

lec 1 & lec 2.

Recall: Lusin's thm: $f: A \rightarrow \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$. Suppose:

1. μ a Borel regular measure on $\mathbb{R}^n$.
2. A is μ -measurable and $\mu(A) < \infty$.
3. $f: A \rightarrow \mathbb{R}^m$ is μ -measurable.

Then $\forall \varepsilon > 0$, $\exists K \subseteq A$ compact s.t. $f|_K$ is cts and $\mu(A-K) < \varepsilon$.

Thm 1. Let μ be Borel-regular on $\mathbb{R}^n$. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be μ -measurable. Assume $A \subseteq \mathbb{R}^n$ is μ -measurable s.t. $\mu(A) < \infty$. Then $\forall \varepsilon > 0$, $\exists$ cts function $\bar{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ s.t.

$$\mu\{x \in A: f(x) \neq \bar{f}(x)\} < \varepsilon$$

pf. By Lusin's thm, $\exists K \subseteq A$ cpt s.t. $\mu(A-K) < \varepsilon$ and $f|_K$ cts. Also by Evans Thm 1.13, we can extend the function $f|_K$ ctsly to $\mathbb{R}^n$. Let this function be $\bar{f}$. Then

$$\mu\{x \in A: f(x) \neq \bar{f}(x)\} \leq \mu(A-K) < \varepsilon. \quad \square$$

Def. Let X, Y be topological spaces. Then $f: X \rightarrow Y$ is Borel measurable if for any Borel measure μ on X .

Rmk. Cts functions are Borel measurable, for $U \subseteq Y$ open, f cts, then $f^{-1}(U)$ is open, thus Borel.

Thm 2 (Egoroff's Thm). Let μ be a measure on $\mathbb{R}^n$. Let $f_k(x): \mathbb{R}^n \rightarrow \mathbb{R}^m$ be μ -measurable. Let $A \subseteq \mathbb{R}^n$ be μ -measurable s.t. $\mu(A) < \infty$. Let $f_k \rightarrow f$ μ -a.e. in A . Then $\forall \varepsilon > 0$, $\exists B \subseteq A$, μ -measurable, s.t.

1. $\mu(A-B) < \varepsilon$
2. $f_k \rightarrow f$ uniformly in B .

pf. $\forall i, j \in \mathbb{N}$, let

$$C_{ij} = \bigcup_{k=j}^{\infty} \left\{ x: |f_k(x) - f(x)| > \frac{1}{2^i} \right\}.$$

So $x \in C_{ij}$ means $\exists k \geq j$ s.t. $|f_k(x) - f(x)| > \frac{1}{2^i}$. Then clearly $C_{i,j+1} \subseteq C_{ij}$. Then $\mu(A) < \infty$ so $\{A \cap C_{ij}\}_{j=1}^{\infty}$ is a sequence of nested sets. Then

$$\lim_{j \rightarrow \infty} \mu(A \cap C_{i,j}) = \mu\left(A \cap \left(\bigcap_{j=1}^{\infty} C_{i,j}\right)\right) = 0.$$

In our def, all measure zero set is measurable, so all measure spaces (Ω, μ) is complete. Thus if $f_k \rightarrow f$ μ -a.e. and f_k are μ -measurable, then f is μ -measurable $\Rightarrow C_{i,j}$ μ -measurable.

Last step: $x \in \bigcap_{j=1}^{\infty} C_{i,j} \Rightarrow \exists$ infinitely many indices $j \in \mathbb{N}$ s.t. $|f_j(x) - f(x)| > \frac{1}{2^i}$. So $f_k(x) \not\rightarrow f(x)$. But this is a null set by assumption. Thus $\forall \varepsilon > 0, \exists N_i$ s.t.

$$\mu(A \cap C_{i,N_i}) < \frac{\varepsilon}{2^i}.$$

Now denote $B = A - \bigcup_{i=1}^{\infty} C_{i,N_i}$. Then

$$\mu(A - B) \leq \sum_{i=1}^{\infty} \mu(A \cap C_{i,N_i}) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

If $x \in B \Rightarrow x \notin C_{i,N_i} \Rightarrow$ whenever $n \geq N_i, |f_n(x) - f(x)| \leq \frac{1}{2^i}$. Thus $f_n(x) \rightarrow f(x)$ uniformly on B . $\square$

Covering Theorems.

Notation. $\bar{B}_r(x) = \{y \in \mathbb{R}^n : |x - y| \leq r\}$. $\hat{B}_r(x) = \{y \in \mathbb{R}^n : |x - y| \leq sr\}$.

Def. (1) A collection $\mathcal{K}$ of closed ball in $\mathbb{R}^n$ is a **cover** of a set $A \in \mathbb{R}^n$, if $A \subseteq \bigcup_{\bar{B} \in \mathcal{K}} \bar{B}$.

(2) $\mathcal{K}$ is a **fine cover** of A , if in addition to (1), we have $\inf \{ \text{diam } \bar{B} : x \in \bar{B}, \bar{B} \in \mathcal{K} \} = 0 \quad \forall x \in A$.

i.e. every pt is covered by smaller and smaller balls.

Thm 1 (Vitali's Covering thm I). Let $\mathcal{K}$ be any collection of nondegenerate (i.e. different from a pt) closed balls in $\mathbb{R}^n$ with $\sup \{ \text{diam } \bar{B} : \bar{B} \in \mathcal{K} \} < \infty$. Then $\exists$ **countable** family G of **disjoint** balls in $\mathcal{K}$ s.t.

$$\bigcup_{\bar{B} \in \mathcal{K}} \bar{B} \subseteq \bigcup_{\bar{B} \in G} \hat{\bar{B}}.$$

Rmk. $\mathcal{K}$ may be uncountable & balls in $\mathcal{K}$ may overlap.

pf. Denote $D = \sup_{\bar{B} \in \mathcal{K}} (\text{diam } \bar{B})$. Define

$$\mathcal{K}_j = \{ \bar{B} \in \mathcal{K} : \frac{D}{2^j} < \text{diam } \bar{B} \leq \frac{D}{2^{j-1}} \}$$

Then $\mathcal{K}_1 = \{ \bar{B} \in \mathcal{K} : \frac{D}{2} < \text{diam } \bar{B} \leq D \}$ includes the largest balls in the family. Now define subfamilies $G_j \subseteq \mathcal{K}_j$ as follows:

(a) G_1 is any maximal disjoint collections of balls in $\mathcal{K}_1$.

[This means G_1 is a collection of disjoint balls in $\mathcal{K}_1$ s.t. if we add one more ball to G_1 from $\mathcal{K}_1$, the balls are no longer disjoint.]

Existence of G_1 proof: HW problem.

(b) Assuming we have chosen the maximal disjoint collections $G_1, \dots, G_{k-1}$ subsets of $\mathcal{K}_1, \dots, \mathcal{K}_{k-1}$. Choose $G_k \subseteq \mathcal{K}_k$ to be any maximal disjoint collection of

$$\{ \bar{B} \in \mathcal{K}_k : \bar{B} \cap \bar{B}' = \emptyset, \forall \bar{B}' \in \bigcup_{i=1}^{k-1} G_i \}.$$

Now, define

$$G = \bigcup_{k=1}^{\infty} G_k.$$

(1) All of G_k countable: they are families of disjoint balls in $\mathbb{R}^n$ and each ball contains a rational number $\Rightarrow$ at most countably many balls in G_k . Thus G is at most countable family of disjoint balls in $\mathcal{K}$.

(2) For any $\bar{B} \in \mathcal{K}$, $\exists \hat{\bar{B}}' \in G$ s.t. $\bar{B} \cap \hat{\bar{B}}' = \emptyset$ and $\bar{B} \subseteq \hat{\bar{B}}'$.

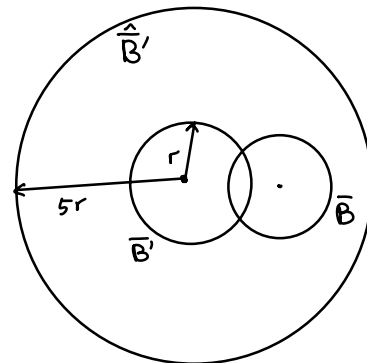
pf. Let $\bar{B} \in \mathcal{K}$. $\exists i \in \mathbb{N}$ s.t. $\bar{B} \in \mathcal{K}_i : D2^{-i} < \text{diam } \bar{B} \leq D2^{-(i-1)}$.

By maximality of G_i , $\exists \bar{B}' \in \bigcup_{k=1}^i G_k$ s.t. $\bar{B} \cap \bar{B}' \neq \emptyset$. Also,

$$\bar{B}' \in \bigcup_{k=1}^i G_k \subseteq \bigcup_{k=1}^i \mathcal{K}_k \Rightarrow \text{diam } \bar{B}' \geq \frac{D}{2^i} \geq \frac{1}{2} \text{diam } \bar{B}$$

So $\text{diam } \bar{B}' \geq \frac{1}{2} \text{diam } \bar{B}$ and $\bar{B} \cap \bar{B}' \neq \emptyset$. But together they imply $\bar{B} \subseteq \hat{\bar{B}}'$.

[$\text{diam } \bar{B}' \geq \frac{1}{2} \text{diam } \bar{B}$ implies $\bar{B}$ is at most $\text{diam } \bar{B} \leq 2 \text{diam } \bar{B}'$ away from $\bar{B}'$. So if we blow up $\bar{B}'$ 5 times, it will certainly contain $\bar{B}$.]



This proves the claim. $\square$

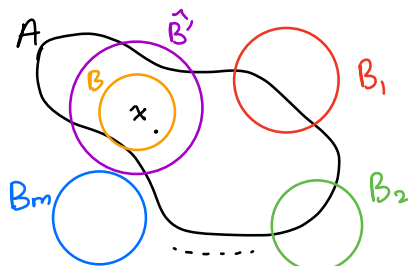
From now on, in all covering thm, balls B will be closed, i.e. $\bar{B} \rightarrow B$ now.

Thm 2 (Vitali's Covering Thm II). Let the family $\mathcal{K}$ of closed balls be a fine cover of the set $A \subseteq \mathbb{R}^n$ s.t. $\sup \{ \text{diam } B : B \in \mathcal{K} \} < \infty$.

Then $\exists$ a family $G \subseteq \mathcal{K}$ of disjoint balls s.t. for each finite subset $\{B_1, B_2, \dots, B_n\} \subseteq \mathcal{K}$, one has

$$A - \bigcup_{i=1}^n B_i \subseteq \bigcup_{B \in G - \{B_1, \dots, B_n\}} \hat{B}.$$

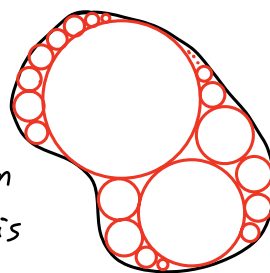
pf. Choose the family G of disjoint balls in $\mathcal{F}$ as in Thm 1.
 Assume $\{B_1, \dots, B_m\} \subseteq \mathcal{F}$. Now if $A - \bigcup_{i=1}^m B_i = \emptyset$, done.
 So Assume $x \in A - \bigcup_{i=1}^m B_i$. Then $x \notin \bigcup_{i=1}^m B_i$. Because
 $x \in A$ and $\mathcal{F}$ a fine cover of A , $\exists B \in \mathcal{F}$ st. $x \in B$, $\text{diam } B < \varepsilon$.
 i.e. $x \in B$ and $B \cap \bigcup_{i=1}^m B_i = \emptyset$.



From the proof of Thm 1, $\exists B' \in G$ st. $B' \cap B \neq \emptyset$. This
 implies that $B' \neq B_1, B_2, \dots, B_m$. Also we have $A \subseteq \bigcup_{B \in \mathcal{F}} B$
 and by the maximality (look at pf of thm 1) and the
 choice of B' we have $B \subseteq B'$. Thus $x \in B \subseteq B'$ with
 $x \in A - \bigcup_{i=1}^m B_i$ and $B' \in G - \{B_1, \dots, B_m\}$. Done. $\square$

Claim. Any open set in $\mathbb{R}^n$ can be filled up by disjoint closed balls
 in $\mathbb{R}^n$ in Lebesgue measure.

Thm 3. Let $U \subseteq \mathbb{R}^n$ be open and $\delta > 0$. There exists
 a countable collection G of disjoint closed balls in
 U st. $\text{diam } B < \delta$, $\forall B \in G$, and when $\mu = m$ is
 the Lebesgue measure,
 $m(U - \bigcup_{B \in G} B) = 0$.

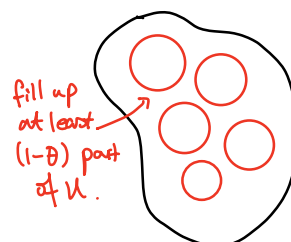


pf. First assume $\mu(U) < \infty$, and $1 - \frac{1}{5^n} < \theta < 1$.

(1) Claim: $\exists$ a finite collection $B_1, B_2, \dots, B_m$ of disjoint closed balls in U
 st. $\mu(U - \bigcup_{i=1}^m B_i) \leq \theta \cdot \mu(U)$.

pf. Consider the fine cover $\mathcal{F}$ of U as follows:
 $\forall x \in U$, as U is open, $\exists \delta_x > 0$ st. $B_{\delta_x}(x) \subseteq U$.
 Then for x consider the local collection

$$\mathcal{F}_x = \{B_r(x) \subseteq U : 0 < r < \min(\delta_x, \delta)\}.$$



Then of course $U \subseteq \mathcal{K} = \bigcup_{x \in U} \mathcal{K}_x$. Also, $\mathcal{K}$ is a fine cover, evidently.
 And $\text{diam } B \leq \delta, \forall B \in \mathcal{K}$. Thus $\mathcal{K}$ is a fine cover of U with uniformly bounded radius. Then apply Vitali's Covering thm I:
 $\exists G \subseteq \mathcal{K}$ countable + disjoint s.t.

$$U \stackrel{\substack{\uparrow \\ (B \subseteq U \\ \forall B \in \mathcal{K})}}{=} \bigcup_{B \in G} B \subseteq \bigcup_{B \in G} \hat{B} \quad \text{let } G = \{B_1, B_2, \dots\}$$

$$\text{Then } \mu(U) \leq \sum_{i=1}^{\infty} \mu(\hat{B}_i) = 5^n \cdot \sum_{i=1}^{\infty} \mu(B_i).$$

Balls are closed (measurable) and disjoint:

$$\mu\left(\underbrace{\bigcup_{i=1}^{\infty} B_i}_{\substack{\text{a Borel set inside } U \\ \downarrow}}\right) = \sum_{i=1}^{\infty} \mu(B_i) \geq \frac{1}{5^n} \mu(U).$$

a Borel set inside U
 $\downarrow$

$$\mu\left(U - \bigcup_{i=1}^{\infty} B_i\right) = \mu(U) - \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \mu(U) \cdot \left(1 - \frac{1}{5^n}\right) < \theta \cdot \mu(U).$$

On the other hand, balls are Lebesgue measurable,

$$\mu\left(U - \bigcup_{i=1}^{\infty} B_i\right) = \lim_{m \rightarrow \infty} \mu\left(U - \bigcup_{i=1}^m B_i\right) < \theta \cdot \mu(U)$$

$$\Rightarrow \exists M_1 \in \mathbb{N} \text{ s.t. } \mu\left(U - \bigcup_{i=1}^{M_1} B_i\right) < \theta \cdot \mu(U). \text{ And } B_i \subseteq U. \quad \blacksquare$$

(2) The set $\bigcup_{i=1}^{\infty} B_i$ is closed (here is when we need balls to be finitely many).

Thus $U_1 = U - \bigcup_{i=1}^{M_1} B_i$ is open.

Apply the claim to open set U_1 : $\exists B_{M_1+1}, \dots, B_{M_2}$ closed disjoint balls that are subsets of U_1 s.t.

$$\mu\left(U_1 - \bigcup_{i=M_1+1}^{M_2} B_i\right) < \theta \cdot \mu(U_1).$$

Do this step infinitely and get open sets $U_1, U_2, \dots$ and $m_1 < m_2 < m_3 < \dots$

In the n th step:

$$\mu\left(U_n - \underbrace{\bigcup_{i=m_{n-1}+1}^{m_n} B_i}_{\text{"}}\right) < \theta \cdot \mu(U_n)$$

$$\Rightarrow \mu(U_{n+1}) < \theta \mu(U_n) \quad U_0 = U.$$

$$\Rightarrow \mu(U_n) < \theta^n \cdot \mu(U_0) = \theta^n \mu(U).$$

Since $U - \bigcup_{i=1}^{\infty} B_i \subseteq U_n, \forall n,$

$$\mu\left(U - \bigcup_{i=1}^{\infty} B_i\right) \leq \mu(U_n) < \theta^n \mu(U) \xrightarrow{n \rightarrow \infty} 0 \quad (\theta < 1).$$

$$\Rightarrow \mu\left(U - \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

By constructions, the balls $\{B_i\}_{i=1}^{\infty}$ are disjoint.

If $\mu(U) = \infty$, apply the previous result to the open sets

$$U \cap \underbrace{\{x : m < |x| < m+1\}}_{\text{open}} \quad m=0, 1, \dots$$

What we may have are spheres of radii $r=0, 1, \dots$. But they have measure zero.

□

Rmk. Vitali's covering Thm works for Lebesgue measures, or for measure that have the following property:

Def (Doubling property). Let μ Borel on $\mathbb{R}^n$, if $\exists C_1, C_2 > 1$ s.t.
 $\forall x \in \mathbb{R}^n, \forall r > 0, \mu(B_{C_1 r}(x)) \leq C_2 \cdot \mu(B_r(x))$, then μ is said to have the **doubling property**.

Then in the proof of the second Vitali thm, instead of making balls 5 times bigger, one can make it C times bigger for "good" C .

If μ doesn't have doubling property, Vitali fails. Better thm:

Thm 4 (**Besicovitch Thm**). $\exists$ a constant $N_n \in \mathbb{N}$, depending only on the dim of the space n , s.t. If $\mathcal{F}$ is any collection of nondegenerate closed balls in $\mathbb{R}^n$ with $\sup\{\text{diam } B : B \in \mathcal{F}\} < \infty$, and if we denote $A = \{x : \exists r > 0 \text{ s.t. } B_r(x) \in \mathcal{F}\}$ as the set of centers of the balls in $\mathcal{F}$, then $\exists N_n$ countable collections $G_1, G_2, \dots, G_{N_n}$ of disjoint balls in $\mathcal{F}$, s.t.

$$A \subseteq \bigcup_{i=1}^{N_n} \bigcup_{B \in G_i} B.$$

~~pf~~ Omitted. See book.

□

Thm. Let μ be a Borel measure on $\mathbb{R}^n$, and $\mathcal{F}$ a collection of nondegenerate closed balls in $\mathbb{R}^n$. Let A denote the set of all the centers of the balls in $\mathcal{F}$: $A = \{x \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } B_r(x) \in \mathcal{F}\}$. Assume $\mu(A) < \infty$ and $\inf\{r : B_r(a) \in \mathcal{F}\} = 0$ for all $a \in A$.

(slightly stronger than $\mathcal{F}$ a fine cover of A)

Then for each open set $U \subseteq \mathbb{R}^n$, there exists a countable collection $G \subseteq \mathcal{F}$ of disjoint balls in $\mathcal{F}$ s.t.

$$1) \bigcup_{B \in G} B \subseteq U \quad (\forall B \in G, B \subseteq U).$$

$$2) \mu\left((A \cap U) - \bigcup_{B \in G} B\right) = 0 \quad (\text{disjoint balls in } G \text{ cover } A \cap U \text{ in measure}).$$

pf. Let θ be a constant s.t. $1 - \frac{1}{N_n} < \theta < 1$ where N_n is the constant from the Besicovitch covering Thm.

Claim. $\exists$ a finite collection $\{B_1, B_2, \dots, B_{m_1}\}$ of disjoint balls in $\mathcal{F}$ s.t.

$$1) B_i \subseteq U, \text{ for each } i=1, 2, \dots, m_1.$$

$$2) \mu(A \cap U - \bigcup_{i=1}^{m_1} B_i) \leq \theta \mu(A \cap U).$$

the balls $B_1, \dots, B_{m_1}$ cover at

least the $(1-\theta)$ part of $A \cap U$. Therefore one would succeed iterating.

pf of claim. Consider the collection

$$\mathcal{F}_1 = \{B \in \mathcal{F} : \text{diam } B \leq 1, B \subseteq U\}.$$

Since $\inf\{r : B_r(a) \in \mathcal{F}\} = 0$ for every $a \in A \cap U$, and U open, $\mathcal{F}_1 \neq \emptyset$. And it has the property that $\sup\{\text{diam } B : B \in \mathcal{F}_1\} \leq 1 < \infty$. Therefore, apply Besicovitch covering Thm to $\mathcal{F}_1$, which gives the following: $\exists G_1, \dots, G_{N_n}$ of disjoint balls in $\mathcal{F}_1$ s.t.

$$\text{new set of centers by def of } \mathcal{F}_1 \leftarrow (A \cap U) \subseteq \bigcup_{i=1}^{N_n} \bigcup_{B \in G_i} B \quad (\text{each } B \in U)$$

Thus by subadditivity,

$$\mu(A \cap U) \leq \sum_{i=1}^{N_n} \mu\left(\bigcup_{B \in G_i} B \cap (A \cap U)\right)$$

Therefore $\exists$ index $j \in \{1, 2, \dots, N_n\}$, we have

$$\mu(A \cap U) \leq N_n \cdot \mu\left(\bigcup_{B \in G_j} B \cap (A \cap U)\right) \rightarrow \text{maximizes this measure.}$$

$$\Rightarrow \mu\left(A \cap U \cap \bigcup_{B \in G_j} B\right) \geq \frac{1}{N_n} \mu(A \cap U).$$

Next step in the book is WRONG: Evans Thm 1.6 DOES NOT APPLY!

Let $G_j = \{B_1, B_2, B_3, \dots\}$. Then

$$\mu(A \cap u \cap \bigcup_{k=1}^{\infty} B_k) \geq \frac{1}{N_n} \mu(A \cap u).$$

Lemma. $\mu(A \cap u \cap \bigcup_{k=1}^{\infty} B_k) = \lim_{m \rightarrow \infty} \mu(A \cap u \cap \bigcup_{k=1}^m B_k)$

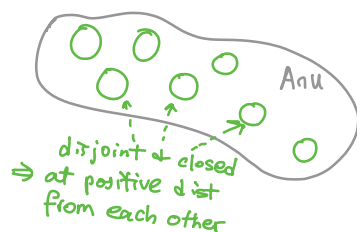
Not straightforward, for A may not be μ -measurable.

Recall one HW problem on Carathéodory criterion:

$$\mu \text{ is Borel on } \mathbb{R}^n \Leftrightarrow \mu(A \cup B) = \mu(A) + \mu(B) \text{ whenever } \text{dist}(A, B) > 0.$$

Now,

$$\mu(A \cap u \cap \bigcup_{k=1}^m B_k) = \mu((A \cap u \cap \bigcup_{k=1}^{m-1} B_k) \cup (A \cap u \cap B_m))$$



$$= \mu(A \cap u \cap \bigcup_{k=1}^{m-1} B_k) + \mu(A \cap u \cap B_m)$$

$$= \sum_{k=1}^m \mu(A \cap u \cap B_k).$$

$$\text{Then } \lim_{m \rightarrow \infty} \mu(A \cap u \cap \bigcup_{k=1}^m B_k) \geq \lim_{m \rightarrow \infty} \sum_{k=1}^m \mu(A \cap u \cap B_k)$$

$$= \sum_{k=1}^{\infty} \mu(A \cap u \cap B_k) \stackrel{\text{subadditivity}}{=} \mu\left(\bigcup_{k=1}^{\infty} A \cap u \cap B_k\right)$$

$$\Rightarrow \lim_{m \rightarrow \infty} \mu(A \cap u \cap \bigcup_{k=1}^m B_k) \geq \mu\left(A \cap u \cap \bigcup_{k=1}^{\infty} B_k\right).$$

Now, $\leq$ is obvious by $M \subseteq N \Rightarrow \mu(M) \leq \mu(N)$. Q.E.D. for Lemma ■

$$\text{Now, } \mu(A \cap u \cap \bigcup_{k=1}^{\infty} B_k) = \mu(A \cap u \cap \bigcup_{B \in G_j} B)$$

$$\geq \frac{1}{N_n} \mu(A \cap u)$$

$$> (1-\theta) \mu(A \cap u).$$

So $\exists$ index $m_1 \in \mathbb{N}$ st.

$$\mu(A \cap u \cap \bigcup_{k=1}^{m_1} B_k) \geq (1-\theta) \mu(A \cap u).$$

As the set $\bigcup_{k=1}^{m_1} B_k$ is measurable, we have

$$\mu(A \cap u) = \mu(A \cap u \cap \bigcup_{k=1}^{m_1} B_k) + \mu(A \cap u - \bigcup_{k=1}^{m_1} B_k)$$

$$\Rightarrow \mu(A \cap u - \bigcup_{k=1}^{m_1} B_k) \leq \theta \mu(A \cap u).$$

This finishes the proof of the main claim. ■

Now iterate. Set $U_1 = U - \bigcup_{k=1}^{m_1} B_k$ which is open. Now consider the collection

$$\mathcal{F}_2 = \{B \in \mathcal{F} : \text{diam } B \leq 1, B \subseteq U_1\}.$$

If the set $A \cap U_1 = \emptyset$, then proof is finished for $\mu(A \cap U_1) = 0$.

Otherwise, $\mathcal{F}_2 \neq \emptyset$, apply same procedure to $\mathcal{F}_2$. Then this gives

$B_{m_1+1}, \dots, B_{m_2} \in \mathcal{F}_2$ disjoint balls s.t.

$$\begin{aligned} \mu(U \cap A - \bigcup_{i=1}^{m_2} B_i) &= \mu(U_1 \cap A \cap \bigcup_{i=m_1+1}^{m_2} B_i) \leq \theta \mu(U_1 \cap A) \\ &= \theta \mu((U - \bigcup_{i=1}^{m_1} B_i) \cap A) \\ &= \theta \mu(U \cap A - \bigcup_{i=1}^{m_1} B_i) \\ &\leq \theta^2 \mu(A \cap U). \end{aligned}$$

By iterating we construct $B_1, B_2, \dots$, s.t.

$$\mu(U \cap A - \bigcup_{i=1}^{\infty} B_i) = 0 \quad \text{for} \quad \mu(U \cap A - \bigcup_{i=1}^{m_n} B_i) \leq \theta^n \mu(U \cap A) \leq \theta^2 \mu(A) \xrightarrow{n \rightarrow \infty} 0.$$

Also $B_i \subseteq U$, $i=1, 2, \dots$ by construction. □

Differentiation of Radon Measure

Idea: Let μ, ν be two Radon measures on $\mathbb{R}^n$. What is the density of ν with respect to μ at each pt $x \in \mathbb{R}^n$?

Def. Let μ, ν be two Radon measures on $\mathbb{R}^n$. Define, for every $x \in \mathbb{R}^n$,

$$\bar{D}_\mu \nu(x) = \begin{cases} \limsup_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \text{ for all } r > 0 \\ +\infty & \text{if } \mu(B_r(x)) = 0 \text{ for some } r > 0. \end{cases}$$

$$D_\mu \nu(x) = \begin{cases} \liminf_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \text{ for all } r > 0 \\ +\infty & \text{if } \mu(B_r(x)) = 0 \text{ for some } r > 0. \end{cases}$$

the quantity $\bar{D}_\mu \nu(x)$ is called the upper derivative (density) of the measure ν with respect to μ . Similarly $D_\mu \nu(x)$ is called the lower derivative (density) of the measure ν with respect to μ .

Def. We say ν is differentiable with respect to μ at $x \in \mathbb{R}^n$, if

$$D_\mu \nu(x) = \bar{D}_\mu \nu(x).$$

Then we denote $D_\mu \nu(x) = \bar{D}_\mu \nu(x) = D_\mu \nu(x)$, and call it the derivative (density) of ν w.r.t. μ at $x \in \mathbb{R}^n$.

Goal: (1) Study the existence of $D_\mu \nu(x)$ given Radon measures μ, ν .
 (2) Can we recover ν by integrating $D_\mu \nu(x)$ by integrating w.r.t. μ ? - Answer: YES.

lem 1. Let $\alpha \in (0, +\infty)$ be fixed. Then

$$(i) \ A \subseteq \{x \in \mathbb{R}^n : D_\mu \nu(x) \leq \alpha\} \Rightarrow \nu(A) \leq \alpha \mu(A).$$

$$(ii) \ A \subseteq \{x \in \mathbb{R}^n : \bar{D}_\mu \nu(x) \leq \alpha\} \Rightarrow \nu(A) \geq \alpha \mu(A).$$

Prmk. In both cases, we do not require A is $\mu + \nu$ measurable.

pf. Case 1. (i) Assume $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$. Fix $\varepsilon > 0$ and any open set U containing A . $\forall x \in A$, consider the family $\mathcal{F}_x = \{B_r(x)\}$ st.

$$(1) \ B_r(x) \subseteq U.$$

$$(2) \ \nu(B_r(x)) \leq (\alpha + \varepsilon) \mu(B_r(x))$$

$$(3) \ r \leq 1.$$

For any $x \in A$, for $\underline{D}_\mu v(x) = \liminf_{r \rightarrow 0^+} \frac{v(B_r(x))}{\mu(B_r(x))} \leq \alpha$, we have
 $\forall x \in A$, one can find a sequence of balls with radii $\rightarrow 0$ satisfying
 (2). But when radii is small, (1) & (3) are satisfied as well. Thus
 $\forall x \in A$, $\inf \{r: B_r(x) \in \mathcal{K}_x\} = 0$. If we consider

$$\mathcal{K} = \bigcup_{x \in A} \mathcal{K}_x.$$

Then apply Thm 1.28 of Evans shows
 $\exists G \in \mathcal{K}$, $G = \{B_1, B_2, \dots\}$ s.t.

$$v\left(A \cap U - \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

(We also implicitly use $v(A) \leq v(\mathbb{R}^n) < \infty$)

$$\begin{aligned} \Rightarrow \text{Thus } v\left(A - \bigcup_{i=1}^{\infty} B_i\right) &= 0 \Rightarrow v(A) \leq v\left(A - \bigcup_{i=1}^{\infty} B_i\right) + v\left(\bigcup_{i=1}^{\infty} B_i\right) \\ &= v\left(\bigcup_{i=1}^{\infty} B_i\right) \quad \left[\begin{array}{l} \text{balls closed} \rightarrow \text{measurable} \\ \text{and disjoint} \end{array} \right] \\ &= \sum_{i=1}^{\infty} v(B_i) \quad [B_i \in \mathcal{K}] \\ &\leq (\alpha + \varepsilon) \sum_{i=1}^{\infty} \mu(B_i) \\ &= (\alpha + \varepsilon) \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \quad [B_i \mu\text{-measurable}] \\ &\leq (\alpha + \varepsilon) \mu(U). \end{aligned}$$

Thus $v(A) \leq (\alpha + \varepsilon) \mu(U)$, $\forall \varepsilon > 0$. Letting $\varepsilon \rightarrow 0$: $v(A) \leq \alpha \cdot \mu(U)$
 true for any U open set containing A . But μ is Radon, so

$$\mu(A) = \inf \{ \mu(U) : A \subseteq U, U \text{ open} \}.$$

Thus taking the inf: $v(A) \leq \alpha \mu(A)$.

(ii) Similarly.

Case 2.

We proved the case when $\mu(\mathbb{R}^n), v(\mathbb{R}^n) < \infty$. In general, we consider the annuli

$$R_k = \{x \in \mathbb{R}^n : k-1 \leq |x| < k\}.$$

These are Borel sets and thus are μ, v measurable.

Claim: $\mu(A) = \sum_{k=1}^{\infty} \mu(A \cap R_k)$, $v(A) = \sum_{k=1}^{\infty} v(A \cap R_k)$.

(pf of claim: $A = A \cap \bigcup_{k=1}^{\infty} R_k = \bigcup_{k=1}^{\infty} A \cap R_k$. So by subadditivity $\mu(A) \leq \sum \mu(A \cap R_k)$.
 On the other hand, as R_1 is μ -measurable, $\mu(A) = \mu(A \cap R_1) + \mu(A - R_1) = \mu(A \cap R_1) + \mu(A \cap \bigcup_{k=2}^{\infty} R_k)$.)

Thus inductively $\mu(A) = \sum_{k=1}^m \mu(A \cap R_k) + \mu(A \cap \bigcup_{k=m+1}^{\infty} R_k) \geq \sum_{k=1}^m \mu(A \cap R_k)$ for any m . Let $m \rightarrow \infty$,
 We have $\sum_{k=1}^{\infty} \mu(A \cap R_k) \leq \mu(A)$. This implies the equality. $\blacksquare$

Now as μ, ν are Radon measures, $\mu(R_k), \nu(R_k) < \infty$. Then by case 1,
 $\nu(A \cap R_k) \leq \alpha \mu(A \cap R_k) \Rightarrow \sum \nu(A \cap R_k) \leq \alpha \sum \mu(A \cap R_k) \Rightarrow \nu(A) \leq \alpha \mu(A)$ for
 (i), and similarly for (ii). $\square$

Rmk. $\mathbb{R}^n$ is σ -finite w.r.t. radon measure.

Thm (Differentiation of Radon measures).

Let μ, ν be radon measures on $\mathbb{R}^n$. Then

- (1) The derivative $D_\mu \nu(x)$ exists and is finite μ -a.e. on $\mathbb{R}^n$.
- (2) The derivative $D_\mu \nu(x)$ is μ -measurable.

pf. WLOG, we can assume that $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$ by restricting to the annuli R_k as before.

(1) Step 1. $D_\mu \nu(x)$ is finite μ -a.e. Let

$$A_\infty = \{x \in \mathbb{R}^n : \bar{D}_\mu \nu(x) = +\infty\} \quad (\text{By def } \bar{D}_\mu \nu(x) \geq 0).$$

$$A_\alpha = \{x \in \mathbb{R}^n : \bar{D}_\mu \nu(x) \geq \alpha\}$$

Then $A_\infty \subseteq A_\alpha, \forall \alpha > 0$. By the Lemma, $\nu(A_\alpha) \geq \alpha \mu(A_\alpha)$ by Lem (ii).

Thus

$$0 \leq \mu(A_\infty) \leq \mu(A_\alpha) \leq \frac{1}{\alpha} \nu(A_\alpha) \leq \frac{1}{\alpha} \nu(\mathbb{R}^n) \xrightarrow{\alpha \rightarrow +\infty} 0.$$

Thus $\mu(A_\infty) = 0$.

Step 2. Denote for $0 \leq a < b < \infty$,

$$A(a,b) = \{x \in \mathbb{R}^n : a \leq \underline{D}_\mu \nu(x) < b \leq \bar{D}_\mu \nu(x)\}.$$

Claim: $\mu(A(a,b)) = 0$.

pf. By the Lem, $\nu(A(a,b)) \leq a \mu(A(a,b))$ and $\nu(A(a,b)) \geq b \mu(A(a,b))$.

But $a < b$, so $\mu(A(a,b)) = 0$. Also, this implies $\nu(A(a,b)) = 0$.

The set where the derivative $D_\mu \nu(x)$ does not exist or is infinite is

$$A \subseteq A_\infty \cup \bigcup_{0 \leq a < b < \infty} A(a,b) \Rightarrow \mu(A) = 0.$$

This proves (1).

(2). We prove $D_\mu \chi(x)$ is μ -measurable. Fix $r > 0$. Set

$$g(x) = \mu(B_r(x)), \quad \forall x \in \mathbb{R}^n.$$

Claim: For any fixed $r > 0$, the function $g(x): \mathbb{R}^n \rightarrow \mathbb{R}$ is μ -measurable

In fact $g(x)$ is upper semicontinuous everywhere in $\mathbb{R}^n$. Or if $x_k \rightarrow x$, $x_k, x \in \mathbb{R}^n$, then $\limsup_k g(x_k) \leq g(x)$. And any lower semi-continuous function in $\mathbb{R}^n$ is Borel measurable. Thus any upper semi-cts (=lower semi-cts) is Borel measurable. Thus g is μ -measurable, for g is everywhere upper semi-cts.

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Thus, we want to show $g(x) = \mu(B_r(x))$ is upper-semi-cts μ -a.e. in $\mathbb{R}^n$.

pf. Assume $x_k \rightarrow x$, $x_k, x \in \mathbb{R}^n$, and denote

$$f_k(y) = \chi_{B_r(x_k)}(y) \quad f(y) = \chi_{B_r(x)}(y).$$

(I) First, we show $x_k \rightarrow x \Rightarrow \limsup_{k \rightarrow \infty} f_k(y) \leq f(y)$, $\forall y \in \mathbb{R}^n$.

pf. If $y \in B_r(x) = \{w: |w-x| < r\}$. Then $f(y) = 1$, and of course

$$f_k(y) \leq 1, \forall y \in \mathbb{R}^n \Rightarrow \limsup_{k \rightarrow \infty} f_k(y) \leq f(y) \quad \forall y \in B_r(x).$$

Assume $y \notin B_r(x)$. Thus $|y-x| = r + \delta$ with $\delta > 0$. Now if $|x-x_k| < \frac{\delta}{2}$,

then $|y-x_k| \geq |y-x| - |x-x_k| = r + \delta - |x-x_k| \geq r + \frac{\delta}{2}$. Therefore

$y \notin B_r(x_k) \Rightarrow f_k(y) = 0$. But as $x_k \rightarrow x$, for k big enough,

$$|x-x_k| < \delta/2 \Rightarrow \forall \text{ such } k, f_k(y) = 0 \Rightarrow \limsup_{k \rightarrow \infty} f_k(y) = 0 = f(y).$$

Thus we proved the inequality

$$\limsup_{k \rightarrow \infty} f_k(y) \leq f(y), \quad \forall y \in \mathbb{R}^n. \quad (*)$$

(II) Apply Fatou's Lemma to the seq of functions $\{1 - f_k(y)\}_{k=1}^\infty$, $y \in B_{2r}(x)$.

So

$$\int_{B_{2r}(x)} \liminf_{k \rightarrow \infty} (1 - f_k(y)) d\mu \leq \liminf_{k \rightarrow \infty} \int_{B_{2r}(x)} (1 - f_k(y)) d\mu.$$

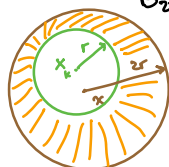
For $k \in \mathbb{N}$ big enough, $x_k \in B_r(x) \Rightarrow f_k(y) = \chi_{B_r(x)}(y) \leq \chi_{B_{2r}(x)}(y)$.

On one hand,

$$\liminf_{k \rightarrow \infty} (1 - f_k) = 1 - \limsup_{k \rightarrow \infty} f_k \geq 1 - f, \quad \forall y \in \mathbb{R}^n.$$

Thus

$$\mu(B_{2r}(x)) - \mu(B_r(x)) = \int_{B_{2r}(x)} (1 - f) d\mu \leq \int_{B_{2r}(x)} \liminf_{k \rightarrow \infty} (1 - f_k) d\mu$$



$$\leq \liminf_{k \rightarrow \infty} \int_{B_{2r}(x)} (1 - f_k(y)) d\mu.$$

$$= \liminf_{k \rightarrow \infty} \mu(B_{2r}(x)) - \mu(B_r(x_k))$$

$$= \mu(B_{2r}(x)) - \limsup_{k \rightarrow \infty} \mu(B_r(x_k)).$$

As $\mu(B_{2r}(x)) < \infty$ since μ Radon, we have

$$\limsup_{k \rightarrow \infty} \mu(B_r(x_k)) \leq \mu(B_r(x))$$

$$\Rightarrow \limsup_{k \rightarrow \infty} g(x_k) \leq g(x).$$

So g is upper-semi-cts in $\mathbb{R}^n$. Thus $g(x) = \mu(B_r(x))$ is Borel-measurable. Similarly $h(y) = \nu(B_r(y))$ is Borel measurable. So they are μ -measurable.

For any $r > 0$ define

$$f_r(x) = \begin{cases} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \\ +\infty & \text{if } \mu(B_r(x)) = 0 \end{cases}.$$

Claim: For any fixed r , $f_r(x)$ is μ -measurable.

pf. Let $X_r = \{x \in \mathbb{R}^n : \mu(B_r(x)) = 0\}$
 $= \{x \in \mathbb{R}^n : g(x) = 0\}$

and $g(x)$ is upper semi-cts and $g \geq 0$. Then X_r is Borel by upper semi-cts of g . Then

$$f_r(x) = \begin{cases} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } x \in \mathbb{R}^n - X_r \\ +\infty & \text{if } x \in X_r \end{cases}.$$

Also, we see $0 < r_1 < r_2 \Rightarrow X_{r_2} \subseteq X_{r_1}$. We proved $D_\mu \nu(x)$ exists and is finite μ -a.e. Thus let

$$A = \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \text{ exists and finite} \right\}$$

Then $\mu(\mathbb{R}^n - A) = 0$. Note if $x \in A$,

$$\begin{aligned} D_\mu \nu(x) &= \lim_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} = \lim_{k \rightarrow \infty} \frac{\nu(B_{\frac{1}{k}}(x))}{\mu(B_{\frac{1}{k}}(x))} \\ &= \lim_{k \rightarrow \infty} f_{\frac{1}{k}}(x). \end{aligned}$$

Can extend f_k to a measurable function by putting anything on $\mathbb{R}^n - A$.

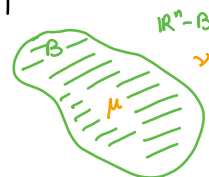
Now, $f_k(x)$ is μ -measurable when $x \in A$, and since $\mu(\mathbb{R}^n - A) = 0$, this does not affect the measurability. Thus $D_\mu \nu(x)$ is measurable μ -a.e. in $\mathbb{R}^n$. □

Def. Let μ, ν be Borel measures on $\mathbb{R}^n$. Then

(1) The measure ν is absolutely continuous wrt μ , if $\mu(A)=0$ implies $\nu(A)=0$, $\forall A \subseteq \mathbb{R}^n$. Write: $\nu \ll \mu$.

(2) The measures μ, ν are mutually singular if $\exists$ a Borel set $B \subseteq \mathbb{R}^n$, s.t. $\mu(\mathbb{R}^n - B) = \nu(B) = 0$. Write: $\nu \perp \mu$.

[μ concentrated in B , ν in $\mathbb{R}^n - B$. They complement each other in $\mathbb{R}^n$].



Thm (Radon-Nikodym thm I). Let μ, ν be radon measures on $\mathbb{R}^n$ s.t. $\nu \ll \mu$. Then we have

$$\nu(A) = \int_A D_\mu \nu(x) d\mu$$

for all μ -measurable sets $A \subseteq \mathbb{R}^n$.

pf. Let A be μ -measurable.

Claim: If $\nu \ll \mu$, then if $A \subseteq \mathbb{R}^n$ is μ -measurable, then A must be ν measurable.

pf of claim. μ radon, so is Borel regular. Therefore, there exists B Borel set s.t. $A \subseteq B$, and $\mu(A) = \mu(B)$. Since $A \in M_\mu$,

$$\mu(B) = \mu(A \cap B) + \mu(B - A) \Rightarrow \mu(B - A) = 0.$$

Since $\nu \ll \mu$, we have $\nu(B - A) = 0$. Thus $(B - A) \in M_\nu$.

Thus $A = B - (B - A)$ is ν measurable. ■

Fix any $t > 0$, consider the sets

$$A_m = \{x \in A : t^m \leq D_\mu \nu(x) < t^{m+1}\} \quad m \in \mathbb{Z}$$

Note $\lim_{m \rightarrow \infty} t^m = \infty$, $\lim_{m \rightarrow -\infty} t^m = 0$. For $(t^m > 0)$. Then

$$A = \bigcup_{m=-\infty}^{\infty} A_m \cup \underbrace{\{x \in A : D_\mu \nu(x) = 0\}}_{:= Z} \cup \underbrace{\{x \in A : D_\mu \nu(x) = \infty\}}_{:= I}.$$

Then A_m are disjoint by def, and I, Z, A_m mutually disjoint.

$$A = \bigcup_{n=-\infty}^{\infty} A_m \cup Z \cup I.$$

Also, by def. A_m, Z, I are μ -measurable, because $D_\mu \nu(x)$ is

μ measurable: $A_m = (D_\mu v)^{-1} [t^m, t^{m+1})$. Also, Z, I are μ -measurable. Thus they are also ν measurable. Since they are disjoint, so

$$\nu(A) = \sum_{m=-\infty}^{\infty} \nu(A_m) + \nu(I) + \nu(Z).$$

Step 1. Examine $\nu(I), \nu(Z), \int_I D_\mu v(x) d\mu, \int_Z D_\mu v(x) d\mu$.

Recall: integrals $\int_A f(x) d\mu$ is countably additive in the following cases:

1) When $f(x) \geq 0, \forall x \in A$ (monotone conv thm);

2) When $\int_A |f| d\mu < \infty$ (dominated conv thm).

Now because of countable additivity and $D_\mu v(x) \geq 0, \forall x \in A$,

$$\int_A D_\mu v(x) d\mu = \sum_{m=-\infty}^{\infty} \int_{A_m} D_\mu v(x) d\mu + \int_I D_\mu v(x) d\mu + \int_Z D_\mu v(x) d\mu$$

Now, by previous results, $\mu(I) = 0$, so $\int_I D_\mu v(x) d\mu = 0$. Also, on Z , $D_\mu v(x) = 0$, we have $\int_Z D_\mu v(x) d\mu = 0$. Now, for $\nu \ll \mu \Rightarrow \mu(I) = 0$ implies $\nu(I) = 0$. So

$$\nu(I) = \int_I D_\mu v(x) d\mu \quad (1)$$

By the lemma 1 before, $D_\mu v(x) = 0 \Rightarrow D_\mu v(x) \leq 0 \Rightarrow \nu(Z) \leq \alpha \mu(Z)$

$\forall \alpha > 0$, and WLOG $\mu(Z) < \infty$ (otherwise restrict to annuli). Thus

$\nu(Z) = 0$. Thus

$$\nu(Z) = \int_Z D_\mu v(x) d\mu \quad (2)$$

Step 2. Compare $\nu(A_m)$ and $\int_{A_m} D_\mu v(x) d\mu$.

We have $A_m = \{x \in A: t^m \leq D_\mu v(x) \leq t^{m+1}\}, m = -\infty, \dots, +\infty$.

$\uparrow$ -truncation
Method

$$\int_{A_m} D_\mu v(x) d\mu \leq \int_{A_m} t^{m+1} d\mu = t^{m+1} \mu(A_m)$$

$$\int_{A_m} D_\mu v(x) d\mu \geq \int_{A_m} t^m d\mu = t^m \mu(A_m).$$

Thus

$$\sum_{m=-\infty}^{\infty} t^m \mu(A_m) \leq \sum_{m=-\infty}^{\infty} \int_{A_m} D_\mu v(x) d\mu \leq t \cdot \sum_{m=-\infty}^{\infty} t^m \mu(A_m).$$

By the lemma 1 again, $t^m \mu(A_m) \leq \nu(A_m) \leq t^{m+1} \mu(A_m)$. Thus

$$\sum_{m=-\infty}^{\infty} t^m \mu(A_m) \leq \sum_{m=-\infty}^{\infty} \nu(A_m) \leq t \sum_{m=-\infty}^{\infty} t^m \mu(A_m).$$

$$\int_A D_\mu \nu d\mu = \sum_{m=-\infty}^{\infty} \int_{A_m} D_\mu \nu d\mu \leq t \cdot \sum_m t^m \mu(A_m) \leq t \sum_m \nu(A_m) = t \nu(A).$$

On the other hand,

$$\int_A D_\mu \nu d\mu = \sum_m \int_{A_m} D_\mu \nu d\mu \geq \sum_m t^m \mu(A_m) \stackrel{(*)}{\geq} \frac{1}{t} \sum_m \nu(A_m) = \frac{1}{t} \nu(A).$$

Thus
$$\frac{1}{t} \nu(A) \leq \int_A D_\mu \nu(x) d\mu \leq t \nu(A). \quad \forall t > 1.$$

If $\nu(A) = \infty$, then $\int_A D_\mu \nu(x) d\mu = \infty$. Then equality holds. If $\nu(A) < \infty$, let $t \rightarrow 1^+$, we get that

$$\nu(A) = \int_A D_\mu \nu(x) d\mu. \quad \square$$

Thm (Change of measure formula). Let μ, ν be Radon measures on $\mathbb{R}^n$, s.t. $\nu \ll \mu$. Then set $A \subseteq \mathbb{R}^n$ μ -measurable, and $f: A \rightarrow \mathbb{R}$ integrable, one has

(measurable by def)
$$\int_A f(x) d\nu = \int_A f(x) D_\mu \nu(x) d\mu.$$

(When $f(x) = 1$, this is Radon-Nikodym Thm)

pf. WLOG, let $f: A \rightarrow [0, \infty]$, i.e. $f \geq 0$. Otherwise write $f = f^+ - f^-$. Since $D_\mu \nu(x) \geq 0$, the general case will follow from the case $f \geq 0$.

Lem. let X be a set, μ be a measure on X . Assume $f: X \rightarrow [0, \infty]$

Then $\exists$ μ -measurable sets $A_1, \dots, A_k, \dots \subseteq X$ s.t.

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}.$$

Note A_k need not be disjoint.

pf of Lem. Evans Thm 1.12. ■

Rmk. Let μ, ν be Radon measures on $\mathbb{R}^n$ s.t. $\nu \ll \mu$. We proved that

$A \in \mathcal{M}_\mu \Rightarrow A \in \mathcal{M}_\nu$. Then it follows that if $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is μ -measurable, then it is also ν -measurable. Thus both sides of $\int_A f d\nu = \int_A f D_\mu \nu d\mu$ makes sense.

By the lemma, $\exists \mu$ -measurable sets $A_k \subseteq A$, $k=1,2,\dots$ s.t.

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x) \quad A_k \in \mathcal{M}_\nu.$$

Thus

$$\begin{aligned} \int_A f(x) d\nu &= \int_A \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k} d\nu \stackrel{(\text{Monotone Conv. Thm})}{=} \sum_{k=1}^{\infty} \frac{1}{k} \int_A \chi_{A_k} d\nu \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \nu(A_k) \stackrel{(R.N.)}{=} \sum_{k=1}^{\infty} \frac{1}{k} \int_A \chi_{A_k} D_\mu \nu(x) d\mu \\ &\stackrel{(\text{Dominated Conv. Thm})}{=} \int_A \left(\sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k} \right) D_\mu \nu(x) d\mu \\ &= \int_A f(x) D_\mu \nu(x) d\mu. \quad \square \end{aligned}$$

Thm (Lebesgue Decomposition Thm). Let μ, ν be Radon measures on $\mathbb{R}^n$. Then

(i) ν can be decomposed into the sum of two Radon measures $\nu = \nu_{ac} + \nu_s$ s.t. $\nu_{ac} \ll \mu$, and $\nu_s \perp \mu$. i.e. $\nu(A) = \nu_{ac}(A) + \nu_s(A)$, $\forall A \subseteq \mathbb{R}^n$.

(ii) $D_\mu \nu = D_\mu \nu_{ac}$ and $D_\mu \nu_s(x) = 0$ μ -a.e. and therefore:

$$\nu(A) = \int_A D_\mu \nu(x) d\mu + \nu_s(A)$$

for any Borel set $A \subseteq \mathbb{R}^n$. In that case ν_{ac} is called the **absolute continuous part** of ν and ν_s the **singular part**.

Pf. Assume wlog that $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$, otherwise work in annuli. Next, define $\mathcal{E} = \{A \subseteq \mathbb{R}^n : A \text{ Borel and } \mu(\mathbb{R}^n - A) = 0\}$.

Consider the quantity $\inf_{A \in \mathcal{E}} \nu(A) := m$.

Claim: The $\inf_{A \in \mathcal{E}} \nu(A)$ is reachable.

Pf: $m < \infty$ by $\nu(\mathbb{R}^n) < \infty$. Thus choose a seq $\{B_k\}_{k=1}^{\infty}$ of Borel set s.t. (1) $B_k \in \mathcal{E}$; (2) $m \leq \nu(B_k) < m + \frac{1}{k}$. Define now $B = \bigcap_{k=1}^{\infty} B_k$. Then B is Borel. Now, $\mu(\mathbb{R}^n - B) = \mu(\mathbb{R}^n - \bigcap_{k=1}^{\infty} B_k) \leq \sum_{k=1}^{\infty} \mu(\mathbb{R}^n - B_k) = 0$. So $B \in \mathcal{E}$. Now, $B \subseteq B_k \Rightarrow \nu(B) \leq \nu(B_k) < m + \frac{1}{k} \forall k$. Also $B \in \mathcal{E} \Rightarrow \nu(B) \geq m$. Let $k \rightarrow \infty$ gives $\nu(B) = m$. ■

Now define:

$$\nu_{ac} = \nu \llcorner B \quad \text{restricted to } \dots \quad \nu_s = \nu \llcorner (\mathbb{R}^n - B)$$

Because B is Borel, it is ν -measurable. So $\forall A \subseteq \mathbb{R}^n$,

$$\nu(A) = \nu(A \cap B) + \nu(A - B) = \nu(A \cap B) + \nu(A \cap (\mathbb{R}^n - B))$$

$$= \nu_{ac}(A) + \nu_s(A).$$

Need to show $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$. Assume otherwise: $\exists A \subseteq \mathbb{R}^n$ s.t. $\mu(A) = 0$ but $\nu_{ac}(A) > 0$. Then

$$\mu(A) = \mu(A \cap B) + \mu(A - B) = 0.$$

On the other hand, both measures $\nu_{ac} = \nu \llcorner B$ and $\nu_s = \nu \llcorner (\mathbb{R}^n - B)$ have the same properties as ν , thus they are Radon. Write

$$\nu_{ac}(A) = \nu_{ac}(A \cap B) = \nu(A \cap B) > 0$$

by def. Denote $C = A \cap B$. Then $C \subseteq B$. And $\mu(C) = 0$ and $\nu(C) > 0$.

Claim: Can assume WLOG A is Borel.

pf: μ, ν_{ac} are Radon, so are Borel regular. Let $A \subseteq \mathbb{R}^n$. We can choose two Borel sets $A_1 \subseteq \mathbb{R}^n$ s.t.

$$(1) A \subseteq A_1.$$

$$(2) \mu(A_1) = \mu(A) = 0.$$

Then $A_1 \cap B$ Borel, $\mu(A_1) = 0$ and $A \subseteq A_1 \Rightarrow \nu(A_1) \geq \nu(A) > 0$.

So WLOG let $A = A_1$. ■

Then the claim implies $C = A \cap B$ is Borel. Then C is ν measurable.

So

$$\begin{aligned} \nu(B) &= \nu(B \cap C) + \nu(B - C) = \nu(C) + \nu(B - C) \\ \Rightarrow \nu(B - C) &= \nu(B) - \nu(C) < \nu(B) \quad \text{for } \nu(C) > 0. \end{aligned}$$

Note that $B - C \in \mathcal{E}$:

$$\begin{aligned} \mu(\mathbb{R}^n - (B - C)) &= \mu(\mathbb{R}^n \cap (B \cap C^c)^c) \\ &= \mu(\mathbb{R}^n \cap (B^c \cup C)) \\ &= \mu((\mathbb{R}^n \cap B^c) \cup (\mathbb{R}^n \cap C)) \\ &\leq \mu(\mathbb{R}^n - B) + \mu(C) = 0 \end{aligned}$$

$$\begin{aligned} \mu(C) &= \mu(A) - \mu(A \cap B) \\ &= 0 \end{aligned}$$

for $\mu(A) = 0$ by assumption.

So $C - B \in \mathcal{E}$. But we got $\nu(B - C) < \nu(B)$ is a contradiction, as B by assumption minimizes $\nu(\cdot)$. But $B - C$ gives a smaller value.

Thus $\nu_{ac} \ll \mu$.

Next, show $\nu_s \perp \mu$. By def, $\nu_s = \nu \llcorner (\mathbb{R}^n - B)$. Then $\nu_s(B) = 0$. Also by the choice, B is a Borel set in $\mathcal{E}$. So $\mu(\mathbb{R}^n - B) = 0$. Thus $\nu_s \perp \mu$.

Now prove $D_\mu \nu_s(x) = 0$, μ -a.e. in $\mathbb{R}^n$. Fix any $\alpha > 0$. Consider

$$C_\alpha = \{x \in B : D_\mu \nu_s(x) \geq \alpha\}. \quad (\mu(\mathbb{R}^n - B) = 0).$$

Then by the lemma, $\nu_s(C_\alpha) \geq \alpha \cdot \mu(C_\alpha)$. Recall $\nu_s = \nu \llcorner (\mathbb{R}^n - B)$. As $\nu_s = \nu \llcorner (\mathbb{R}^n - B)$ and $C_\alpha \subseteq B \Rightarrow \nu_s(C_\alpha) = 0$. Then the inequality above gives $0 \geq \alpha \mu(C_\alpha)$ for $\alpha > 0 \Rightarrow \mu(C_\alpha) = 0 \quad \forall \alpha > 0$. Then it follows $D_\mu \nu_s(x) = 0$ μ -a.e. in $\mathbb{R}^n$.

We know $D_\mu \nu(x)$ exists μ -a.e. in $\mathbb{R}^n$. $\nu = \nu_{ac} + \nu_s$ gives

$$D_\mu \nu(x) = D_\mu \nu_{ac}(x) \quad \mu\text{-a.e. in } \mathbb{R}^n.$$

Now for any Borel set $A \subseteq \mathbb{R}^n$,

$$\begin{aligned} \nu(A) &= \nu_{ac}(A) + \nu_s(A) \stackrel{\text{RN}}{=} \int_A D_\mu \nu_{ac}(x) d\mu + \nu_s(A) \\ &\stackrel{\uparrow}{=} \int_A D_\mu \nu(x) d\mu + \nu_s(A) \\ &\quad (D_\mu \nu = D_\mu \nu_{ac} \text{ } \mu\text{-a.e.}) \end{aligned}$$

□

Notation: Let X be a topological space and μ a measure on X . Then

$$L^1(X) = \left\{ f: X \rightarrow [-\infty, \infty] : f \in M_\mu(X), \int_X |f(x)| d\mu < \infty \right\}. \quad \leftarrow \text{summable functions}$$

$$L^1_{loc}(X) = \left\{ f: X \rightarrow [-\infty, \infty] : f \in M_\mu(X), \int_K |f(x)| d\mu < \infty, \forall K \subseteq X \text{ compact} \right\}. \quad \leftarrow \text{locally summable functions}$$

For any $1 < p < \infty$, let

$$L^p(X) = \left\{ f: X \rightarrow [-\infty, \infty] : f \in M_\mu(X), \int_X |f(x)|^p d\mu < \infty \right\}$$

$$L^p_{loc}(X) = \left\{ f: X \rightarrow [-\infty, \infty] : f \in M_\mu(X), \int_K |f(x)|^p d\mu < \infty, \forall K \subseteq X \text{ compact} \right\}.$$

Def let μ be a measure on $\mathbb{R}^n$ and $E \subseteq \mathbb{R}^n$ be a μ -measurable subset. Then the average of f over E is defined to be

$$\frac{1}{\mu(E)} \int_E f(x) d\mu := \int_E f d\mu \quad \text{and } 0 < \mu(E) < \infty.$$

for any μ -measurable function $f: E \rightarrow [-\infty, \infty]$ for which $\int_E f(x) d\mu$ is defined. $\checkmark$

Lec 6. Differentiation Theorems

Goal:

Thm (Lebesgue-Besicovitch Differentiation Thm). Let μ be a Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\lim_{r \rightarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(y) d\mu = f(x) \quad \mu\text{-a.e. in } \mathbb{R}^n.$$

Cor 1. Let μ be a measure on $\mathbb{R}^n$ and let $f: \mathbb{R}^n \rightarrow [-\infty, \infty]$ be μ -measurable s.t. $\forall A \subseteq \mathbb{R}^n$ μ -measurable set

$$\int_A f(x) d\mu = 0,$$

then $f(x) = 0$ μ -a.e. on $\mathbb{R}^n$.

pf. Define $A_n = \{x \in \mathbb{R}^n : f(x) \geq \frac{1}{n}\}$ and $B_n = \{x \in \mathbb{R}^n : f(x) \leq -\frac{1}{n}\}$. Then A_n & B_n are μ -measurable for f is μ -measurable. Thus

$$\int_{A_n} f(x) d\mu = \int_{B_n} f(x) d\mu = 0.$$

Now

$$0 = \int_{A_n} f(x) d\mu \geq \int_{A_n} \frac{1}{n} d\mu = \frac{1}{n} \mu(A_n) \geq 0$$

So $\mu(A_n) = 0$. In the same way,

$$0 = \int_{B_n} f(x) d\mu \leq \int_{B_n} \left(-\frac{1}{n}\right) d\mu = -\frac{\mu(B_n)}{n} \Rightarrow \mu(B_n) = 0.$$

Since $\mu(A_n) = \mu(B_n) = 0 \forall n$. Now

$$A = \{x \in \mathbb{R}^n : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} (A_n \cup B_n) \Rightarrow \mu(A) = 0.$$

□

Cor 2. Let μ be a Borel measure on $\mathbb{R}^n$. Let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$ and $f(x) \geq 0$ μ -a.e. Define $\nu: 2^{\mathbb{R}^n} \rightarrow [0, +\infty]$ as

(1) If $B \subseteq \mathbb{R}^n$ is Borel, then $\nu(B) = \int_B f(x) d\mu$.

(2) For any subset $A \subseteq \mathbb{R}^n$, define $\nu(A) = \inf \left\{ \nu(B) : A \subseteq B, B \text{ is Borel subset of } \mathbb{R}^n \right\}$.

Then:

(i) $\nu: 2^{\mathbb{R}^n} \rightarrow [0, +\infty]$ is in fact a Radon measure.

(ii) If μ is Radon, then $\nu \ll \mu$ and $D_\mu \nu(x) = f(x)$ μ -a.e. in $\mathbb{R}^n$.

pf. (i) • $\phi \subseteq \mathbb{R}^n$ is Borel so $\nu(\phi) = \int_{\phi} f(x) d\mu = 0$. So $\nu(\phi) = 0$.

• Assume $A, A_1, A_2, \dots, A_m, \dots \subseteq \mathbb{R}^n$ s.t. $A \subseteq \bigcup_{k=1}^{\infty} A_k$. Then by def, fix $\varepsilon > 0$, $\exists B_k \subseteq \mathbb{R}^n$ Borel s.t. $A_k \subseteq B_k$ and $\nu(A_k) > \nu(B_k) - \frac{\varepsilon}{2^k}$. Then $A \subseteq \bigcup A_k \subseteq \bigcup B_k$. So by def $\nu(A) \leq \nu(\bigcup B_k)$ for $\bigcup B_k$ Borel containing A and $\nu(A)$ is the inf of such sets. Then notice

$$\begin{aligned} \nu(A) &\leq \nu\left(\bigcup_{k=1}^{\infty} B_k\right) = \int_{\bigcup_{k=1}^{\infty} B_k} f(x) d\mu = \int f(x) \cdot \chi_{\bigcup B_k} d\mu \\ &\leq \int_{\mathbb{R}^n} f(x) \sum_{k=1}^{\infty} \chi_{B_k} d\mu \\ &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f(x) \chi_{B_k} d\mu \quad (\text{MCT}) \\ &= \sum_{k=1}^{\infty} \int_{B_k} f(x) d\mu = \sum_{k=1}^{\infty} \nu(B_k) \leq \sum_{k=1}^{\infty} \nu(A_k) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\nu(A) \leq \sum_{k=1}^{\infty} \nu(A_k)$. Thus ν is a measure.

Now we prove ν is a Borel measure by Carathéodory criterion. Then let $A_1, A_2 \subseteq \mathbb{R}^n$ s.t. $\text{dist}(A_1, A_2) = d > 0$. WTS $\nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$. Because ν is a measure on $\mathbb{R}^n$, subadditivity gives $\nu(A_1 \cup A_2) \leq \nu(A_1) + \nu(A_2)$. So remains to prove $\nu(A_1 \cup A_2) \geq \nu(A_1) + \nu(A_2)$. By def, we fix $\varepsilon > 0$. Then we choose a Borel set B s.t. $A_1 \cup A_2 \subseteq B$ and

$$\nu(A_1 \cup A_2) \geq \nu(B) - \varepsilon.$$

Consider $B_1 = \bar{A}_1 \cap B$ and $B_2 = \bar{A}_2 \cap B$.

Since $B_i = (\text{closed set}) \cap (\text{Borel}) = \text{Borel}$, and

$$\text{dist}(B_1, B_2) = \text{dist}(\bar{A}_1, \bar{A}_2) = \text{dist}(A_1, A_2) > 0$$

for $A_i \subseteq B_i \subseteq \bar{A}_i$ and $\text{dist}(\bar{A}_1, \bar{A}_2) = \text{dist}(A_1, A_2) = d > 0$. So $B_1 \cap B_2 = \emptyset$. Then

$$\nu(B_1 \cup B_2) = \int_{B_1 \cup B_2} f(x) d\mu = \int_{B_1} f(x) d\mu + \int_{B_2} f(x) d\mu = \nu(B_1) + \nu(B_2).$$

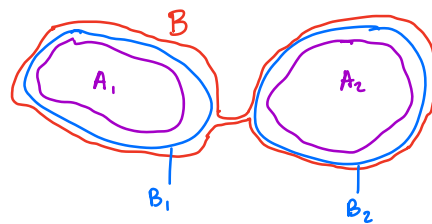
Since $B_1, B_2 \subseteq B$, $\nu(B) \geq \nu(B_1 \cup B_2)$ by monotonicity. Also

$$\nu(B) \leq \nu(A_1 \cup A_2) + \varepsilon. \quad (A_i \subseteq B_i)$$

Thus $\nu(A_1 \cup A_2) + \varepsilon > \nu(B_1 \cup B_2) = \nu(B_1) + \nu(B_2) \geq \nu(A_1) + \nu(A_2)$.

This proves that ν is Borel.

Now we prove ν is Borel regular. WTS $\forall A \subseteq \mathbb{R}^n$, $\exists B$ Borel s.t.



$A \subseteq B$ and $\nu(A) = \nu(B)$. By def, $\forall A \subseteq \mathbb{R}^n$, $\nu(A) = \inf \{ \nu(B) : A \subseteq B \text{ Borel} \}$.

Thus choose $B_k \supseteq A$ Borel s.t. $\nu(A) \geq \nu(B_k) - \frac{1}{k}$. Let $B = \bigcap B_k \supseteq A$ and B is Borel. Thus $\nu(A) \leq \nu(B)$. On the other hand, $B \subseteq B_k \Rightarrow \nu(B) \leq \nu(B_k) \leq \nu(A) + \frac{1}{k}$. $\forall k$. Now this implies $\nu(B) \leq \nu(A)$. So $\nu(A) = \nu(B)$.

Finally, let $K \subseteq \mathbb{R}^n$ be a compact set, then

$$\nu(K) = \int_K f(x) d\mu < \infty \quad \text{for } f \in L^1_{loc}(\mathbb{R}^n, \mu).$$

So ν is Radon. This^K proves (i).

- (ii) Assume μ is Radon. We show that $\nu \ll \mu$. Let $A \subseteq \mathbb{R}^n$ s.t. $\mu(A) = 0$. Since μ is Borel-regular, $\exists B \subseteq \mathbb{R}^n$ s.t. $A \subseteq B$, $\mu(A) = \mu(B)$. Then $\mu(B) = 0$. But B is Borel, so

$$\nu(B) = \int_B f(x) d\mu = 0 \quad \text{for integrating on a measure zero set.}$$

Then $\nu(A) \leq \nu(B) = 0 \Rightarrow \nu(A) = 0 \Rightarrow \nu \ll \mu$.

Claim. If $A \subseteq \mathbb{R}^n$ is μ -measurable, then

$$\nu(A) = \int_A f(x) d\mu.$$

Now this is true not only for Borel set, but any $A \in \mathcal{M}_\mu(\mathbb{R}^n)$.

pf of Claim. Assume $A \in \mathcal{M}_\mu(\mathbb{R}^n)$. Then $\exists B$ Borel s.t. $A \subseteq B$

and $\mu(A) = \mu(B)$. Then

$$\mu(B) \underset{A \in \mathcal{M}_\mu}{=} \mu(A) + \mu(B-A) \Rightarrow \mu(B-A) = 0.$$

$(A, B-A \in \mathcal{M}_\mu)$

$$\begin{aligned} \Rightarrow \nu(B) = \int_B f(x) d\mu &\stackrel{\downarrow}{=} \int_A f(x) d\mu + \int_{B-A} f(x) d\mu \\ &= \int_A f(x) d\mu. \end{aligned}$$

Also $\nu \ll \mu \Rightarrow \nu(B-A) = 0$. Then $\nu(A) \leq \nu(B)$ and $\nu(B) \leq \nu(A) + \nu(B-A)$
 $\dots = \nu(A) \Rightarrow \nu(A) = \nu(B) = \int_A f(x) d\mu.$ ▣

Finally, by R-N Thm, $\forall A \in M_\mu(\mathbb{R}^n)$,

$$\int_A D_\mu v(x) d\mu = v(A) = \int_A f(x) d\mu.$$

$$\text{So } \int_A [D_\mu v(x) - f(x)] d\mu = 0, \quad \forall A \in M_\mu.$$

So by Lem 1, $D_\mu v(x) - f(x) = 0$ μ -a.e. □

Lec 7. Lebesgue-Besicovitch Differentiation Theorem.

Thm (Lebesgue-Besicovitch Differentiation Thm). Let μ be a Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\lim_{r \rightarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(y) d\mu = f(x) \quad \mu\text{-a.e. in } \mathbb{R}^n.$$

pf. Write $f = f^+ - f^-$. Define measures $\nu^+, \nu^- : 2^{\mathbb{R}^n} \rightarrow [0, +\infty]$ as follows:

$$\nu^+(B) = \int_B f^+(x) d\mu \quad \nu^-(B) = \int_B f^-(x) d\mu$$

for any Borel set $B \subseteq \mathbb{R}^n$. For any $A \subseteq \mathbb{R}^n$, let

$$\nu^+(A) = \inf \left\{ \nu^+(B) : A \subseteq B \subseteq \mathbb{R}^n, B \text{ Borel} \right\}$$

$$\nu^-(A) = \inf \left\{ \nu^-(B) : A \subseteq B \subseteq \mathbb{R}^n, B \text{ Borel} \right\}$$

Since $f \in L^1_{loc}(\mathbb{R}^n, \mu)$, for all compact $K \subseteq \mathbb{R}^n$,

$$\int_K f(x) d\mu = \int_K f^+(x) d\mu + \int_K f^-(x) d\mu.$$

Both integrals are finite. So $f^+, f^- \in L^1_{loc}(\mathbb{R}^n, \mu)$. By Lem 2 last time, $\nu^+, \nu^- : 2^{\mathbb{R}^n} \rightarrow [0, +\infty]$ are Radon measures and

$$1) \nu^\pm \ll \mu,$$

$$2) D_\mu \nu^\pm(x) = f^\pm(x), \quad \mu\text{-a.e. } x \in \mathbb{R}^n.$$

Thus for μ -a.e. $x \in \mathbb{R}^n$,

$$D_\mu \nu^\pm(x) = f^\pm(x) = \lim_{r \rightarrow 0} \frac{\nu^\pm(B_r(x))}{\mu(B_r(x))} = \lim_{r \rightarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f^\pm(x) d\mu \quad \mu\text{-a.e. } x \in \mathbb{R}^n$$

But

$$\lim_{r \rightarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(x) d\mu = \lim_{r \rightarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} (f^+ - f^-)(x) d\mu = f^+(x) - f^-(x) = f(x) \quad \mu\text{-a.e. } x \in \mathbb{R}^n. \quad \square$$

Thm (Lebesgue points for Radon Measures).

Let $1 \leq p < \infty$ and μ be a Radon measure on $\mathbb{R}^n$. Then $\forall$ locally p -integrable function $f \in L^p_{loc}(\mathbb{R}^n, \mu)$ one has

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - f(x)|^p d\mu = 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n. \quad (*)$$

- Points $x \in \mathbb{R}^n$ satisfying $(*)$ are called **Lebesgue points** of f . Note this is stronger than the last Thm:

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - f(x)|^p d\mu = 0 \Rightarrow \left| \int_{B_r(x)} f(y) d\mu - f(x) \right| = \left| \int_{B_r(x)} f(y) d\mu - \int_{B_r(x)} f(x) d\mu \right| = \left| \int_{B_r(x)} |f(y) - f(x)|^p d\mu \right| \xrightarrow{r \rightarrow 0} 0.$$

$f(x)$ is constant, y is variable

Lem. For $a, b \geq 0$ and $1 \leq p < \infty$ one has

$$(a+b)^p \leq 2^{p-1} (a^p + b^p).$$

If $a, b \in \mathbb{R}$, then

$$|a+b|^p \leq 2^{p-1} (|a|^p + |b|^p).$$

pf. Study $f(t) = 2^{p-1} (a^p + t^p) - (a+t)^p$ for extremums in the interval $t \in [0, \infty)$. Then we see $t=a$ is the only pt of minimum. $\square$

Cor. $|f(y) + g(y)|^p \leq 2^{p-1} (|f(y)|^p + |g(y)|^p)$. In particular if $f, g \in L^p_{loc}(\mathbb{R}^n, \mu)$, then $f+g \in L^p_{loc}(\mathbb{R}^n, \mu)$.

pf of Thm.

e.g. just $\mathbb{Q}$.

Choose a dense sequence $\{\gamma_k\} \subseteq \mathbb{R}$. Then for each k , we have $f(y) - \gamma_k \in L^p_{loc}(\mathbb{R}^n, \mu)$, $|f(y) - \gamma_k|^p \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then by the first thm, $\exists A_k \subseteq \mathbb{R}^n$ st. $\mu(A_k) = 0$ and

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - \gamma_k|^p d\mu = |f(x) - \gamma_k|^p \quad \forall x \in \mathbb{R}^n - A_k.$$

Denote $A = \bigcup_{k=1}^{\infty} A_k$. Then $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k) = 0$. Also,

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - \gamma_k|^p d\mu = |f(x) - \gamma_k|^p \quad \forall x \in \mathbb{R}^n - A.$$

Fix $\varepsilon > 0$. Choose γ_k st. $|f(x) - \gamma_k|^p < \varepsilon/2^p$ because $\{\gamma_k\}$ is dense in $\mathbb{R}$.

$$\Rightarrow \limsup_{r \rightarrow 0} \int_{B_r(x)} |f(y) - f(x)|^p d\mu = \limsup_{r \rightarrow 0} \int_{B_r(x)} |f(y) - \gamma_k + \gamma_k - f(x)|^p d\mu$$

$\limsup (a_n + b_n) \leq \limsup (a_n) + \limsup (b_n)$

$$\begin{aligned}
&\leq \limsup_{r \rightarrow 0} 2^{p-1} \int_{B_r(x)} |f(y) - r_k|^p d\mu + \limsup_{r \rightarrow 0} 2^{p-1} \int_{B_r(x)} |f(x) - r_k|^p d\mu \\
&= 2^{p-1} |f(x) - r_k|^p + \lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - r_k|^p d\mu \\
&= 2 \cdot 2^{p-1} |f(x) - r_k|^p = 2^p |f(x) - r_k|^p < \varepsilon. \quad \forall x \in \mathbb{R}^n - A.
\end{aligned}$$

Thus $\forall x \in \mathbb{R}^n - A$, $\lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - f(x)|^p d\mu = 0$.

□

Thm. (Differentiation with noncentered balls)

Let $1 \leq p < \infty$ and $f \in L^p_{loc}(\mathbb{R}^n, dx)$ where $dx = \mu$ is the n -dim Lebesgue measure. Then

$$\lim_{\substack{r \rightarrow \infty \\ x \in B_r(z)}} \int_{B_r(z)} |f(y) - f(x)|^p dy = 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

pf. let $x \in B_r(z)$. Then

$$\int_{B_r(z)} |f(y) - f(x)|^p dy = \frac{1}{|B_r(z)|} \int_{B_r(z)} |f(y) - f(x)|^p dy$$

Because $x \in B_r(z) \subseteq B_{2r}(x)$, we have

$$\begin{aligned}
\int_{B_r(z)} |f(y) - f(x)|^p dy &\leq \int_{B_{2r}(x)} |f(y) - f(x)|^p dy \\
\Rightarrow \int_{B_r(z)} |f(y) - f(x)|^p d\mu &\leq \frac{1}{|B_r(z)|} \int_{B_{2r}(x)} |f(y) - f(x)|^p d\mu \\
&= \frac{2^n}{|B_{2r}(x)|} \int_{B_{2r}(x)} |f(y) - f(x)|^p d\mu \\
&= 2^n \cdot \int_{B_{2r}(x)} |f(y) - f(x)|^p d\mu \rightarrow 0
\end{aligned}$$

for a.e. $x \in \mathbb{R}^n$. In conclusion

$$\lim_{\substack{r \rightarrow \infty \\ x \in B_r(z)}} \int_{B_r(z)} |f(y) - f(x)|^p dy = 0 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

Cor. Let $1 \leq p < \infty$ and $f \in L^p_{loc}(\mathbb{R}^n, \mathcal{L}^n)$. Then for a.e. $x \in \mathbb{R}^n$,

$$\lim_{\substack{r \rightarrow \infty \\ x \in B_r(z)}} \int_{B_r(z)} f(y) dy = f(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

pf. $f \in L^p_{loc}(\mathbb{R}^n, \mathcal{L}^n) \Rightarrow f \in L^1_{loc}(\mathbb{R}^n, \mathcal{L}^n)$. Apply the previous Thm and it implies the weaker version by Δ -ineqn. $\square$

Rmk.

$$\begin{aligned} \int_K |f(x)| d\mu &= \int_{K \cap \{|f(x)| > 1\}} |f(x)| d\mu + \int_{K \cap \{|f(x)| \leq 1\}} |f(x)| d\mu \\ &\leq \int_{K \cap \{|f(x)| > 1\}} |f(x)|^p d\mu + \int_{K \cap \{|f(x)| \leq 1\}} 1 d\mu \\ &\leq \int_K |f(x)|^p d\mu + \mu(K) < \infty \end{aligned}$$

for μ Radon
for $f \in L^p_{loc}(\mathbb{R}^n, \mu)$.

• Points of Density 1 and 0.

Thm. Let $E \subseteq \mathbb{R}^n$ be Lebesgue measurable, i.e. $E \in \mathcal{M}_{\mathcal{L}^n}(\mathbb{R}^n)$. Then

1) $\lim_{r \rightarrow 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1$ for a.e. $x \in E$ these are called pts of density 1.

2) $\lim_{r \rightarrow 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 0$ for a.e. $x \in \mathbb{R}^n - E$. these are called pts of density 0.

Rmk: $|\cdot| = \mathcal{L}^n(\cdot)$

pf. Let $f(x) = \chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c \end{cases}$ is Lebesgue measurable and locally integrable wrt $\mathcal{L}^n$. Apply Lebesgue-Besicovitch Differentiation Thm to $f = \chi_E$. Then

$$f(x) = \lim_{r \rightarrow 0} \int_{B_r(x)} f(y) dy = \lim_{r \rightarrow 0} \frac{|B_r(x) \cap E|}{|B_r(x)|}$$

and $f(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c \end{cases}$.

$\square$

• Approximate Limit & Continuity.

Def. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lebesgue measurable. Then the vector $l \in \mathbb{R}^m$ is said to be approximate limit of f at $x \in \mathbb{R}^n$, if for every $\varepsilon > 0$,

$$\lim_{r \rightarrow 0} \frac{|B_r(x) \cap \{f - l > \varepsilon\}|}{|B_r(x)|} = 0. \quad \text{writ a.p. lim}_{y \rightarrow x} f(y) = l.$$

lec 8. Approximate Limit and Approximate Continuity

Thm. Approximate limit is unique if it exists.

pf. Otherwise $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lebesgue measurable and $l_1 \neq l_2 \in \mathbb{R}^m$ are two limit.
Let $|l_1 - l_2| = \varepsilon > 0$. Now, for any $y \in \mathbb{R}^n$,

$$|f(y) - l_1| + |f(y) - l_2| \geq |l_1 - l_2| = \varepsilon.$$

Thus for any $y \in \mathbb{R}^n$, either $|f(y) - l_1| \geq \frac{\varepsilon}{2}$ or $|f(y) - l_2| \geq \frac{\varepsilon}{2}$. Thus

$$\begin{aligned} B_r(x) &\subseteq \{y \in B_r(x) : |f(y) - l_1| \geq \frac{\varepsilon}{2}\} \cup \{y \in B_r(x) : |f(y) - l_2| \geq \frac{\varepsilon}{2}\} \\ &\equiv A_1(r) \cup A_2(r) \end{aligned}$$

$$\Rightarrow |B_r(x)| \leq |A_1(r)| + |A_2(r)| \quad \forall r > 0. \quad (\Delta)$$

But

$$\text{ap} \lim_{y \rightarrow x} f(y) = l_1 \Rightarrow \lim_{r \rightarrow 0} \frac{|A_1(r)|}{|B_r(x)|} = 0. \quad (1)$$

$$\text{ap} \lim_{y \rightarrow x} f(y) = l_2 \Rightarrow \lim_{r \rightarrow 0} \frac{|A_2(r)|}{|B_r(x)|} = 0. \quad (2)$$

(1) + (2) contradicts (Δ) . □

Def. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lebesgue measurable. Then f is called **approximately continuous** at $x \in \mathbb{R}^n$ if $\text{ap} \lim_{y \rightarrow x} f(y) = f(x)$, i.e.

$$\lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} |B_r(x) \cap \{y \in \mathbb{R}^n : |f(y) - f(x)| \geq \varepsilon\}| = 0 \quad \forall \varepsilon > 0.$$

Thm. Every Lebesgue measurable function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is approximately continuous $\mathcal{L}^n$ -a.e.

pf. We show $\exists$ a sequence $\{K_\ell\}$ of compact disjoint sets $K_\ell \subseteq \mathbb{R}^n$ s.t.

$$|\mathbb{R}^n - \bigcup_{\ell=1}^{\infty} K_\ell| = 0 \quad \text{and} \quad f|_{K_\ell} \text{ is continuous, } \ell = 1, 2, \dots$$

By Lusin's thm, $\exists K_1 \subseteq B_1$ (unit ball) s.t. $|B_1 - K_1| \leq 1$ and $f|_{K_1}$ is cts.

Let $B_\ell = \{x \in \mathbb{R}^n : |x| \leq \ell\}$, $\ell = 1, 2, 3, \dots$

And $K_1, \dots, K_{\ell-1}$ is chosen. Then choose

- $K_L \subseteq B_L - \bigcup_{i=1}^{L-1} K_i$. ← Borel set
- $|B_L - \bigcup_{i=1}^{L-1} K_i| \leq 1/L$.
- $f|_{K_L}$ is continuous.

This will imply that

$$|\mathbb{R}^n - \bigcup_{L=1}^{\infty} K_L| = 0.$$

Proof. For any ball $B \in \mathbb{R}^n$, choose $L_0 \in \mathbb{N}$ s.t. $B \subseteq B_{L_0}$. Then $B \subseteq B_L$ $\forall L \geq L_0$. Then

$$|B - \bigcup_{i=1}^{\infty} K_i| \leq |B_L - \bigcup_{i=1}^{\infty} K_i| \leq |B_L - \bigcup_{i=1}^L K_i| \leq \frac{1}{L} \quad \forall L \geq L_0.$$

$$\Rightarrow |B - \bigcup_{i=1}^{\infty} K_i| = 0. \quad (\text{let } L \rightarrow \infty).$$

$$\Rightarrow |\mathbb{R}^n - \bigcup_{i=1}^{\infty} K_i| = \left| \bigcup_{i=1}^{\infty} B_i - \bigcup_{i=1}^{\infty} K_i \right| \leq \sum_{i=1}^{\infty} |B_i - \bigcup_{j=1}^{\infty} K_j| = 0. \quad \blacksquare$$

We have proven almost all points of K_i are pts of density 1, $\forall i$. Thus $\exists C_i \subseteq K_i$ s.t. $|C_i| = 0$ and

$$\lim_{r \rightarrow 0} \frac{|K_i \cap B_r(x)|}{|B_r(x)|} = 1 \quad \forall x \in K_i - C_i.$$

$$\Rightarrow \left| \bigcup_{i=1}^{\infty} C_i \right| \leq \sum_{i=1}^{\infty} |C_i| = 0, \text{ let } C = \bigcup_{i=1}^{\infty} C_i.$$

Now, if $A = \bigcup_{i=1}^{\infty} (K_i - C_i)$. Then

$$|\mathbb{R}^n - A| = \left| \mathbb{R}^n - \bigcup_{i=1}^{\infty} (K_i - C_i) \right| \leq \left| \mathbb{R}^n - \bigcup_{i=1}^{\infty} K_i \right| + \left| \bigcup_{i=1}^{\infty} C_i \right| = 0.$$

We'll prove f is approximately cts at any point $x \in A$, and this will be $\mathbb{R}^n$ -a.e. in $\mathbb{R}^n$. Pick $x \in A = \bigcup_{i=1}^{\infty} (K_i - C_i)$. $\exists N \in \mathbb{N}$ s.t. $x \in K_N - C_N$.

So x is a density 1 pt for K_N . So

$$\lim_{r \rightarrow 0} \frac{|B_r(x) \cap K_N|}{|B_r(x)|} = 1.$$

Since f cts on K_N , $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$.

For $0 < r < \delta$,

$$B_r(x) \cap \{y \in \mathbb{R}^n : |f(y) - f(x)| \geq \varepsilon\} \subseteq B_r(x) \cap (\mathbb{R}^n - K_N)$$

$$\Rightarrow |B_r(x) \cap \{y \in \mathbb{R}^n : |f(y) - f(x)| \geq \varepsilon\}| \leq |B_r(x) \cap K_N^c| = |B_r(x)| - |B_r(x) \cap K_N|$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{|B_r(x) \cap \{y \in \mathbb{R}^n : |f(y) - f(x)| \geq \varepsilon\}|}{|B_r(x)|}$$

$$\leq \lim_{r \rightarrow 0} \frac{|B_r(x)| - |B_r(x) \cap K_N|}{|B_r(x)|}$$

$$= 1 - \lim_{r \rightarrow 0} \frac{|B_r(x) \cap K_N|}{|B_r(x)|} = 1 - 1 = 0$$

$$\Rightarrow \lim_{y \rightarrow x} f(y) = f(x) \quad \forall x \in A, \quad |\mathbb{R}^n - A| = 0.$$

□

Product Measures & Fubini's Theorem.

Def. Let $X, Y \neq \emptyset$ and μ, ν be measure on X, Y respectively. Define the function $\mu \times \nu: 2^{X \times Y} \rightarrow [0, +\infty]$ as follows:

$$(\mu \times \nu)(S) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : S \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{M}_\mu(X), B_i \in \mathcal{M}_\nu(Y) \right\}.$$

Then $\mu \times \nu: 2^{X \times Y} \rightarrow [0, +\infty]$ is a measure on $X \times Y$. This is called the **product measure** of μ and ν .

Def. Let (X, μ) be a measure space. Let $f: X \rightarrow [-\infty, \infty]$ be μ -measurable. Then f is called **σ -finite** wrt μ if the set $\{x \in X : f(x) \neq 0\}$ is σ -finite wrt μ .

Rmk. Any summable function is σ -finite. Fix $\varepsilon > 1$.

pf. Let $f \in L^1(X)$. Then $\{x \in X : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in X : \frac{1}{n} \leq |f(x)| < n\} \cup \{f(x) = \infty\} \cup \{f(x) = -\infty\}$.

Each sets have finite measure (the latter two has measure zero). And

$$\mu(A_n) \leq \frac{1}{\varepsilon^{n-1}} \int_X |f(x)| d\mu < \infty \Rightarrow f \text{ is } \sigma\text{-finite wrt } \mu.$$

Thm (Fubini).

THEOREM 1.22 (Fubini's Theorem). Let μ be a measure on X and ν a measure on Y .

- (i) Then $\mu \times \nu$ is a regular measure on $X \times Y$, even if μ and ν are not regular.
- (ii) If $A \subseteq X$ is μ -measurable and $B \subseteq Y$ is ν -measurable, then $A \times B$ is $(\mu \times \nu)$ -measurable and

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B).$$

- (iii) If $S \subseteq X \times Y$ is σ -finite with respect to $\mu \times \nu$, then the cross section

$$S_y := \{x \mid (x, y) \in S\}$$

is μ -measurable for ν -a.e. y ,

$$S_x := \{y \mid (x, y) \in S\}$$

is ν -measurable for μ -a.e. x , $\mu(S_y)$ is ν -integrable, and $\nu(S_x)$ is μ -integrable. Moreover,

$$(\mu \times \nu)(S) = \int_Y \mu(S_y) d\nu(y) = \int_X \nu(S_x) d\mu(x).$$

- (iv) If f is $(\mu \times \nu)$ -integrable and f is σ -finite with respect to $\mu \times \nu$ (in particular, if f is $(\mu \times \nu)$ -summable), then the mapping

$$y \mapsto \int_X f(x, y) d\mu(x)$$

is ν -integrable, the mapping

$$x \mapsto \int_Y f(x, y) d\nu(y)$$

is μ -integrable, and

$$\begin{aligned} \int_{X \times Y} f d(\mu \times \nu) &= \int_Y \left[\int_X f(x, y) d\mu(x) \right] d\nu(y) \\ &= \int_X \left[\int_Y f(x, y) d\nu(y) \right] d\mu(x). \end{aligned}$$

Now,

Fine Properties of Single Variable Functions

- 1) Monotone Functions
- 2) Functions of Bounded Variation (BV).
- 3) Absolutely Continuous Functions (AC).
- 4) Lipschitz Functions (Lip).

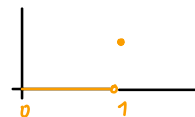
Monotone Functions

Thm 1. Let $f: [a, b] \rightarrow \mathbb{R}$ be monotone increasing or decreasing. Then

- 1) f is differentiable a.e. in $[a, b]$ w.r.t $\mathcal{L}$, and is Lebesgue measurable.
- 2) $f'(x)$ is summable in $[a, b]$ and $\int_a^b f'(x) dx \leq f(b) - f(a)$ if f is increasing.

Rmk. Equality in (2) might fail!

e.g. $f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x \in 1 \end{cases}$



Then $f'(x) = 0$ a.e.

$$\int_0^1 f'(x) dx = 0 < f(1) - f(0) = 1.$$

lem. (Proven in 201A). Let $G: \mathbb{R} \rightarrow \mathbb{R}$ be a monotone increasing function that is right-continuous:

- 1) $G \uparrow$ in $(-\infty, +\infty)$
- 2) $G(x) = \lim_{y \rightarrow x^+} G(y) \quad \forall x \in (-\infty, \infty).$

Then G give rise to a Borel-regular (in fact Radon) measure on $\mathbb{R}$ defined as:

- 1) $\nu(a, b] = G(b) - G(a)$
- 2) $\nu(A) = \inf \left\{ \sum_{k=1}^{\infty} G(b_k) - G(a_k) : A \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k] \text{ and } (a_k, b_k] \text{ disjoint} \right\}.$

pf. Proved that ν is Borel.

Borel-regular: Denote $I_k = [a_k, b_k]$. Then

$$\nu(A) = \inf \left\{ \sum_{k=1}^{\infty} \nu(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k, I_k \text{ disjoint} \right\}.$$

Take a sequence of $\{I_k^m\}$ s.t.

$$\sum_{k=1}^{\infty} \nu(I_k^m) - \frac{1}{m} \leq \nu(A) \leq \sum_{k=1}^{\infty} \nu(I_k^m).$$

Set $\bigcup_{k=1}^{\infty} I_k^m = B_m$, which is Borel, and

1) $A \subseteq B_m$

2) $\nu(B_m) - \frac{1}{m} \leq \nu(A) \leq \nu(B_m)$. $m=1, 2, \dots$

Let $B = \bigcap_{m=1}^{\infty} B_m$. Then $A \subseteq B$, and

$$\nu(B) \leq \nu(B_m) \leq \nu(A) + \frac{1}{m} \quad m \rightarrow \infty \Rightarrow \nu(B) \leq \nu(A)$$

$$\nu(A) \leq \nu(B) \text{ for } A \subseteq B \Rightarrow \nu(A) = \nu(B) \Rightarrow \nu \text{ Borel regular.}$$

Radon: Let $K \subseteq \mathbb{R}$ be compact. Then $K \subseteq [a, b]$ for some $a < b$. Then $\nu(K) \leq \nu(a, b] = G(b) - G(a) < \infty$. $\square$

Notation: $f(x+) = \lim_{h \rightarrow 0+} f(x+h)$
 $f(x-) = \lim_{h \rightarrow 0+} f(x-h)$

Lec X. Monotone Function.

Thm 2. Same as Thm 1, but w/ additional condition that f is right-cts:
 $\lim_{y \rightarrow x+} f(y) = f(x), \forall x$.

lem 1. The set of discontinuities of any monotone function $f: \mathbb{R} \rightarrow \mathbb{R}$ is at most countable.

pf. WLOG f is increasing (otherwise consider $-f$).
 Let $x \in \mathbb{R}$ be a pt of discontinuity of f . Then $f(x+) > f(x-)$. In addition, if $y > x$ then $f(y-) \geq f(x+)$. Consider $\{f(x-), f(x+)\}_{x \text{ disc.}}$.
 Then these are disjoint, nondegenerate intervals... Each contains a rational. Since $\mathbb{Q}$ is countable, done. $\square$

lem 2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be increasing and right-cts. Define $\nu: 2^{\mathbb{R}} \rightarrow [0, +\infty]$ as follows:

1) $\nu(a, b] = f(b) - f(a)$.

2) For any $A \subseteq \mathbb{R}$, let

$$\nu(A) = \inf \left\{ \sum_{k=1}^{\infty} f(b_k) - f(a_k) : A \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k], (a_k, b_k] \text{ disjoint} \right\}.$$

Then ν is a Radon measure on $\mathbb{R}$.

pf. (I) ν is a measure:

• $\phi \in \bigcup_{k=1}^{\infty} (k, k+\varepsilon_k) \Rightarrow \nu(\phi) \leq \sum_{k=1}^{\infty} f(k+\varepsilon_k) - f(k)$. Choose ε_k small s.t.
 $f(k+\varepsilon_k) - f(k) \leq \varepsilon/2^k \Rightarrow \nu(\phi) \leq \varepsilon \Rightarrow \nu(\emptyset) = 0$.

• Assume $A \subseteq \bigcup_{k=1}^{\infty} A_k$. Prove $\nu(A) \leq \sum_{k=1}^{\infty} \nu(A_k)$. $\downarrow$ disjoint
 Fix $\varepsilon > 0$. For each k , choose a cover $A_k \subseteq \bigcup_{i=1}^{\infty} (a_k^i, b_k^i]$ s.t.

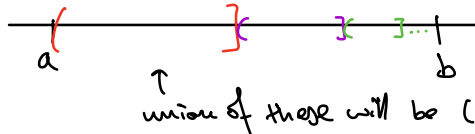
$$\nu(A_k) \leq \sum_{i=1}^{\infty} f(b_k^i) - f(a_k^i) - \varepsilon/2^k.$$

For each pair (k, i) , choose $c_k^i > b_k^i$ s.t. $f(c_k^i) < f(b_k^i) + \varepsilon/2^{i+k}$ since f is right continuous. Then $\therefore$ open

$$A \subseteq \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} (a_k^i, c_k^i) = \bigcup_{\ell=1}^{\infty} (a_{\ell}, c_{\ell}) \leftarrow \text{disjoint.}$$

$$\nu(A_k) \geq \sum_{i=1}^{\infty} (f(c_k^i) - f(a_k^i)) - \frac{\varepsilon}{2^{k-1}}.$$

Claim 1. $\nu(a, b) = f(b-) - f(a).$

Pf.  $\Rightarrow \nu(a, b) \leq f(b-) - f(a)$

↑
union of these will be (a, b)

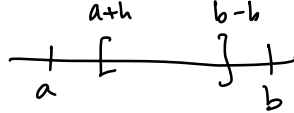
and $\nu(a, b) \leq \sum \dots = f(b-) - f(a)$

Let $(a, b) \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k)$ s.t. $\nu(a, b) > \sum_{k=1}^{\infty} (f(b_k) - f(a_k)) - \delta$. Choose $h, h_k > 0$ s.t.

$$g(a+h) - g(a) < \delta.$$

$$g(b-) - g(b-h) < \delta$$

$$g(b_k + h_k) - g(b_k) < \frac{\delta}{2^k}$$



Then

$$g(b-) - g(a) - 2\delta \leq \dots$$

$$\dots g(b-h) - g(a+h) \leq \sum_{k=1}^N (g(b_k + h_k) - g(a_k))$$

$$[a+h, b-h] \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k + h_k).$$

compact $\rightarrow \subseteq \bigcup_{k=1}^N (a_k, b_k + h_k)$

↑
disjoint.

$$g(b-) - g(a) - 2\delta \leq \sum_{k=1}^{\infty} (g(b_k + h_k) - g(a_k)) \leq \sum_{k=1}^{\infty} (g(b_k) - g(a_k)) + \delta$$

$$g(b-) - g(a) \leq \sum_{k=1}^{\infty} (g(b_k) - g(a_k)) + 3\delta = \nu(a, b) + 4\delta$$

$$\Rightarrow \nu(a, b) \geq g(b-) - g(a).$$

///

Claim 2. Let $A \subseteq \bigcup_{\ell=1}^{\infty} (a_{\ell}, c_{\ell})$ disjoint. Then $\nu(A) \leq \sum_{\ell=1}^{\infty} (f(c_{\ell}-) - f(a_{\ell}))$.

Pf. ...

Lec 14. Monotone Functions Again

2) ν -Borel: Just prove any $(a, b]$ is measurable.

$$\nu(A) \leq \nu(A \cap (a, b]) + \nu(A - (a, b]) \quad \text{since } \nu \text{ is a measure.}$$

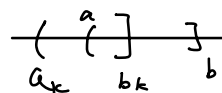
Fix $\varepsilon > 0$, $\exists$ cover $A \subseteq \bigcup_k (a_k, b_k]$ s.t.

$$\nu(A) \geq \sum_k (f(b_k) - f(a_k)) - \varepsilon.$$

$$\begin{cases} A \cap (a, b] \subseteq \bigcup_k (a_k, b_k] \cap (a, b] \\ A - (a, b] \subseteq \bigcup_k (a_k, b_k] - (a, b] \end{cases} \Leftrightarrow \text{union of half open half closed intervals.}$$

$$\Rightarrow \nu(A \cap (a, b]) \leq \sum_{k=1}^{\infty} (f(\cdot) - f(\cdot))$$

$$\nu(A - (a, b]) \leq \sum_{k=1}^{\infty} (f(\cdot) - f(\cdot))$$



$$\Rightarrow \nu(A \cap (a, b]) + \nu(A - (a, b]) \leq \sum_{k=1}^{\infty} (f(b_k) - f(a_k)) \leq \nu(A) + \varepsilon \quad \forall \varepsilon > 0.$$

Done.

//

Cor. $\nu(\{a\}) = f(a) - f(a-).$

$$\nu(a, b] = f(b) - f(a)$$

$$\nu(a, b) = f(b-) - f(a).$$

$$\nu[a, b] = \nu(a, b) \cup \{b\}$$

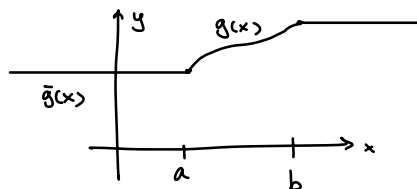
Radon - Obvious.

□

[pf of Thm 2:] let $g: [a, b] \rightarrow \mathbb{R}$ be increasing & right-cts. extend g

as

$$\bar{g}(x) = \begin{cases} g(a) & x < a \\ g(x) & x \in [a, b] \\ g(b) & x > b \end{cases}$$



Identify $\bar{g}$ w/ g . $\bar{g}$ will generate a Radon measure (as we proved) ν s.t.

$$\nu(c, d] = g(d) - g(c) \quad \forall (c, d] \subseteq \mathbb{R}.$$

By Lebesgue Decomposition thm,

$$\nu = \nu_{ac} + \nu_s \quad \text{w/} \quad \nu_{ac} \ll \lambda, \quad \nu_s \perp \lambda. \quad \text{Lebesgue } \mathcal{L}$$

1. The derivative $D_\lambda \nu_{ac}(x)$ exists and is finite λ -a.e.

2. $D_\lambda \nu_s(x) = 0$ λ -a.e.

3. $\forall$ Borel set $A \subseteq \mathbb{R}$, $\nu_{ac}(A) = \int_A D_\lambda \nu_{ac}(x) dx$.

The function $D_\lambda v_{ac}(x)$ is nonnegative and is in $L^1_{loc}(\mathbb{R}, \lambda)$.

$$\int_{(a,b)} f(y) d\gamma \rightarrow f(x)$$

Claim. Let $x \in \mathbb{R}$ be a Lebesgue pt of $D_\lambda v_{ac}(x)$ and let $D_\lambda v_s(x) = 0$.

Then $g'(x) = D_\lambda v_{ac}(x)$.

pf. Let $h > 0$.

$$\frac{g(x+h) - g(x)}{h} = \frac{v(x, x+h)}{h} = \frac{v_{ac}(x, x+h)}{h} + \frac{v_s(x, x+h)}{h}$$

$$= \frac{1}{h} \int_{[x, x+h]} D_\lambda v_{ac}(x) dx + \frac{v_s(x, x+h)}{h}$$

$$= \frac{1}{h} \int_{[x, x+h]} D_\lambda v_{ac}(x) dx + \frac{v_s(x, x+h)}{h}$$

Lebesgue Diff Thm
for non-centered
balls

$$\xrightarrow{h \rightarrow 0} D_\lambda v_{ac}(x) + 0$$

↑
x Lebesgue pt
of $D_\lambda v_{ac}$

$$0 \leq \frac{v_s(x, x+h)}{h} \leq 2 \cdot \frac{v_s[x-h, x+h]}{2h} \xrightarrow{h \rightarrow 0_+} 2 D_\lambda v_s(x) = 0.$$

So $g'(x) \rightarrow D_\lambda v_{ac}(x)$. Same for the left derivative.

Since almost every $x \in \mathbb{R}$ is a Lebesgue pt of $D_\lambda v_{ac}$, g is differentiable.

///

WTS $\int_a^b g'(x) dx \leq g(b) - g(a)$.

$$\begin{aligned} \int_a^b g'(x) dx &= \int_{[a,b]} g'(x) dx = \int_{[a,b]} D_\lambda v_{ac}(x) dx = v_{ac}(a,b) \\ &= v(a,b] - v_s(a,b] \\ &= g(b) - g(a) - v_s(a,b]. \end{aligned}$$

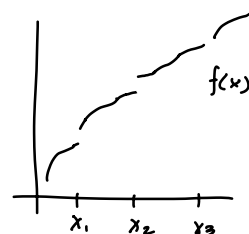
$f \upharpoonright g : [a,b] \rightarrow \mathbb{R}$

$\mathcal{D}$

pf of Thm 1. Let $\{x_1, \dots, x_n, \dots\}$ be the discontinuities of f (lem 1).

Define $g(x) = f(x+)$ ← resetting the value.

$$g(x) = \begin{cases} f(x), & x \notin \mathcal{D} \\ f(x_k+), & x = x_k \in \mathcal{D} \end{cases}$$



Assume $g'(x)$ exists and $x \notin \mathcal{D}$. Then:

Claim: $f'(x)$ exists and $f'(x) = g'(x)$. The Thm 1 will follow

pf of claim. Assume $g'(x) = L \in \mathbb{R}$ is finite. Assume $x \in (a, b)$. Fix $\varepsilon > 0$. $\exists \delta > 0$,

$$\text{s.t.} \quad \left| \frac{g(x+h) - g(x)}{h} - L \right| < \varepsilon \quad \text{whenever } |h| < \delta.$$

$$\Rightarrow (L - \varepsilon)|h| < |g(x+h) - g(x)| < (L + \varepsilon)|h|$$

(I) $h > 0$: $(L - \varepsilon)h < g(x+h) - g(x) < (L + \varepsilon)h$ whenever $0 < h < \delta$.

Since $x \notin \mathcal{D}$, $f(x) = g(x)$. Fix $h \in (0, \delta)$. $\exists y \in (x + (1-\varepsilon)h, x+h)$ s.t. f is cts at y , since $\mathcal{D}$ is at most countable. So

$$f(x+h) - f(x) = f(x+h) - g(x)$$

$$\geq f(y) - g(x)$$

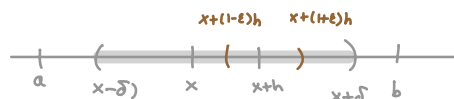
$$\geq g(y) - g(x)$$

$$(|y-x| < \delta)$$

$$\geq (L - \varepsilon)(y - x)$$

$$> (L - \varepsilon)(x + (1-\varepsilon)h - x)$$

$$= (L - \varepsilon)(1 - \varepsilon)h.$$



$$\Rightarrow \frac{f(x+h) - f(x)}{h} \geq (L - \varepsilon)(1 - \varepsilon) \quad \forall h \in (0, \delta).$$

$$\Rightarrow \liminf_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \geq (L - \varepsilon)(1 - \varepsilon).$$

To get an upper bound, choose $y \in (x+h, x + (1+\varepsilon)h)$ s.t. f is cts at y . Then

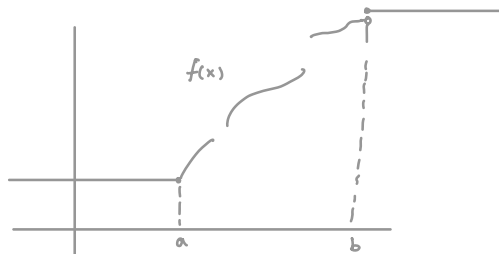
$$\limsup_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \leq (1 + \varepsilon)(L + \varepsilon).$$

$$\text{let } \varepsilon \rightarrow 0, \quad L \leq \liminf_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \leq \limsup_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} < L$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x) = L.$$

RIGHT derivative $\equiv f'(x+)$

(II) $h < 0$: Analogous to (I).



Since $f'(x) = g'(x)$ a.e. (Thm 2).

$$\int_a^b f'(x) dx = \int_a^b g'(x) dx \leq g(b) - g(a) = f(b) - f(a) \leq f(b) - f(a).$$

Same because we extend $g: [a, b] \rightarrow \mathbb{R}$ to $g: \mathbb{R} \rightarrow \mathbb{R}$ by above. So $g(b) = f(b)$ $\square$

Lec 15. Function of Bounded Variation.

Def. A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be a **function of bounded variation** (BV function) if $\exists M > 0$ s.t. for any partition P of $[a, b]$:

$$P: a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

we have

$$V_a^b(f, P) := \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq M.$$

The **total variation** of f on $[a, b]$ is

$$V_a^b(f) = \sup_P V_a^b(f, P).$$

Ex. If $f: [a, b] \rightarrow \mathbb{R}$ is monotone, then $V_a^b(f, P) = |f(b) - f(a)| \quad \forall P$. So

$V_a^b(f) = |f(b) - f(a)|$ also. Thus

$$\{\text{monotone } f: [a, b] \rightarrow \mathbb{R}\} \subseteq BV[a, b].$$

Thm 1. Let $f, g \in BV[a, b]$. Then

(P1) $f \pm g \in BV[a, b]$, $V_a^b(f \pm g) \leq V_a^b(f) + V_a^b(g)$.

(P2) $cf \in BV[a, b]$, $V_a^b(cf) = |c| \cdot V_a^b(f)$.

(P3) $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in [a, b]$. (BV functions are bounded)

(P4) $f \cdot g \in BV[a, b]$. If $|f|, |g| \leq M$, then $V_a^b(fg) \leq M(V_a^b(f) + V_a^b(g))$.

(P5) Let $c \in [a, b]$. Then $f \in BV[a, b] \iff f \in BV[a, c] \cap BV[c, b]$. Moreover,

$$V_a^b(f) = V_a^c(f) + V_c^b(f) \quad \forall c \in [a, b].$$

(P6) $V_a^b(f) \geq f(b) - f(a)$.

(P7) The function $V(x) = V_a^x(f): [a, b] \rightarrow \mathbb{R}$ is increasing in $[a, b]$.

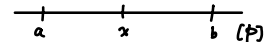
pf.

(P1).
$$\begin{aligned} V_a^b(f+g, P) &= \sum_i |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq \sum_i |f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})| \\ &\leq V_a^b(f) + V_a^b(g). \end{aligned}$$

Now take $\sup_P$ on both sides. Similar for $f-g$.

(P2) $V_a^b(cf, P) = |c| \sum_i |f(x_i) - f(x_{i-1})| = |c| \cdot V_a^b(f, P)$. Take $\sup$ on both sides.

$$\begin{aligned}
 (P3). \quad \forall x \in [a, b]. \quad \text{Write } |f(x)| &\stackrel{\Delta}{\leq} |f(a)| + |f(a) - f(x)| \\
 &\leq |f(a)| + |f(a) - f(x)| + |f(x) - f(b)| \\
 &= |f(a)| + V_a^b(f, P) \leq |f(a)| + V_a^b(f).
 \end{aligned}$$



(P4). $\forall P$, we have

$$\begin{aligned}
 V_a^b(f, P) &= \sum_i |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
 &= \sum_i |f(x_i)g(x_i) - f(x_{i-1})g(x_i) + f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})| \\
 &\leq \sum_i |g(x_i)| \cdot |f(x_i) - f(x_{i-1})| + |f(x_{i-1})| \cdot |g(x_i) - g(x_{i-1})| \\
 &\leq M \sum_i |f(x_i) - f(x_{i-1})| + |g(x_{i-1})| \\
 &\leq M \left(V_a^b(f) + V_a^b(g) \right).
 \end{aligned}$$

(P5). By adding a pt in the partition, the variation of f may only increase. This is obvious from Δ -inequality.

($\Rightarrow$): Let $f \in BV[a, b]$. We prove:

$$V_a^b(f) = V_a^c(f) + V_c^b(f). \Rightarrow f \in BV[a, c] \cap BV[c, b].$$

Now:

$$V_a^c(f, P) \leq V_a^b(f, P \cup \{b\})$$

$\nwarrow$ just add one pt to partition P of $[a, c]$ to get partition $P' = P \cup \{b\}$ of $[a, b]$.

$$\Rightarrow V_a^c(f, P) = \sup_{P: [a, c]} V_a^c(f, P) \leq V_a^b(f) < \infty.$$

$\Rightarrow f \in BV[a, c]$. Similarly $f \in BV[c, b]$.

To prove $V_a^b(f) = V_a^c(f) + V_c^b(f)$: Fix $\varepsilon > 0$. $\exists P_1: [a, c], P_2: [c, b]$ s.t.

$$V_a^c(f, P_1) > V_a^c(f) - \varepsilon$$

$$V_c^b(f, P_2) > V_c^b(f) - \varepsilon.$$

$$\Rightarrow V_a^b(f, P_1 \cup P_2) = V_a^c(f, P_1) + V_c^b(f, P_2) > V_a^c(f) + V_c^b(f) - 2\varepsilon.$$

$$\Rightarrow V_a^b(f) > V_a^c(f) + V_c^b(f) - 2\varepsilon \xrightarrow{\varepsilon \rightarrow 0} V_a^b(f) \geq V_a^c(f) + V_c^b(f).$$

Now choose $P: [a, b]$ s.t. $\check{V}_a^b(f) < \check{V}_a^b(f, P) + \varepsilon$.

Let P' denote $P \cup \{c\}$. Then

$$\begin{aligned} \check{V}_a^b(f) &< \check{V}_a^b(f, P) + \varepsilon \leq \check{V}_a^b(f, P') + \varepsilon \\ &= \check{V}_a^c(f, P_1) + \check{V}_c^b(f, P_2) + \varepsilon \\ &\leq \check{V}_a^c(f) + \check{V}_c^b(f) + \varepsilon. \end{aligned}$$

$$\xRightarrow{\varepsilon \rightarrow 0} \check{V}_a^b(f) \leq \check{V}_a^c(f) + \check{V}_c^b(f). \quad \text{So get the equality.} \quad ///$$

($\Leftarrow$): Similar argument. Just prove $f \in BV[a, b]$, then repeat the argument above. //

(P6). Obvious. Just Δ -inequality.

(P7). Let $y > x$ in $[a, b]$. Then

$$\check{V}_a^y(f) = \check{V}_a^x(f) + \check{V}_x^y(f) \geq \check{V}_a^x(f).$$

□.

Thm 2. Assume $f \in BV[a, b]$. Then f is a difference of two increasing functions in $[a, b]$.

pf. $f(x) = \check{V}_a^x(f) - (\check{V}_a^x(f) - f(x)).$

$\boxed{y > x}$: $\check{V}_a^y(f) - f(y) \geq \check{V}_a^x(f) - f(x) \Leftrightarrow V(y) - V(x) \geq f(y) - f(x) \Leftrightarrow \check{V}_x^y(f) \geq f(y) - f(x)$
 $\uparrow$
 this is (P6).

Let $A = \{f: [a, b] \rightarrow \mathbb{R} : f \uparrow [a, b]\}$.

Then $BV[a, b] \subseteq A - A$. Since increasing functions are BV functions,

So $A - A \subseteq BV[a, b] - BV[a, b] \underset{(P1)}{\subseteq} BV[a, b]$

So as a set,

$$BV[a, b] = A - A.$$

Note. A is not a vector space ($-A \cap A = \emptyset$). But $BV[a, b]$ is a vector space by (P1) and (P2).

Lipschitz Functions

Def. A function $f: [a, b] \rightarrow \mathbb{R}$ is **Lipschitz** if $\exists M > 0$ s.t. $|f(x) - f(y)| \leq M|x - y|$ $\forall x, y \in [a, b]$. This class is denoted by $\text{Lip}[a, b]$.

Lemma 1. $\text{Lip}[a, b] \subseteq \text{BV}[a, b]$.

pf. Let $f \in \text{Lip}[a, b]$. Then $\exists M > 0$ s.t. $|f(x) - f(y)| \leq M|x - y|$, $x, y \in [a, b]$.
Then let P be a partition of $[a, b]$. We have

$$\bigvee_a^b(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq M \sum_{i=1}^n |x_i - x_{i-1}| = M(b-a). \quad \checkmark \quad a = x_0 < x_1 < \dots < x_n = b$$

So $f \in \text{BV}[a, b]$. □

Cor. Lipschitz function in $[a, b]$ are a.e. differentiable in $[a, b]$.

pf. Since if $f \in \text{BV}$, it is a difference of two increasing functions f_1, f_2 , and increasing functions are a.e. differentiable. So **BV functions are a.e. differentiable. So $\text{Lip} \subseteq \text{BV}$ are a.e. differentiable.** □

Absolutely Continuous Functions.

Def. A function $f: [a, b] \rightarrow \mathbb{R}$ is **absolutely continuous** in $[a, b]$ if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. whenever $(a_k, b_k) \subseteq [a, b]$ are disjoint w/ $\sum_{k=1}^{\infty} (b_k - a_k) < \delta$, then

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \varepsilon. \quad \text{Write } f \in \text{AC}[a, b].$$

Rmk. If f is a.b. cts, $b - a < \delta$, then $|f(b) - f(a)| < \varepsilon$. So f is uniformly continuous on $[a, b]$.

Goal:

Thm 1. Let $f \in \text{AC}[a, b]$. Then $f'(x)$ exists and is finite a.e. in $[a, b]$.

Also:
$$\int_a^b f'(x) dx = f(b) - f(a).$$

This implies $f'(x)$ is Lebesgue summable in $[a, b]$: $\int_a^b |f'(x)| dx < \infty$.

Thm 2 (Properties). Let $f, g \in AC[a, b]$. Then

(P1) $f \pm g, cf \in AC[a, b]$.

(P2) $AC[a, b] \subseteq BV[a, b]$.

(P3) $f \cdot g \in AC[a, b]$

pf (P1): Def + linearity

///

(P2): Choose $\varepsilon = 1$. Then $\exists \delta > 0$ s.t. $\sum_k (b_k - a_k) < \delta$, then $\sum_k |f(b_k) - f(a_k)| < 1$.

Let P be any partition of $[a, b]$.

Choose N s.t. $(b-a)/N < \delta$. Let $a = y_0, y_1, \dots, y_N = b$ be these endpoints.

Then

$$\begin{aligned} \bigvee_a^b(f, P) &= \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq \bigvee_a^b(f, P \cup \{y_1, \dots, y_N\}). \\ &= \sum_{k=1}^N \bigvee_a^b(f, P[y_{k-1}, y_k]) \\ &\leq N \cdot 1 = N. \quad (\text{each interval in } P[y_{k-1}, y_k] \text{ is less than } \delta) \end{aligned}$$

Note this bound $N = (b-a)/\delta$ is constant and P is arbitrary. So take

$\sup_P$ on both sides $\Rightarrow f \in BV[a, b]$.

///

(P3): (P2) $\Rightarrow f, g$ are bounded by $M > 0$. Fix $\varepsilon > 0$, $\exists \delta_1, \delta_2$ for f, g s.t.

Choose $\delta = \min(\delta_1, \delta_2)$. Let $(a_k, b_k)_{k=1}^\infty \subseteq [a, b]$ be disjoint s.t.

$$\sum_{k=1}^\infty (b_k - a_k) < \delta.$$

Then

$$\begin{aligned} \sum |f(b_k)g(b_k) - f(a_k)g(a_k)| &= \sum |f(b_k)g(b_k) - f(b_k)g(a_k) + f(b_k)g(a_k) - f(a_k)g(a_k)| \\ &\leq \sum (|f(b_k)| \cdot |g(b_k) - g(a_k)| + |g(a_k)| \cdot |f(b_k) - f(a_k)|) \\ &\leq M \cdot \sum |g(b_k) - g(a_k)| + M \sum |f(b_k) - f(a_k)| \\ &\leq 2M\varepsilon. \end{aligned}$$

///

pf of Thm 1. $f \in AC \Rightarrow f \in BV \Rightarrow f = f_1 - f_2$ w/ $f_i \uparrow \Rightarrow f'(x) = f'_1(x) - f'_2(x)$ a.e.

Note: $f'_i(x) \geq 0$ if it exists. So:

(I) $f'(x)$ is $\mathcal{L}$ -measurable since $f'_i(x)$ is. ← '!

$$\begin{aligned} \text{(II)} \quad \int_a^b |f'(x)| dx &\leq \int_a^b |f'_1(x)| dx + \int_a^b |f'_2(x)| dx \\ &= \int_a^b f'_1(x) dx + \int_a^b f'_2(x) dx \\ &\leq f_1(b) - f_1(a) + f_2(b) - f_1(b) < \infty. \end{aligned}$$

Last Equality:

Step 1. Since $f \in AC \subseteq BV$, $f = v(x) - (v(x) - f(x))$, $v(x) = \overset{x}{V}(f)$.

Step 2. $f \in AC \Rightarrow \overset{x}{V}_a(f) = v(x) \in AC$.

pf. Given $\varepsilon > 0$. Let $\delta > 0$ be the one for f from def. Let (a_k, b_k) be disjoint s.t. $\sum_{k=1}^{\infty} b_k - a_k < \delta$. Then

$$\sum_{k=1}^{\infty} |v(b_k) - v(a_k)| = \sum_k \overset{b_k}{V}_{a_k}(f) := S.$$

Fix $N \in \mathbb{N}$. Consider $(a_1, b_1), \dots, (a_N, b_N)$. $\forall k \in \{1, 2, \dots, N\}$, $\exists$ partition P_k of $[a_k, b_k]$ s.t. $\overset{b_k}{V}_{a_k}(f, P_k) > \overset{b_k}{V}_{a_k}(f) - \varepsilon/2^k$. Then P_k give rise to new intervals $(a_i^k, b_i^k)_{i=1}^{m_k}$, $m_k \in \mathbb{N}$. Then

$$\overset{b_k}{V}_{a_k}(f) \leq \sum_{i=1}^{m_k} |f(b_i^k) - f(a_i^k)| + \varepsilon/2^k \quad \forall k.$$

$$\begin{aligned} \text{So } \sum_{k=1}^N \overset{b_k}{V}_{a_k}(f) &< \sum_{k=1}^N \sum_{i=1}^{m_k} |f(b_i^k) - f(a_i^k)| + \varepsilon \\ &< \varepsilon + \varepsilon = 2\varepsilon \quad (\text{each } (a_i^k, b_i^k) \subseteq (a_k, b_k)) \end{aligned}$$

Let $N \rightarrow \infty$, $S < 2\varepsilon$. So $v(x) \in AC[a, b]$. //

Step 3. $f = \underbrace{v(x)}_{AC \nmid \uparrow} - \underbrace{(v(x) - f(x))}_{AC \nmid \uparrow}$. So wlog $f \uparrow$ and $f \in AC$.

Then the radon measure ν generated by f is Absolutely Cts wrt $\mathcal{L} = \lambda$.

pf. Assume $A \subseteq \mathbb{R}$ w/ $\lambda(A) = 0$. Fix $\varepsilon > 0$, choose $\delta > 0$ that works for f in def. $\exists$ disjoint cover $\bigcup_{k=1}^{\infty} (a_k, b_k) \supseteq A$ s.t.

$$\sum_{k=1}^{\infty} (b_k - a_k) < \delta \quad (\lambda(A) = 0).$$

inf of all such sums

$$v(A) \leq \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \varepsilon \Rightarrow v(A) = 0 \Rightarrow v \ll \lambda. \quad //)$$

← proved before

Step 4. So in Lebesgue decomp: $v = v_{ac}$. Since $f'(x) = D_{\lambda} v_{ac}(x) = D_{\lambda} v(x)$.

Now

$$v(a, b] = f(b) - f(a)$$

$$\stackrel{R.N}{=} \int_{(a, b]} D_{\lambda} v(x) dx = \int_{[a, b]} D_{\lambda} v(x) dx = \int_a^b f'(x) dx.$$

□

lec 13.

Thm 1 (Integral representation of AC functions).

A function $f: [a, b] \rightarrow \mathbb{R}$ is absolutely cts in $[a, b] \iff \exists g \in L^1([a, b])$ s.t.

$$f(x) = f(a) + \int_a^x g(t) dt. \quad \forall x \in [a, b].$$

Moreover, $f'(x) = g(x)$ a.e. in $[a, b]$.

see pt of Thm 1 last time.

pf. ($\Rightarrow$). Already proven. If $f \in AC$, then $f' \in L^1([a, b])$ s.t. $f(x) - f(a) = \int_a^x f'(t) dt$.

The last ineq is because $f \in AC[a, b] \Rightarrow f \in AC[a, x] \quad \forall x \in [a, b]$.

($\Leftarrow$). Assume $g \in L^1([a, b])$ and $f(x) = f(a) + \int_a^x g(t) dt$. Let $(a_k, b_k)_{k=1}^{\infty} \subseteq [a, b]$

disjoint. Then

$$\sum |f(b_k) - f(a_k)| = \sum \left| \int_{a_k}^{b_k} g(t) dt \right| \leq \sum \int_{a_k}^{b_k} |g(t)| dt = \int_{\bigcup_{k=1}^{\infty} (a_k, b_k)} |g(t)| dt$$

By absolute continuity of integral:

$$\sum (b_k - a_k) < \delta \Rightarrow \int_{\bigcup_k (a_k, b_k)} |g(t)| dt < \varepsilon$$

$$\Rightarrow f \in AC[a, b].$$

← endpoints has measure 0, doesn't matter.

NTS $f'(x) = g(x)$. Assume $x \in (a, b)$. Let $h > 0$ small. So $x+h \in (a, b)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \int_x^{x+h} g(t) dt. \quad (*)$$

Assume x is s.t. $f'(x)$ exists (a.e. in $[a,b]$). $g(t) \in L^1[a,b]$. So a.e. x is a Lebesgue pt for g . So wlog also assume x is a Lebesgue pt for g .

So let $h \rightarrow 0^+$ in (*). Then we get

$$f'(x) = g(x).$$

Although LHS is technically the right derivative, but since $f'(x)$ exists at x , $D^+ f(x) = f'(x)$. □

BV Functions Again

Thm2. Let $f \in BV[a,b]$ and f is cts. Then $v(x) = \check{V}_a(f)$ is cts in $[a,b]$.

pf. Let $x \in (a,b)$. We'll prove $v(x-) = v(x)$ and $v(x+) = v(x)$.

Given $\varepsilon > 0$. $\exists P: a = x_0 < x_1 < \dots < x_n = x$ s.t. $\check{V}_a(f) < \check{V}_a(f, P) + \varepsilon$. Now assume $y \in (x_{n-1}, x_n)$. Then

$$v(x) = \check{V}_a(f) < \check{V}_a(f, P) + \varepsilon \leq \check{V}_a(f, P \cup \{y\}) + \varepsilon \leq \check{V}_a(f) + |f(y) - f(x)| + \varepsilon.$$

$$\Rightarrow v(x) - v(y) < \varepsilon + |f(x) - f(y)|.$$

$$\xRightarrow{y \rightarrow x^-} 0 \leq v(x) - v(x-) \leq \varepsilon, \forall \varepsilon > 0 \Rightarrow v(x-) = v(x).$$

(This also proves $v(x)$ is left cts at b). For right continuity, Choose a partition

$P: x = x_0 < \dots < x_n < b$ s.t. $\check{V}_x(f) < \check{V}_x(f, P) + \varepsilon$. Choose $y \in (x, x_1)$. Now,

$$\check{V}_x(f) < \check{V}_x(f, P) + \varepsilon \leq \check{V}_x(f, P \cup \{y\}) + \varepsilon \leq \check{V}_y(f) + |f(x) - f(y)| + \varepsilon$$

$$\Rightarrow 0 \leq v(y) - v(x) < |f(x) - f(y)| + \varepsilon$$

$$\xRightarrow[\varepsilon \rightarrow 0]{y \rightarrow x^+} v(x+) = v(x).$$

(This also proves $v(x)$ is right cts at a). □

Cantor Function.

Thm3. $\exists$ a function $f: [0,1] \rightarrow \mathbb{R}$ s.t.

1) $f \uparrow$

2) $f \in C[0,1]$.

3) $f'(x) = 0$ a.e.

4) $f(0) = 0, f(1) = 1$.

5) $f \notin AC[0,1]$.

pf. Step 1: Cantor Set.

Denoted by C . We have: $\lambda(C) = 0$.

$\forall x \in [0,1]$, $x = \sum_{k=1}^{\infty} \frac{b_k}{3^k}$, $b_k \in \{0,1,2\}$. e.g.

$$\begin{cases} \frac{1}{3} = \frac{1}{3} + \frac{0}{3^2} + \dots \\ \frac{1}{3} = \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \dots \end{cases} \quad 1 = \sum_{k=1}^{\infty} \frac{2}{3^k}.$$

Claim. $x \in C \iff x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k}$, $a_k \in \{0,1\}$.

pf.

Define:

$$C(x) = \begin{cases} \sum_{k=1}^{\infty} \frac{a_k}{2^k}, & x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k} \in C. \\ \sup_{\substack{y \leq x \\ y \in C}} C(y), & y \notin C. \end{cases}$$

Step 1. $C(x)$ increasing on C .

$x, y \in C$, $x < y$. Let $x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k} < y = \sum_{k=1}^{\infty} \frac{2c_k}{3^k}$. WTS $\sum_{k=1}^{\infty} \frac{a_k}{2^k} \leq \sum_{k=1}^{\infty} \frac{c_k}{2^k}$

Claim. $C_1 > a_1$. If $C_1 < a_1$, then

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{c_k}{3^k} - \sum_{k=1}^{\infty} \frac{a_k}{3^k} &= \frac{C_1 - a_1}{3} + \sum_{k=2}^{\infty} \frac{c_k - a_k}{3^k} \\ &\leq -\frac{1}{3} + \sum_{k=2}^{\infty} \frac{1}{3^k} < 0. \end{aligned}$$

Lec 15.

Thm 1. Let $f \in BV[a, b]$. Then $\int_a^b |f'(x)| dx \leq \frac{b}{a}(f)$.

pf. Recall $v(x) = \tilde{V}_x(f)$ is increasing in $[a, b]$.

$$|f(x) - f(y)| \leq |\tilde{V}_x(f) - \tilde{V}_y(f)| = |v(x) - v(y)| \quad \forall x, y \in (a, b).$$

Fix $x \in (a, b)$, let $y \in (a, b)$, $y \neq x$. So

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq \left| \frac{v(y) - v(x)}{y - x} \right|. \quad (*)$$

Assume both f and v are differentiable at x (since diff. a.e. in (a, b) since $f \in BV$, $v \uparrow$). Let $y \rightarrow x$ in $(*)$ so

$$|f'(x)| \leq |v'(x)| = v'(x) \quad v'(x) \geq 0.$$

Thus

$$\int_a^b |f'(x)| dx \leq \int_a^b v'(x) dx \stackrel{\substack{\text{Fund. Thm} \\ \text{for increasing} \\ \text{function}}}{\leq} v(b) - v(a) = \frac{b}{a}(f).$$

□

Thm 2. Let $f \in AC[a, b]$. Then $\int_a^b |f'(x)| dx = \frac{b}{a}(f)$.

pf. Choose any partition $\mathcal{P}: a = x_0 < x_1 < \dots < x_n = b$. f is AC.

$$\begin{aligned} \frac{b}{a}(f, \mathcal{P}) &= \sum_{k=1}^n |f(x_k) - f(x_{k-1})| = \sum_{k=1}^n \left| \int_{x_{k-1}}^{x_k} f'(t) dt \right| \leq \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f'(t)| dt \\ &= \int_a^b |f'(t)| dt \end{aligned}$$

Since $\mathcal{P}$ is arbitrary, take $\sup \mathcal{P}$ gives $\frac{b}{a}(f) \leq \int_a^b |f'(x)| dx$.

□

Prop. Let $f \in C[a, b]$ and differentiable in (a, b) s.t. $|f'(x)| \leq M \quad \forall x \in (a, b)$.

Then $f \in \text{Lip}[a, b]$. In particular $f \in C' \Rightarrow f \in \text{Lip}$.

pf. Pick any $x, y \in [a, b]$. By mean value thm, $\exists c \in (x, y)$ s.t.

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

□

Rmk. $C' \subseteq \text{Lip} \subseteq AC \subseteq BV$.

We know $f \in AC \Rightarrow \int_a^b |f'(x)| dx = V_a^b(f)$.

Prop. If $f \in BV$ and $\int_a^b |f'(x)| dx = V_a^b(f) \Rightarrow f \in AC$.

$$\text{Pf. } \int_a^x |f'(t)| dt \leq V_a^x(f) \quad \textcircled{1} \quad \int_y^b |f'(t)| dt \leq V_y^b(f) \quad \textcircled{2} \quad \text{since } f \in BV.$$

$$\Rightarrow \int_x^y |f'(x)| dx \leq V_x^y(f) \quad \textcircled{3} \quad \forall x, y \in [a, b].$$

But $\textcircled{1} + \textcircled{2} + \textcircled{3}$: $\int_a^b |f'(x)| dx \leq V_a^b(f)$. Actually " $=$ " holds

$$\Rightarrow \boxed{\int_x^y |f'(x)| dx = V_x^y(f) \quad \forall x, y \in [a, b]}!$$

Let $(a_k, b_k) \subseteq (a, b)$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| &\leq \sum_{k=1}^{\infty} V_{a_k}^{b_k}(f) = \sum_{k=1}^{\infty} \int_{a_k}^{b_k} |f'(t)| dt \\ &= \int_{\bigcup_{k=1}^{\infty} (a_k, b_k)} |f'(t)| dt. \end{aligned}$$

Since integral is AC, if $m(\bigcup_{k=1}^{\infty} (a_k, b_k)) < \delta$,

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| \leq \int_{\bigcup_{k=1}^{\infty} (a_k, b_k)} |f'(t)| dt < \varepsilon.$$

□

lec 16.

Functional Analysis

• Metric Space

Thm 1. Let X be a metric space and $A \subseteq X$. A is compact $\Leftrightarrow \forall \{x_n\} \subseteq A$,
 $\exists n_k \uparrow \in \mathbb{N}$ s.t. $x_{n_k} \rightarrow x \in A$ (sequentially compact).

Cor. Any compact set is bounded and closed in a metric space.

Thm 2. Let X be a metric space. Let $K_1 \supseteq K_2 \supseteq \dots \supseteq K_n \supseteq \dots$ be a nested sequence of compact sets. Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. If in addition, $\text{diam}(K_n) \rightarrow 0$, then $\bigcap_{n=1}^{\infty} K_n = \{x\}$ is a single point.

pf. Choose any point $x_n \in K_n$. Since $K_n \subseteq K_1, \forall n$, $\{x_n\}_n$ is a seq in K_1 . Since K_1 is compact, $\exists n_k \uparrow$ s.t. $x_{n_k} \rightarrow x \in K_1$. $n_1 < n_2 < \dots < n_k < \dots$. So $n_k \geq k$. Fix $m \in \mathbb{N}$, then $x_{n_k} \in K_m \forall k \geq m$. So x is a limit point for each K_m , so $x \in K_m \forall m \Rightarrow x \in \bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

$\text{diam}(A) = \sup_{x,y \in A} \rho(x,y)$. Assume $x,y \in \bigcap_{n=1}^{\infty} K_n$ and $\text{diam}(K_n) \rightarrow 0$, then $x,y \in K_n \forall n$.
Let $n \rightarrow \infty$, $\rho(x,y) \leq \text{diam}(K_n) \rightarrow 0 \Rightarrow x=y$. $\square$

Def. Let X be a metric space. A mapping $f: X \rightarrow X$ is a **contraction** if $\exists \alpha \in (0,1)$ s.t. $\rho(f(x), f(y)) \leq \alpha \cdot \rho(x,y) \forall x,y \in X$.

Thm 3. Let X be a complete metric space and $f: X \rightarrow X$ be a contraction. Then f has a unique fixed point.

pf. Pick any $x_0 \in X$ and define $x_n = f^{(n)}(x_0) = \overbrace{f(f(f \dots f(x_0)))}^{n \text{ times}} = f(x_{n-1})$.

Then $\{x_n\}$ is Cauchy and its limit is the unique fixed pt.

• Baire Category Theorem.

Def. Let X be a metric space, $A \subseteq X$. A is said to be **nowhere dense** in X if for any ball $B_R(x) \subseteq X$, $\exists B_R(y) \subseteq B_R(x)$ s.t. $A \cap B_R(y) = \emptyset$.

Def. (X, ρ) m.s. $A \subseteq X$. Then A is

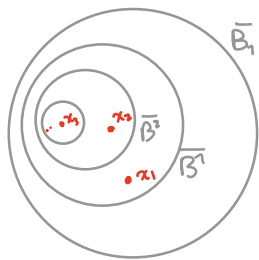
- (1) of Baire first category set if $A = \bigcup_{n=1}^{\infty} A_n$, w/ A_n nowhere dense in X .
 (2) of Baire second category otherwise.

Lec 17.

Thm 1 (Baire Category Thm). Any complete metric space X is of second Baire category set.

p.f. Assume X is of first Baire category. Thus $X = \bigcup_{n=1}^{\infty} A_n$ where A_n is nowhere dense in X . Since A_1 is nowhere dense in X , Consider

- 1) $\bar{B}_1 = \{x \in X : |x| \leq 1\}$, $\exists \bar{B}^1 \subseteq \bar{B}_1$ s.t. $\bar{B}^1 \cap A_1 = \emptyset$. — can take closed ball instead of open.



Also, assume that $\text{diam}(\bar{B}^1) \leq 1$ (otherwise make it smaller).

- 2) $\exists \bar{B}^2 \subseteq \bar{B}^1$ s.t. $\bar{B}^2 \cap A_2 = \emptyset$ $\text{diam}(\bar{B}^2) \leq \frac{1}{2}$.

$\vdots$

- n) $\exists \bar{B}^n \subseteq \bar{B}^{n-1}$ s.t. $\bar{B}^n \cap A_n = \emptyset$ $\text{diam}(\bar{B}^n) \leq \frac{1}{n}$.

Thus we get a seq $\bar{B}^1 \supseteq \bar{B}^2 \supseteq \dots \supseteq \bar{B}^n \supseteq \bar{B}^{n+1} \supseteq \dots$ s.t.

(I) $\text{diam} \bar{B}^n < \frac{1}{n}$

(II) $A_n \cap \bar{B}^n = \emptyset$.

In general metric space,
 $\bar{B}^n$ may not be compact.

Claim. $\bigcap_{n=1}^{\infty} \bar{B}^n \neq \emptyset$.

Pick $x_n \in \bar{B}^n$. Then $\{x_n\}$ is Cauchy: if $m > n$, $x_m \in \bar{B}^m \subseteq \bar{B}^n$, $x_n \in \bar{B}^n$. So $\rho(x_n, x_m) \leq \text{diam} \bar{B}^n \leq \frac{1}{n}$. So $\rho(x_m, x_n) \rightarrow 0$ when $m, n \rightarrow \infty$. Since X is complete, $x_n \rightarrow x \in X$. If $m > n$, then $x_m \rightarrow x$. Then since $x_m \in \bar{B}^n$, $\forall m > n$. Thus x is a limit pt of $\bar{B}^n$. So $x \in \bar{B}^n \forall n \in \mathbb{N}$. //

Thus $x \notin A_n$, $n=1, 2, 3, \dots$ for $A_n \cap \bar{B}^n = \emptyset$. Thus

$$x \notin \bigcup_{n=1}^{\infty} A_n = X$$

$\Rightarrow \text{fals.}$

□

Def 1. (X, ρ) a metric space. $A \subseteq X$. Then A is **dense** in X if $\bar{A} = X$.

Where $\bar{A} = A \cup \{\text{limit pts of } A\}$.

Denote $\text{int}(A) = A^\circ = \{x \in A : \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A\}$.

Def 2. Let $(X, \rho), (Y, d)$ be metric spaces, and $f: X \rightarrow Y$. Then f is called an **isometry** if f is bijective and $d(f(x_1), f(x_2)) = \rho(x_1, x_2)$. X, Y are called **isometric** if there exists an isometry $f: X \rightarrow Y$.

Def 3. Let (X, ρ) be a metric space (Y, d) is called a **completion** of X if

- 1) $X \subseteq Y$ and $\rho(x_1, x_2) = d(x_1, x_2) \quad \forall x_1, x_2 \in X$;
- 2) X is dense in Y ;
- 3) Y is complete.

Thm 2. Any metric space admits a unique completion (up to isometry).

• Normed Spaces and Bounded Linear Operators

$(\mathbb{C} \text{ or } \mathbb{R})$

Def 4. A vector space X over K is a **normed space** w/ norm $\|\cdot\|: X \rightarrow \mathbb{R}$, if

- 1) $\|x\| \geq 0 \quad \forall x \in X$ and $\|x\| = 0 \Leftrightarrow x = 0$.
- 2) $\|\alpha x\| = |\alpha| \cdot \|x\|, \quad \forall \alpha \in K, x \in X. \quad \Rightarrow (\|x\| = \|-x\|)$
- 3) $\|x+y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X. \quad (\Delta\text{-ineqn})$

Rmk. A normed space X is a metric space w/ $\rho(x, y) = \|x - y\|$.

A complete normed space is called a **Banach space**.

Def 6. Let X, Y be normed spaces. Then a mapping $T: X \rightarrow Y$ is called

- 1) **linear** if $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2) \quad \forall \alpha, \beta \in K, x_1, x_2 \in X$.
- 2) **continuous** at $x \in X$ if $\lim_{x \rightarrow x_0} T(x) = T(x_0) \Leftrightarrow \|Tx - Tx_0\|_Y \rightarrow 0 \text{ if } \|x - x_0\|_X \rightarrow 0$.
- 3) continuous at X if T is continuous at $x, \quad \forall x \in X$.

Def 7. Let X, Y be normed spaces, $T: X \rightarrow Y$ a linear mapping. Then T is **bounded** if $\exists M > 0$ s.t. $\|Tx\|_Y \leq M \cdot \|x\|_X, \quad \forall x \in X$. In that case, the **operator**

norm of T , denoted by $\|T\|$, is

$$\|T\| := \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} \leq M.$$

Thm 3. Let X, Y be normed spaces, and $T: X \rightarrow Y$ linear. Then

1) $\|T\| = \sup_{\|x\|_X=1} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X \leq 1} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X=1} \|Tx\|_Y$, when T is bounded.

2) T is bounded $\Leftrightarrow T$ is cts at 0.

3) T is cts at 0 $\Leftrightarrow T$ is cts in X .

4) T is bounded $\Leftrightarrow T$ maps bounded sets in X to bounded sets in Y .

Lec 18. Bounded Linear Operators

Thm 1. Let X, Y be normed spaces and $T: X \rightarrow Y$ linear. Then TFAE:

- (1) T is cts at zero;
- (2) T is cts in X ;
- (3) T is bounded.

pf. (1) $\Leftrightarrow$ (2): Let $x, h \in X$.

$$\|T(x+h) - Tx\| \xrightarrow{\|h\| \rightarrow 0} 0 \Leftrightarrow \|Th\| \xrightarrow{\|h\| \rightarrow 0} 0 = \|T(0)\|. \quad ///$$

(1) $\Rightarrow$ (3): Continuity at zero implies $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|h\| \leq \delta$ implies $\|Th\| \leq \varepsilon$.

Choose $\varepsilon = 1$. Then $\exists \delta > 0$ s.t. $\|h\| \leq \delta$ implies $\|Th\| \leq 1$. So

$$\|h\| = \delta \Rightarrow \|Th\| \leq 1$$

If $x \neq 0$, then $\|x\| > 0$, write $x = \frac{x}{\|x\|} \cdot \delta \cdot \frac{1}{\delta} \|x\| = \frac{\|x\|}{\delta} \cdot \left(\delta \frac{x}{\|x\|} \right)$. Then

$$\|Tx\| = \left\| \frac{\|x\|}{\delta} T\left(\delta \cdot \frac{x}{\|x\|}\right) \right\| = \frac{\|x\|}{\delta} \left\| T\left(\delta \cdot \frac{x}{\|x\|}\right) \right\| \leq \frac{1}{\delta} \|x\|.$$

Therefore

$$\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} \leq \frac{1}{\delta}.$$

So T is bounded. ///

(3) $\Rightarrow$ (1): Let $\|T\| \leq M < \infty$. Then $\|Tx\| \leq M \|x\|$, $\forall x \in X$. So as $\|x\| \rightarrow 0$, clearly $\|Tx\| \rightarrow 0$. ///

□

Thm 2. Let X, Y be normed spaces and $T: X \rightarrow Y$ be linear and bounded.

Then

$$\|T\| := \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{\|x\| \leq 1 \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \quad (1)$$

$$= \sup_{\substack{\|x\| < 1 \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \quad (2)$$

$$= \sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|. \quad (3)$$

pf. To prove (3), for any $x \in X \setminus \{0\}$, write

$$x = \|x\| \cdot \frac{x}{\|x\|} \Rightarrow Tx = \|x\| \cdot T\left(\frac{x}{\|x\|}\right) \Rightarrow \frac{\|Tx\|}{\|x\|} = T\left(\frac{x}{\|x\|}\right).$$

Therefore

$$\sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} = \sup_{x \in X \setminus \{0\}} T\left(\frac{x}{\|x\|}\right) = \sup_{\|y\|=1} \|Ty\|.$$

(1) & (2) are left as exercise. □

Thm 3 (Uniform Boundedness Principle / Banach-Steinhaus Thm).

Let X be a Banach space (complete normed space) and let Y be a normed space. Let the family of operators $\mathcal{F} = \{T_\lambda\}_{\lambda \in \Lambda} = \{T\}$, $T: X \rightarrow Y$ linear operator be st.

$$\sup \{ \|Tx\| : T \in \mathcal{F} \} < \infty, \quad \forall x \in X. \quad (*)$$

Then the operators in $\mathcal{F}$ are uniformly bounded: $\exists M > 0$ st. $\|T\| \leq M$, $\forall T \in \mathcal{F}$.

pf: For any $n \in \mathbb{N}$, consider the set

$$A_n = \{x \in X : \|Tx\| \leq n, \forall T \in \mathcal{F}\}.$$

By (*), we have

$$X = \bigcup_{n=1}^{\infty} A_n.$$

Since X is Banach, by BCT, at least one of the sets A_n is NOT nowhere dense in X . So A_n is dense in a closed ball $B_R(x) = \{y \in X : \|y-x\| \leq R\}$.

Claim 1. A_n is closed $\forall n \in \mathbb{N}$.

Indeed, let $\{x_k\} \subseteq A_n$, $x_k \rightarrow x$. Then $x_k \in A_n \Rightarrow \|Tx_k\| \leq n \quad \forall T \in \mathcal{F}$.

Fix $T \in \mathcal{F}$, then $\|Tx_k\| \leq n$, $k=1,2,3,\dots$. Let $k \rightarrow \infty$, then $\|Tx_k\| \rightarrow \|Tx\|$.

Then $\|Tx\| \leq n$ (T bdd $\Rightarrow$ cts). Thus

$$\|Tx\| \leq n, T \in \mathcal{F} \Rightarrow x \in A_n. \quad ///$$

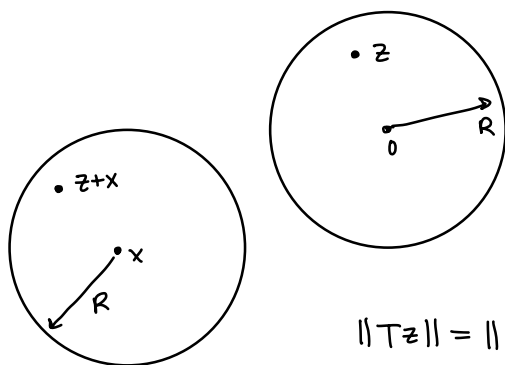
Thus:

1) A_n is closed.

2) A_n is dense in $B_R(x)$.

Then $\overline{A_n} = A_n \supseteq B_R(x)$. Thus

$$\forall y \in B_R(x), \|Ty\| \leq n, \quad \forall T \in \mathcal{F} \quad (\star)$$



Assume $z \in \overline{B_R(0)}$. Then $\|z\| \leq R$.

Thus $z+x \in \overline{B_R(y)}$. Then

$$\begin{aligned} \Rightarrow \|T(z+x)\| &\leq N, \quad \|Tx\| \leq N. \\ &\forall T \in \mathcal{F}. \end{aligned}$$

Therefore:

$$\begin{aligned} \|Tz\| &= \|T(x+z) - Tx\| \\ &\leq \|T(x+z)\| + \|Tx\| \leq 2N. \end{aligned}$$

$$\forall z \in \overline{B_R(0)}, T \in \mathcal{F}.$$

From the linearity of $T \in \mathcal{F}$, $\forall \|y\|=1$, write $y = \frac{1}{R} \cdot Ry$, $Ry \in \overline{B_R(0)}$.

$$\begin{aligned} \|Ty\| &= \left\| \frac{1}{R} T(Ry) \right\| = \frac{1}{R} \|T(Ry)\| \leq \frac{2N}{R}. \\ &= \frac{2N}{R} \|y\| \end{aligned}$$

$\forall \|y\|=1$. Thus

$$\|T\| \leq \frac{2N}{R}, \quad \forall T \in \mathcal{F}.$$

□

Def 1. Let X, Y be normed spaces. $T: X \rightarrow Y$. Then T is said to be **open** if it maps open subsets of X to open subsets of Y .

Observation. Assume T is linear in addition, Then $T(X)=Y$, i.e. T is onto.

pf. Take $B_1(0) = \{x \in X: \|x\| < 1\}$ in X . By assumption that T is open, we have $T(B_1) = V \subseteq Y$ is open. Since $0 \in V$, $\exists B_R(0) \subseteq V$ where $R > 0$. Thus $\overset{Y}{B_R(0)} \subseteq \overset{X}{T(B_1(0))}$.

By linearity,

$$T(B_n(0)) = n \cdot T(B_1(0)) \supseteq n \cdot B_R(0) = B_{n \cdot R}(0).$$

So the image of T covers everything:

$$T(X) = T\left(\bigcup_{n=1}^{\infty} B_n(0)\right) = \bigcup_{n=1}^{\infty} T(B_n(0)) \supseteq \bigcup_{n=1}^{\infty} B_{n \cdot R}(0) = \overset{Y}{Y}.$$

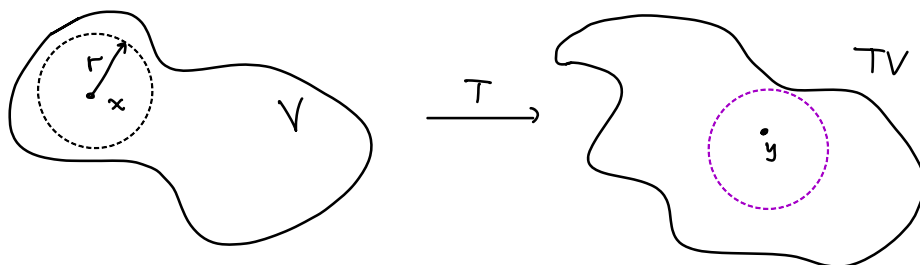
□

Thm 4 (Open Mapping Thm). Let X, Y be Banach spaces, $T: X \rightarrow Y$ be linear, bounded and onto: $TX = Y$. Then T is open.

pf: Denote: $U :=$ unit ball in X .

Claim: It is sufficient to prove TU contains a neighborhood of the origin in Y .

If $B_R(0) \subseteq TU$, $R > 0$, then let $V \subseteq X$ be open. Take $y \in TV$. Then $y = Tx$, $x \in V$. Since V is open, $\exists B_r(x) \subseteq V$, $r > 0$.



Then:

$$T(B_r(x)) = T(x + B_r(0)) = Tx + r(TB_r(0)) = Tx + rTU.$$

By assumption $Tx + TU$ contains a nbh of $Tx = y$. So $T(B_r(0))$ contains a nbh of y , $\forall y \in TV \Rightarrow T$ is open.

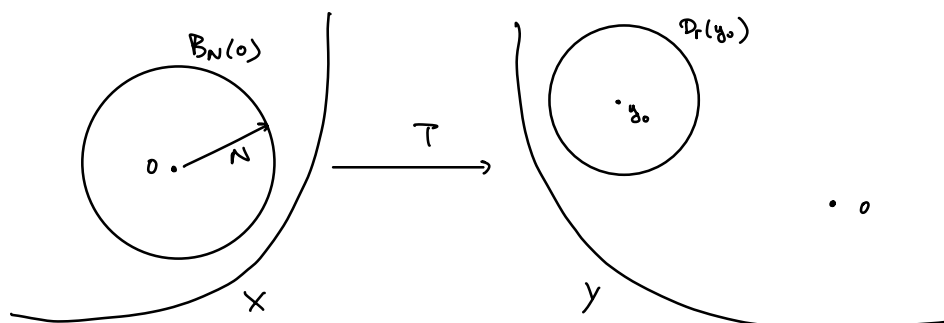
$B =$ balls in X $D =$ balls in Y

NTs: $T(B_1(0))$ contains a nbh of 0 in Y .

$$X = \bigcup_{n=1}^{\infty} nB_1(0).$$

$$\Rightarrow T(X) = Y = \bigcup_{n=1}^{\infty} T(nB_1(0)).$$

By BCT (Y is complete), one of $T(nB_1(0))$ is not nowhere dense in Y . So $\exists N \in \mathbb{N}$ s.t. $T(NB_1(0)) = T(B_N(0))$ is dense in a ball $D_r(y_0) \subseteq Y$.



In any vector space, $B_R(y_0) - B_R(y_0) = B_{2R}(0)$. So

$$\begin{aligned} T(B_N(0) - B_N(0)) &= T(B_{2N}(0)) \\ &= T(B_N(0)) - T(B_N(0)). \end{aligned} \quad (*)$$

But $(*)$ is dense in $D_R(y_0) - D_R(y_0) = D_{2R}(0)$. So it is also dense in $D(0)$ by scaling. Again by scaling $T(B_1(0))$ is dense in $D_{r/N}(0) = D_c(0)$.

Claim 2: $D_c(0) \subseteq T(B_2(0))$.

Pick $y \in D_c(0)$, then $\exists x_1 \in B_1(0)$ s.t. $\|y - Tx_1\| < c/2$, $\|x_1\| \leq 1$.

Choose $x_2 \in B_{1/2}(0)$ s.t. $\|y - Tx_1 - Tx_2\| < c/2^2$, $\|x_2\| \leq \frac{1}{2}$. — scaling by linearity & continuity of T

$\vdots$

We have chosen $x_1, \dots, x_n$ s.t. $\|x_k\| \leq \frac{1}{2^{k-1}}$, $k=1, 2, \dots, n$, and

$$\|y - Tx_1 - Tx_2 - \dots - Tx_n\| < \frac{c}{2^n}.$$

Choose $B_{1/2^n}(0)$ s.t.

$$\left\| y - \sum_{k=1}^n Tx_k - Tx_{n+1} \right\| < \frac{c}{2^{n+1}}.$$

This is due to the fact that $T(\frac{1}{2^n}B(0))$ is dense in $\frac{1}{2^n}D_c(0)$.

$S_n = \sum_{k=1}^n x_k$ is a Cauchy seq (check) in X .

X Banach $\Rightarrow S_n \rightarrow s$.

$$\|s\| \leq \sum_{k=1}^{\infty} \|x_k\| = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

So $s \in B_2(0)$. On the other hand,

$$\|y - Ts_n\| < \frac{c}{2^{n-1}} \Rightarrow \|y - Ts\| = 0 \Rightarrow y = Ts \in T(B_2(0)). \quad ///$$

Lec 19. Hahn-Banach Theorem.

Def. Let X be a vector space over K ($\mathbb{R}$ or $\mathbb{C}$).

1) The function $g: X \rightarrow \mathbb{R}$ is called a **quasi-seminorm** if

$$(i) \quad g(x+y) \leq g(x) + g(y) \quad \forall x, y \in X.$$

$$(ii) \quad g(\lambda x) = \lambda g(x) \quad \forall \lambda > 0, x \in X$$

2) $g: X \rightarrow \mathbb{R}$ is called a **seminorm** if in addition to (i) & (ii), we have

$$(iii) \quad g(\lambda x) = |\lambda| g(x), \quad \forall \lambda \in K, x \in X.$$

Rmk. If g is a seminorm, then g is nonnegative. Since by (iii) $g(0) = 0$.

Also (iii) implies $g(x) = g(-x)$.

$$2g(x) = g(x) + g(-x) \stackrel{(i)}{\geq} g(0) = 0.$$

Thm 1. (Real Version*) Let X be a real v.s. and let $g: X \rightarrow \mathbb{R}$ be a quasi-seminorm. Suppose $Y \subseteq X$ is a subspace, $\varphi: Y \rightarrow \mathbb{R}$ is a linear functional s.t.

$\varphi(y) \leq g(y) \quad \forall y \in Y$. Then φ admits a linear extension on X still

dominated by g :

$$\exists \tau: X \rightarrow \mathbb{R} \text{ s.t. } \tau|_Y = \varphi \text{ and } \tau(x) \leq g(x) \quad \forall x \in X.$$

Pf. Case 1: $\text{Codim}(Y) = 1$, i.e. $X = Y + \lambda x_0$, $x_0 \notin Y$, $\lambda \in \mathbb{R}$. Let

$$\tau(y + \lambda x_0) = \tau(y) + \lambda \tau(x_0) = \varphi(y) + \lambda \tau(x_0).$$

So NT define $\tau(x_0)$ s.t. $\tau \leq g$ on X , i.e.

$$\varphi(y) + \lambda \tau(x_0) \leq g(y + \lambda x_0) \quad \forall y \in Y, \lambda \in \mathbb{R}.$$

if $\lambda > 0$:	$\varphi\left(\frac{y}{\lambda}\right) + \tau(x_0) \leq g\left(\frac{y}{\lambda} + x_0\right)$
if $\lambda < 0$:	$\varphi\left(\frac{y}{\lambda}\right) + \tau(x_0) \geq -g\left(-\frac{y}{\lambda} - x_0\right)$

Denote $y/\lambda = z \in Y$.