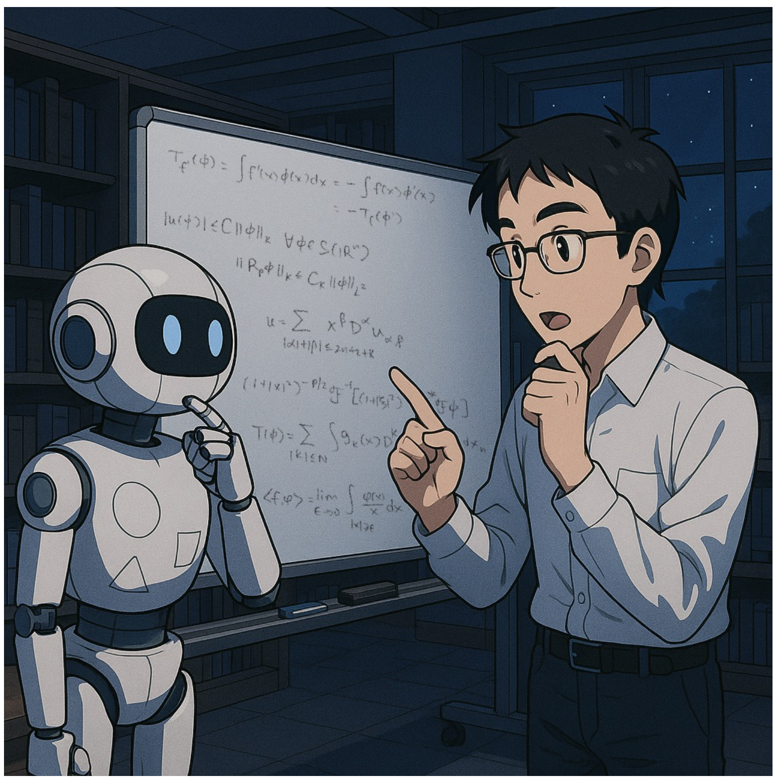
# SOLUTIONS TO SELECTED PROBLEMS ON FUNCTIONAL ANALYSIS

WRITTEN BY

TIANYI WANG



FROM

REED AND SIMON VOL I

# Preface

This is a collection of problems on functional analysis that I have done during the spring 2022 quarter. At UCSB we have a course on functional analysis (course number math 201C), which covers Hilbert spaces, Banach spaces, $L^p$ theory, locally convex spaces and distribution theory (really basic), and bounded operators. The textbook we use is Reed and Simon, *Modern Methods of Mathematical Physics I: Functional Analysis*, and we covered chapter 2,3,5,6.

From my perspective, Reed and Simon is not the best textbook for functional analysis, although it was highly recommended to me by one of my closest professors. The problem of this book is that, in my opinion, too many results are left unproven and are left as exercises. And for those theorem that are proven, they are not written out in details. Instead, the authors assume that many arguments are just obvious to the reader, while in reality it may not be. Granted, the authors are top in their field (and I still have deep respect for them), but sometimes it is much more difficult to appreciate how hard mathematics can be for beginners than the mathematics itself. Therefore, if you are frustrated because you cannot understand a proof in this book, I've been there. My suggestion would be: take a deep breath, go to office hours and ask questions, or discuss with friends, or simply read another book. I highly recommend the following two textbooks on this subject:

- Richard Bass, *Real Analysis for Graduate Students.*
- Sheldon Axler, *Measure, Integration & Real analysis.*

"One who reads the textbook and cannot do problems have learned nothing", said the great physicists J.J.Sakurai. Therefore, I tried to do as many problems as I could, partially for preparation of the final exam. This file is a collection of solutions to all the problems that I have done from Reed and Simon. I also included all the homework problems, which are also from Reed and Simon. Homework problems are indicated by *. Please forgive my arbitrary use of $\varepsilon$ and $\epsilon$. And please do email me at tianyiwang@ucsb.edu if you find any mistakes.

Finally, I want to give special thanks to my classmates Nick Sun and Xingzhe Li. Nick and Xingzhe helped solving a lot of difficult problems. We had very inspirational discussions everyday, and we also had fun preparing finals for real analysis (201) sequence together for the whole year. Thank you for being such amazing friends.

Tianyi Wang
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# Chapter 1: Preliminaries

**Problem 1.4*.** Let $\{x_n\}$ be a Cauchy sequence of metric space $(X, d)$. Suppose that for some subsequence $\{x_{n_k}\}$, $x_{n_k}$ converges to $x_0$ as $k \to \infty$. We want to show that $x_n \to x_0$ too.

Fix $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, there exists $N > 0$ such that for all $m, n \geq N$, we have $d(x_m, x_n) < \varepsilon/2$. Also, since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ that converges to $x_0$, this means that there exists $M_1 > 0$, such that for all $k \geq M_1$, we have $d|x_{n_k}, x_0) < \varepsilon/2$. Also, by Cauchy condition, there exists $M_2 > 0$ such that for all $k \geq M_2$, we have $n_k \geq N$, so that $d(x_m, x_{n_k}) < \varepsilon/2$. Choose $x_{n_k}$ such that $k \geq \max(M_1, M_2)$. Then we have that

$$d(x_{n_k}, x_0) < \frac{\varepsilon}{2} \quad \text{and} \quad n_k \geq N.$$

So, for all $m \geq N$, we have that

$$d(x_m - x_0) \leq d(x_m - x_{n_k}) + d(x_{n_k} - x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the claim.

**Problem 1.28*.** We want to construct a sequence $f_n(x)$ of bounded functions on $[0, 1]$ converging to zero in $L^1$, but $f_n$ converges at no point in $[0, 1]$.Take

$$\begin{aligned}
f_1 &= \chi_{[0,1]} \\
f_2 &= \chi_{[0,\frac{1}{2}]} \\
f_3 &= \chi_{[\frac{1}{2},1]} \\
f_4 &= \chi_{[0,\frac{1}{4}]} \\
f_5 &= \chi_{[\frac{1}{4},\frac{1}{2}]} \\
f_6 &= \chi_{[\frac{1}{2},\frac{3}{4}]} \\
f_7 &= \chi_{[\frac{3}{4},1]} \\
f_8 &= \chi_{[0,\frac{1}{8}]} \\
&\vdots
\end{aligned}$$

That is, we first divide $[0, 1]$ into $2^0 = 1$ pieces, and let $f_1$. be the characteristic function on that piece. We then divide $[0, 1]$ into $2^1$ equal pieces, and let the next $2^1 = 2$ functions to be characteristic function on each piece; next we divide $[0, 1]$ into $2^2$ equal pieces, and let the next $2^2 = 4$ functions to be characteristic function on each of the $2^2$ pieces; after that we divide $[0, 1]$ to $2^3$ equal pieces, and let the next $2^3 = 8$ functions to be characteristic function on each of the $2^3$ pieces...

Clearly $|f_n| \leq 1$. So this is a bounded sequence. It is also clear that

$$\int_{[0,1]} |f_1| = 1$$
$$\int_{[0,1]} |f_2| = 1/2$$
$$\int_{[0,1]} |f_3| = 1/2$$
$$\int_{[0,1]} |f_4| = 1/4$$
$$\int_{[0,1]} |f_5| = 1/4$$
$$\int_{[0,1]} |f_6| = 1/4$$
$$\int_{[0,1]} |f_7| = 1/4$$
$$\int_{[0,1]} |f_8| = 1/8$$
$$\vdots$$

So that $\|f_n\|_1 = \int_{[0,1]} |f_n| \to 0$. But also, fix $N > 0$ large for every $x \in [0,1]$, we have $f_n(x) = 0$ and $f_n(x) = 1$ infinitely many times for $n \geq N$. Thus the sequence $\{f_n\}$ is nowhere convergent.

# Chapter 2: Hilbert Spaces

**Problem 2.2.**

(a) To show that simple functions are dense in $L^2[0,1]$, let $f \in L^2[0,1]$. Choose natural number $M$ such that

$$\int_{A_M^c} |f(x)|^2 dx \leq \frac{1}{N}$$

where

$$A_M = \{x \in [0,1] : |\operatorname{Re} f| \leq M, |\operatorname{Im} f| \leq M\}.$$

This can be done since the sequence $\int_{A_M} |f|^2$ increases to $\int_{[0,1]} |f|^2 < \infty$ as $M$ increases. Then denote

$$C_{kl} = \left\{x \in [0,1] : \frac{k}{N} < \operatorname{Re} f \leq \frac{k+1}{N}, \frac{l}{N} < \operatorname{Im} f \leq \frac{l+1}{N}\right\}$$

where $k, l = -MN, -MN+1, ..., MN-1$. Note that $C_{kl}$ are measurable and disjoint, and $A_m = \bigcup_{k,l} C_{kl}$. Now let our simple function be

$$S_N(x) = \sum_{k,l=-MN}^{MN-1} \left(\frac{k}{N} + i\frac{l}{N}\right) \chi_{C_{kl}}(x).$$

Then on $A_M^c$, we have $S_N(x) \equiv 0$. On $A_M$, we have

$$|S_N(x) - f(x)|^2 \leq \left|\frac{1}{N}(1+i)\right|^2 = \frac{2}{N^2}.$$

Therefore we have

$$\|f - S_N\|_{L^2[0,1]}^2 = \int_{A_M} |f - S_N|^2 + \int_{A_M^c} |f|^2 \leq \frac{2}{N^2} + \frac{1}{N}$$

which goes to zero as $N$ goes to infinity. This proves the claim.

(b) Note that a **step function** is a function of the form $\varphi(x) = \sum_{i=1}^{N} c_i \chi_{(a_i, a_{i+1})}$ where $\{a_i\}_{i=1}^{N+1}$ form a partition of $[0,1]$. First, for simplicity, suppose the simple function is of the form $\chi_B$. Then by the regularity of the Lebesgue measure, we have

$$\mu(B) = \inf\left\{\mu\left(\bigcup_{l=1}^{n} (a_l, b_l)\right) : (a_l, b_l) \text{ disjoint and } B \subset \bigcup_{l=1}^{n} (a_l, b_l)\right\}.$$

Thus we can choose $\bigcup_{l=1}^{n}(a_l, b_l)$ such that $B \subset \bigcup_{l=1}^{n}(a_l, b_l)$ and

$$\mu(B) - \varepsilon \leq \mu\left(\bigcup_{l=1}^{n}(a_l, b_l)\right) \leq \mu(B) + \varepsilon.$$

Then $\chi_{\bigcup_{l=1}^{n}(a_l,b_l)}$ is a step function where the partition is given by the points $a_l, b_l$. Now

$$\begin{aligned}
\left\|\chi_B - \chi_{\bigcup_{l=1}^{n}(a_l,b_l)}\right\|_{L^2[0,1]}^2 &= \int_0^1 |\chi_B - \chi_{\bigcup_{l=1}^{n}(a_l,b_l)}|^2 d\mu \\
&= \int_{\bigcup_{l=1}^{n}(a_l,b_l)-B} |\chi_B - \chi_{\bigcup_{l=1}^{n}(a_l,b_l)}|^2 d\mu \\
&\leq 4\mu\left(B - \bigcup_{l=1}^{n}(a_l, b_l)\right) \leq 4\varepsilon.
\end{aligned}$$

Hence we see that in this case we can approximate $\chi_B$ arbitrarily closely by a step function. If now our simple function is of the form

$$S(x) = \sum_{i=1}^{n} c_i \chi_{A_i}$$

where each $A_i$s are disjoint and measurable, then for each $\chi_{A_i}$ choose $\chi_{\bigcup_{l=1}^{N_i}(a_l^i,b_l^i)}$ as above, and consider the function

$$\varphi(x) = \sum_{i=1}^{n} c_i \chi_{\bigcup_{l=1}^{N_i}(a_l^i,b_l^i)}(x).$$

We can turn $\varphi$ into a step function whose partition is naturally given by $\{a_l^i, b_l^i\}$ for all the $l$ and $i$. By triangle inequality, we have that

$$\|S - \varphi\|_{L^2[0,1]} \leq \sum_{i=1}^{n} \left\|c_i\chi_{A_i} - c_i\chi_{\bigcup_{l=1}^{N_i}(a_l^i,b_l^i)}\right\|_{L^2[0,1]} \leq 2\sqrt{\varepsilon}\left(\sum_{i=1}^{n} |c_i|\right).$$

Since $\varepsilon$ is arbitrary, we are done.

(c) Let our step function be

$$\varphi(x) = \sum_{i=1}^{n} c_i \chi_{(a_i,a_{i+1})}(x).$$

Then construct a continuous function $f$ as follows: Require $f$ to take the same value as $\varphi$ on $(a_i+\varepsilon, a_{i+1}-\varepsilon)$, and extend $f$ continuously to the entire $[0,1]$ and we can do it in such a way that on the interval $[a_i-\varepsilon, a_i+\varepsilon]$, we have $|c_{i-1}| \leq |f| \leq |c_i|$ if $|c_{i-1}| \leq |c_i|$ WLOG. Let $M = \max_{1\leq i\leq n} |c_{i+1}| - |c_i|$ and $K = \bigcup_{i=1}^n [a_i-\varepsilon, a_i+\varepsilon]$. Then

$$\|\varphi - f\|^2_{L^2[0,1]} = \int_K |\varphi - f|^2 d\mu \leq M^2\mu(K) \leq 2\varepsilon M^2.$$

Since $\varepsilon$ is arbitrary, we see that a step function can be arbitrarily approximated by continuous functions. The rest of the claim follows straightforwardly.

**Problem 2.4.**

(a) We want to show that in every inner product space $X$, the following identity (parallelogram law) holds:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \forall x, y \in X.$$

This follows from computation:

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= (x+y, x+y) + (x-y, x-y) \\ &= (x,x) + (x,y) + (y,x) + (y,y) + (x,x) - (x,y) - (y,x) + (y,y) \\ &= 2(x,x) + 2(y,y) = \|x\|^2 + \|y\|^2.\end{aligned}$$

(b) We want to show that that the parallelogram law characterizes inner product space. Namely, suppose that the parallelogram law holds on a normed linear space $X$, then one can define an inner product $(\cdot,\cdot)$ on $X$ in a way that $\|x\| = (x,x)^{1/2}$ for all $x \in X$. Hint: Consider the polarization identity

$$(x,y) = \frac{1}{4}\big[(\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)\big].$$

Suppose that the norm satisfies the parallelogram identity. We define

$$(x,y) = \frac{1}{4}\big[(\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)\big].$$

We need to check that this indeed defines an inner product.

- **conjugate symmetry.** To check that this inner product satisfies conjugate symmetry, we have that

$$\begin{aligned}\overline{(x,y)} &= \frac{1}{4}\left[\overline{(\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)}\right] \\ &= \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 - i\|x-iy\| + i\|x+iy\|^2\big) \\ &= \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 - i\|(-i)(ix+y)\|^2 + i\|i(-ix+y)\|^2\big) \\ &= \frac{1}{4}\big(\|y+x\|^2 - \|y-x\|^2 - i\|y+ix\|^2 + i\|y-ix\|^2\big) \\ &= (y,x).\end{aligned}$$

- **linearity in the first argument.** We want to show that $(x+y,z) = (x,z)+(y,z)$. We first consider the real part. We have that

$$\begin{aligned}\|x+y+z\|^2 &= 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2 \\ &= 2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2\end{aligned}$$

where the second equality follows by exchanging $x$ and $y$. Since $A = B$ and $A = C$ implies $A = (B+C)/2$, we have that

$$\|x+y+z\|^2 = \|x\|^2+\|y\|^2+\|x+z\|^2+\|y+z\|^2-\frac{1}{2}\|x-y+z\|^2-\frac{1}{2}\|y-x+z\|^2.$$

Replacing $z$ by $-z$ in the last equation gives that

$$\|x+y-z\|^2 = \|x\|^2+\|y\|^2+\|x-z\|^2+\|y-z\|^2-\frac{1}{2}\|x-y-z\|^2-\frac{1}{2}\|y-x-z\|^2.$$

Therefore, we have that

$$\begin{aligned}\mathrm{Re}(x+y,z) &= \frac{1}{4}[\|x+y+z\|^2 - \|x-y-z\|^2] \\ &= \frac{1}{4}\left(\|x+z\|^2 - \|x-z\|^2\right) + \frac{1}{4}\left(\|y+z\|^2 - \|y-z\|^2\right) \\ &= \mathrm{Re}[(x,z)+(y,z)]\end{aligned}$$

where in the last step the negative terms cancel out because $\|-w\| = \|w\|$. Now replacing $z$ by $iz$ we get

$$\mathrm{Im}(x+y,z) = \mathrm{Im}[(x,z)+(y,z)].$$

Therefore, this proves the linearity in the first slot.

- **Positive definiteness.** Through some calculation it is easy to see that

$$(x, x) = \frac{1}{4}[\|x + x\|^2 - \|x - x\|^2 - i\|x + ix\|^2 + i\|x - ix\|^2] = \|x\|^2 \geq 0$$

  and $(x, x) = 0$ if and only if $x = 0$ by definition of the norm. Notice that we used the fact that $\|1 + i\| = \|1 - i\|$ as they are conjugates.

- **Homogeneity.** We would like to prove that $(\lambda x, y) = \lambda(x, y)$ for all $\lambda \in \mathbb{C}$. Observe that $(ix, y) = i(x, y)$. Therefore it suffices to prove the case where $\lambda \in \mathbb{R}$.

  Clearly the statement holds when $\lambda = -1$ and by additivity and induction for all $\lambda \in \mathbb{N}$, hence for all $\lambda \in \mathbb{Z}$. Suppose $\lambda = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0$. Let $x' = \frac{x}{q}$. Then

$$q(\lambda x, y) = q(px', y) = p(qx', y) = p(x, y).$$

  So dividing by $q$ gives that

$$(\lambda x, y) = \lambda(x, y) \text{ for all } \lambda \in \mathbb{Q}.$$

  To extend this result to the reals, we show that for fixed $x, y \in X$ the map $t \mapsto \frac{1}{t}(tx, y)$ defined on $\mathbb{R}\backslash\{0\}$ is continuous. If this is the case, then since we know for all $t \in \mathbb{Q}$, this function $\eta(t) = (x, y)$ is a constant, and by the continuity and the fact that rationals are dense over reals, we get that $\eta(t) = \frac{1}{t}(tx, y) = (x, y)$ for all $t \in \mathbb{R}\backslash\{0\}$, i.e. $(tx, y) = t(x, y)$ for all $t \in \mathbb{R}\backslash\{0\}$. And since the equality clearly holds when $t = 0$, we are done.

  Therefore, it remains to prove the continuity of the map $t \mapsto \frac{1}{t}(tx, y)$. The map $t \mapsto \frac{1}{t}$ is continuous, so it remains to show that $t \mapsto (tx, y)$ is continuous when $x, y$ is fixed. Note that this map is a composition

$$t \mapsto tx \mapsto (tx, y).$$

  The first map $t \mapsto tx$ is continuous with respect to the norm on $X$, since $x$ is fixed. So our goal now is to show that for fixed $y$, the map

$$x \mapsto (x, y)$$

  is continuous. We will show that

$$|(f, g)| \leq \sqrt{2}\|f\|\|g\|.$$

If we look at the real part, we have that

$$\begin{aligned}
\mathrm{Re}(f,g) &= \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right) \\
&= \frac{1}{4}\left(2\|f+g\|^2 - \|f+g\|^2 - \|f-g\|^2\right) \\
&= \frac{1}{4}\left(2\|f+g\|^2 - 2\|f\|^2 - 2\|g\|^2\right) \\
&\leq \frac{1}{4}\left(2(\|f\|+\|g\|)^2 - 2\|f\|^2 - 2\|g\|^2\right) \\
&= \frac{1}{4}\left(2\|f\|^2 + 4\|f\|\|g\| + 2\|g\|^2 - 2\|f\|^2 - 2\|g\|^2\right) \\
&= \|f\|\|g\|
\end{aligned}$$

Replacing $g$ with $-ig$ and noting that $\|-ig\| = \|g\|$, we also have

$$\mathrm{Im}(f,g) \leq \|f\|\|g\|.$$

Therefore,

$$|(f,g)| = \sqrt{(\mathrm{Re}(f,g))^2 + (\mathrm{Im}(f,g))^2} \leq \sqrt{2}\|f\|\|g\|.$$

This proves the claim. And this already proves that $x \mapsto (x,y)$ is continuous for fixed $y$, since by choosing $\|x-x'\| < \frac{\varepsilon}{\sqrt{2}\|y\|}$, we will have

$$|(x,y)-(x',y')| = |(x-x',y)| \leq \sqrt{2}\|y\|\|x-x'\| < \varepsilon.$$

Therefore, the composition

$$t \mapsto tx \mapsto (tx,y)$$

is continuous for fixed $x, y$. Then since product of continuous maps is continuous, the map

$$\eta(t) = \frac{1}{t}(tx,y)$$

is continous, as desired.

**Problem 2.5.** If we let $c_n = (x_n, x)$, then we have that

$$\left(x_k, x - \sum_{n=1}^{N} c_n x_n\right) = 0 \text{ for } k = 1, ..., N.$$

Then for any $c_n'$, $1 \le n \le N$, we have

$$\begin{aligned}\left\| x - \sum_{n=1}^{N} c_n' x_n \right\|^2 &= \left\| \sum_{n=1}^{N} (c_n - c_n') x_n + x - \sum_{n=1}^{N} c_n x_n \right\|^2 \\ &= \sum_{n=1}^{N} |c_n - c_n'|^2 + \left\| x - \sum_{n=1}^{N} c_n x_n \right\|^2\end{aligned}$$

which is clearly minimized when $c_n' = c_n$ for $1 \le n \le N$.

**Problem 2.6.** $\mathscr{M}^\perp$ is a linear subspace follows straightforwardly from definition. To show that $\mathscr{M}^\perp$ is closed: Suppose $z \notin \mathscr{M}^\perp$, then there exists some $x \in \mathscr{M}$ such that $(x, z) = 1$. This is always possible since $\mathscr{M}$ is a linear subspace of $\mathscr{H}$. Then take $z'$ such that

$$\|z - z'\| < \frac{1}{2\|x\|}$$

we get that

$$(x, z') = (x, z) - (x, z - z') \ge \frac{1}{2}$$

by Schwarz inequality. Thus $(\mathscr{M}^\perp)^c$ is open, hence $\mathscr{M}^\perp$ is closed. Now we show that

$$\overline{\mathscr{M}} = (\mathscr{M}^\perp)^\perp.$$

First notice that we clearly have $\mathscr{M} \subset (\mathscr{M}^\perp)^\perp$. Thus $\overline{\mathscr{M}} \subset (\mathscr{M}^\perp)^\perp$. To prove the reverse inclusion, we argue by contradiction. Let $y \in (\mathscr{M}^\perp)^\perp$. Since $\overline{M}$ is a closed linear subspace of $\mathscr{H}$, by the lemma and projection theorem on pg 42 of the textbook, we can choose $z \in \overline{M}$ that is closest to $z$. Assume, for the sake of contradiction, that $y \neq z$. Then again, by the projection theorem,

$$y - z \in (\overline{\mathscr{M}})^\perp \subset \mathscr{M}^\perp.$$

The first inclusion follows from the proof of the projection theorem on pg 42. But then

$$(y, y - z) = (y - z, y - z) = \|y - z\|^2 > 0$$

which shows that $y \notin (\mathscr{M}^\perp)^\perp$, a contradiction to our assumption, as desired.

**Problem 2.9*.** Let $\mathscr{M}$ be a subspace of a Hilbert space $\mathscr{H}$, and $f : \mathscr{M} \to \mathbb{C}$ a bounded linear functional on $\mathscr{M}$ with bound $C$. We want to prove that there is a

**unique** extension of $f$ to a bounded linear functional on $\mathscr{H}$ with the same bound $C$.

The Hahn-Banach theorem guarantees the existence of such an extension $F$ of $f$ on the entire Hilbert space $\mathscr{H}$ with the same bound $C$. It remains to show that this extension is unique. Since $F$ need to agree with $f$ on $\mathscr{M}$, it is unique on $\overline{\mathscr{M}}$. This is because $\mathscr{M}$ is dense in $\overline{M}$ and $F$ is a bounded linear operator so it is continuous, and the value of a continuous linear operator is determined by its value on a dense subset. But since $\overline{\mathscr{M}}$ is a closed subspace of $\mathscr{H}$, it is itself a Hilbert space. Now, by Riesz representation theorem there exists a vector $u \in \overline{\mathscr{M}}$ such that

$$F(v) = (u, v)$$

for all $v \in \overline{\mathscr{M}}$. This uniquely determines $F$ on the entire $H$, namely

$$F(v) = (u, v)$$

for all $v \in \mathscr{H}$. To see that this extension is unique, suppose there is another extension $F'$ of $f$ onto $\mathscr{H}$ with the same bound $C$. Then $\|F'\| = \|F\| = C$. And suppose

$$F' = (u', v)$$

for all $v \in H$. Then since $F' = F$ on $\overline{\mathscr{M}}$, for any $w \in \overline{\mathscr{M}}$ we have

$$0 = F(w) - F'(w) = (u - u', w).$$

Therefore $u - u' \in (\overline{\mathscr{M}})^{\perp}$. Now by Riesz theorem

$$\|F'\| = \|u'\| = \|u - (u - u')\| = \sqrt{\|u\|^2 + \|u - u'\|^2}.$$

In the last step we have used the fact that $u \in \overline{\mathscr{M}}$ and $u - u' \in (\overline{M})^{\perp}$. Since $\|F\| = \|u\| = C$ and $\|F'\| = \|F\|$, we must have

$$\sqrt{\|u\|^2 + \|u - u'\|^2} = \|u\|.$$

Thus

$$\|u - u'\| = 0.$$

This shows that $u = u'$ and $F = F'$. So the extension is indeed unique.

**Problem 2.11.** The claim is that the set

$$Q := \left\{ \sum_{j=1}^{n} (p_j + iq_j)\chi_{(a_j,b_j)} : n \in \mathbb{N}, p_j, q_j, a_j, b_j \in \mathbb{Q} \right\} \subset L^2(\mathbb{R})$$

is a dense subset. First, just like problem 2 part (a) and part (b), one can show that simple function are dense in $L^2(\mathbb{R})$, and step functions are dense in simple functions in $L^2(\mathbb{R})$. Thus it suffices to prove that any step function can be approximated arbitrarily by elements in $Q$.

We first prove that $a\chi_{\bigcup_{l=1}^n(a_l,b_l)}$ can be approximated by elements in $Q$ arbitrarily for some $a \in \mathbb{C}$. By density, we can find $q \in \mathbb{Q} + i\mathbb{Q}$ such that $|a - q| < \varepsilon$. For each $(a_l, b_l)$, we can find an interval with rational endpoints $(a_l', b_l') \subset (a_l, b_l)$, such that $\mu((a_l, b_l) - (a_l', b_l')) < \varepsilon/2n$. Denoting

$$K = \bigcup_{l=1}^n (a_l, b_l) - \bigcup_{l=1}^n (a_l', b_l'),$$

we have

$$\begin{aligned}
&\left\| a\chi_{\bigcup_{l=1}^n(a_l,b_l)} - q\chi_{\bigcup_{l=1}^n(a_l',b_l')} \right\|_{L^2}^2 \\
&= \int_K |a\chi_{\bigcup_{l=1}^n(a_l,b_l)} - q\chi_{\bigcup_{l=1}^n(a_l',b_l')}|^2 d\mu \\
&\leq \int_K (|a\chi_{\bigcup_{l=1}^n(a_l,b_l)} - a\chi_{\bigcup_{l=1}^n(a_l',b_l')}| + |a\chi_{\bigcup_{l=1}^n(a_l',b_l')} - q\chi_{\bigcup_{l=1}^n(a_l',b_l')}|)^2 d\mu \\
&\leq \int_K (2|a| + \varepsilon)^2 d\mu \\
&\leq (2|a| + \varepsilon)^2 \mu(K) < (2|a| + \varepsilon)^2 \varepsilon.
\end{aligned}$$

Therefore, we have shown that $a\chi_{\bigcup_{l=1}^n(a_l,b_l)}$ can be approximated arbitrarily by elements in $Q$. Now it follows from our result and from the argument of problem 2 part (b), any step function can be approximated by elements in $Q$ by triangle inequality. Hence simple function can be approximated by elements in $Q$ by triangle inequality. Finally it follows from problem 2 part (a) that $Q$ is dense in $L^2(\mathbb{R})$.

**Problem 2.13.** A Hilbert space is separable if and only if it admits a countable orthonormal basis. Thus it suffices to construct a Hilbert space with an uncountable orthonormal basis. Consider the space

$$\mathscr{H} = \left\{ f : [0,1] \to \mathbb{C} : \#\{x : f(x) \neq 0\} \text{ is countable and } \sum_{f(x)\neq 0} |f(x)|^2 < \infty \right\}.$$

This is a Hilbert space with the inner product

$$(f, g) = \sum_x \overline{f(x)} g(x)$$

which is well-defined by the Cauchy-Schwarz inequality. An uncountable orthonormal basis of $\mathcal{H}$ would be $\{e_x\}_{x\in[0,1]}$ where

$$e_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y. \end{cases}$$

This set is clearly linearly independent, and is a basis because for any $f \in \mathscr{H}$ one has

$$f = \sum_{x \in [0,1]} f(x) e_x.$$

Note that the sum is countable by definition. Finally, this is equivalent to constructing an inseparable Hilbert space using the direct sum, because it is easy to see that

$$\mathscr{H} \cong \bigoplus_{x \in [0,1]} \mathbb{C}.$$

**Problem 2.16*.** We want to show that the unit ball in an infinite dimensional Hilbert space contains infinitely many **disjoint** balls of radius $\sqrt{2}/4$.

Since every Hilbert space admits a countable orthonormal basis $\mathcal{B}$, we can extract a countable orthonormal set $\{a_n\}_{n=1}^{\infty}$ from $\mathcal{B}$. Now set

$$B_n = B_{\frac{\sqrt{2}}{4}}\left(\frac{1}{2} a_n\right) \qquad n = 1, 2, 3...$$

We claim that this set of balls are what we want. First, they are contained in the unit ball: For if $x \in B_n$ for any $n$, then

$$\|x - 0\| = \|x - a_n/2 + a_n/2\| \leq \|x - a_n/2\| + \|a_n/2\| < \frac{\sqrt{2}}{4} + \frac{1}{2} < 1.$$

Furthermore, if $n \neq m$, then we claim that $B_n \cap B_m$. Otherwise, let $z \in B_n \cap B_m$. Then

$$\|a_n/2 - a_m/2\| \leq \|a_n/2 - z\| + \|a_m/2 - z\| < \frac{\sqrt{2}}{2}.$$

So

$$\|a_n/2 - a_m/2\|^2 < \frac{1}{2}.$$

But since $a_n, a_m$ are orthonormal, we have that

$$\|a_n/2 - a_m/2\|^2 = \|a_n/2\|^2 + \|a_m/2\|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

This is a contradiction. This completes the proof.

**Problem 2.25.** Let $M = \mathbb{R}$ with Borel sigma algebra on $\mathbb{R}$. Let $\mu$ be the counting measure on $\mathbb{R}$. Then

$$L^2(M, d\mu) = \left\{ f : \mathbb{R} \to \mathbb{C} : f(x) \neq 0 \text{for countably many } x \text{ and } \sum_x |f(x)|^2 < \infty \right\}.$$

Now set

$$f_y(x) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

Then it is easy to check that $f_y$ is an orthonormal basis: It is spanning, linearly independent, and $(f_y, f_z) = \int f_y f_z d\mu = 0$ if $y \neq z$ because $f_y f_z = 0$ of $y \neq z$, and $(f_y, f_y) = \int f_y^2 d\mu = 1$.

**Problem 2.27.** Since $\mathscr{M}$ is a closed subspace in $\mathscr{H}$, it is complete, hence a Hilbert space itself. By Theorem II.5, we can choose an orthonormal basis $\{\varphi_\alpha\}_{\alpha \in A}$ of $\mathscr{M}$. By Zorn's lemma, there exists an orthonormal basis $\mathcal{B}$ of $\mathcal{H}$ containing $\{\varphi_\alpha\}_{\alpha \in A}$. Then for any $x \in \mathscr{H}$ we can write

$$x = \sum_\alpha c_\alpha \varphi_\alpha + \sum_{\psi_\beta \in \mathcal{B} - \{\varphi_\alpha\}_{\alpha \in A}} c_\beta \psi_\beta =: z + w.$$

Since $\{\varphi_\alpha\}_{\alpha \in A}$ is already an orthonormal basis of $\mathscr{M}$, it must be the case that all the $\psi_\beta$ are inside $\mathscr{M}^\perp$. Therefore we are done.

# Chapter 3: Banach Spaces

**Problem 3.1*.** Let $(M, \mu)$ be a measure space. Show that $L^\infty(M, \mu)$ is a Banach space.

Let $\|g\|_\infty$ denote the essential sup of $g$, i.e.

$$\|g\|_\infty = \inf_A \left( \sup_{x \in M \setminus A} |g(x)| \right) \text{ where } \mu(A) = 0.$$

Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(M, \mu)$. Then let $\varepsilon > 0$ be given, there exists $N$ such that $\|f_n - f_m\|_\infty < \varepsilon$ for all $m, n \geq N$. Put

$$F_{m,n} = \{x \in M : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}.$$

Then by definition we have $\mu(F_{m,n}) = 0$. Set

$$F = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty F_{m,n}.$$

Then

$$\mu(F) \leq \sum_{n=1}^\infty \sum_{m=1}^\infty \mu(F_{m,n}) = 0.$$

Then put

$$\begin{aligned} E &= \bigcap_{m,n \in \mathbb{N}} \{x \in X : |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty\} \\ &= \{x \in X : |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty \, \forall m, n \in \mathbb{N}\}. \end{aligned}$$

It is straightforward to see that

$$E = F^c$$

and $\mu(E^c) = \mu(F) = 0$. Notice that $E$ is measurable. Then for every $x \in E$, the sequence $\{f_n(x)\} \subset \mathbb{C}$ is a Cauchy sequence. Therefore, for every $x \in E$, define $f(x) = \lim_{n\to\infty} f_n(x)$. And for every $x \in E^c$, we can define $f(x) = 0$. Then since $E$ is measurable, we have that

$$f(x) = \lim_{n\to\infty} f_n(x)\chi_E(x)$$

is measurable. To show that $f_n$ converges to $f$ in essential sup, choose $N > 0$ such that for all $n, m \geq N$

$$\|f_n - f_m\|_\infty < \varepsilon.$$

Fix $n$ and let $m \to \infty$ we have that

$$\|f_n - f\|_\infty < \varepsilon$$

for $n \geq N$. Therefore $f_n \to f$ in essential sup. Finally, notice that

$$\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty < \varepsilon + \|f_n\|_\infty < \infty.$$

This shows that $f \in L^\infty(M, \mu)$. Therefore, $L^\infty(M, \mu)$ is a Banach space.

**Problem 3.2. Warning:** The set of sequences with rational coefficient is uncountable! To see this, consider a sequence $(s_n)$ where $s_n = \pm 1$. Then $(s_n) \mapsto (s_n/n)$ gives an injection from the set of binary sequences to a subset of the sequence with rational coefficients. And the set of binary sequences have cardinality $|\{-1, 1\}^{\mathbb{N}}| = |[0, 1]|$ which is uncountable. Hence the set of all rational sequences is not a countable set.

(a) Separability of $\ell_p$: Let $x \in \ell_p$. The set of finite sequences is dense in $\ell_p$: We can approximate $x$ by $x^{(n)}$ where $x_i^{(n)} = x_i$ if $i \leq n$ and $x_i^{(n)} = 0$ if $i > n$. Then we see that

$$\left\|x - x^{(n)}\right\|_{\ell_p}^p = \sum_{k>n} |x_k|^p \to 0$$

as $n$ goes to infinity. Then we can approximate finite sequences by finite sequences of rational numbers (which form a countable set). That is, consider the set

$$\mathcal{Q} = \{r = (r_n) : r_n \in \mathbb{Q} + i\mathbb{Q}, r_n \neq 0 \text{ for finitely many } n\}.$$

Let $x^{(n)}$ be finite sequence in $\ell_p$ with $x_i^{(n)} = 0$ for all $i > n$. Then for each $i \leq n$ choose $r_i \in \mathbb{Q} + i\mathbb{Q}$ such that $|x_i^{(n)} - r_i| < \varepsilon/2^{ip}$. Then take $r^{(n)} = (r_k)$ with $r_k = 0$ for all $k > n$. Then

$$\left\|x^{(n)} - r^{(n)}\right\|_{\ell_p}^p = \sum_{k=1}^{\infty} |x_k^{(n)} - r_k|^p \leq \varepsilon^p.$$

This shows that $\mathcal{Q}$ is dense in $\ell_p$. And we note that

$$\mathcal{Q} = \bigcup_{k \geq 1} \{(x_1, ..., x_k, 0, 0, 0, ...) : x_i \in \mathbb{Q} + i\mathbb{Q}\}$$

is countable. Hence this shows that $\ell_p$ is separable.

Separability of $c_0$: Again we show $\mathcal{Q}$ is dense in $c_0$. Let $x \in c_0$. Then there exists $N$ such that $|x_k| < \varepsilon$ for all $k \geq N$. Then choose $r \in \mathcal{Q}$ by $r = (r_1, ..., r_N, 0, ...)$ such that $|x_k - r_k| < \varepsilon$. Then clearly $\|x - r\|_\infty < \varepsilon$. This proves the claim.

Inseparability of $\ell_\infty$: For each subset $I \subset \mathbb{N}$, define $e_I \in \ell_\infty$ by

$$(e_I)_i = \begin{cases} 1 & i \in I \\ 0 & i \notin I \end{cases}.$$

Then we see that $\|e_I - e_J\|_\infty = 1$ whenever $I \neq J$. Then the set

$$\mathcal{B} = \left\{ B_{1/2}(e_I) : I \subset \mathbb{N} \right\}$$

is an uncountable set with disjoint open balls. If $\mathcal{A}$ is a dense set in $\ell_\infty$, then each element in $\mathcal{B}$ contains at least one element in $\mathcal{A}$. And since the balls are disjoint, $\mathcal{A}$ must be uncountable. Hence $\ell_\infty$ is not separable.

(b) We first observe that if $p < q$, then $\ell_p \subset \ell_q$. To see this, let $a \in \ell_p$. Then

$$\|a\|_{\ell_q}^q = \sum_{n=1}^\infty |a_n|^q = \sum_{n=1}^\infty |a_n|^{q-p}|a_n|^p \leq \left(\sup_n |a_n|\right)^{q-p} \left(\sum_{n=1}^\infty |a_n|^p\right).$$

So we have

$$\|a\|_{\ell_q}^q \leq \|a\|_\infty^{q-p}\|a\|_{\ell_p}^p < \infty$$

since $a \in \ell_p$ and $\ell_p \subset \ell_\infty$ for all $p$. Thus $a \in \ell_q$. Therefore, it suffices to show that $s \subset \ell_1$. Now let $a \in s$. Since $\lim_{n\to\infty} n^p a_n = 0$ for all positive integers $p$, there exists $N$ such that $|a_n| < 1/n^2$ for all $n > N$. Then we have

$$\|a\|_{\ell_1} = \sum_{n=1}^\infty |a_n| \leq N \max_{1\leq k\leq N} |a_k| + \sum_{n=N+1}^\infty \frac{1}{n^2} < \infty.$$

Therefore $a \in \ell_1$. This proves the claim.

**Problem 3.3*.** We want to prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.

Let $X$ be a Banach space, and suppose we have a sequence $\{x_n\}$ such that

$$\sum_{n=1}^\infty \|x_n\| < \infty.$$

Then we have

$$\left\|\sum_{n=k}^{k+m} x_n\right\| \leq \sum_{n=k}^{k+m} \|x_n\| \to 0$$

as $k \to \infty$, so the sequence of partial sums $\left\{\sum_{n=1}^{N} x_n : N \in \mathbb{N}\right\}$ is a Cauchy sequence. Since $X$ is a Banach space, the series $\sum_{n=1}^{\infty} x_n$ converges in $X$.

Conversely, suppose we have a Cauchy sequence $\{x_n\}$. We can then choose a subsequence $\{x_{n_k}\}$ such that

$$\|x_{n_{k+1}} - x_{n_k}\| < 1/2^k.$$

Then the series

$$\sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| < \infty.$$

By assumptipn the series

$$\sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k})$$

converges to some limit $x$. But notice the $K$th partial sum of this series is $x_{n_{K+1}} - x_{n_1}$. Therefore $\{x_{n_k}\}$ converges to $x + x_{n_1}$. Since now we have a Cauchy sequence $\{x_n\}$ with a convergent subsequence $\{x_{n_k}\}$, Problem 1 of HW1 says that $\{x_n\}$ converges in $X$. So $X$ is a Banach space.

**Problem 3.4.** It suffices to prove that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to the standard norm $\|\cdot\|_e$ on $\mathbb{R}^n$. Let $v = x_1 e_1 + \cdots + x_n e_n$ where $\{e_i\}$ is a basis for $\mathbb{R}^n$. Then by the triangle inequality of the norm and Cauchy Schwarz inequality for the real numbers we get:

$$\|v\| \leq \sum_{j=1}^{n} |x_j| \|e_j\| \leq \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} \left(\sum_{j=1}^{n} \|e_j\|^2\right)^{1/2} = C\|v\|_e$$

where $C = (\sum_{j=1}^{n} \|e_j\|^2)^{1/2}$. Now to prove the other inequality, consider the set $K = \{v : \|v\|_e = 1\}$ which is compact in the Euclidean topology. We claim that the function $x \mapsto \|x\|$ is continuous in the Euclidean topology: if $x_k \to x$ in the Euclidean topology, then we have $|\|x_k\| - \|x\|| \leq \|x_k - x\| \leq C\|x_k - x\|_e$ by triangle inequality. This proves the continuity of the map $x \mapsto \|x\|$. Since $K$ is compact, let $m > 0$ be the minimum of $\|v\|$ on $K$. Now let $v \neq 0$ be any vector and let $u = v/\|v\|_e \in \mathbb{R}^n$. Then $\|u\|_e = 1$. So $\|u\| \geq m$. Hence

$$\|v\| \geq m\|v\|_e.$$

Therefore we have that

$$m\|v\|_e \leq \|v\| \leq C\|v\|_e$$

for any $v \neq 0$, and the result holds trivially when $v = 0$. This concludes the proof.

**Problem 3.5.** Let $f \in C_\infty(\mathbb{R})$. Then consider the sequence of functions

$$f_n(x) = \begin{cases} f(x) & |x| < M \\ f(n)(n+1-x) & x \in [n, n+1] \\ f(-n)(x+n+q) & x \in [-n-1, -n] \\ 0 & |x| > n+1. \end{cases}$$

Then clearly $f_n \to f$ in the infinity norm (essential sup) and $f_n \in C_\kappa((R))$.

**Problem 3.6.** First, we have

$$|\Lambda(\{a_k\}_{k=1}^\infty)| = \left|\sum_{k=1}^\infty \lambda_k a_k\right| \leq \sum_{k=1}^\infty |\lambda_k a_k| \leq \|a\|_\infty \sum_{k=1}^\infty |\lambda_k|$$

for $\ell_1 \subset c_0$. Therefore we have shown that

$$\|\Lambda\| \leq \sum_{k=1}^\infty |\lambda_k|.$$

On the other hand, consider the sequence $a^{(n)} = \{a_k^n\}_{k=1}^\infty$ where $a_k^n = e^{-i \arg \lambda_k}$ if $k \leq n$ and $a_k^n = 0$ if $k > n$. Then it is easy to verify that $a^{(n)} \in c_0$ for all $n$ and $\left\|a^{(n)}\right\|_\infty = 1$ for all $n$. Then we see that

$$\|\Lambda\| \geq |\Lambda(a^{(n)})| = \left|\sum_{k=1}^n \lambda_k e^{-i \arg \lambda_k}\right| = \sum_{k=1}^n |\lambda_k|$$

for all $n$. Hence taking $n \to \infty$ give the desired inequality. Thus we conclude that

$$\|\Lambda\| = \sum_{k=1}^\infty |\lambda_k|.$$

**Problem 3.7.** $\ell_\infty = (\ell_1)^*$: First we show $\ell_\infty \subset (\ell_1)^*$. Let $\lambda = \{\lambda_k\} \in \ell_\infty$, where $\|\lambda\|_\infty = \sup_k |\lambda_k|$. Then define

$$\Lambda(\{a_k\}) = \sum_{k=1}^{\infty} \lambda_k a_k$$

for $\{a_k\} \in \ell_1$. Then $\Lambda$ is a dual function on $\ell_1$ and it is not hard to show $\|\Lambda\| = \|\lambda\|_\infty$. Next we show $(\ell_1)^* \subset \ell_\infty$. Let $\Lambda \in (\ell_1)^*$. Then let $e_k = (0, 0, ...0, 1, 0, 0, ...) \in \ell_1$ be the sequence with a 1 in the $k$th slot and zero everywhere else. Then this is a basis for $\ell_1$ and define $\lambda_k = \Lambda e_k$. Then the sequence $\lambda = \{\lambda_k\} \in \ell_\infty$ because $\|\lambda\|_\infty \leq \|\Lambda\|$. Conversely, if $x = \{\alpha_k\} \in \ell_1$, then

$$|\Lambda x| = \left|\Lambda\left(\sum_{k=1}^{\infty} \alpha_k e_k\right)\right| = \left|\sum_{k=1}^{\infty} \alpha_k \Lambda(e_k)\right| \leq \sum_{k=1}^{\infty} |\alpha_k| \cdot |\Lambda(e_k)| \leq \|\lambda\|_\infty \|x\|_{\ell_1}.$$

Therefore we have $\|\Lambda\| \leq \|\lambda\|_\infty$. Hence $\|\Lambda\| = \|\lambda\|_\infty$. Therefore we conclude that there exists an isometric isomorphism $\ell_\infty \cong (\ell_1)^*$.

$\ell_\infty^* \neq \ell_1$: It is easy to see that $\ell_1 \subset \ell_\infty^*$. For any $\lambda = \{\lambda_k\} \in \ell_1$, we can define $\Lambda \in \ell_\infty^*$ as

$$\Lambda(\{a_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \lambda_k a_k.$$

But we claim that $\ell_\infty^* \not\subset \ell_1$, i.e. there is a linear functional on $\ell_\infty$ that does not arise from the construction above. To see this, let

$$Z = \left\{a \in \ell_\infty : \lim_{n\to\infty} a_n = 0\right\} = c_0 \subset \ell_\infty.$$

Then $Z$ is clearly a subspace of $\ell_\infty$. Let $y = (1, 1, 1, ...) \in \ell_\infty$. Then clearly we see that $\mathrm{dist}(y, Z) = 1$. By Corollary 3 on pg 77 of the textbook (Problem 3.10), there exists a linear function $\Lambda \in \ell_\infty^*$ such that $\Lambda(Z) = 0$ and $\Lambda(y) = 1$, moreover $\|\Lambda\| \leq 1$. For the sake of contradiction, suppose

$$\Lambda(\{y_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} a_k y_k$$

for some $\{a_k\}_{k=1}^{\infty} \in \ell_1$. Then since $\Lambda(Z) = 0$, we see that $\Lambda(e_k) = 0$ for all $k$ where $e_k$ only has 1 in the $k$th slot and 0 everywhere else. But then $\Lambda(e_k) = a_k = 0$ for all $k$, yet

$$\Lambda(y) = \sum_{k=1}^{\infty} a_k = 1.$$

This is the desired contradiction.

**Problem 3.8*.**

(a) Let $V$ be the span of $C(\mathbb{C})$ plus some particular member in $L^\infty(\mathbb{R})$ that is not in $C(\mathbb{R})$, for example the indicator function $\chi_{[0,1]}$. Then any element in $V$ can be written as $v = f + t\chi_{[0,1]}$, where $f \in C(\mathbb{R})$ and $t \in \mathbb{R}$. Define a linear functional on $V$ by

$$\varphi(f + t\chi_{[0,1]}) = t.$$

We first show that $\varphi$ is a bounded linear functional on $V$.

To see this, we notice that $C(\mathbb{R})$ is a closed subspace of $L^\infty(\mathbb{R})$: Let $f_n \in C(\mathbb{R})$ and $f_n \to f$ in $L^\infty$. Since $(f_n)$ is a Cauchy sequence, for all $\epsilon > 0$ there exists $N$ such that $m, n \geq N$ implies $\|f_m - f_n\|_\infty < \epsilon$. This means that $|f_m(x) - f_n(x)| < \epsilon$ a.e. But since $f_m - f_n$ is continuous, if $|f_m(x) - f_n(x)| \geq \epsilon$ for some $x \in \mathbb{R}$, then by continuity we would have $|f_m(x) - f_m(x)| \geq \epsilon$ for a neighborhood around $x$, which is of positive measure, contradicting the assumption. Hence we conclude that $(f_n)$ actually converges uniformly to $f$. Thus the uniform limit $f$ is continuous. Now fix $\epsilon > 0$, choose $N$ such that $\|f_N - f\|_\infty < \epsilon$. Then

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq \epsilon + M < \infty \qquad \text{a.e.}$$

where $M = \|f_N\|_\infty$. But since $f$ is continuous, we again apply the neighborhood argument to conclude that $|f(x)| \leq \epsilon + M$ for all $x$. Thus $f \in C(\mathbb{R})$.

Now we show that $\varphi$ is bounded. Since $C(\mathbb{R})$ is closed in $L^\infty(\mathbb{R})$, let $d = \operatorname{dist}(\chi_{[0,1]}, C(\mathbb{R}))$. Then we have

$$\|f + t\chi_{[0,1]}\| = |t| \cdot \|t^{-1}f + \chi_{[0,1]}\| \geq |t| \cdot d.$$

Therefore

$$|\varphi(f + t\chi_{[0,1]})| = |t| \leq d^{-1}\|f + t\chi_{[0,1]}\|.$$

Hence

$$\|\varphi\| \leq \frac{1}{d}.$$

Thus $\varphi$ is bounded. Also, $\varphi$ is clearly a nonzero linear function (e.g. nonzero at $\chi_{[0,1]}$) that vanishes on $C(\mathbb{R})$. Then we can extend $\varphi$ to the entire $L^\infty(\mathbb{R})$ by Hahn-Banach theorem.

(b) Define the linear functional $\lambda : C(\mathbb{R}) \to \mathbb{C}$ by $\lambda(f) = f(0)$. This is clearly linear. It is also bounded: Since if $f \in C(\mathbb{R})$ and $\|f\|_\infty = M$, then $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Therefore,

$$\|\lambda(f)\| = |f(0)| \leq M = \|f\|_\infty.$$

So $\|\lambda\| < 1$ is bounded. Then we can use Hahn Banach-theorem to extend $\lambda$ to a linear functional $\Lambda$ defined on the entire $L^\infty(\mathbb{R})$ such that $\|\Lambda\| = \|\lambda\|$.

To show that $(L^\infty)^* \neq L^1$, we first remark that every function $g \in L^1$ give rise to a linear functional on $L^\infty$ given by

$$\lambda_g(f) = \int_{\mathbb{R}} fg d\mu.$$

By Hölder inequality we have

$$|\lambda_g(f)| \leq \|g\|_1 \|f\|_\infty.$$

so $\|\lambda_g\| \leq \|g\|_1$ is bounded hence continuous. On the other hand, consider the function

$$f(x) = \begin{cases} 0 & \text{if } g(x) = 0 \\ \frac{\overline{g(x)}}{|g(x)|} & \text{if } g(x) \neq 0. \end{cases}$$

Then

$$\|\lambda_g\| \geq |\lambda_g(f)| = \|g\|_1.$$

So we have

$$\|\lambda_g\| = \|g\|_1.$$

Thus $g \mapsto \lambda_g$ is an isometry of $L^1$ into $(L^\infty)^*$. But by our previous work, we know there is a linear functional $\Lambda$ on $L^\infty(\mathbb{R})$ such that $\Lambda(f) = f(0)$ for any $f \in C(\mathbb{R})$. We claim that $\Lambda \neq \lambda_g$ for any $g \in L^1$. Otherwise, we would have

$$\Lambda(f) = f(0) = \int fg d\mu$$

for all bounded continuous function $f$ and for some $g \in L^1$. Now, given any $x_0 \neq 0$ and $\epsilon < |x_0|$, it is possible to find continuous function $f_k$ with support in $[x_0 - \epsilon, x_0 + \epsilon]$ which converge pointwise to $\chi_{[x_0-\epsilon, x_0+\epsilon]}$ and $0 \leq f_k(x) \leq 1$ for all $x$. Since $|f_n g| \leq |g|$, then by the dominated convergence theorem,

$$0 = \frac{1}{2\epsilon} \int f_k g d\mu \to \frac{1}{2\epsilon} \int g \chi_{[x_0-\epsilon, x_0+\epsilon]} d\mu = \frac{1}{2\epsilon} \int_{x_0-\epsilon}^{x_0+\epsilon} g d\mu.$$

If $x_0$ is a Lebesgue point for $f$, such an intergral on the right hand side must converge to $g(x_0)$ when $\epsilon \to 0$, by the Lebesgue differentiation theorem. Hence $g(x_0) = 0$ for all Lebesgue points $x_0 \neq 0$. And since almost every point is a Lebesgue point, we conclude that

$$g(x) = 0 \text{ a.e.}$$

But this shows that

$$\Lambda(f) = \int fgd\mu = 0 \text{ for all } f \in L^\infty.$$

Hence $\Lambda = 0$. But this is a contradiction, for $\Lambda$ acting on the constant function with value 1 should give 1. This shows that

$$(L^\infty)^* \neq L^1.$$

**Problem 3.9.** (Compare with Problem 2.9) Let $\mathscr{M}$ be a subspace (not necessarily closed) of the Hilbert space $\mathscr{H}$. Then $\overline{\mathscr{M}}$ is itself a Hilbert space. Let $\lambda \in \mathscr{M}^*$. Riesz representation theorem says there exists $v_\lambda \in \overline{\mathscr{M}}$ such that $\lambda(x) = (v_\lambda, x)$ for all $x \in \mathscr{M}$. Thus all possible continuous extensions of $\lambda$ are of the form

$$\lambda'(x) = (v_\lambda + w, x) \qquad w \in \mathscr{M}^\perp.$$

**Problem 3.10.** Apply the Hahn-Banach theorem with $p(x) = \operatorname{dist}(x, Z) \leq \|x\|$ since $0 \in Z$, and $Y = \{z + \alpha y : z \in Z, \alpha \in \mathbb{C}\}$, and $\lambda(z + \alpha y) = \alpha d$.

**Problem 3.11.** We will show that there is a linear functional $\lambda$ on $\ell_\infty(\mathbb{R})$ so that $\liminf_{n\to\infty} a_n \leq \lambda(\{a_n\}_{n=1}^\infty) \leq \limsup_{n\to\infty} a_n$ for all $\{a_n\}_{n=1}^\infty \in \ell_\infty(\mathbb{R})$. Since $\ell_1 \subset \ell_\infty$, we first define $\lambda$ on $\ell_1$ by

$$\lambda(\{a_n\}_{n=1}^\infty) = \lim_{N\to\infty} \frac{a_1 + \cdots + a_N}{N}.$$

Since $\{a_n\}_{n=1}^\infty \in \ell_1$, we have that $\sum_{n=1}^\infty |a_n| < \infty$, so $\lambda \equiv 0$ on $\ell_1$. Also, since $|a_n| \to 0$ for $\{a_n\}_{n=1}^\infty \in \ell_1$, we have $\liminf_{n\to\infty} a_n \leq \lambda(\{a_n\}_{n=1}^\infty) \leq \limsup_{n\to\infty} a_n$ since both sides of the inequality are zero. Now $\lambda$ is dominated by two sublinear functions (namely lim sup and lim inf) on $\ell_1$. Hence we can extend $\lambda$ to the entire $\ell_\infty$ by Hahn-Banach theorem without violating the inequality. This completes the proof.

**Problem 3.12.** Let $X = C[0,1]$ and $M = \{f \in C[0,1] : f(0) = 0\}$. We will show that $X/M = \mathbb{C}$. First, we shall show that

$$\|[f]\| = |f(0)|$$

where the LHS is the quotient norm defined on pg 79 of the textbook. To see this, let $f \in C[0,1]$. Then for any $m \in M$, we have

$$\|f - m\|_\infty = \sup_{x \in [0,1]} |f(x) - m(x)| \geq |f(0) - m(0)| = |f(0)|.$$

Thus

$$\|[f]\| = \inf_{m \in M} \|f - m\|_\infty \geq |f(0)|.$$

Conversely, define the function $m(x) = f(0) - f(x)$. Then clearly $m \in M$ and $\|f - m\|_\infty = |f(0)|$. Therefore $\|[f]\| = \inf_{m' \in M} \|f - m'\|_\infty \leq \|f - m\|_\infty = |f(0)|$. This proves the equality of norm above. Then consider the map $\varphi : X \to \mathbb{C}$ by

$$\varphi(f) = f(0).$$

Clearly $\varphi$ is a surjection on $\mathbb{C}$ and $\ker \varphi = M$. So by the first isomorphism theorem for vector spaces, we have that $X/\ker\varphi \cong \operatorname{im}\varphi$, i.e. $X/M \cong \mathbb{C}$. And the equation of norms that we have just proved shows that $\varphi$ is also an isometry. This shows that we can make the identification $X/M = \mathbb{C}$.

**Problem 3.13.** (Hellinger-Toeplitz Theorem) Let $A : \mathscr{H} \to \mathscr{H}$ be a linear operator that is defined everywhere and that $(x, Ay) = (Ax, y)$ for all $x, y \in \mathscr{H}$, we will show that $A$ is bounded.

Define $\varphi_y : \mathscr{H} \to \mathbb{C}$ by $\varphi_y(x) = (Ax, y)$. Then consider the family

$$\mathscr{F} = \{\varphi_y : \|y\| = 1\}.$$

Then take $x \in \mathscr{H}$, we have

$$\sup_{\|y\|=1} |\varphi_y(x)| = \sup_{\|y\|=1} |(Ay, x)| = \sup_{\|y\|=1} |(y, Ax)| \leq \|Ax\| < \infty.$$

Hence the uniform boundedness principle applies and we have that

$$C := \sup_{T \in \mathscr{F}} \|T\| < \infty.$$

Then we have that

$$\|Ax\|^2 = (Ax, Ax) = \|x\|(A(x/\|x\|), Ax) = \|x\|\varphi_{x/\|x\|}(Ax) \leq \|x\|C\|Ax\|.$$

Dividing both sides by $\|Ax\|$ gives that $\|A\| \leq C$. This concludes the proof.

**Problem 3.14*.** Let $X$ be a Banach space. We want to give an example of an everywhere-defined but discontinuous linear functional $T$. Show directly that the graph of $T$ is not closed.

Let

$$X = \ell_\infty = \left\{ (a_n)_{n=1}^\infty : \|a\|_\infty = \sup_n |a_n| < \infty \right\}.$$

Then the sequence $\delta_k$, which is 0 everywhere else and is 1 at the $k$th entry, i.e.

$$\delta_k = (0, 0, ..., \underbrace{1}_{k\text{th entry}}, 0, 0, ...)$$

is a linarly independent set.[1] By the axiom of choice, there exists a basis $\mathcal{B}$ of $X$ containing the set of all $\delta_k$s. Define the linear function $T$ by defining its value on all the basis element $\mathcal{B}$ to be

$$T(\delta_k) = k \qquad k = 1, 2, 3...$$

and define $T(b) = 0$ for all the other basis element $b \in \mathcal{B}$. Then extend $T$ to $X$ by linearity. By construction,$T$ is clearly discontinuous, for it is unbounded:

$$\|T\| = \sup_{\|a\|_\infty = 1} |T(a)| \geq |T(\delta_k)| = k$$

for any $k = 1, 2, 3...$ Next, we show that its graph $\Gamma(T) = \{(a, Ta) : a \in X\} \subset X \times \mathbb{C}$ is not closed. To show this, it suffices to find a sequence $a_n \in X$ such that $a_n \to a$ and $Ta_n \to y$ but $Ta \neq y$. To this end, consider

$$a_n = \frac{1}{n}\delta_n = (0, 0, ..., \underbrace{\frac{1}{n}}_{n\text{th entry}}, 0, 0, ...).$$

Then we see that $a_n$ converges to the 0, the sequence of all zeros in $\ell_\infty$-norm, and

$$T(a_n) = \frac{1}{n}T(\delta_n) = \frac{1}{n}n = 1.$$

So $Ta_n \to 1$. But

$$T(0) = 0 \neq 1.$$

This shows that $\Gamma(T)$ is not closed.

[1] By this we mean for all linear combinations involving finitely many terms of this set is zero, then all the coefficients of this linear combination must be zero.

**Problem 3.17.** Use induction on the number of slots the linear functional $B : X \times \cdots X \to \mathbb{C}$ has. The base case where $n = 2$ is proven by the corollary on pg 81 of the textbook. Repeat the same argument as in the proof of the corollary for the inductive case $n = k+1$ with inductive hypothesis that the statement is true for $n = k$.

**Problem 3.18.** The same argument in problem 3.13 works, except now we have $\sup_{\|y\|=1} |\varphi_y(x)| = \sup_{\|y\|=1} |(Ay, x)| = \sup_{\|y\|=1} |(y, Bx)| \leq \|Bx\| < \infty$. The uniform boundedness principle again applies and by the same argument this shows that $A$ is bounded. Reversing the role of $A$ and $B$ shows that $B$ is also bounded.

**Problem 3.19*.** Let $X$ be a Banach space in either of the norms $\|\cdot\|_1$ or $\|\cdot\|_2$. Suppose that $\|\cdot\|_1 \leq C\|\cdot\|_2$ for some constant $C$. We want to prove that there is a constant $D$ such that $\|\cdot\|_2 \leq D\|\cdot\|_1$.

Consider the map between Banach spaces

$$\begin{aligned} \varphi : (X, \|\cdot\|_2) &\to (X, \|\cdot\|_1) \\ x &\mapsto x. \end{aligned}$$

$\varphi$ is bounded since it is continuous at 0: Fix $\epsilon > 0$, since $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$, choose $\delta = \epsilon/C$ will show that $\|\varphi(x)\|_2 < \epsilon$ as long as $\|x\|_1 < \delta$. Also, $\varphi$ is clearly surjective (in fact bijective). So by the open mapping theorem, we see that $\varphi$ is an open mapping. Now we claim that $\varphi^{-1}$ is bounded, then we have

$$\frac{\|x\|_2}{\|x\|_1} \leq \|\varphi^{-1}\|$$

which is the desired constant $D$ (If $x = 0$ then any constant will do). Since for all open sets $U \subset X$, the set $\varphi(U) \subset X$ is open, then $(\varphi^{-1})^{-1}(U)$ is open in $Y$. So $\varphi^{-1}$ is a continuous map for the preimage of open set under it is open. Hence $\|\varphi^{-1}\| < \infty$. This concludes the proof.

**Problem 3.20.** In one point space $X = \{x\}$, the only set whose closure has empty interior is the empty set, so Baire category theorem still holds. Notice that although $\overline{\{x\}} = \{x\}$, but in one-point space, $\{x\}$ is open so $\{x\}^o = \{x\}$ which is not empty.

**Problem 3.22*.** For part (a), We want to prove that a Banach space $X$ is reflexive if and only if its dual $X^*$ is reflexive. (Hint: If $X \neq X^{**}$, find a bounded linear functional on $X^{**}$ which vanishes on $X$.)

To prove the claim, we need a lemma first:

**Lemma.** Let $J \in \mathcal{L}(X, X^{**})$ be the map $x \mapsto j_x$ where $j_x(\varphi) = \varphi(x)$. Then the range of $J$ is a closed subspace of $X^{**}$.

*Proof of Lemma.* Take a convergent sequence $(j_n)_{n=1}^{\infty} \subset J(X)$. WLOG say $j_n = j_{x_n}$ for some $x_n \in X$. By Theorem III 4 in the textbook we know $J$ is an isometry, so

$$\|j_{x_n} - j_{x_m}\| = \|j_{x_n - x_m}\| = \|x_n - x_m\|.$$

Hence $(j_{x_n})_{n=1}^{\infty}$ converges if and only if $(x_n)_{n=1}^{\infty}$ converges. Say $x_n \to x$, then we want to show that $j_{x_n} \to j_x$. Suppose $j_{x_n} \to k \in X^{**}$, to show that $k = j_x$, we notice that

$$\|j_x - k\| \leq \|j_x - j_{x_n}\| + \|j_{x_n} - k\| = \|x - x_n\|_X + \|j_{x_n} - k\|.$$

Taking $n \to \infty$ we see that $j_x = k$. Thus $J(X) \subset X^{**}$ is closed. $\square$

Now we prove our claim:

If $X$ is reflexive, then $X = X^{**}$. Hence $X^* = (X^{**})^* = X^{***}$. Therefore $X^*$ is reflexive. Now suppose $X^*$ is reflexive, but $X$ is not, i.e. $JX \subsetneq X^{**}$. Then there exists $\varphi \in X^{**}$ such that $\varphi \notin JX$. First, we define a linear functional $\zeta$ on $\mathrm{Span}(JX, \varphi) = \{j_x + t\varphi : x \in X, t \in \mathbb{R}\}$ by

$$\zeta(j_x + t\varphi) = t.$$

Clearly this is a nonzero linear functional. Also $\zeta \equiv 0$ on $JX$. Finally, since $JX$ is closed in $X^{**}$ by the lemma, we can apply the argument in the proof of Problem 3.8 part (a) to show that $\zeta$ is a bounded linear operator. Extending $\zeta$ to $X^{**}$ using Hahn-Banach theorem gives a linear functional $\xi \in X^{***}$ that is nonzero but vanishes on $JX$.

Now since $X^*$ is reflexive, there exists $\theta \in X^*$ such that $\xi = j_\theta$. Then for all $x \in X$:

$$0 = \xi(j_x) = j_\theta(j_x) = j_x(\theta) = \theta(x).$$

But then $\theta(x) = 0$ for all $x \in X$ and $\theta \in X^*$. This means that $\theta \equiv 0$. But then

$$\|\xi\| = \|j_\theta\| = \|\theta\| = 0.$$

So $\xi \equiv 0$ on $X^{***}$, which is the desired contradiction.

**Problem 3.23.** Let $X$ be a Hilbert space and $\mathscr{M}$ be a closed subspace. The map natural map $\pi : \mathscr{M}^\perp \to X/\mathscr{M}$ defined by $\pi(x) = [x]$ is clearly a surjection by the projection theorem in Hilbert space. Clearly $\ker \pi = 0$. Thus by the first isomorphism theorem of vector space we have $\mathscr{M}^\perp \cong X/\mathscr{M}$. It remains to show that $\pi$ is an isometry. Now, by definition we clearly have

$$\|[x]\| = \inf_{m \in \mathscr{M}} \|x - m\| = \operatorname{dist}(x, \mathscr{M}) = \|x\|$$

where the last step is because we assumed $x \in \mathscr{M}^\perp$. This concludes the proof.

**Problem 3.24.** It is not hard to show that $\ell$ is a surjection onto $\mathbb{R}$. Thus by the first isomorphism theorem we have $X/\ker \ell \cong \mathbb{R}$. (Need $\ell$ nonzero.)

# Chapter 5: Locally Convex Spaces

**Problem 5.1*.**

(a) Let $\{\rho_i\}_{i=1}^n$ be the seminorms that generates the topology of a locally convex space. Then we consider the seminorm defined by

$$\rho(x) = (\rho_1 + \cdots + \rho_n)(x).$$

This is actually a norm, for if $\rho(x) = 0$, then since $\rho_i(x) \geq 0$ for all $i$, we must have $\rho_i(x) = 0$ for all $i$. Since $\{\rho_i\}$ separate points, we must have $x = 0$. We show that $\rho$ generates the same topology on this locally convex space. Clearly $\rho_i \leq \rho$ for each $i$, and $\rho \leq \rho_1 + \cdots + \rho_n$. So by the Proposition on pg 126 of the textbook, this shows that $\rho$ and $\{\rho_i\}$ are equivalent seminorms, i.e. they generates the same topology.

(b) If $X$ is a locally convex space that has a topology generated by a single norm $\rho$, then $\{x \in X : \rho(x) < \epsilon\}$ is a bounded neighborhood of 0.

Conversely, suppose 0 has a bounded neighborhood $U$. Since $X$ is locally convex, by definition, we can find an open set $V \subset U$ and containing 0 such that $V$ is absorbing (i.e. $\bigcup_{t>0} tV = X$) and absolutely convex (i.e. if $x, y \in V$ and $|\alpha| + |\beta| = 1$ then $\alpha x + \beta y \in V$). Consider the Minkowski functional of $V$: $\varphi : X \to [0, \infty)$ by

$$\varphi(x) = \inf \{t \geq 0 : x \in tV\}.$$

Then we have the following Lemma:

> **Lemma.** $\varphi : X \to [0, \infty)$ is a continuous seminorm on $X$, and furthermore
>
> $$V' = \{x \in X : \varphi(x) < 1\}$$
>
> is an open set contained in $U$.
>
> *Proof of Lemma.* For continuity, we note that $V$ is a neighborhood of 0, and we refer the reader to This link. We remark that since $V$ is absolutely convex, it is balanced (circled) and convex, hence the Lemma on pg 128 of the textbook applies. For a proof we refer to This file. The lemma on the book shows that $V' \subset V$. Now, since $\varphi$ is continuous and $V' = \varphi^{-1}([0, 1))$ which is the preimage of an open set in $[0, \infty)$, we see that $V'$ is open. $\square$

We now return to our question. We claim that $\varphi$ is the norm that generates the topology on $X$. Suppose $q : X \to [0, \infty)$ is a continuous seminorm on $X$. Then the set $W = \{x \in X : q(x) < 1\}$ is a neighborhood of 0. Since $U$ is bounded, we can find $\epsilon > 0$ such that

$$W \supset \epsilon U \supset \epsilon V'.$$

Then if $x \in \epsilon V'$, then $x = \epsilon y$ for some $y \in V'$. Then we have

$$\varphi(x) = \epsilon\varphi(y) < \epsilon$$

where the last inequality come from the fact that $y \in V'$ and from the Lemma. This shows that if $\varphi(x) < \epsilon$ then we have $q(x) < 1$. Now let $y \in X$ be any element, and we set

$$\alpha = \frac{1}{\epsilon}\varphi(y).$$

Pick $\delta > 0$. Then we have

$$\frac{1}{\epsilon}\varphi\left(\frac{y}{\alpha + \delta}\right) = \frac{\alpha}{\alpha + \delta} < 1.$$

So $\varphi((a + \delta)^{-1}y) < \epsilon$. Therefore, we must have

$$q\left(\frac{y}{a + \delta}\right) = \frac{1}{a + \delta}q(y) < 1.$$

After some algebra we have that

$$q(y) < a + \delta = \frac{1}{\epsilon}\varphi(y) + \delta.$$

Since $\delta$ is arbitrary, we have that

$$q(y) \leq \frac{1}{\epsilon}\varphi(y)$$

for all $y \in X$. Suppose now we have a sequence $\{x_n\} \subset X$ that converges to 0 in the topology generated by the seminorms of $X$, then

$$\lim_{n \to \infty} \frac{1}{\epsilon}\varphi(x_n) = 0$$

since $\varphi$ is continuous in this topology by the lemma. By the inequality above we have that

$$\lim_{n \to \infty} q(x_n) = 0.$$

Since topology generated by a single seminorm is metrizable so sequential continuity implies continuity, $q$ is continuous in the topology generated by $\varphi$. Hence topology generated by $\varphi$ is finer than the original topology. But $\varphi$ is already continuous in the original topology, the original topology is finer than the topology generated by $\varphi$. Thus they are literally the same topology. Thus the topology of $X$ can be generated by a single seminorm. By (a), this shows that the topology of $X$ can be generated by a single norm.

**Problem 5.21*.** Let $\mathcal{S}$ be the Schwarz space (functions of rapid decay), and $\delta$ be the Dirac delta function. Prove directly that the derivative of $\delta$, denoted by $\delta'$, is in $\mathcal{S}'$ (the dual space of $\mathcal{S}$). Prove that $\delta'$ does not come from a measure.

We now proceed to the proof. From the definition of the weak derivative, we have

$$\delta'(f) = -f'(0).$$

Clearly $\delta'$ is linear on $\mathcal{S}(\mathbb{R})$, and notice that

$$|\delta'(f)| = |-f'(0)| \leq \sup_{x\in\mathbb{R}} |f'(x)| \leq \|f\|_{0,1}$$

according to the notation of the textbook. Then by theorem V.2, this shows that $\delta'$ is continuous, hence bounded linear functional on $\mathcal{S}(\mathbb{R})$. This proves that $\delta' \in \mathcal{S}'(\mathbb{R})$.

Now argue by contradiction: Suppose there is a measure $\mu$ that give rise to $\delta'$, i.e. we have

$$\delta'(f) = \int f d\mu$$

for every $f \in \mathcal{S}(\mathbb{R})$. Then, we claim that there exists a function $\varphi \in \mathcal{S}(\mathbb{R})$ such that

- $\varphi'(0) = 1$;
- $\varphi(x)$ is of compact support. This immediately implies that $\varphi \in \mathcal{S}(\mathbb{R})$. Therefore, $\varphi$ is bounded by $\|\varphi\|_\infty$.

If we assume such a function $\varphi$ exists (proven on the next page), then take

$$f_n(x) = \frac{1}{n}\varphi(nx).$$

Then notice that if we take a nonnegative smooth function $\eta(x)$ such that $\eta \equiv 1$ on a ball around the origin containing $\mathrm{supp}(\varphi)$ of radius $R$, and is zero outside of a ball of radius $R + \varepsilon$ for some $\varepsilon > 0$, then we see that

$$|f_n(x)| \leq \|\varphi\|_\infty \eta(x).$$

Then by dominated convergence theorem,

$$\lim_{n\to\infty}\int f_n d\mu = 0.$$

However,

$$\int f_n d\mu = \delta'(f_n) = -f_n'(0) = -1$$

for all $n$. This is the desired contradiction.

It remains to prove that $\varphi$ exists, and also the function $\eta$ described above exists. To show the existence of $\varphi$, take a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(x) = \begin{cases} 1 & |x| \le 1 \\ \text{in between } [0,1] & 1 < |x| < 2 \\ 0 & |x| \ge 2. \end{cases}$$

Such a function $\psi$ can be constructed via

$$h(t) = \frac{f(2-t)}{f(2-t) + f(t-1)}$$

where

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0. \end{cases}$$

Then it can be shown that $f$ is smooth (Lee smooth manifold Lemma 2.20), and $h(t) \equiv 1$ for all $t \le 1$, and $h(t) \in (0,1)$ for all $1 < t < 2$, and $h(t) \equiv 0$ for all $t \ge 2$ (Lee Lemma 2.21). Then simply take

$$\psi(x) = h(|x|).$$

Then it is easy to verify that $\psi$ is smooth with the desired properties, from the fact that $h$ is smooth and the result we arrived above. Then we take

$$\varphi(x) = x\psi(x).$$

We see that $\operatorname{supp}(\varphi) \subset \operatorname{supp}(\psi) \subset [-2,2]$. Therefore $\varphi \in \mathcal{S}(\mathbb{R})$, and we have that

$$\varphi'(x) = x\psi'(x) + \psi(x).$$

So $\varphi'(0) = \psi(0) = 1$. This shows the existence of $\varphi$. The construction above also shows that if we take

$$\eta(x) = \frac{f(R+\varepsilon-x)}{f(R+\varepsilon-x) + f(x-R)}$$

for $R > 2$, then $\eta(x)$ has the desired properties.

# Chapter 6: Bounded Operators

**Problem 6.1.** Let $\ell \in Y^*, T_n$, let $T \in \mathscr{L}(X,Y)$, and let $x \in X$. Then we have

$$|\ell(T_n x) - \ell(Tx)| \leq \|\ell\| \cdot \|T_n x - Tx\| \leq \|\ell\| \cdot \|T_n - T\| \cdot \|x\|.$$

From the first inequality we see that strong topology is stronger than weak topology, and from the second inequality we see that uniform operator topology is stronger than than strong topology.

**Problem 6.3*.** Let $X$ and $Y$ be Banach spaces. We want to prove that if $T_n \in \mathcal{L}(X,Y)$ and $\{T_n x\}$ is a Cauchy sequence for each $x \in X$, then there exists $T \in \mathcal{L}(X,Y)$ so that $T_n \to T$ in the strong operator topology.

Since for each $x \in X$, the sequence $\{T_n x\}$ is a Cauchy sequence in a Banach space $Y$, we have that for every $x \in X$, the sequence $\{T_n x\}$ converges in $Y$. Therefore, define pointwisely

$$Tx = \lim_{n\to\infty} T_n x.$$

Then we see that for every $x \in X$, we have $\|T_n x - Tx\|_Y \to 0$ by definition. Therefore we conclude that $T_n \to T$ in the strong operator topology. It remains to show that $T \in \mathcal{L}(X,Y)$. We first prove that $T$ is linear. This is easy: Let $x, y \in X$ and $\lambda \in \mathbb{C}$, we have

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty} (T_n x + T_n y) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) = T(x) + T(y).$$

The second to last step is because we know both limit exist. The other equality follows similarly:

$$T(\lambda x) = \lim_{n\to\infty} T_n(\lambda x) = \lim_{n\to\infty} (\lambda T_n x) = \lambda \lim_{n\to\infty} T_n(x) = \lambda T(x).$$

This proves the linearity. To prove the boundedness of $T$, we notice that for each $x \in X$, the sequence $\{\|T_n x\|_Y\}$ is bounded: This is because for every $x \in X$ and every $n$,

$$\|T_n x\| \leq \|T_n x - T_x\| + \|Tx\|.$$

We can find a sufficiently large $N$ such that $\|T_n x - T_x\| < 1$ for all $n \geq N$. Then the sequence $\|T_n x\|$ is bounded by

$$C_x = \max\{\|T_1 x\|, ..., \|T_{N-1} x\|, 1 + \|T_x\|\}.$$

Therefore by the uniform boundedness principle, the family $\{\|T_n\|\}$ is bounded by some constant $C < \infty$. Therefore,

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \lim_{n\to\infty} \|T_n x\| \leq \sup_{\|x\|=1} \lim_{n\to\infty} \|T_n\| \cdot \|x\|_X \leq C < \infty.$$

This shows that $T$ is bounded, hence $T \in \mathcal{L}(X, Y)$. This concludes the proof.

**Problem 6.5*.** Let $T_t$ be an operator on $L^2(\mathbb{R})$ with $T_t\varphi(x) := \varphi(x+t)$. What is the norm of $T_t$? To what operator does $T_t$ converges as $t \to \infty$ and in what topology?

We first calculate the norm of $T_t$:

$$\begin{aligned}
\|T_t\| &= \sup_{\|\varphi\|_2=1} \|T_t\varphi\|_2 \\
&= \sup_{\|\varphi\|_2=1} \left( \int_{-\infty}^{\infty} |\varphi(x+t)|^2 dx \right)^{1/2} \\
&= \sup_{\|\varphi\|_2=1} \left( \int_{-\infty}^{\infty} |\varphi(y)|^2 dy \right)^{1/2} \\
&= \sup_{\|\varphi\|_2=1} \|\varphi\|_2 = 1.
\end{aligned}$$

Note that we used the change of variable $y = x + t$. Also, we claim that

$$T_t \to 0 \text{ in weak operator topology.}$$

To prove this claim, we first need

> **Lemma.** Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. The set of complex-valued continuous function on $\mathbb{R}$ with compact support, denoted by $C_c(\mathbb{R})$, is dense in $L^p(\mathbb{R}, \mu)$.
>
> *Proof of Lemma.* We first show that for each measurable set $A$ with $\mu(A) < \infty$, the characteristic function $\chi_A$ can be approximated by functions in $C_c(\mathbb{R})$ in $L^p$ norm. Let $\varepsilon > 0$. Since the Lebesgue measure is regular, one can find an open set $U$ and a compact set $K$ such that $K \subset A \subset U$ and
>
> $$\mu(U - K) = \mu(U) - \mu(K) < \varepsilon^p/2^p.$$
>
> For a proof of this statement see this link. Then since $\mathbb{R}$ is locally compact and Hausdorff, we can apply Urysohn's lemma for locally compact

Hausdorff spaces and get a function $f \in C_c(\mathbb{R})$ such that $0 \leq f \leq 1$, and $f|_K \equiv 1$, and supp$f \subset U$. It takes some work to show that $f$ is indeed compactly supported. We refer the reader to this link. Therefore,

$$\int |\chi_A - f|^p d\mu = \int_{U-K} |\chi_A - f|^p d\mu < \varepsilon^p.$$

So we see that

$$\|\chi_A - f\|_p < \varepsilon.$$

Thus $\chi_A$ can be approximated by functions in $C_c(\mathbb{R})$. It follows then that all simple functions of the form

$$s = \sum_{i=1}^{n} c_i \chi_{A_i}$$

where each $A_i$ has a finite measure, and they are mutually disjoint, can be approximated by a compactly supported function. But we know simple functions are dense in $L^p(\mathbb{R})$. So for every function $g \in L^p(\mathbb{R})$, take a sequence of simple functions $s_n$ converging to $g$ in $L^p$, and for each $n$ take a sequence of functions $f_{nj}$ in $C_c(\mathbb{R})$ converging to $s_n$. Then by a diagonal argument and triangle inequality we see that the sequence $\{f_{nn}\}_{n=1}^{\infty}$ converges to $g$ in $L^p$. This proves the Lemma. □

To actually prove our claim, note that $L^2(\mathbb{R})$ is a Hilbert space with $(L^2)^* = L^2$ by the Riesz representation theorem. Therefore we can identify a linear functional with $(\psi, \cdot) : L^2 \to \mathbb{C}$ for some $\psi \in L^2(\mathbb{R})$.

First, suppose that $\psi \in C_c(\mathbb{R})$. Since $\psi$ is compactly supported, we may assume supp$\psi \subset [-M, M]$ for some $M > 0$. Then, we have that

$$\begin{aligned}
|(\psi, T_t\varphi)| &= \left|\int_{-\infty}^{\infty} \overline{\psi(x)}\varphi(x+t)dx\right| \\
&\leq \int_{-\infty}^{\infty} |\overline{\psi}(x)\varphi(x+t)|dx \\
&= \int_{-\infty}^{\infty} |\psi(x)\chi_{[-M,M]}(x)\varphi(x+t)|dx \\
&\leq \left(\int_{-\infty}^{\infty} |\psi(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |\varphi(x+t)\chi_{[-M,M](x)}|^2 dx\right)^{1/2} \\
&= \|\psi\|_2 \cdot \left(\int_{-M}^{M} |\varphi(x+t)|^2 dx\right)^{1/2} = \|\psi\|_2 \cdot \left(\int_{-M-t}^{M-t} |\varphi(x)|^2 dx\right)^{1/2}.
\end{aligned}$$

Since $\varphi \in L^2(\mathbb{R})$, given $\varepsilon > 0$, there exists a large enough $t > 0$ such that

$$\int_{-M-t}^{M-t} |\varphi(x)|^2 dx < \frac{\varepsilon^2}{\|\psi\|_2^2}.$$

Then for this $t$ we have

$$|(\psi, T_t\varphi)| < \varepsilon.$$

Therefore, we have shown that for every $\psi \in C_c(\mathbb{R})$, we have

$$\lim_{t\to\infty} (\psi, T_t\varphi) = 0.$$

Now let $\psi \in L^2(\mathbb{R})$ be arbitrary. By the Lemma, there exists a sequence $\psi_n$ in $C_c(\mathbb{R})$ such that $\psi_n \to \psi$ in $L^2$. We claim that

$$(\psi, T_t\varphi) = \lim_{n\to\infty} (\psi_n, T_t\varphi).$$

This is because

$$\begin{aligned}
|(\psi, T_t\varphi) - (\psi_n, T_t\varphi)| &= |(\psi - \psi_n, T_t\varphi)| \\
&= \left| \int_{-\infty}^{\infty} \overline{(\psi - \psi_n)(x)} \varphi(x+t) dx \right| \\
&\le \int_{-\infty}^{\infty} |(\psi - \psi_n)(x)\varphi(x+t)| dx \\
&\le \|\psi - \psi_n\|_2 \cdot \left( \int_{-\infty}^{\infty} |\varphi(x+t)|^2 dx \right)^{1/2} \\
&= \|\psi - \psi_n\|_2 \cdot \|\varphi\|_2 \to 0
\end{aligned}$$

as $n \to \infty$. Then we claim that

$$\lim_{t\to\infty} (\psi, T_t\varphi) = 0.$$

Let $\varepsilon > 0$ be given. Since we see that

$$|(\psi, T_t\varphi) - (\psi_n, T_t\varphi)| \le \|\psi - \psi_n\|_2 \cdot \|\varphi\|_2$$

and the RHS is independent of $t$, we can choose $n$ large enough such that

$$|(\psi, T_t\varphi) - (\psi_n, T_t\varphi)| < \frac{\varepsilon}{2}$$

for all $t$. Now choose $t$ large enough such that

$$|(\psi_n, T_t\varphi)| < \frac{\varepsilon}{2}.$$

Then for this $t$, we see that

$$|(\psi, T_t\varphi)| \leq |(\psi, T_t\varphi) - (\psi_n, T_t\varphi)| + |(\psi_n, T_t\varphi)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the claim.

**Problem 6.7*.** Consider the operator $A : \ell_2 \to \ell_2$ defined by $(A\xi)_n = \frac{1}{n}\xi_n$. More explicitly,

$$A(\xi_1, \xi_2, \xi_3, ...) = \left(\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, ...\right).$$

Then since $\|A\xi\|_2 = (\sum_{k=1}^{\infty} \frac{1}{n^2}|\xi_n|^2)^{1/2} \leq (\sum_{k=1}^{\infty} |\xi_n|^2)^{1/2} = \|\xi\|_2$, we have $\|A\| \leq 1$. So $A$ is bounded. Now, consider $\eta \in \ell_2$ given by $\eta = (1, \frac{1}{2}, \frac{1}{3}, ...)$. We see that $\eta \notin$ Range$(A)$, since $(1, 1, ...) \notin \ell_2$. On the other hand, we have $\eta^n = (1, \frac{1}{2}, ..., n, 0, 0, ...) = A(1, 1, ..., 1, 0, ...)$ for all $n$. So clearly $\eta^n \in$ Range$(A)$. And we see that $\eta^n \to \eta$ in $\ell_2$ norm. Hence the range of $A$ is not closed.

Assuming now that $T$ is an operator on a Banach space, and suppose $Tx_n \to y$. Then $\{Tx_n\}$ is a Cauchy sequence. Since $T$ is an isometry, we have that $\{x_n\}$ is also a Cauchy sequence. Since we are in a Banach space, so say $x_n \to x$. Then again by the assumption that $T$ is an isometry, we have that $Tx_n \to Tx$. Since limits are unique, $y = Tx$. This shows that $y \in$ Range$(T)$ and that Range$(T)$ is closed.

**Problem 6.8.** Suppose $A$ is a self-adjoint bounded operator on a Hilbert space. Let $A\varphi = \lambda\varphi$ with $\|\varphi\| \neq 0$. Then we have

$$\lambda(\varphi, \varphi) = (A\varphi, \varphi) = (\varphi, A\varphi) = (\varphi, \lambda\varphi) = \overline{\lambda}(\varphi, \varphi).$$

Therefore $\lambda = \overline{\lambda}$. Hence $\lambda$ must be real. Suppose now $A\psi = \mu\psi$ is another eigenvector equation with $\mu \neq \lambda$. Then we have

$$(\mu - \lambda)(\psi, \varphi) = (A\psi, \varphi) - (\psi, A\varphi) = 0.$$

In the last step we used the fact that $A$ is self adjoint, and $A$ has real eigenvectors as we showed before. Since $\mu - \lambda \neq 0$, we must have $(\psi, \varphi) = 0$. Hence eigenvectors corresponding to distinct eigenvalues are orthogonal.

**Problem 6.9*.**

(a) Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|.$$

Hint: Note that

$$\mathrm{Re}(x, Ay) = \frac{1}{4}[(x+y, A(x+y)) - (x-y, A(x-y))].$$

Then using

$$|(z, Az)| \leq \|z\|^2 \sup_{\|u\|=1} |(u, Au)|$$

and the parallelogram law to prove that

$$|(x, Ay)| \leq \sup_{\|u\|=1} |(u, Au)|$$

if $\|x\| = \|y\| = 1$.

We first prove the equations displayed in the hint. The first equality follows from direct computation and the fact that $A^* = A$. The second inequality is simply because $|(z/\|z\|, Az/\|z\|)| \leq \sup_{\|u\|=1} |(u, Au)|$ and we can take out a factor of $\|z\|^{-2}$ on the left and multiply it to the right. Finally, suppose $\|x\| = \|y\| = 1$, and set

$$\eta = \sup_{\|u\|=1} |(u, Au)|.$$

By the first two equations, we get that

$$\begin{aligned} \mathrm{Re}(x, Ay) &= \frac{1}{4}[(x+y, A(x+y)) - (x-y, A(x-y))] \\ &\leq \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2\right)\eta \\ &= \frac{1}{4}(2\|x\|^2 + 2\|y\|^2)\eta = \eta. \end{aligned}$$

In the second step we used the second equation from the hint, and in the third step we used the parallelogram law. This shows that for every pair of unit vectors $x, y$, we have the inequality

$$\mathrm{Re}(x, Ay) \leq \eta.$$

Now write $(x, Ay) = |(x, Ay)|e^{i\theta}$ for some $\theta$. Then $e^{i\theta}y$ is another unit vector. So we get that

$$\eta \geq \mathrm{Re}(x, Ae^{i\theta}y) = \mathrm{Re}(e^{-i\theta}(x, Ay)) = \mathrm{Re}|(x, Ay)| = |(x, Ay)|.$$

Therefore, for every pair of unit vectors $x, y$, we have that

$$|(x, Ay)| \leq \eta.$$

This proves the last inequality of the hint. Also notice that

$$\|Ax\| = \sup_{\|y\|=1} |(Ax, y)|.$$

The RHS is clearly smaller than $\|Ax\|$ by Cauchy-Schwarz inequality. Setting $y = x/\|x\|$ gives $|(Ax, y)| = |(Ax, Ax/\|x\|)| = \|Ax\|$. Hence the LHS is smaller than the RHS. Then if $\|x\| = 1$, we have that

$$\|Ax\| = \sup_{\|y\|=1} |(Ax, y)| = \sup_{\|y\|=1} |(x, Ay)| \leq \eta$$

by the previous inequality. Therefore,

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \leq \eta.$$

This proves the claim.

(b) Find an example which shows that the conclusion of (a) need not be true if $A$ is not self-adjoint.

For an example where the statement fails if $A$ is not self-adjoint, consider $\mathcal{H} = \mathbb{C}^2$ with the usual norm and inner product. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This operator is not self adjoint since

$$A^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq A.$$

Let $\varphi = (x, y) \in \mathcal{H}$ be a vector, then

$$\sup_{\|\varphi\|=1} |(\varphi, A\varphi)| = \sup_{|x|^2+|y|^2=1} |x\overline{y}| = \sup_{|x|^2+|y|^2=1} |x| \cdot |y| = \sup_{a^2+b^2=1} ab$$

where $a, b$ are non-negative real numbers. And it is easy to verify that its value is $1/2$. Hence we have that

$$\sup_{\|\varphi\|=1} |(\varphi, A\varphi)| = \frac{1}{2}.$$

But on the other hand,

$$\|A\| = \sup_{\|\varphi\|=1} |A\varphi| = \sup_{|x|^2+|y|^2=1} |y| = 1.$$

Thus the statement fails when $A$ is not self-adjoint.

**Problem 6.10*.** Show that the spectral radius of the Volterra integral operator

$$(Tf)(x) = \int_0^x f(y)dy$$

as a map on $C[0,1]$, with the supremum norm, is equal to zero. What is the norm of $T$?

We claim that $\|T\| = 1$. To see one inequality, notice that

$$\|T\| = \sup_{\|f\|_\infty=1, f\in C[0,1]} \|Tf\|_\infty = \sup_{\|f\|_\infty=1} \sup_{x\in[0,1]} \left| \int_0^x f(y)dy \right| \le \sup_{\|f\|_\infty=1} \|f\|_\infty = 1.$$

To see the other inequality, notice that $\chi_{[0,1]} \in C[0,1]$ and its norm is 1. Therefore,

$$\|T\| \ge \|T\chi_{[0,1]}\|_\infty = \sup_{x\in[0,1]} \left| \int_0^y \chi_{[0,1]}(y)dy \right| = 1.$$

Therefore $\|T\| = 1$. We claim that $r(T) = 0$. To do this, we first show that, by induction

$$(T^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-r)^{n-1} f(r)dr. \tag{*}$$

Note that when $n = 1$, (*) is clearly true by definition. Suppose now (*) is true for $(T^n f)(x)$, to complete the inductive step, we need to check that

$$(T^{n+1} f)(x) = \frac{1}{n!} \int_0^x (x-r)^n f(r)dr.$$

Now

$$(T^{n+1} f)(x) = \int_0^x (T^n f)(y)dy = \int_0^x \int_0^y \frac{1}{(n-1)!} (y-r)^{n-1} f(r)drdy.$$

So we only need to show that

$$\int_0^x \int_0^y (y-r)^{n-1} f(r)drdy = \frac{1}{n} \int_0^x (x-r)^n f(r)dr. \tag{**}$$

The natural way to check (**) is by applying Fubini's theorem and exchange the integrals. We notice that since $x, y \in [0,1]$ and $f \in C[0,1]$, and we are working on the domain

$$\Omega = \{(r,y) : 0 \leq r \leq y, 0 \leq y \leq x\} \subset [0,1]^2.$$

Therefore,

$$\begin{aligned}
\int_0^x \int_0^y |(y-r)^{n-1} f(r)| dr dy &\leq \|f\|_\infty \int_0^x \int_0^y |(y-r)^{n-1}| dr dy \\
&= \|f\|_\infty \int_0^x \int_0^y (y-r)^{n-1} dr dy \\
&= \|f\|_\infty \int_0^x \left[ -\frac{(y-r)^n}{n} \right]_{r=0}^{r=y} dy \\
&= \|f\|_\infty \int_0^x \frac{y^n}{n} dy \\
&\leq \frac{\|f\|_\infty}{n} < \infty.
\end{aligned}$$

Hence the Fubini theorem applies and we may change the order of integration. Evaluating the left hand side of (**) now gives

$$\begin{aligned}
\int_0^x \int_0^y (y-r)^{n-1} f(r) dr dy &= \int_0^x \int_r^x (y-r)^{n-1} f(r) dy dr \\
&= \int_0^x \left[ \frac{(y-r)^n}{n} f(r) \right]_{y=r}^{y=x} dr \\
&= \frac{1}{n} \int_0^x (x-r)^n f(r) dr.
\end{aligned}$$

This proves (**). Consequently we have proven (*) by induction. Now using (*) we can do the following estimate:

$$\begin{aligned}
|(T^n f)(x)| &= \left| \frac{1}{(n-1)!} \int_0^x (x-r)^{n-1} f(r) dr \right| \\
&\leq \frac{\|f\|_\infty}{(n-1)!} \frac{x^n}{n} \leq \frac{\|f\|_\infty}{n!}
\end{aligned}$$

where in the last step we used that $|x| \leq 1$. Since this holds for every $x \in [0,1]$, we conclude that

$$\|T^n\| \leq \frac{1}{n!}.$$

Now since $C[0,1]$ is a Banach space with the sup norm, we have that

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} \leq \lim_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0.$$

This proves the claim. To show that $(1/n!)^{1/n} \to 0$ as $n \to \infty$, note that by using the AM-GM inequality, one has

$$\left(\frac{1}{n!}\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} \leq \frac{1}{n}\left(1 + \int_1^n \frac{1}{t}dt\right) = \frac{1+\ln(n)}{n}$$

for $n \geq 1$. And the right hand side clearly goes to zero as $n$ goes to infinity.

**Problem 6.14*.**

(a) Suppose $A_n \geq 0$ are positive operators and $A_n \to A$ in norm. Then we want to show that $\sqrt{A_n} \to \sqrt{A}$ in norm as well. Since positive operators are bounded, we may assume without loss of generality $\|A_n\|, \|A\| \leq 1$ (otherwise just normalize the operator). Then we can expand $\sqrt{A}$ as follows (Lemma on pg 195 of the textbook):

$$\sqrt{A} = \sum_{n=0}^{\infty} c_n (I - A)^n$$

since $\|I - A\| \leq 1$. Then we have that

$$\begin{aligned}\left\|\sqrt{A_k} - \sqrt{A}\right\| &= \left\|\sum_{n=0}^{\infty} c_n[(I - A_k)^n - (I - A)^n]\right\| \\ &\leq \sum_{n=0}^{\infty} |c_n|(\|(I - A_k)^n\| + \|(I - A)^n\|) \\ &\leq \sum_{n=0}^{\infty} 2|c_n| < \infty.\end{aligned}$$

In the last step, we have used the lemma on pg 195 again since $\sqrt{1-z}$ converges absolutely at $|z| = 1$. Then $\sum_{n=0}^{\infty} 2|c_n|$ will later serve as a dominating function for $\|\sqrt{A_k} - \sqrt{A}\|$ where $k = 1, 2, 3....$

Now we show that if $\|B_n - B\| \to 0$ as $n \to \infty$, then $\left\|B_n^k - B^k\right\| \to 0$ for all $k$. We show this by induction: $k = 1$ is trivial, and for the inductive step we have

$$\begin{aligned}\left\|B_n^k - B^k\right\| &= \left\|B_n^k - B_n^{k-1}B + B_n^{k-1}B - B^k\right\| \\ &\leq \|B_n\|^{k-1}\|B_n - B\| + \|B\|\left\|B_n^{k-1} - B^{k-1}\right\| \to 0.\end{aligned}$$

The first term goes to zero by assumption and the second term goes to zero by inductive hypothesis. Then we can apply[2] the dominated convergence theorem and conclude that

$$\begin{aligned}\lim_{k\to\infty}\left\|\sqrt{A_k} - \sqrt{A}\right\| &\leq \lim_{k\to\infty}\sum_{n=0}^{\infty}|c_n| \cdot \|(I - A_k)^n - (I - A)^n\| \\ &= \sum_{n=0}^{\infty}\lim_{k\to\infty}|c_n| \cdot \|(I - A_k)^n - (I - A)^n\| = 0.\end{aligned}$$

Therefore we have shown that $\sqrt{A_k} \to \sqrt{A}$ in norm.

(b) Now suppose $A_n \to A$ strongly, and $A_n \geq 0$. We want to show that $\sqrt{A_n} \to \sqrt{A}$ strongly. Since $A_n x \to Ax$ for all $x$, there exists $N$ such that $\|A_n x\| \leq \|Ax\| + \varepsilon$ for all $n \geq N$. Then the family $\{\|A_n x\| : n \in \mathbb{N}\}$ is bounded by $\max\{\|A_1 x\|, ..., \|A_N x\|, \|Ax\| + \varepsilon\}$. Hence by the uniform boundedness principle, $\|A_n\| < \infty$ for all $n$. Again without loss of generality we may assume $\|A_n\|, \|A\| \leq 1$. Fix $x$, we again have

$$\left\|\sqrt{A_k}x - \sqrt{A}x\right\| \leq \sum_{n=0}^{\infty} 2|c_n| \cdot \|x\|.$$

Again, one can show that if $\|B_k x - Bx\| \to 0$ then $\|B_k^n x - B^n x\| \to 0$ for all $n$ by induction. The argument is exactly the same so we won't repeat the derivation here. Then applying dominated convergence theorem again we get

$$\lim_{k\to\infty}\left\|\sqrt{A_k}x - \sqrt{A}x\right\| \leq \lim_{k\to\infty}\sum_{n=0}^{\infty}|c_n| \cdot \|(I - A_k)^n x - (I - A)^n x\| = 0.$$

This shows that $\sqrt{A_k}x \to \sqrt{A}x$ for all $x$. Thus $\sqrt{A_k} \to \sqrt{A}$ strongly. This completes the proof.

[2] This is exactly the dominated convergence theorem applied to the measure which is the sum of the point-mass measures at each integer.

**Problem 6.15.**

(a) Let $A_n \to A$ in norm, we want to prove that $|A_n| \to |A|$ in norm. First, since the map $T \mapsto T^*$ is an isometric isomorphism, we have that $\|T\| = \|T^*\|$. Therefore $A_n^* \to A^*$ as well. Therefore,

$$\begin{aligned}\|A_n^* A_n - A^* A\| &\leq \|A_n^* A_n - A_n^* A\| + \|A_n^* A - A^* A\| \\ &\leq \|A_n^*\| \cdot \|A_n - A\| + \|A_n^* - A^*\| \cdot \|A\| \to 0.\end{aligned}$$

Thus $A_n^* A_n \to A^* A$ in norm. Since $T^* T \geq 0$, by problem 14 part (a), we have that $\sqrt{A_n^* A_n} \to \sqrt{A^* A}$ in norm, i.e. $|A_n| \to |A|$ in norm.

(b) The argument is similar to part (a), and we then use positivity of $T^* T$ and apply problem 14 part (b).

(c) If a map $f : X \to Y$ is weakly continuous, then given any weakly convergent sequence $x_n \xrightarrow{w} x$, we must also have $T(x_n) \xrightarrow{w} T(x)$. To show that $|\cdot| : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ is not weakly continous, consider the shift operator $T_n : \ell_2 \to \ell_2$ defined by

$$T_n(a_1, a_2, ...) = (\underbrace{0, \cdots, 0}_{n}, a_1, a_2, ...).$$

Then $T_n \xrightarrow{w} 0$ by example on pg 184 of the textbook. And example on pg 185 shows that

$$T_n^*(a_1, a_2, ...) = (a_{n+1}, a_{n+2}, ...).$$

Therefore $T_n^* T_n = I$. But this shows that the absolute value function maps $|T_n| = \sqrt{T_n^* T_n}$ to $I$ for all $n$. If it were weakly continuous, we should have gotten $|T_n| \xrightarrow{w} 0$.

**Problem 6.17.** Let $\sigma_1, \sigma_2, \sigma_3$ be defined as in problem 6.16. Then consider $A = \sigma_3 + \sigma_1$, and let $B = \sigma_3 + I$, then $A - B = \sigma_1 - I$. This is the desired counterexample.

**Problem 6.19.** We have

$$\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

**Problem 6.20*.** It is a topological fact that if $X$ is reflexive, then every bounded sequence $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_k}\}$. Since $T$ maps weakly convergent sequences into norm convergent sequences, then $\{Tx_{n_k}\}$ is a (norm) convergent subsequence.

Here is an alternative (and much more concrete solution):

**Problem 6.20*.** Let $X$ be a reflexive Banach space and $T : X \to X$ a bounded linear operator. Furthermore assume $T$ takes weakly convergent sequences into norm convergent sequences. Then we want to show that $T$ is compact. We first present a lemma:

> *A closed subspace $Y$ of a reflexive Banach space $X$ is itself reflexive.*

Assuming that this lemma is given, we now move on to the proof. First, *we assume that $X^*$ is separable.* Then $X$ is also separable. Let $\{x_n\} \subset X$ be a bounded sequence, i.e. $\sup_n |x_n| \leq M$. It suffices to find a weakly convergent subsequence of $\{x_n\}$ (and $T$ will map it to a norm convergent subsequence). Since $X^*$ is separable, then there exists a countable dense set $\{l_m\} \subset X^*$. First, consider $\{l_1(x_n)\}$. This is a bounded sequence in $\mathbb{C}$ because $|l_1(x_n)| \leq \|l_1\| \cdot \|x_n\| \leq M\|l_1\| < \infty$. By Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{l_1(x_{n_k})\}$. Now take a convergent subsequece for $l_2$ from this subsequence and do so infinitely for all $l_m$. Use diagonal argument, we obtain a sequence which we still call $\{x_{n_k}\}$, such that $l_m(x_{n_k})$ converges for all $m$.

By denseness, for all $l \in X^*$, there exists $\varepsilon > 0$ such that $\|l - l_m\| < \varepsilon$ for some $m$. Hence we have

$$\begin{aligned} |l(x_{n_k}) - l(x_{n_j})| &\leq |l(x_{n_k}) - l_m(x_{n_k})| + |l_m(x_{n_k}) - l_m(x_{n_j})| + |l_m(x_{n_j}) - l(x_{n_j})| \\ &\leq \|l - l_m\| \cdot \|x_{n_k}\| + \|l - l_m\| \cdot \left\|x_{n_j}\right\| + |l_m(x_{n_k}) - l_m(x_{n_j})| \\ &\leq 2M\varepsilon + \varepsilon \end{aligned}$$

for sufficiently large $k, j$. This shows that $\{l(x_{n_k})\}$ is a Cauchy sequence. To prove that the weak limit of $\{x_{n_k}\}$ exists, define

$$y(l) = \lim_{k\to\infty} l(x_{n_k})$$

for all $l \in X^*$. This is a well-defined, bounded linear functional on $X^*$ because $|y(l)| \leq \|l\|M$. Since $X$ is reflexive, there exists $x \in X$ such that $Jx = y$ ($J$ is defined in Theorem III.4 of the textbook). Then we have

$$\lim_{k\to\infty} l(x_{n_k}) = y(l) = Jx(l) = l(x)$$

for all $l \in X^*$. Thus $x$ is the weak limit of $x_{n_k}$.

In general, $X$ need not be separable. Let $Y = \overline{\text{span}\{x_n\}}$ where $\{x_n\}$ is a bounded sequence. Then $Y$ is separable because the span of $x_n$ with coefficients in $\mathbb{Q} + i\mathbb{Q}$ is a dense subset in $Y$. Also by the previous lemma, $Y$ is reflexive. This shows that $\lim_{k\to\infty} l(x_{n_k}) = l(x)$ for all $l \in Y^*$, just like the separable case. This concludes the proof, because we only need to consider $l|_Y$.

**Problem 6.22.** Note that if we set $G = \sum_{j=1}^r F_j(x)H_j(y)$, where $F_j, H_j \in C([a,b])$, then the integral operator

$$(G'f)(x) = \int_a^b G(x,y)f(y)dy$$

is of finite rank $r$. Suppose $K \in C([a,b]^2)$. By Stone-Weierstrass theorem, the set $\mathcal{A}$ of functions of the form $\sum_{j=1}^r F_j(x)H_j(y)$ is dense in $C([a,b]^2)$. Therefore we can find a sequence $K_n \in \mathcal{A}$ that converges to $K$ in operator norm. Also, each operator $K_n'$ is of finite rank. Finally, we have that

$$\begin{aligned}
|(T - K_n')(f)(x)| &= \left|\int_a^b (K - K_n)(x,y)f(y)dy\right| \\
&\le \int_a^b |(K - K_n)(x,y)f(y)|dy \\
&\le \|K - K_n\|_\infty \|f\|_\infty (b-a).
\end{aligned}$$

Therefore, we have that

$$\|T - K_n'\| \le \|K - K_n\|_\infty \to 0$$

as $n \to \infty$. This concludes the proof.

**Problem 6.25.** Let $K \in L^2(M \times M, d\mu \otimes d\mu)$. We will show that the integral operator

$$(A_K\varphi)(x) = \int_M K(x,y)\varphi(y)d\mu(y)$$

on $L^2(M, d\mu)$ is well defined and $\|A_K\| \le \|K\|_{L^2}$. Since $\varphi \in L^2(M, d\mu)$, by Hölder inequality we have

$$\int_M |K(x,y)\varphi(y)|d\mu(y) \le \left(\int_M |K(x,y)|^2 d\mu(y)\right)^{1/2} \|\varphi\|_{L^2}$$

for every $x \in X$. Therefore integrating again we get

$$\begin{aligned}
\int_M \left(\int_M |K(x,y)\varphi(y)|d\mu(y)\right)^2 d\mu(x) &\le \left(\int_M \int_M |K(x,y)|^2 d\mu(y)d\mu(x)\right) \|f\|_{L^2}^2 \\
&= \|K\|_{L^2(M\times M, d\mu\otimes d\mu)}^2 \|f\|_{L^2}^2 < \infty.
\end{aligned}$$

Note that the equality follows from the fact that we can exchange the order of integration since $|K(x,y)|^2$ is measurable on $M \otimes M$ and we can apply Tonelli's theorem (See, for example, Axler's measure theory theorem 5.28). This shows that $\int_M |K(x,y)\varphi(y)|d\mu(y)$ is finite for $\mu$-almost every $x \in X$. Also we have shown that $A_K\varphi \in L^2(M, d\mu)$, and the operator $A_K$ is well-defined. Notice that the inequality above also shows that

$$\|A_K\varphi\|_{L^2} \leq \|K\|_{L^2(M\times M, d\mu\otimes d\mu)}\|\varphi\|_{L^2}.$$

Hence we have shown that $\|A_K\| \leq \|K\|_{L^2}$.

**Problem 6.41.** Let $P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. We have $P^2 = P$ but $P^* \neq P$.

**Problem 6.45*.**

(a) Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$. Suppose $A$ is an operator such that

$$\sup_{\substack{\psi\in\{\phi_1,\ldots,\phi_n\}^{\perp} \\ \|\psi\|=1}} \|A\psi\| \to 0 \qquad \text{as } n \to \infty.$$

Then the claim is that the sequence of finite rank operators

$$A_n = \sum_{j=1}^{n} (\phi_j, \cdot)A\phi_j$$

converges to $A$ in norm. To see this, notice that

$$\|A - A_n\| = \sup_{\|\psi\|=1} \|A\psi - A_n\psi\| = \sup_{\|\psi\|=1} \left\| A\left(\psi - \sum_{j=1}^{n}(\phi_j, \psi)\phi_j\right)\right\|.$$

Now if $\psi \in \text{span}\{\phi_1, ..., \phi_n\}$, then since we have an orthonormal basis, then we have $\psi - \sum_{j=1}^{n}(\phi_j, \psi)\phi_j = 0$. Hence the right hand side will simply be zero. Thus without loss of generality, we may assume that $\psi \in \{\phi_1, ..., \phi_n\}^{\perp}$. Then, this shows that we have exactly

$$\|A - A_n\| = \sup_{\substack{\psi\in\{\phi_1,\ldots,\phi_n\}^{\perp} \\ \|\psi\|=1}} \|A\psi\| \to 0 \qquad \text{as } n \to \infty.$$

Hence we have shown that $A_n \to A$ in operator norm. Since each $A_n$ is of finite rank, $A_n$ is compact for all $n$. Then by Theorem VI.12 of the textbook, $A$ is compact.

(b) There is a mistake in the problem. The correct formulation should be: Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ and $A$ be compact. Then we want to show that
$$\sup_{\substack{\psi\in\{\phi_1,\ldots,\phi_n\}^{\perp}\\ \|\psi\|=1}} \|A\psi\| \to 0 \qquad \text{as } n\to\infty.$$
Note that the condition that $\|\psi\| = 1$ is necessary. This follows directly from the proof of Theorem VI.13 of the textbook: if we define
$$\lambda_n = \sup_{\substack{\psi\in\{\phi_1,\ldots,\phi_n\}^{\perp}\\ \|\psi\|=1}} \|A\psi\|$$
then $\lambda_n$ is exactly $\|A - A_n\|$ in the previous part, hence $\lambda_n \to 0$.

**Problem 6.46*.**

(a) Since $A \geq 0$, we know $A$ is self-adjoint. Also we are given that $A$ is compact. We want to show that $\sqrt{A}$ is also compact. By Hilbert-Schmidt theorem (Theorem VI.16 in textbook), there exists an orthonormal basis $\{x_n\}$ of $\mathscr{H}$ such that $Ax_n = \lambda_n x_n$ and $\lambda_n \to 0$. Then we have that
$$Ax = A\left(\sum_{n=1}^{\infty}(x,x_n)x_n\right) = \sum_{n=1}^{\infty}(x,x_n)Ax_n = \sum_{n=1}^{\infty}\lambda_n(x,x_n)x_n.$$
We then claim that
$$\sqrt{A}x = \sum_{n=1}^{\infty}\sqrt{\lambda_n}(x,x_n)x_n.$$
This is because
$$\begin{aligned}\sqrt{A}(\sqrt{A}x) &= \sum_{n=1}^{\infty}\sqrt{\lambda_n}(\sqrt{A}x,x_n)x_n\\ &= \sum_{n=1}^{\infty}\sqrt{\lambda_n}\left(\sum_{m=1}^{\infty}\sqrt{\lambda_m}(x,x_m)x_m, x_n\right)x_n\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{\lambda_n}\sqrt{\lambda_m}(x,x_m)\delta_{mn}x_n\\ &= \sum_{n=1}^{\infty}\lambda_n(x,x_n)x_n = Ax.\end{aligned}$$

Since we know square root of an operator is unique, this proves the claim. Now let

$$B_N = \sum_{n=1}^{N} \sqrt{\lambda_n}(\cdot, x_n)x_n.$$

This is a sequence of finite rank operators, hence compact. We claim that $B_N \to \sqrt{A}$ in norm, hence by Theorem VI.12 of the textbook, $\sqrt{A}$ will be compact. Now since $\lambda_n \to 0$, let $M_N = \sup_{n \geq N+1}\{\lambda_n\}$. Then

$$\begin{aligned}\left\|(B_N - \sqrt{A})x\right\|^2 &= \left\|\sum_{n=N+1}^{\infty} \sqrt{\lambda_n}(x, x_n)x_n\right\|^2 \\ &= \sum_{n=N+1}^{\infty} \lambda_n |(x, x_n)|^2 \leq M_N \|x\|^2.\end{aligned}$$

This shows that $\|B_N - \sqrt{A}\| \leq M_N$. Now since $\lambda_n \to 0$, we have that $M_N \to 0$ as $N \to \infty$. This concludes the proof.

(b) Since $B - A \geq 0$, we have $((B-A)x, x) \geq 0$ for all $x \in \mathscr{H}$. Hence $(Bx, x) \geq (Ax, x)$ for all $x$. Since $\sqrt{A}, \sqrt{B} \geq 0$, they are self-adjoint. Hence $(Bx, x) \geq (Ax, x)$ tells us that

$$\|\sqrt{B}x\| \geq \|\sqrt{A}x\|$$

for all $x$. Now let $\{x_n\} \subset \mathscr{H}$ be bounded. Then since $\sqrt{B}$ is compact by part (a), we have a convergent subsequence $\{\sqrt{B}x_{n_k}\}$. Thus for $k, j$ sufficiently large we have

$$\|\sqrt{B}x_{n_k} - \sqrt{B}x_{n_j}\| < \varepsilon.$$

But on the other hand

$$\|\sqrt{A}x_{n_k} - \sqrt{A}x_{n_j}\| = \|\sqrt{A}(x_{n_k} - x_{n_j})\| \leq \|\sqrt{B}(x_{n_k} - x_{n_j})\| < \varepsilon$$

for $k, j$ sufficiently large. Therefore $\{\sqrt{A}x_{n_k}\}$ is Cauchy in $\mathscr{H}$, hence converges. Thus $\sqrt{A}$ is compact. Since $\sqrt{A}$ is compact and bounded (by self-adjointness), we have $A = \sqrt{A}\sqrt{A}$ is also compact. This concludes the proof.