M_2} e(E_2).$$

This completes the proof. $\square$

Now we study another version of Poincaré-Hopf theorem, which explains the mysterious name Euler class:

Theorem 7.44. *Let M be a compact, oriented manifold. Then*

$$\int_M e(TM) = \chi(M).$$

Proof. Let $\Delta \subset M \times M$ be the diagonal of M, and $N\Delta \subset M \times M$ the normal bundle of Δ in $M \times M$. Let $\mathcal{N}(\Delta)$ be a tubular neighborhood of Δ. Then consider the following commutative diagram

$$\begin{tikzcd}
TM \arrow[r, "\psi_1"] & T\Delta \arrow[r, "\psi_2"] & N\Delta \arrow[r, "\psi_3"] & \mathcal{N}(\Delta) \\
M \arrow[u, "s_1"] \arrow[r] & \Delta \arrow[u, "s_2"] \arrow[r, "\mathrm{id}_\Delta"'] & \Delta \arrow[u, "s_3"'] \arrow[r, "\mathrm{id}_\Delta"'] & \Delta \arrow[u, "s_4"']
\end{tikzcd}$$

where $\psi_1(x, v) = (x, x, v, v)$, $\psi_2(x, x, v, v) = (x, x, -v, v)$, and $\psi_3 : N(\Delta) \to \mathcal{N}(\Delta)$ is an orientation preserving bundle isomorphism, whose existence is guaranteed by the tubular neighborhood theorem. It is easy to verify that each of the blocks in the diagram above satisfies the conditions in the previous lemma. Hence by repeatedly applying the lemma above, we get

$$\int_M e(TM) = \int_\Delta e(T\Delta) = \int_\Delta e(N\Delta) = \int_\Delta e(\mathcal{N}(\Delta)).$$

Now, $s_4 : \Delta \to \mathcal{N}(\Delta)$ is homotopic to the zero section (inclusion), and $\tau(\mathcal{N}(\Delta))$ may also be taken as the Poincaré dual $\mathrm{PD}(\Delta)$ of Δ in $\mathcal{N}(\Delta)$ by Corollary 6.95. Therefore, by the definition of the Poincaré dual, we get that

$$\begin{aligned}\int_\Delta e(\mathcal{N}(\Delta)) = \int_\Delta i_\Delta^*\tau(\mathcal{N}(\Delta)) = \int_\Delta i_\Delta^* \mathrm{PD}(\Delta) &= \int_{\mathcal{N}(\Delta)} \mathrm{PD}(\Delta) \wedge \mathrm{PD}(\Delta) \\ &= \chi(\mathcal{N}(\Delta)) = \chi(M),\end{aligned}$$

where the last equality is because $\mathcal{N}(\Delta)$ deformation retracts onto Δ, which is diffeomorphic to M, and the Euler characteristic is a homotopy invariant.

This concludes the proof. $\square$

Next, we interpret Euler class as obstruction class to finding nowhere vanishing sections of E on our manifold M:

Theorem 7.45. *Let E be an oriented rank n vector bundle on a compact oriented connected n-dimensional manifold M. If there is a nowhere vanishing section $\sigma : M \to E$ of E, then $e(E)$ vanishes in $H^n(M)$.*

Proof. Let $\sigma : M \to E$ be such a nowhere vanishing section. Let $\pi : E \to M$ be the projection. Since $\pi \circ \sigma = \mathrm{id}_M$, we conclude that σ is injective, in fact it is an embedding onto its image $\sigma : M \to \sigma(M)$.

Now let $\sigma_0 : M \to E$ be the zero section. Note that $\dim \sigma(M) = \dim \sigma_0(M) = n$, which means they have complementary dimension in E (whose dimension is $2n$). Since σ is nowhere vanishing, we must have that $I(\sigma(M), \sigma_0(M)) = 0$. Using Theorem 7.18 and Theorem 6.95, we have that

$$0 = I(\sigma(M), \sigma_0(M)) = \int_E \mathrm{PD}(\sigma(M)) \wedge \tau(E) = \pm \int_{\sigma(M)} \tau(E),$$

where the last equality is by the definition of Poincaré dual. Since $\sigma : M \to \sigma(M)$ is an embedding, we have that

$$0 = \int_{\sigma(M)} \tau(E) = \int_M \sigma^* \tau(E) = \int_M e(E).$$

Finally, since M is orientable, integration gives an isomorphism $H^n(M) \to \mathbb{R}$. Therefore, the cohomology class of $e(M)$ must vanish in $H^n(M)$. $\square$

Remark 7.46. In fact, the assumption that $\operatorname{rank} E = n$ in the statement of the theorem above is redundant. However, our proof using intersection theorem breaks down if this assumption is removed, because we need this assumption to deduce that $\sigma(M), \sigma_0(M)$ have complementary dimension in E, so that we can use intersection theory.

This gives another proof of the Hairy Ball theorem:

Proof of Hairy Ball Theorem (Again). We know $e(TS^{2n})$ is the obstruction to the existence of nowhere vanishing vector field on S^{2n}. Since Theorem 7.44 gives

$$2 = \chi(S^{2n}) = \int_{S^{2n}} e(TS^{2n}),$$

which is nonzero, the theorem follows from Theorem 7.45. $\square$

To conclude this section, we will show that if $s : M \to E$ is any section that intersects the zero section of E transversely, then the Euler class $e(E)$ is the Poincaré dual $\mathrm{PD}(Z(s))$ of the zeros of the section s. To prove the desired statement, we first need a lemma.

Lemma 7.47. *Let $\pi : E \to M$ be any vector bundle and Z the zero locus of a transversal section. Then Z is a submanifold of M and its normal bundle in M is $N_{Z/M} \cong E|_Z$.*

Proof. Let $s : M \to E$ be a generic section that intersects with the zero section $S_0 \cong M$ transversely. We denote S the image of $s : M \to E$. Then $M \cong S$. Since S is transverse to S_0, we know $S \cap S_0$ is a submanifold of S, hence also a submanifold of M via the identification $S \cong M$.

Since E is locally trivial, its tangent bundle on S_0 has the canonical decomposition $T_E|_{S_0} = E|_{S_0} \oplus T_{S_0}$ (See Figure 7.4). By transversality of $S \cap S_0$, we have $T_S + T_{S_0} = T_E = E \oplus T_{S_0}$. Therefore, the projection $T_S \to E$ is surjective, and the kernel is precisely $T_{S_0} \cap T_S = T_{S \cap S_0}$. Using the identification $Z \cong S \cap S_0$, we have an exact sequence

$$0 \to T_Z \to T_S|_Z \to E|_Z \to 0.$$

Hence we conclude that $E|_Z \cong N_{Z/S} \cong N_{Z/M}$ because $S \cong M$. $\square$

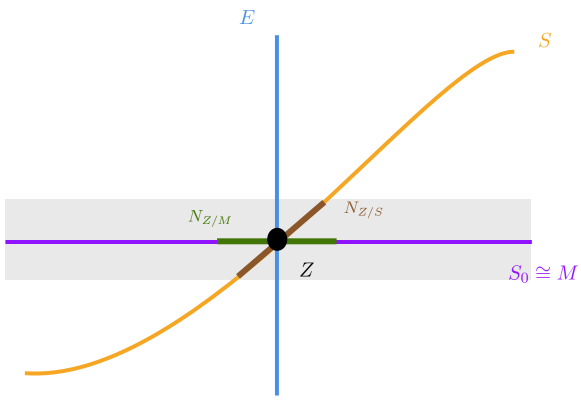


Figure 7.4: An illustration. First, note that when restricted to S_0 we have a decomposition $T_E|_{S_0} = E|_{S_0} \oplus T_{S_0}$. Here, Z is a single point, and $N_{Z/M} \cong N_{Z/S} \cong E$. The support of the Thom class is contained in a thin strip near $M \cong S_0$, which is shaded in gray in the figure.

In Lemma 7.47, if E and M are both oriented, then the zero locu Z of a transversal section is naturally an oriented manifold, oriented in such a way that $E|_Z \oplus T_Z = T_M|_Z$. Now we are ready to prove

Theorem 7.48. *Let $\pi : E \to M$ be a vector bundle of rank r over a smooth manifold M of dimension n. Let $s : M \to E$ be a section that intersects $M = \sigma_0(M)$ transversely. Then*

$$e(E) = \mathrm{PD}(Z(s)) \in H^r(M).$$

Proof. We will identify M with the image S_0 of the zero section. By Lemma 7.47, we know $Z = Z(s)$ is a closed oriented submanifold and $N_{Z/M} = E|_Z$. Moreover, the exact sequence in the proof of Proposition 7.47 shows that we also have $N_{Z/S} = E|_Z$, which is not surprising since S is diffeomorphic to M. The normal bundles $N_{Z/M}$ and $N_{Z/S}$ will be identified with the tubular neighborhoods of Z in M and in S respectively, as Figure 7.4 suggests.

Let us choose the Thom class $\tau(E)$ to have support so close to the zero section (we can do this by Corollary 6.95 and Corollary 6.96), so that $\tau(E)$ restricted to the tubular neighborhood $N_{Z/S}$ in S has compact support in the vertical direction. We now show that $e(E) = s^*\tau(E)$ is the Thom class of the tubular neighborhood $N_{Z/M}$ in M.

Let E_p, S_p, M_p be the fibers of $E|_Z \cong N_{Z/S} \cong N_{Z/M}$ respectively above the point $p \in Z$. Because $\tau(E)$ has compact support in S_p, in fact $s^*\tau(E)$ has compact support in

M_p. It suffices to check that fiber integration along M_p of $e(E)$ is 1. First, we note that $s : M_p \to S_p$ is an orientation preserving diffeomorphism. Therefore, we have

$$\int_{M_p} e(E) = \int_{M_p} s^*\tau(E) = \int_{S_p} \tau(E).$$

Next, we note that E_p is homotopic to S_p modulo the region in E where $\tau(E)$ is zero (see Figure 7.4). Therefore,

$$\int_{S_p} \tau(E) = \int_{E_p} \tau(E) = \pi_*\tau(E)|_p = 1.$$

Hence, we have shown that $e(E) = s^*\tau(E)$ is the Thom class of $N_{Z/M}$. By Theorem 6.94, this is precisely the Poincaré dual $\mathrm{PD}(Z(s))$ to Z in E. $\square$

7.8 Problems

1. (Hopf invariant) Given a smooth map $f : S^{2n-1} \to S^n$, we pick a volume form $\alpha \in \Omega^n(S^n)$ such that $\int_{S^n} \alpha = 1$. Then $f^*\alpha \in \Omega^n(S^{2n-1})$ is closed. Indeed,

$$d(f^*\alpha) = f^*(d\alpha) = 0,$$

since α is closed in S^n. As $H^n(S^{2n-1}) = 0$ ($n > 1$ by assumption), there exists $\omega \in \Omega^{n-1}(S^{2n-1})$ such that $d\omega = f^*\alpha$. Then we define the ***Hopf invariant*** $H(f)$ of f to be

$$H(f) := \int_{S^{2n-1}} \omega \wedge d\omega.$$

Show that:

   (a) $H(f)$ is well-defined. That is, the definition of the Hopf invariant $H(f)$ is independent of the choices of α and of ω.

   (b) For odd n, the Hopf invariant is 0.

   (c) Homotopic maps have the same Hopf invariant.

2. Let $f : S^{2n-1} \to S^n$ be a smooth map, and let $H(f)$ be its Hopf-invariant defined as in the previous problem.

   (a) Let $g : S^{2n-1} \to S^{2n-1}$ be a smooth map. Show that $H(fg) = (\deg g)H(f)$.

   (b) Let $g : S^n \to S^n$ be a smooth map. Show that $H(gf) = (\deg g)^2 H(f)$.

3. Show that a compact manifold M admits a nowhere vanishing vector field if and only if $\chi(M) = 0$.

4. Show that the only compact surfaces that admit a nowhere vanishing vector field are the torus and the Klein bottle.

Figure 7.5: René Thom (1923-2002).

René Thom (1923-2002) won the Fields medal in 1958 for his contribution to topology. In an interview, Thom discussed a question with two anthropologists. When talking about why ancient people preserved fire, one anthropologist said it was because fire could provide warmth; another said it was because fire could be used to roast fresh meat. But Thom said: because when night falls, the flickering firelight, dazzling and graceful, is the most beautiful of all.

Beauty is the eternal pursuit of mathematicians.

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