NOTES ON DIFFERENTIAL EQUATIONS

WRITTEN BY

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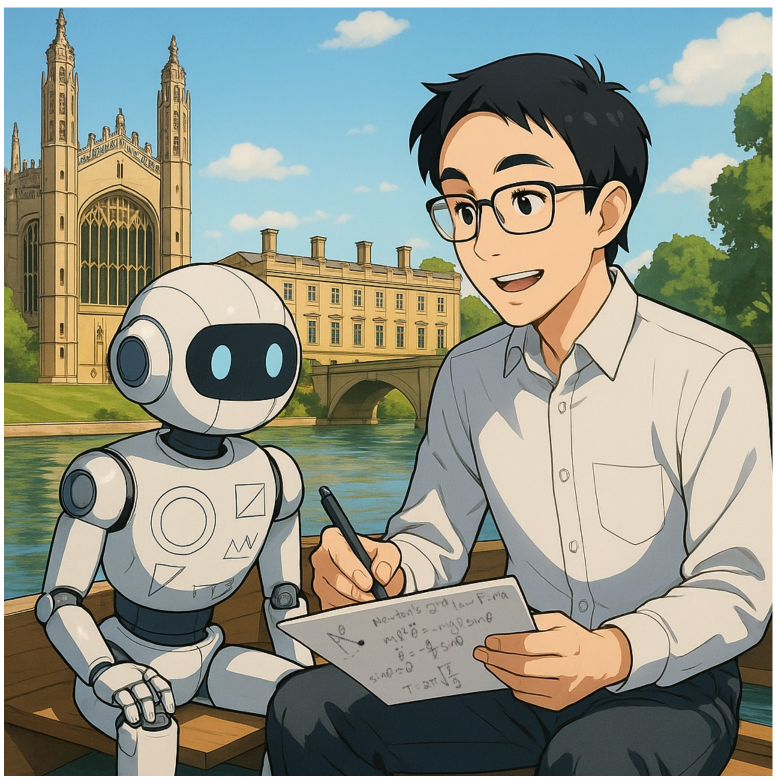

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Contents

Preface

I assume you already know how to solve simple ordinary differential equations using seperation of variables.

This note is not going to be a detailed repeat of the book. This is merely an abstract of some important ideas in the book. To ensure you really understand the material, you need to do lots of problems in the book. I recommend you doing those examples with explanations in the book, which should give you enough instruction and practice.

This note is for my classmate, colleague, and friend **Chen Yanxiang**. Thank you for requesting my notes, which made my work meaningful. Thank you for joining the differential equation class. Your mere existence makes the class much more interesting. Thank you for all the encouragements and conversations, which helped me to become a better and more "normal" person.

I know you got discouraged a lot by the level of difficulty of this class, and you probably already begin to hesitate whether to study math at college. But please keep in mind that it is because of people like you that our own mathematical society can become an infinitely better place. Regardless of your future choice, I wish you the very best future, and I thank you for being such an amazing friend.

Tianyi
October 11, 2019

Chapter 1

First Order Differential Equations

1.1 Linear Equations

Consider the first order equation with the following form

$$y' + p(t)y = g(t) \tag{1.1}$$

This equation is not seperable, making it difficult to solve. Note that the first term has a y', and the second term has a y, which naturally reminds us of chain rule. One trick we can use is try to simplify this equation into the following form

$$\frac{d}{dt}[I(t)y] = I(t)g(t) \tag{1.2}$$

Then it immediately follows that

$$y(t) = \frac{1}{I(t)}\int I(t)g(t)dt = \frac{1}{I(t)}(\int_{t_0}^{t} I(t)g(t)dt + c) \tag{1.3}$$

To find such a function $I(t)$, we first do the multiplication

$$I(t)y' + I(t)p(t)y = I(t)g(t) \tag{1.4}$$

Notice that in order to make 1.2 true, we must have

$$\frac{d}{dt}I(t) = I(t)p(t) \tag{1.5}$$

Notice that 1.5 is a seperable equation. Therefore

$$\ln|I(t)| + c = \int p(t)dt \tag{1.6}$$

$$I(t) = \pm e^{-c} \exp\left(\int p(t)dt\right) = A\exp\left(\int p(t)dt\right) \tag{1.7}$$

But we are looking for a particular function $I(t)$ that makes 1.2 true, not the general solution. Therefore, for convinience, we let $A = 1$. Therefore, we have find such a function $I(t)$ and we find solution to 1.1.

Theorem 1.1.1. *To solve equations with the form $y' + p(t)y = g(t)$, multiply by an integrating factor $I(t)$ where*

$$I(t) = \exp\left(\int p(t)dt\right)$$

Then the solution to this equation is

$$y(t) = \frac{1}{I(t)}(\int_{t_0}^{t} I(t)g(t)dt + c)$$

Note that some equations have the form

$$a(t)y' + b(t)y = c(t)$$

But this is mathematically equivalent as $y' + p(t)y = g(t)$ where

$$p(t) = \frac{b(t)}{a(t)} \quad g(t) = \frac{c(t)}{a(t)}$$

The technique of multiplying an integrating factor on both sides allow us to solve first order linear differential equations. In the next section, we will see that it actually helps us solve even more complicated equations.

1.2 Exact Equations and Integrating Factors

There are some first order differential equations that are more difficult to solve than those in the last section. Suppose y is a function of x described by the differential equations of the following form

$$M(x, y) + N(x, y)y' = 0 \tag{1.8}$$

Now, this equation involves both the independent variable x and the function y itself. Therefore, it is harder to solve. again, using the idea from the last section, it would be really nice if we can write 1.8 into the following form

$$\frac{d}{dx}\psi(x, y) = 0 \tag{1.9}$$

It is worth noticing that y is also a function of x. Therefore, we can let $y = \phi(x)$, so 1.9 becomes

$$\frac{d}{dx}\psi[x, \phi(x)] = 0 \tag{1.10}$$

Then it is obvious that

$$\psi(x, y) = c$$

In order for 1.10 to be true, according to chain rule, we have

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0 \tag{1.11}$$

By looking at 1.8, we know that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \quad \frac{\partial \psi}{\partial y}(x, y) = N(x, y) \tag{1.12}$$

This means that if we can find such a function ψ that satisfy 1.12, then we can solve 1.8.

Theorem 1.2.1. *Let the functions M, N, M_y, and N_y be continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. Then $M(x, y) + N(x, y)y' = 0$ is an exact differential equation in R, and there exists a function ψ satisfying 1.8 and 1.12 if and only if*

$$M_y(x, y) = N_x(x, y) \tag{1.13}$$

Proof. We already proved that if there exists a function ψ such that 1.12 is true, then 1.8 is satisfied.

Computing M_y and N_x from 1.12, we obtain $M_y(x, y) = \psi_{xy}(x, y), \quad N_x(x, y) = \psi_{yx}(x, y)$ Since M_y, N_x are continuous, it follows that ψ_{xy} and ψ_{yx} are also continuous. Therefore, $\psi_{xy} = \psi_{yx}$, thus concluding the proof. □

Sometimes, 1.13 can not be satified. For example, consider the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

where $M_y \neq N_x$. In this case if we multiply on both sides of the equation an integrating factor μ such that $(\mu M)_y = (\mu N)_x$, we can solve the equation. Then we have

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0$$

which gives

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0 \tag{1.14}$$

In order to simplify the task of solving 1.14, we turn this partial differential equation into an ordinary differential equation by considering the special case. Specifically, we consider the case where μ is only a function of x or y. Suppose that $\mu(x)$ is only a function of x, then $\mu_y = 0$, and we have

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \tag{1.15}$$

In the example given above, we have

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

which suggests the integrating factor is $\mu(x) = x$. Then the equation becomes

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$$

which gives

$$\phi(x, y) = x^3y + \frac{1}{2}x^2y^2$$

Therefore the solution to this equation is

$$x^3y + \frac{1}{2}x^2y^2 = c$$

Chapter 2

Linear Algebra Needed in Differential Equation

The definitions, theorems, and corollaries listed below are some basic concepts in linear algebra. Here we omit the proof. If you have trouble understanding them, I suggest you read Gilbert Strang's *Introduction to Linear Algebra*. Alternatively, you can watch MIT OpencourseWare. The course is called Linear Algebra (18.06).

2.1 Initial Value Problem

In differential equation, we will sometimes encounter initial value problem. Suppose that we found a general solution to one differential equation. Let the solution be $c_1 y_1(t) + c_2 y_2(t)$. We want to find values for c_1, c_2. We know initial conditions y_0, y_0'. All we need to do is to solve c_1, c_2 from the following system of equation:

$$\begin{aligned} c_1 y_1 (t_0) + c_2 y_2 (t_0) &= y_0 \\ c_2 y_1' (t_0) + c_2 y_2' (t_0) &= y_0' \end{aligned} \tag{2.1}$$

However, when we have multiple linearly independent solutions $y_1, y_2, ...y_n$, we need to solve n number of constants through two linear equations, which is difficult for sufficiently large n. Therefore, we need to use linear algebra to solve this systems.

we can write the system in matrix form

$$\underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} y_0 \\ y_0' \end{bmatrix}}_{U} \tag{2.2}$$

or symbolically we can write

$$WC = U$$

where W is called the Wronskian. We want to find the vector C. To do so, we need to multiply on the both side of the equation the inverse of matrix W, which is denoted by W^{-1}. Then

$$(W^{-1}W)C = C = W^{-1}U$$

But not every matrix W has an inverse. We now introduce determinant and inverse criteria.

Definition 2.1.1. *For a 2*2 matrix* $W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *the determinant of* W, *denoted by* $\det W$, *is defined as* $\det W = ad - bc$.

Theorem 2.1.1. *For matrix* W *to be invertible, that is for* W^{-1} *to exist, the following must be true*

$$\det W \neq 0$$

Otherwise, the matrix W *is not invertible or singular.*

Corollary 2.1.1. *For* c_1, c_2 *in 2.1 to be unique, we must have* $\det W \neq 0$.

Now, we introduce two more definition and theorem in linear algebra.

Definition 2.1.2. *For convenience, we introduce notations*

$$W_1 = \begin{bmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{bmatrix} \quad W_2 = \begin{bmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{bmatrix}$$

As a helpful memorization tip, you can think of W_1 *as the matrix* W *with its first row being replaced by vector* U, *and* W_2 *as the matrix* W *with its second row being replaced by vector* U.

Theorem 2.1.2. ***Cramer's Law****: For the systems defined by 2.1, we can solve the value of* c_1, c_2 *by using the following rule:*

$$c_1 = \frac{\det W_1}{\det W} \quad c_2 = \frac{\det W_2}{\det W}$$

Therefore, for the systems above, we can calculate the value of the constants:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

$$c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2' \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

These tedious calculation and effort may seem like redundant when we can just solve 2.1 by directly solving it. However, consider the case where you have the solutions of a differential equation being $y_1, y_2, ...y_n$. Then we have to solve

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) + \ldots + c_{n-1} y_{n-1}(t_0) + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \ldots + c_{n-1} y_{n-1}'(t_0) + c_n y_n'(t_0) &= y_0' \end{aligned}$$

With the knowledge of matrix operation, we can easily handle large datasets like this. Therefore, it would be nicer if we can write it in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) & \ldots & y_{n-1}(t_0) & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \ldots & y_{n-1}'(t_0) & y_n'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

and then we can simply do operations similar to

$$WC = U$$

However, calculating determinants of large matrix can be painful. But at least it gives us a systematic way of solving this time of problems. For more information about determinants and calculating higher-order determinant, check the sources provided at the beginning of the chapter.

Chapter 3

Second Order Homogeneous Differential Equations

3.1 Homogeneous Equations

the most general form of a second order ODE is the following

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

Let us simplify the situation and try to solve the following equation

$$ay'' + by' + cy = 0 \tag{3.1}$$

where $a, b, c \in \mathbb{R}$. Equation 3.1 is called a homogeneous equation. Now, the solutions of such equations are clearly dependent on the value of a, b, c. Let us now derive the solutions of such equations first in a general form.

3.2 Distinct Roots

We define the descriminant using parameters in 3.1. Let the descriminant $\Delta = b^2 - 4ac$. We first consider the case where $\Delta > 0$. Before we start, we first claim some rules:

1. If y_1 is a solution to 3.1, then the multiple of this solution $c_1 y_1$ is also a solution to this equation, where $c_1 \in \mathbb{R}$.

Substitute $c_1 y_1$ back to the equation, and the constant c_1 cancels out. □

2.If y_1, y_2 are two distinct solutions of 3.1, then $c_1 y_1 + c_2 y_2$ is also a solution to the equation, where $c_1, c_2 \in \mathbb{R}$.

Substitute $c_1 y_1 + c_2 y_2$ into the equation, we have

$$c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) = c_1(0) + c_2(0) = 0$$

which satisfies the equation. Therefore, $c_1 y_1 + c_2 y_2$ is the most general solution for 3.1. Similarly, if $y_1, y_2, ...y_n$ are solutions to the equation, then

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n = \sum_{i=1}^{n} c_i y_i \tag{3.2}$$

is the general solution of the equation. We call 3.2 the *linear combination* of $y_1, y_2, ...y_n$. □

Remark 3.2.1. *From rule 2, we know that for a differential equation, if we have all distinct solutions $y_1, y_2, ...y_n$, we can find a general solution by taking the linear combination of these solutions. If the solutions are distinct, we call those solutions linearly independent.*

Now we begin to find solutions to 3.1 where $\Delta > 0$. Intuitively, we guess that $e^{\lambda x}$ is a solution to this equation. We want to find λ. Substitute our guess solution back in, we have

$$a\lambda^2 + b\lambda + c = 0 \tag{3.3}$$

This is called the characteristic equation of equation 3.1. In order for our guess to be the solution, λ must satisfy 3.3. Since we have two distinct solutions to 3.3, we have

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Therefore, two solutions to the differential equation are $e^{\lambda_1 x}, e^{\lambda_2 x}$. Therefore, the general solution is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

3.2.1 A Linear Algebra Point of View (Optional)

We have the linear systems of equations

$$\begin{cases} y'' = -\frac{b}{a}y' - \frac{c}{a}y \\ y' = y' \end{cases}$$

This system can be written in matrix form

$$\frac{d}{dx}\begin{bmatrix} y' \\ y \end{bmatrix} = \begin{bmatrix} -b/a & -c/a \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y' \\ y \end{bmatrix} \tag{3.4}$$

The 2*2 matrix on the right hand side is the multiplicative matrix $\mathbf{A}$. Taking the derivative of $(y', y)^T$ is equivalent as multiplying $\mathbf{A}$ with $(y', y)^T$.

The solution is

$$\left(y', y\right)^T = c_1 e^{\lambda_1 x}\mathbf{v}_1 + c_2 e^{\lambda_2 x}\mathbf{v}_2$$

where λ_1, λ_2 are two eigenvalues of the matrix $\mathbf{A}$. And $\mathbf{v}_1, \mathbf{v}_2$ are two eigenvectors. It is easy to calculate that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \mathbf{v} = (\lambda, 1)^T \tag{3.5}$$

Therefore, the solution (when the eigenvalues are distinct) to 3.1 is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{3.6}$$

3.3 Repeated Roots

When we have repeated eigenvalues, the solution to 3.1 will be slightly different than 3.6. Here we give a intuitive but non-rigorous proof by perturbation. We will change the parameters of 3.1 by a slight amount. Let's say we change the initial values by ϵ amount. We hope that we can still express the solution of 3.1 by 3.6. More explicitly, consider the following equation

$$(a + \epsilon)y'' + (b + \epsilon)y' + (c + \epsilon)y = 0 \tag{3.7}$$

We hope the solution of 3.7 can be expressed as

$$y(x) = c_1 e^{\lambda_1(\epsilon)x} + c_2 e^{\lambda_2(\epsilon)x} \tag{3.8}$$

We can rewrite 3.8 as

$$y(x) = \gamma_1(e^{\lambda_1(\epsilon)x} + e^{\lambda_2(\epsilon)x}) + \gamma_2(e^{\lambda_1(\epsilon)x} - e^{\lambda_2(\epsilon)x}) \tag{3.9}$$

which can be further simplified into

$$y(x) = \gamma_1\left(e^{\lambda_1(\epsilon)x} + e^{\lambda_2(\epsilon)x}\right) + \tilde{\gamma}_2\left(\frac{e^{\lambda_1(\epsilon)x} - e^{\lambda_2(\epsilon)x}}{\lambda_1(\epsilon) - \lambda_2(\epsilon)}\right) \tag{3.10}$$

where

$$\tilde{\gamma}_2 = \gamma_2\left(\lambda_1(\epsilon) - \lambda_2(\epsilon)\right) \tag{3.11}$$

Now all we need to do is considering the limit case where $\epsilon \to 0$. We can do this by letting $\lambda_1 = \lambda + \Delta\lambda$, and $\lambda_2 = \lambda$. Then we let $\Delta\lambda \to 0$. Now we consider what will happen to 3.10. The first part will just become some constant times $e^{\lambda x}$. What we are interested is the second part. It is not hard to see that

$$\lim_{\epsilon \to 0} \frac{e^{\lambda_1(\epsilon)x} - e^{\lambda_2(\epsilon)x}}{\lambda_1(\epsilon) - \lambda_2(\epsilon)} = \lim_{\Delta\lambda \to 0} \frac{e^{(\lambda+\Delta\lambda)x} - e^{\lambda x}}{\Delta\lambda} = \frac{d}{d\lambda}e^{\lambda x} = xe^{\lambda x}$$

Therefore, for the second part, we will get some constant times $xe^{\lambda x}$. So the solution to equation 3.1 when $b^2 - 4ac = 0$, or the equation has repeated eigenvalue λ, is

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x} \tag{3.12}$$

In thg proof above, we didn't use a rigorous approach. One confusion that we made is that we use $\tilde{\gamma}_2$ to replace γ_2, which may cause some problems. If you look at 3.11, you may think that as $\lambda_1 \to \lambda_2$, $\tilde{\gamma}_2$ will go to zero. Now we prove that $\tilde{\gamma}_2$ remains a constant even in the limit case.

Proof. First we need to express γ_2 in terms of c_1, c_2. By looking at 3.1 and 3.9, we know that

$$\gamma_2 = \frac{c_1 - c_2}{2} \quad \tilde{\gamma}_2 = \frac{c_1 - c_2}{2}(\lambda_1 - \lambda_2)$$

Let the initial condition at $x = 0$ be y_0. And we need the initial condition of the first derivative, which is y_0'. These initial conditions give

$$c_1 = -\frac{\lambda_2 y_0 - y_0'}{\lambda_1 - \lambda_2} \quad c_2 = \frac{\lambda_1 y_0 - y_0'}{\lambda_1 - \lambda_2}$$

Now we have

$$\tilde{\gamma}_2 = \frac{1}{2}\left[-\frac{\lambda_2 y_0 - y_0'}{\lambda_1 - \lambda_2} - \frac{\lambda_1 y_0 - y_0'}{\lambda_1 - \lambda_2}\right](\lambda_1 - \lambda_2) = \frac{1}{2}\left[2y_0' - (\lambda_1 + \lambda_2)\, y_0\right]$$

We now consider the limit case where $\lambda_1 \to \lambda_2$, or $\lambda_1 = \lambda_2 = \lambda$. This will give us

$$\tilde{\gamma}_2 = y_0' - \lambda y_0$$

which is a constant. □

Remark 3.3.1. *The reason why $\tilde{\gamma}_2$ is a constant is that although we multiply γ_2 by a quantity that is getting smaller and smaller in the limit case, namely $\lambda_1 - \lambda_2$, but γ_2 is getting bigger and bigger in the limit case because it can be expressed a fraction who has $\lambda_1 - \lambda_2$ as the denominator. This denominator is hidden in γ_2, but it cancels out the decreasing effect exactly. Therefore, $\tilde{\gamma}_2$ is a constant even in the limit case.*

3.4 Imaginary Roots

Now we will discuss the case where $\Delta < 0$. In this case, let us denote two solutions by $\lambda_1 = \mu + i\eta, \lambda_2 = \mu - i\eta$. Then the solutions of the equation are

$$\begin{aligned} y_1 &= e^{(\mu+i\eta)x} = e^{\mu x}(\cos \mu x + i \sin \mu x) \\ y_2 &= e^{(\mu-i\eta)x} = e^{\mu x}(\cos \mu x - i \sin \mu x) \end{aligned} \tag{3.13}$$

This equation is obtained by the famous Euler identity $e^{ix} = \cos x + i \sin x$. Now, we will find the general solution to 3.1 when $\Delta < 0$. We first prove the following theorem.

Theorem 3.4.1. *If $y(x) = f(x) + ig(x)$ is a solution to equation 3.1, then both $f(x)$ and $g(x)$ are real-valued solutions for 3.1.*

Proof. We first begin by substituting y into the equation, then we have

$$a(f''(x) + ig''(x)) + b(f'(x) + ig'(x)) + c(f(x) + ig(x)) = 0$$

By slightly rearranging the equation we have that

$$(af''(x) + bf'(x) + cf(x)) + i(ag''(x) + bg'(x) + cg(x)) = 0$$

For a complex number to be zero, its real and imaginary parts have to equal to zero seperately. Therefore, it follows that

$$\begin{aligned} af''(x) + bf'(x) + cf(x) = 0 \\ ag''(x) + bg'(x) + cg(x) \end{aligned}$$

which suggests that both $f(x)$ and $g(x)$ are solutions to the equation. □

Corollary 3.4.1. *If $\Delta < 0$, and $\lambda_1 = \mu + i\eta, \lambda_2 = \mu - i\eta$, the general real-valued solution to 3.1 is*

$$y(x) = e^{\mu x}(c_1 \cos \mu x + c_2 \sin \mu x) \tag{3.14}$$

Remark 3.4.1. *In physics, when we are solving for differential equations of damped oscillations, usually for simple case we will take $c_2 = 0$, taking the solution to be $y(t) = ce^{\mu t} \cos \mu t$.*

3.5 Forced Vibrations

We now consider the case of a oscillating spring with damping and an external force acting on the system. Let the damping coefficient be γ, let the spring mass be m, and the spring constant be k. The external force acting on the system is denoted by $F(t)$. Then the differential equation, by Newton's second law, is

$$my''(t) + \gamma y'(t) + ky(t) = F(t) \tag{3.15}$$

Usually the external force is periodic. Therefore, let the force be $F(t) = F_0 \cos \omega t$. Then the equation becomes

$$my''(t) + \gamma y'(t) + ky(t) = F_0 \cos \omega t \tag{3.16}$$

The general solution to this equation would be the general solution $u_0(t)$ for homogeneous equation $my''(t) + \gamma y'(t) + ky(t) = 0$ plus a particular solution of 3.16, which is denoted by $U(t)$. The general solution u_0 will be similar to equation 3.14. We know that as $t \to \infty$, this solution will die out. In other words, given sufficient time, $U(t)$ dominates. This means, physically, that the frequency of the system ω will eventually be the same as the frequency of $U(t)$. To solve 3.16, we take the entire equation to complex plane and look for the complex solution. After we found such a solution, we take the real part, and it will become the steady state solution U. In complex plane, the equation becomes

$$mz''(t) + \gamma z'(t) + kz(t) = F_0 e^{i\omega t} \tag{3.17}$$

Let the guess solution be $z(t) = Re^{i(\omega t-\delta)}$. It is important to understand why the guess function has frequency ω. It is because that after a sufficient long period of time, the frequency of the system is going to be the same as the input frequency. Substitute the guess back to the equation, we have

$$(-\omega^2 + \gamma i\omega + \omega_0^2)Re^{i(\omega t-\delta)} = \frac{F_0}{m}e^{i\omega t}$$

which simplifies to

$$(-\omega^2 + \gamma i\omega + \omega_0^2)R = \frac{F_0}{m}e^{i\delta} = \frac{F_0}{m}(\cos\delta + i\sin\delta)$$

which gives

$$(-\omega^2 + \omega_0^2)R = \frac{F_0}{m}\cos\delta$$
$$\gamma\omega R = \frac{F_0}{m}\sin\delta$$

Notice that $\cos^2\delta + \sin^2\delta = 1$. So we can immediately get

$$R = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \tag{3.18}$$

Also it is easy to find that

$$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \tag{3.19}$$

And we find the solution to 3.16 is

$$y(t) = A\cos(\omega t - \delta)$$

From equation 3.18, we can get the relation below since $\omega_0 = k/m$. Define $\Gamma = \gamma^2/mk$, and we have

$$\frac{Rk}{F_0} = 1/\left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}\right]^{1/2}$$

We can also find the maximum point of Rk/F_0 by differentiating the expression with respect to ω and set it to zero. We may find that

$$\omega_{\text{max}}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right)$$

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1+\frac{\gamma^2}{8mk}\right)$$

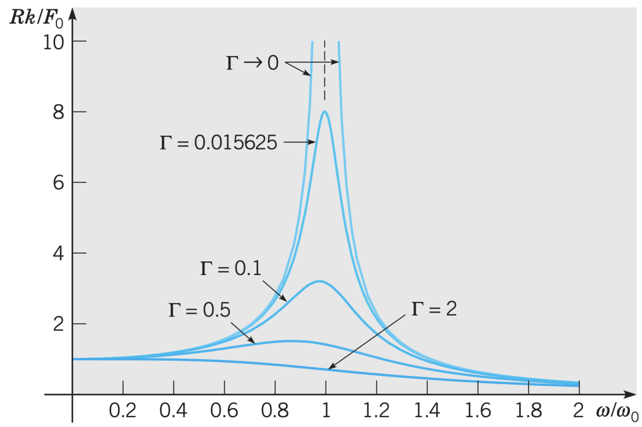

Figure 3.1: Forced vibration with damping: amplitude of steady state response versus frequency of driving force

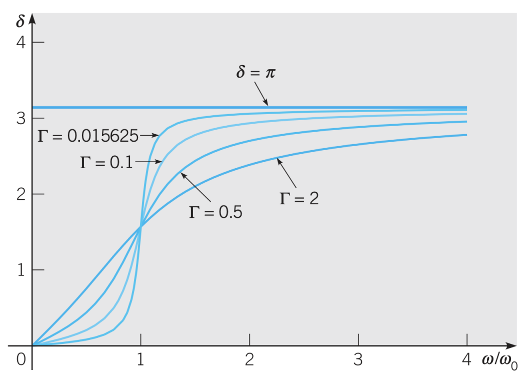

Figure 3.2: Forced vibration with damping: phase of steady state response versus frequency of driving force

Chapter 4

Series Solution of Second Order Linear Equations

4.1 Review of Power Series

Definition 4.1.1. *A power series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *is said to converge at point* x *if*

$$\lim_{m\to\infty}\sum_{n=0}^{m} a_n(x-x_0)^n$$

exists for that x.

Theorem 4.1.1. *For a fixed value of* x, *if*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L$$

exists, then the power series converges absolutely at x *if* $|x-x_0|L < 1$, *and diverges if* $|x-x_0|L > 1$. *If* $|x-x_0|L = 1$, *the test is inconclusive.*

Definition 4.1.2. *A function* f *that has a Taylor series expansion about* $x = x_0$

$$f(x) = \sum_{0}^{\infty}\frac{f^{(n)}(x)}{n!}(x-x_0)^n$$

with a radius of convergence $\rho > 0$ *is said to be analytic at* $x = x_0$.

4.2 Series Solutions Near an Ordinary Point

Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{4.1}$$

We convert this equation to the following equation if $P(x) \neq 0$.

$$y'' + p(x)y' + q(x)y = 0 \tag{4.2}$$

where $p(x) = P(x)/Q(x)$ and $q(x) = Q(x)/R(x)$.

We now introduce our notation for the radius of convergence. For a power series y, we denote its radius of convergence as R_y.

Theorem 4.2.1. *If x_0 is an ordinary point of the differential equation 4.1, that is, if $p(x), q(x)$ are analytic at x_0, then the general solution of 4.1 is*

$$y = \sum_0^\infty a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x) \tag{4.3}$$

The solution y_1, y_2, which are two power series, form a fundamental set of solutions. Further, the radius of divergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q, that is

$$R_{y_1} \geq \min{(R_p, R_q)}$$
$$R_{y_2} \geq \min{(R_p, R_q)}$$

Theorem 4.2.2. *(from complex analysis)There exists a convergent power series expansion about a point $x = x_0$ for the ratio of two polynomial Q/P if $P(x_0) \neq 0$ Further, if there is no common factor to P and Q, then $R_{Q/P}$ is precisely the distance between x_0 and the nearest zero of P.*

4.3 Euler Equations and Regular Singular Points

A relatively simple equation that has singular points is the Euler equation, which has the following forms

$$L[y] = x^2y'' + \alpha xy' + \beta y = 0 \tag{4.4}$$

where α, β are constants. We guess that the solution has the following form

$$y(x) = x^r$$

substitute our guess into equation 4.4 and we get

$$L[x^r] = x^r F(r) = x^r[r(r-1) + \alpha r + \beta] = 0 \tag{4.5}$$

Therefore,

$$r_1, r_2 = \frac{-(\alpha - 1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

which holds for distinct roots. Just like what we did in the second order homogeneous equation section, we need to consider different cases where r_1, r_2 are distinct, repeated, and complex.

For real and distinct roots r_1, r_2, it is obvious that the solution to equation 4.4 has the following form

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2} \tag{4.6}$$

For real and repeated roots, we use the same technique that we used in 2nd order linear equations: We change the solution by ϵ amount, and we let that change goes to zero. We make slight perturbations and we get

$$y(x) = c_1 x^{r_1(\epsilon)} + c_2 x^{r_2(\epsilon)} \tag{4.7}$$

Rearranging the forms, we have

$$y(x) = \gamma_1 (x^{r_1(\epsilon)} + x^{r_2(\epsilon)}) + \gamma_2 (x^{r_1(\epsilon)} - x^{r_2(\epsilon)}) \tag{4.8}$$

Since Eq.(4.7) and Eq.(4.8) have the same span, it is okay for us to make this change. For fixed ϵ, we can go further and make

$$y(x) = \gamma_1 (x^{r_1(\epsilon)} + x^{r_2(\epsilon)}) + \tilde{\gamma_2} \frac{x^{r_1(\epsilon)} - x^{r_2(\epsilon)}}{r_1(\epsilon) - r_2(\epsilon)} \tag{4.9}$$

Now let $r_1 = r + \Delta r$ and $r_2 = r$. We let $\Delta r \to 0$. Then we have the form

$$y(x) = \gamma_1 (x^{r+\Delta r} + x^r) + \tilde{\gamma_2} \frac{x^{r+\Delta r} - x^r}{\Delta r} \tag{4.10}$$

In the limit case, the second part will be

$$\frac{d}{dr}x^r = x^r \ln x$$

Therefore we have the form

$$y(x) = (c_1 + c_2 \ln x)x^r \tag{4.11}$$

Finally, for complex conjugate r_1, r_2. Let $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$. Keep in mind that $x^r = e^{r \ln x}$. We have

$$x^{\lambda \pm i\mu} = x^\lambda[\cos \mu \ln x \pm i \sin \mu \ln x]$$

Therefore, the general solution is

$$y(x) = c_1 x^\lambda \cos \mu \ln x + c_2 x^\lambda \sin \mu \ln x$$

Chapter 5

Laplace Transform

The inspiration of Laplace transform came from power series. Say we have a power series $A(x)$ defined as

$$A(x) = \sum_{n=0}^{\infty} a(n)x^n$$

we can see $a(n)$ as the nth coefficient of the power series, but we can also view $a(n)$ as a function which maps every nonnegative integers to a real value. For example, let $a(n) = 1$, which means it maps every nonnegative integers to 1. Then this power series becomes $A(x) = 1 + x + x^2 + ...$, which converges to $\frac{1}{1-x}$ for all $|x| < 1$. Here we made a weird connection between function $a(n)$ and function $A(x)$. We can write it as

$$a(n) \to A(x)$$

or if you like to be specific,

$$1 \to \frac{1}{1-x}$$

Another example, if $a(n) = \frac{1}{n!}$, then $A(x) = e^x$ for all values of x. Again,

$$\frac{1}{n!} \to e^x$$

Two things worth noticing. First, we changed the domain of function $a(n)$ to $A(x)$ as we made the transformation. But that doesn't matter as long as we can build a one-to-one connection. Second, we start as a function of n and end with a function of x. Now, we notice that the function $a(n)$ is discrete, since it only takes integers as inputs. What, then, is the continuous analogue of this power series? Namely, if we replace $a(n)$ by a continuous variable t into a function $a(t)$. Then the analogue would be

$$\int_0^\infty a(t)x^t dt$$

There are several modifications we want to make. First, let's change $a(t)$ to $f(t)$. Also, we want to make it easier to integrate this function, so notice that $x^t = e^{\ln xt}$.Then we have

$$\int_0^\infty f(t)e^{\ln xt} dt$$

For this transformation to exist, the integral must converge. Previously we saw a power series converge when $|x| < 1$. So we make the same restriction here by letting $0 < x < 1$. Then we let $-s = \ln x$ so that s can be all real positive numbers. Therefore, we define the laplace transform of a function $f(t)$ as

$$\mathcal{L}(f) := \int_0^\infty f(t)e^{-st} dt \tag{5.1}$$

And notice that $\mathcal{L}(f)$ transforms $f(t)$ into a function of s. Therefore it is also conventional to write $\mathcal{L}(f(t)) = F(s)$.

5.1 Definition of Laplace Transform

Definition 5.1.1. *An integral transform is a relation of the form*

$$F(s) = \int_\alpha^\beta K(s,t)f(t)dt$$

where $K(s,t)$ is a given function, called the $kernel$ of the transformation. The relation transforms $f(t)$ into another function $F(s)$, which is called the $transform$ of $f(t)$.

Definition 5.1.2. *Let $f(t)$ be given for $t \geq 0$, and the Laplace transform of f, if exists, is give by*

$$\mathcal{L}[f] := \int_0^\infty f(t)e^{-st} dt$$

It is also common to denote $\mathcal{L}[f]$ as $F(s)$.

Definition 5.1.3. *A function f is said to be of exponential order if there exist constants M and c such that*

$$|f(t)| \leq Me^{ct}$$

for sufficiently large t.

Remark 5.1.1. *Any polynomial is of exponential order. The proof is the following*

$$e^{at} = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} \geq \frac{t^n a^n}{n!} \Longrightarrow t^n \leq \frac{n!}{a^n} e^{at}$$

This suggests that $M = \frac{n!}{a^n}$ and $c = a$.

Theorem 5.1.1. *Let f be a piece-wise continuous function in $[0, \infty)$ and is of exponential order. The Laplace transform $F(s)$ of f exists for $s > c$, where c is a real number that depends on f.*

Proof. Since f is of exponential order, there exists A, M, c such that

$$|f(t)| \leq Me^{ct} \quad \text{for } t \geq A$$

Now we write

$$I = \int_0^\infty f(t)e^{-st}\,dt = I_1 + I_2$$

where

$$I_1 = \int_0^A f(t)e^{-st}\,dt \quad \text{and} \quad I_2 = \int_A^\infty f(t)e^{-st}$$

Since f is piece-wise continuous, I_1 exists. For I_2, we note that for $t \geq A$,

$$\left|e^{-st} f(t)\right| \leq Me^{-(s-c)t}$$

Thus

$$\int_A^\infty \left|f(t)e^{-st}\right| dt \leq M \int_A^\infty e^{-(s-c)t}\,dt \leq M \int_0^\infty e^{-(s-c)t}\,dt = \frac{M}{s-c}, \quad s > c$$

Since I_2 converges absolutely for $s > c$, I_2 converges for $s > c$. Hence $I = I_1 + I_2$ converges for $s > c$. This completes the proof. □

Theorem 5.1.2. *The Laplace operator is linear*

$$\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$$

Theorem 5.1.3. *We now establish the Laplace transform of function $f(t) = e^{at}$. This is the most important formula in Laplace transform, so it deserves to be called as a theorem.*

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad s > c \tag{5.2}$$

5.2 Solution of Initial Value Problems

Now, without loss of generality, we assume function f is continous and of exponential order in the following discussion.

Theorem 5.2.1.

$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0) = sF - f(0)$$

Proof. By definition

$$\mathcal{L}[f'] = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{\frac{df}{dt}dt}_{dv} = \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\int_0^\infty e^{-st} f(t)dt = sF - f(0)$$

□

Corollary 5.2.1. *By Thm.5.2.1, we can see that the Laplace transform of derivatives is recurrent. Replacing f' by f'', we can just as easily get*

$$\mathcal{L}[f''] = s^2F - sf(0) - f'(0)$$

Further generalizing, we have

$$\mathcal{L}[f^{(n)}(t)] = s^nF - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

or in more compact notation

$$\mathcal{L}[f^{(n)}(t)] = s^nF - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$$

where $f^{(0)}(0) = f(0)$.

Now we can see why Laplace transform is useful for solving differential equations. It turns every differential equation into an algebraic equation (as long as the transform exists) if we take Laplace transform on both sides of the equation.

5.3 Step Functions

Definition 5.3.1. *The unit step function denoted by $u_c(t)$ is defined as*

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

Theorem 5.3.1. *If $F(s) = \mathcal{L}[f]$ exists for $s > a \geq 0$ and if c is a positive constant, then*

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s > a$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

Proof.

$$\begin{aligned} \mathcal{L}\{u_c(t) f(t-c)\} &= \int_0^\infty e^{-st} u_c(t) f(t-c) dt \\ &= \int_c^\infty e^{-st} f(t-c) dt \end{aligned}$$

Introducing a new integration variable $\xi = t - c$, we have

$$\begin{aligned} \mathcal{L}\{u_c(t) f(t-c)\} &= \int_0^\infty e^{-(\xi+c)s} f(\xi) d\xi = e^{-cs} \int_0^\infty e^{-s\xi} f(\xi) d\xi \\ &= e^{-cs}F(s) \end{aligned}$$

□

Theorem 5.3.2. *If $F(s) = \mathcal{L}[f]$ exists for $s > a \geq 0$ and if c is a positive constant, then*

$$\mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c), \quad s > a + c$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Proof.

$$\begin{aligned} \mathcal{L}\left\{e^{ct} f(t)\right\} &= \int_0^\infty e^{-st} e^{ct} f(t) dt = \int_0^\infty e^{-(s-c)t} f(t) dt \\ &= F(s-c) \end{aligned}$$

□

5.4 Convolution Integral

Suppose we are given the Laplace transform $F(s), G(s)$ of two functions $f(t), g(t)$. Now it is natural for us to wonder whether there exist a relation between the product of Laplace Transform $F(s)G(s)$ and function $f(t), g(t)$. The convolution integral derived from this question.

We mentioned before that Laplace transform is the continuous analogue of power series. Therefore, it seems reasonable to first look at the products of two power series. Suppose we have

$$A(x) = \sum_{n=0}^{\infty} a(n)x^n \qquad B(x) = \sum_{n=0}^{\infty} b(n)x^n$$

We are interested in its products, whose coefficient is denoted by $c(n)$.

$$C(x) = A(x)B(x) = \sum_{n=0}^{\infty} c(n)x^n$$

Now, we need to find an expression for $c(n)$. We can do this by brute computation. Writing out first few terms we have

$$\begin{aligned} A(x)B(x) =& (a_0 + a_1 x + a_2 x^3 + ...)(b_0 + b_1 x + b_2 x^2 + ...) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + ... \end{aligned} \tag{5.3}$$

Then it is obvious that

$$c(n) = \sum_{k=0}^{n} a(k)b(n-k)$$

Therefore, we suspect that the continous analogue of this power series to be

$$F(s)G(s) = \int_0^{\infty} e^{-st}(f * g)dt$$

where

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau \tag{5.4}$$

This operation of $f * g$ is called convolution. Now we will prove that Eq.(5.4) is indeed true.

Proof. We start from the definition

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi) d\xi \qquad G(s) = \int_0^\infty e^{-s\tau} g(\tau) d\tau$$

Multiplying them together we have

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi) d\xi \int_0^\infty e^{-s\tau} g(\tau) d\tau$$

Since the first integral is independent of the variable τ, we can write it in iterative form

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\tau} g(\tau) \left[\int_0^\infty e^{-s\xi} f(\xi) d\xi \right] d\tau \\ &= \int_0^\infty g(\tau) \left[\int_0^\infty e^{-s(\xi+\tau)} f(\xi) d\xi \right] d\tau \end{aligned}$$

Now we introduce a new variable. Let $\xi = t - \tau$ for fixed τ. Then $d\xi = d\tau$.

$$F(s)G(s) = \int_0^\infty g(\tau) \left[\int_\tau^\infty e^{-st} f(t-\tau) dt \right] d\tau$$

Reversing the order of the integral we have

$$F(s)G(s) = \int_0^\infty e^{-st} \left[\int_0^t f(t-\tau) g(\tau) d\tau \right] dt$$

□

Remark 5.4.1. *The final change of upper and lower limits in the integral inside the square bracket is done by changing the variable of a double integral, which we will not prove here.*

There are several laws regarding the convolution worth noticing.

Theorem 5.4.1.

$$\begin{aligned} f * g &= g * f \\ f * (g_1 + g_2) &= f * g_1 + f * g_2 \\ (f * g) * h &= f * (g * h) \\ f * 0 &= 0 * f = 0 \end{aligned}$$

The first law suggest that convolution is commutative, which is not intuitive. However, by thinking of convolution as the coefficients of power series, then it doesn't matter which function comes first.

5.5 Impulse Functions

Definition 5.5.1. *The unit impulse function, also known as the Dirac delta function, is defined as*

$$\delta(t) = 0 \quad t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Theorem 5.5.1. *Consider the shifted Dirac delta function $\delta(t - t_0)$. Suppose $t_0 > 0$. Then $\delta(t) = 0$ for $t \neq t_0$. Then the Laplace transform of shifted Dirac delta function $\mathcal{L}[\delta(t - t_0)] = e^{-st_0}$.*

Proof. We are looking at

$$\mathcal{L}[\delta(t - t_0)] = \int_0^{\infty} e^{-st}\delta(t - t_0)dt$$

When $t \neq t_0$, anything multiplied with the Dirac delta function is zero. Only at $t = t_0$, the Dirac delta function is scaled by a factor of e^{-st_0}. Therefore, $\mathcal{L}[\delta(t - t_0)] = e^{-st_0}$. □

Remark 5.5.1. *The proof above takes advantage of the linearity of the integral. We already know, $\int_0^\infty \delta(t - t_0)dt = 1$. Therefore, $\int_0^\infty c\delta(t - t_0)dt = c\int_0^\infty \delta(t - t_0)dt = c$.The proof above only changes c to e^{-st_0}.*

Chapter 6

Higher Order Linear Equations

In this chapter we discuss equations with the following form

$$P_0 y^{(n)}(t) + P_1 y^{(n-1)}(t) + ... + P_{n-1} y'(t) + P_n y(t) = G(t) \tag{6.1}$$

When $P_0, P_1, ...P_n$ are constants and $G(t) = 0$, it is called an nth order homogeneous equation. We have discussed case when $n = 2$ thoroughly before. For distinct real eigenvalues $\lambda_1 ... \lambda_n$, the solution has the form

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + ... + c_n e^{\lambda_n t} = \sum_{i=1}^{n} c_i e^{\lambda_i t} \tag{6.2}$$

which also holds for distinct complex eigenvalues.

For repeated eigenvalues $\lambda_1 = \lambda_2 = ... = \lambda_n = \lambda$, the solution is also very similar to the 2nd order solution form that we have established. The solution has the following form

$$y(t) = c e^{\lambda t} + \sum_{i=1}^{n} c_i t^i e^{\lambda t} \tag{6.3}$$

If the eigenvalue $\lambda = \eta + i\mu$ is complex, and is repeated s times, then the complex conjugate $\lambda = \eta - i\mu$ is also repeated s times. Therefore, there are $2s$ real-valued solutions contributed by these repeated roots, which are

$$\sum_{k=0}^{s-1} c_k e^{\eta t} t^i (\cos \mu t + \sin \mu t) \tag{6.4}$$

The complete general solution is obtained by adding Eq.(6.4) with the solutions of the distinct eigenvalues of the form of Eq.(6.2).

After quickly finishing the trivial stuff, we turn to more interesting case when $P_0, P_1, ...P_n$ are constants and $G(t) \neq 0$. Then to obtain the general solution of Eq.(6.1), we need to find a particular solution to Eq.(6.1) and add the general solution for nonhomogeneous equation (which is discussed above).

The book introduces the method of undetermined coefficients, which is boring and ugly. Sometimes, the guess just comes out of nowhere. Here we introduce a different method called operator method. Using this method we can get the answer much faster, and much more elegant as well.

6.1 Finding a Particular Solution of Nonhomogeneous Equations

Definition 6.1.1. *Consider a differential equation with the following form*

$$P_0 y^{(n)}(t) + P_1 y^{(n-1)}(t) + ... + P_{n-1} y'(t) + P_n y(t) = G(t) \tag{6.5}$$

where $P_0, P_1, ...P_n$ *are constants and* $G(t) \neq 0$. *We define the polynomial operator*

$$p(D) = P_0 D^n + P_1 D^{n-1} + ... + P_{n-1} D + P_n \tag{6.6}$$

where $D^n = d^n/d^n t$ *is the differential operator that takes the nth derivative of a function.*

Therefore Eq.(6.5) can be written as

$$p(D)y = G(t)$$

Now we establish some basic rules about how to use this operator.

Theorem 6.1.1. *The polynomial operator is linear.*

$$p(D)(c_1 f + c_2 g) = c_1 p(D) f + c_2 p(D) g$$

Theorem 6.1.2.

$$p(D)e^{at} = p(a)e^{at}$$

Proof. We have, by repeated differentiation

$$De^{at} = ae^{at}, \quad D^2 e^{at} = a^2 e^{at}, \dots, D^k e^{at} = a^k e^{at}$$

Therefore,

$$(P_0 D^n + ... + P_n)e^{at} = (P_0 a^n + ... + P_n)e^{at} = p(a)e^{at}$$

$\square$

Now we introduce one very important rule called the exponential shift rule.

Theorem 6.1.3.

$$p(D)e^{at}u = e^{at}p(D+a)u$$

Proof. We prove this theorem in successive stages. First let $p(D) = D$. Then we have

$$De^{at}u(t) = e^{at}Du(t) + ae^{at}u(t) = e^{at}(D+a)u(t)$$

Now prove that this is true for $p(D) = D^2$

$$\begin{aligned} D^2e^{at}u = D\left(De^{at}u\right) &= D\left(e^{at}(D+a)u\right) \\ &= e^{at}(D+a)((D+a)u) \\ &= e^{at}(D+a)^2u \end{aligned}$$

Similarly for $p(D) = D^3$

$$\begin{aligned} D^3e^{at}u = D\left(D^2e^{at}u\right) &= D\left(e^{at}(D+a)^2u\right) \\ &= e^{at}(D+a)\left((D+a)^2u\right) \\ &= e^{at}(D+a)^3u \end{aligned}$$

Thus this is true for all polynomial operator $p(D) = D^k$. To prove that this is true for $p(D) = P_0D^n + ... + P_n$, we immediately notice that this is self-explanatory by the linearity of the operator D.

□

Now we are ready to find a particular solution with the form

$$p(D)y = e^{at}$$

This is very useful because not only can you solve $G(s) = e^{at}$, but you can also solve forms involving sine or cosine if you allow a to be complex and take the real/imaginary part.

Theorem 6.1.4. *Let $p(D)$ be a polynomial operator and let $p^{(s)}$ be its s-th derivative. Then $p(D)y = e^{at}$, where a is real or complex, has the particular solution*

$$Y_p = \begin{cases} i)\ \frac{e^{at}}{p(a)} & \text{if } p(a) \neq 0 \\ ii)\ \frac{te^{at}}{p'(a)} & \text{if } p(a) = 0 \text{ and } p'(a) \neq 0 \\ iii)\ \frac{t^2e^{at}}{p''(a)} & \text{if } p(a) = p'(a) = 0 \text{ and } p''(a) \neq 0 \\ \cdots & \\ iv)\ \frac{t^se^{at}}{p^{(s)}(a)} & \text{if } a \text{ is an } s\text{ -fold zero} \end{cases}$$

Proof. We first check whether (i) is true. It is obvious by direct substitution that

$$p(D)Y_p = p(D)\frac{e^{at}}{p(a)} = \frac{1}{p(a)}p(D)e^{at} = \frac{p(a)e^{at}}{p(a)} = e^{at}$$

Since (ii) and (iii) are special cases for (iv), we jump right to that. We begin by noting that to say that the polynomial $p(D)$ has a as its s-fold zero is equivalent to

$$p(D) = q(D)(D-a)^s, \quad q(a) \neq 0 \tag{6.7}$$

We will first prove that Eq.(6.7) implies that

$$p^{(s)}(a) = q(a)s! \tag{6.8}$$

To prove this, let k be the degree of $q(D)$ and write $q(D)$ as powers of $(D-a)$

$$\begin{aligned}&q(D) = q(a) + c_1(D-a) + \ldots + c_k(D-a)^k;\ \text{then}\\ &p(D) = q(a)(D-a)^s + c_1(D-a)^{s+1} + \ldots + c_k(D-a)^{s+k}\\ &p^{(s)}(D) = q(a)s! + \ \text{positive powers of}\ D-a\end{aligned}$$

Substitute $D = a$ into $p^{(s)}(D)$ proves Eq.(6.8). We also note that in Eq.(6.7), if we substitute $D \to D-a$, we get

$$p(D+a) = q(D+a)D^s$$

Now using Eq.(6.8), we can easily prove this theorem by direct substitution.

$$\begin{aligned}p(D)\frac{e^{at}t^s}{p^{(s)}(a)} &= \frac{e^{at}}{p^{(s)}(a)}p(D+a)t^s\\ &= \frac{e^{at}}{p^{(s)}(a)}q(D+a)D^s t^s\\ &= \frac{e^{at}}{q(a)s!}q(D+a)s!\\ &= \frac{e^{at}}{q(a)s!}q(a)s! = e^{at}\end{aligned}$$

where the last line comes from

$$q(D+a)s! = \left(q(a) + c_1 D + \ldots + c_k D^k\right)s! = q(a)s!$$

Thus the proof is completed. $\square$

The following example is an excellent demonstration of how to use this theorem.

Example 6.1.1. *Find a particular solution of the equation*

$$y''' - 4y' = t + 3\cos t + e^{-2t}$$

We first split the equation into three parts.

$$y''' - 4y' = t, \quad y''' - 4y' = 3\cos t, \quad y''' - 4y' = e^{-2t}$$

We now find the particular solutions for three equations Y_1, Y_2, Y_3 separately, and then add them together gives the particular solution of the original equation.

We first find Y_1. This requires some inspiration. Since the right hand side is t, we guess it must have some form At^2. So when you take derivative three times, it becomes zero. Taking the derivative once gives the linear term, then we are able to make correspondence. Plug in our guess, we find

$$-8A = 1 \rightarrow A = -\frac{1}{8}$$

Therefore

$$Y_1 = -\frac{1}{8}t^2$$

Then we find Y_2. The right hand side is $3\cos t$. By Euler's identity we know that $3e^{it} = 3\cos t + 3i\sin t$. Therefore, we turn left hand side into $3e^{it}$. After we find the particular solution, which is a function of complex variable, we take the real part and get the cosine.

$$p(D) = D^3 - 4D$$

Then it follows that

$$p(i) = i^3 - 4i = -i - 4i = -5i$$

Therefore, the particular solution should be

$$Z = \frac{3e^{it}}{-5i} = \frac{3}{5}i(\cos t + i\sin t)$$

$$Y_2 = Re(Z) = -\frac{3}{5}\sin t$$

Finally we find Y_3. The left hand side is e^{-2t}. We find $p(-2) = 0$. Therefore, we take the derivative. $p'(s) = 3(-2)^2 - 4 = 8$. Therefore,

$$Y_3 = \frac{1}{8} t e^{-2t}$$

So we find the particular solution to be

$$Y = Y_1 + Y_2 + Y_3 = -\frac{1}{8} t^2 - \frac{3}{5} \sin t + \frac{1}{8} t e^{-2t}$$

We now look at another example.

Example 6.1.2. *Find a particular solution of the following equation*

$$y^{(4)} + 2y'' + y = 3 \sin t - 5 \cos t$$

Recall that

$$\sin(t - \alpha) = \sin t \cos \alpha - \cos t \sin \alpha$$

Therefore we set $\cos \alpha = 3, \sin \alpha = 5$. Then $3 \sin t - 5 \cos t = \sin(t - \alpha)$. We know $e^{i(t-\alpha)} = \cos t - \alpha + i \sin t - \alpha$.Then we set the right hand side to be $e^{i(t-\alpha)}$, and after we are done, we take the imaginary part of the particular solution.

$$p(D) = D^4 + 2D^2 + 1$$

We note that

$$p(i) = i^4 + 2i^2 + 1 = 1 - 2 + 1 = 0$$

Take the derivative again and plug in i.

$$p'(i) = 4i^3 + 4i = -4i + 4i = 0$$

Taking derivative again we find

$$p''(i) = 12i^2 + 4 = -8$$

Therefore the solution is

$$Z = -\frac{1}{8} t^2 e^{i(t-\alpha)}$$

Taking the imaginary part gives the particular solution to the original equation

$$Im(z) = -\frac{1}{8} t^2 \sin t - \alpha = -\frac{3}{8} t^2 \sin t + \frac{5}{8} t^2 \cos t$$

6.2 The Method of Variation of Parameters

Suppose we already know the solution to the homogeneous equation Eq.(6.1) where $G(t) = 0$. Let the solution be

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Now we assume that the solution to the nonhomogeneous equation has similar forms

$$Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t) + \cdots + u_n(t) y_n(t) \tag{6.9}$$

Now that we need to determine what function $u_1, u_2..., u_n$ are. We try to substitute our guess into the equation. We take the first derivative of Eq.(6.9). We get

$$Y' = \left(u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'\right) + \left(u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n\right) \tag{6.10}$$

For simplicfication, we let

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0 \tag{6.11}$$

Then

$$Y' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n' \tag{6.12}$$

We continue to take derivatives and get $Y''...Y^{(n-1)}$. Everytime we set

$$u_1' y_1^{(m)} + u_2' y_2^{(m)} + \cdots + u_n' y_n^{(m)} = 0, \quad m = 1, 2, \ldots, n-2 \tag{6.13}$$

Finally,

$$Y^{(n)} = \left(u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}\right) + \left(u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right) \tag{6.14}$$

By plugging in $Y, Y', ...Y^{(n-1)}, y^{(n)}$ into Eq.(6.1), we eliminate most of the term. Since

$$u_i(P_0 y_i^n + P_1 y_i^{(n-1)} + ... + P_n y_i) = 0$$

for $i = 1, 2, ...n$. It follows that

$$\sum_{i=1}^{n} u_i(P_0 y_i^n + P_1 y_i^{(n-1)} + ... + P_n y_i) = 0$$

Therefore, what we are left with is

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = G(t)$$

Then we add our restrictions and form a system of equations

$$\begin{aligned}
y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0 \\
y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0 \\
y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' &= 0 \\
&\vdots \\
y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} u_n' &= G
\end{aligned} \tag{6.15}$$

By cramer's rule we have

$$u_m'(t) = \frac{G(t) W_m(t)}{W(t)}, \quad m = 1, 2, \ldots, n \tag{6.16}$$

where W_m is the wronskian whose mth column is replaced by column $[0, 0, ...1]^T$. Therefore, the full solution is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int \frac{G(s) W_m(s)}{W(s)} ds \tag{6.17}$$