THE GEOMETRY OF SUPERSYMMETRY: SUPERMANIOFOLDS AND THE INDEX THEOREM

MATHEMATICAL FOUNDATIONS OF THEORETICAL PHYSICS, VOLUME II

WRITTEN BY
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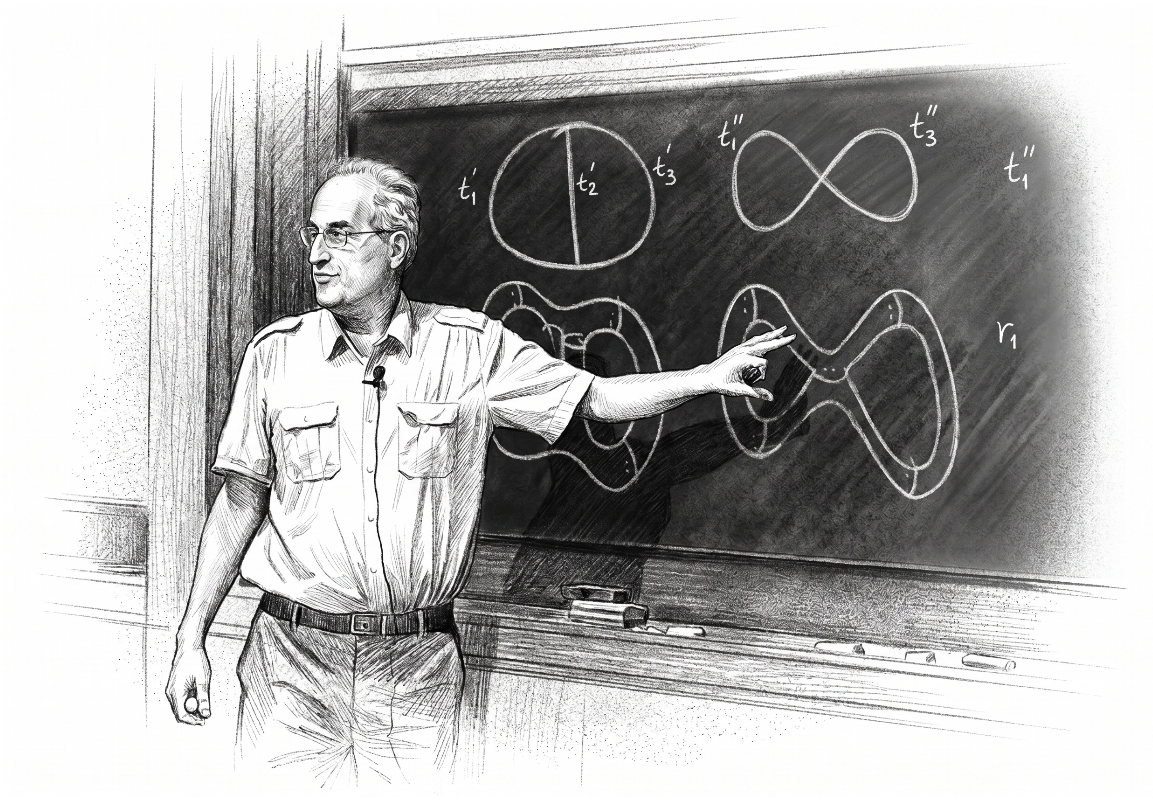

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To Prof. Ron Donagi and Prof. Edward Witten

Preface

This is my study notes on supermanifolds, supergeometry, and supermoduli space, and also my reading notes on serveral papers in relevant subject, mostly the paper by Donagi and Witten [2, 31] and the paper by Donagi and Ott [3]. In the first chapter where we develop the theory of supermanifolds, I will also refer to Witten's notes [5] and Donagi's unfinished textbook [11]. For more detailed study of supermanifolds, supermoduli, and its application in superstring perturbation theory, we refer the reader to the series of legendary notes [5, 6, 7, 8, 9] by Witten.

I am deeply indebted to Ron Donagi for introducing me to this intriguing subject in his mathematical physics class, suggesting me interesting papers such as [2, 3] to read and relevant research projects on supergeometry, and meeting me regularly to answer my questions.

The second part of this book mostly originated from a course on quantum field theory at UPenn taught by Jonathan Heckman, who I also greatly appreciate.

I am also grateful to Nadia Ott for giving a talk at UPenn's math physics joint seminar, which triggered my interest in this subject, and for suggesting a list of books and notes for supergeometry.

Tianyi Wang,
Philadelphia
Mar 2025

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Part I

Supermanifolds and Supermoduli

Chapter 1

Supermanifold

1.1 Basic Definitions

One can define a C^∞-manifold M in two ways: First is the usual way of defining to be a topological space that is locally diffeomorphic to $\mathbb{R}^n$, and with a set of chart transition maps satisfying C^∞-compatibility condition. However, one can also note that the smooth manifold M comes with a sheaf of smooth functions $\mathcal{O}_M$ on M, which algebraic geometers like to refer to as the structure sheaf. Then one can define a C^∞-manifold to be a locally ringed space that is locally isomorphic to the model space $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. The first approach is more intuitive, however the second approach is easier to generalize to supermanifold, which we will define below.

Definition 1.1 (supercommutativity)**.** A $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ is said to be ***supercommutative*** if $\alpha\beta = (-1)^{|\alpha|\cdot|\beta|}\beta\alpha$.

Definition 1.2 (split supermanifold)**.** Now let M be a manifold and V a vector bundle over M. Then we define the ***split supermanifold*** $S := S(M, V)$ to be the locally ringed space $(M, \mathcal{O}_S)$, where $\mathcal{O}_S := \mathcal{O}_M \otimes \bigwedge^\bullet V^*$ is the sheaf of $\mathcal{O}_M$-valued sections of the exterior algebra $\bigwedge^\bullet V^*$. In other words,

$$S(M, V) = (M, \mathcal{O}_M \otimes \bigwedge\nolimits^\bullet V^*).$$

If we interpret V as a locally free sheaf of $\mathcal{O}_M$-modules, then $\mathcal{O}_S$ is simply $\bigwedge^\bullet V^*$. Then $\mathcal{O}_S$ is $\mathbb{Z}_2$-graded (as there is a natural notion of eveness and oddness in the exterior algebra), and supercommutative, and its stalks are local rings.

Example 1.3 (examples of split supermanifolds)**.** The ***affine superspace*** $\mathbb{A}^{m|n}$ is defined by

$$(\mathbb{A}^m, \mathcal{O}_{\mathbb{A}^{m|n}}) = S(\mathbb{A}^m, \mathcal{O}_{\mathbb{A}^m}^{\oplus n}),$$

where for M a manifold $V = \mathcal{O}_M^{\oplus n}$ is the trivial rank n bundle on it.

Here the affine space $\mathbb{A}^n = \mathbb{A}^n_K$ depends on the choice of the base field K. In our case, we almost always consider either $K = \mathbb{R}$ or $K = \mathbb{C}$.

More specifically, let us consider the case where $K = \mathbb{R}$. Then $S(\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^m}^{\oplus n})$ is just the smooth manifold $\mathbb{R}^m$, with a sheaf of supercommutative ring

$$\mathcal{O} : U \mapsto C^\infty(U)[\theta^1, ..., \theta^n],$$

that sends U to the Grassmann algebra with generators $\theta^1, ..., \theta^n$ over $C^\infty(U) = \mathcal{O}_{\mathbb{R}^m}(U)$. This is the structure sheaf. An element of the structure sheaf looks like $f = \sum_I f_I \theta^I$ where

$I = (i_1, ..., i_k) \subset \{1, ..., n\}$ is an index set and $\theta^I = \theta^{i_1}\theta^{i_2}\cdots\theta^{i_k}$. We will call this model space $\mathbb{R}^{m|n}$. Thus $\mathbb{R}^{m|n}$ has even coordinates $x^1, ..., x^m$ and odd coordinates $\theta^1, ..., \theta^n$. For brevity, we will usually write an element f in the structure sheaf as $f(x^\mu, \theta^\nu)$, and we will denote the structure sheaf $\mathcal{O}$ by $\mathcal{O}_{\mathbb{R}^{m|n}}$ if we want to be specific.

Similarly, one can define $\mathbb{C}^{m|n}$.

Definition 1.4. A ***supermanifold*** of dimension $m|n$ is a locally ringed space $(S, \mathcal{O}_S)$ that is locally isomorphic to some model space $S(M, V)$, where $\dim M = m$ and $\operatorname{rank} V = n$. We will usually just refer to this supermanifold as S with the understanding that it also comes with the structure sheaf $\mathcal{O}_S$. The manifold M is called the ***reduced space*** of S and we usually write $|S| = M$.

In particular, a real supermanifold of dimension $m|n$ is a locally ringed space locally isomorphic to $(\mathbb{R}^{m|n}, \mathcal{O}_{\mathbb{R}^{m|n}})$, and a complex supermanifold of dimension $m|n$ is a locally ringed space locally isomorphic to $(\mathbb{C}^{m|n}, \mathcal{O}_{\mathbb{C}^{m|n}})$.

Remark 1.5. In the C^0, C^1, C^∞, or analytic settings, we may as well restrict our local models to affine superspaces $\mathbb{A}^{m|n}$. But in the algebraic setting we muts allow all $S(M, V)$ with affine M, as in the bosonic algebraic case.

Remark 1.6 (morphisms of supermanifolds)**.** In the definition above, by an isomorphism we mean an isomorphism of locally ringed spaces. Recall that a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a map $f : X \to Y$ between topological spaces together with a morphism of sheaves $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$, where $f_*\mathcal{O}_X$ is a sheaf on Y defined to be $\mathcal{O}_X(f^{-1}(U))$ for any open set $U \subset Y$. That is, every morphism f of locally ringed space comes with the data of a ring homomorphism $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, called ***pullback*** on U. With this notion of morphism, supermanifolds form a category.

It is also important to note that the isomorphisms above are isomorphisms of $\mathbb{Z}_2$-graded algebras, but they need not preserve the $\mathbb{Z}$-grading of $\bigwedge^\bullet V^*$. For example, let z be a coordinate on M and θ^i are the fiber coordinates of V. Then z cannot go to $z + \theta_1$ under such isomorphisms, since this changes parity, but it can go to $z + \theta^1\theta^2$.

Theorem 1.7. *Let $f : S \to S'$ be a morphism between supermanifolds. Choose a local coordinate $(x^1, ..., x^m, \theta^1, ..., \theta^n)$ on $U \subset S$ and $(y^1, ..., y^s, \chi^1, ..., \chi^\ell)$ a local coordinates on an open subset containing $f(U) \subset S'$. Then f has the local expression*

$$f(x^\mu, \theta^\nu) = (f^*y^1, ..., f^*y^s, f^*\chi^1, ..., f^*\chi^\ell)$$

where $f^* : \mathcal{O}_{S'} \to f_*\mathcal{O}_S$ is the morphism on locally ringed spaces. Conversely, the information above on all local charts completely determines $f : |S| \to |S'|$ and $f_* : \mathcal{O}_{S'} \to f_*\mathcal{O}_S$.

Proof. To obtain the local expression, note that y^α, χ^β are the (even or odd) coordinate functions on S'. Since the local expression of f is obtained by postcomposing f with the coordinate functions, the theorem follows. Note that in the case that the odd dimensions are zero, we reduce to the case of a map between smooth manifolds. □

Remark 1.8. In the familiar case where $f : M \to M'$ is simply a smooth map between manifolds, we have similar results, where if we take local coordinates $x^1, ..., x^m$ and $y^1, ..., y^n$ containing $U \subset M$ and $f(U) \subset M'$ then we have

$$f(x^1, ..., x^m) = (f^*y^1, ..., f^*y^n).$$

Hence we can also define manifolds as locally ringed space isomorphic to the Euclidean model. However, note that in the case of supermanifolds, there are way more morphisms than ordinary manifolds, because a morphism can mix even and odd part as long as it preserves grading. For example, the map $\varphi : \mathbb{R}^{1|2} \to \mathbb{R}^{1|2}$ defined by

$$\varphi(x, \theta^1, \theta^2) = (x + \theta^1\theta^2, \theta^1, \theta^2)$$

is an isomorphism. But note that $\varphi^* x = x + \theta^1\theta^2$, so it need not be true that in the above expression $f^* y^j$ only involves even coordinates, it can also depends on odd coordinates.

Remark 1.9. [recovering manifolds from supermanifolds] Given a supermanifold $(S, \mathcal{O}_S)$, its structure sheaf $\mathcal{O}_S$ contains an ideal J consisting of all nilpotents. This is the ideal generated by all odd functions. Note that the underlying topological space of S is already given, which we call $M = |S|$. To recover the structure sheaf $\mathcal{O}_M$, we note that $\mathcal{O}_M = \mathcal{O}_S/J$. The dual bundle V^* can also be recovered because $V^* = J/J^2$. Hence from the supermanifold $S = S(M, V)$ we can recover the data of S and V. In other words, a supermanifold S determines its reduced space.

Note that the definition of a supermanifold $(S, \mathcal{O}_S)$ gives $\mathcal{O}_S$ the structure of a sheaf of (real or complex) algebras, and also with a surjective homomorphism to $\mathcal{O}_M$ by setting $\theta^\nu = 0$ for all odd coordinates. In other words, $\mathcal{O}_M = \mathcal{O}_S/J$ induces an exact sequence of sheaves

$$0 \to J \to \mathcal{O}_S \to \mathcal{O}_M \to 0.$$

However, this does not endow $\mathcal{O}_S$ the structure of $\mathcal{O}_M$-modules, because although there are open sets U on which $\mathcal{O}_S$ is isomorphic to $\mathcal{O}_M(U)[\theta^1, ..., \theta^n]$, and multiplication by elements in $\mathcal{O}_M(U)$ is commutative there, but it need not commute for larger open sets. In other words, multiplication by $\mathcal{O}_M$ need not to commute with gluing isomorphisms.

Definition 1.10 (split and projected supermanifold)**.** A supermanifold S is said to be ***split*** if it is globally isomorphic to $S(M, V)$ for some manifold M and some vector bundle $V \to M$. In other words, it is globally isomorphic to some split supermanifold.

A supermanifold S is said to be ***projected*** if there exists a projection map $\pi : S \to M$, such that the inclusion $i : M \to S$, which is identity on the reduced space $i : |M| \to |S|$, and $i^* : \mathcal{O}_S \to \mathcal{O}_M$ given by imposing $\theta^1 = \cdots = \theta^n = 0$, is a section of π, i.e., $\pi i = \mathrm{id}_M$ and $i^*\pi^* = \mathrm{id}_{\mathcal{O}_M}$.

Remark 1.11 (equivalent definition for split and exactness)**.** There are equivalent ways of defining splitness and projectedness, which is used in [12]. Our definition follows from [11].

Let S be a supermanifold with $M = |S|$ its underlying topological space. There is a natural closed immersion $i : M \to S$ of locally ringed spaces that is identity on $M = |S|$ together with the morphism $i^* : \mathcal{O}_S \to i_*\mathcal{O}_M = \mathcal{O}_S/J$ induced by the quotient (it is easy to see that this closed immersion is precisely the inclusion $i : M \to S$ in the definition above, and i^* is the pullback of i). Thus we have an exact sequence of $\mathcal{O}_M$-modules

$$0 \to J \to \mathcal{O}_S \xrightarrow{i^*} \mathcal{O}_S/J_S \to 0.$$

We call this exact sequence the ***structural sequence*** of the supermanifold. We call S ***projected*** if its structural sequence splits, i.e., there exits a morphism $\pi^* : \mathcal{O}_M \to \mathcal{O}_S$ such that $i^*\pi^* = \mathrm{id}_{\mathcal{O}_M}$.

What about splitness? Given a supermanifold S of dimension $m|n$, we have a filtration

$$\mathcal{O}_S = J^0 \supset J \supset J^2 \supset \cdots \supset J^m \supset J^{m+1} = 0.$$

We can then define a sheaf of $\mathbb{Z}$-graded rings

$$\operatorname{Gr}\mathcal{O}_S = \bigoplus_{k=0}^{\infty} J^k/J^{k+1} = \mathcal{O}_M \oplus J/J^2 \oplus J^2/J^3 \oplus \cdots \oplus J^m/J^{m+1}.$$

Then a supermanifold S is said to be ***split***, if there is an isomorphism $\mathcal{O}_S \cong \operatorname{Gr}\mathcal{O}_S$. It is easy to see that if $S \cong S(M,V) = (M, \mathcal{O}_M \otimes \bigwedge^\bullet V^*)$, then clearly S is split in this definition, since $\mathcal{O}_S \cong \mathcal{O}_M \oplus \bigwedge^1 V^* \oplus \bigwedge^2 V^* \oplus \cdots$

Lemma 1.12. *Any split supermanifold S is projected.*

Proof. Indeed, if S is split, then $\mathcal{O}_S \cong \mathcal{O}_M \otimes \bigwedge^\bullet V^*$. Then we have a morphism $\pi^* : \mathcal{O}_M \to \pi_*\mathcal{O}_S$ via embedding $\mathcal{O}_M$ to be the degree zero part. And it is easy to see that $i^*\pi^* = \mathrm{id}_{\mathcal{O}_M}$. Now to construct $\pi : S \to M$ with the desired property, we choose the map of underlying space $|S| \to |M|$ to be identity, and $\pi^* : \mathcal{O}_M \to \pi_*\mathcal{O}_S$ to be the map given above. $\square$

Lemma 1.13. *The product of two split supermanifolds is split.*

Proof. Suppose we have two split supermanifolds with global isomorphisms $S_1 \cong S(M_1, V_1)$ and $S_2 \cong S(M_2, V_2)$. Let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the projections onto the first and second components. Then the pullback bundles $\pi_1^* V_1$ and $\pi_2^* V_2$ are vector bundles on $M_1 \times M_2$. Hence we now consider the vector bundle $V = \pi_1^* M_1 \oplus \pi_2^* M_2$ over $M_1 \times M_2$. Then it is not hard to see that

$$S_1 \times S_2 \cong S(M_1 \times M_2, V) = (M_1 \times M_2, \mathcal{O}_{M_1 \times M_2} \otimes \textstyle\bigwedge^\bullet V^*)$$

because clearly we have an isomorphism between

$$\mathcal{O}_{M_1 \times M_2} \otimes \textstyle\bigwedge^\bullet (\pi_1^* V_1 \oplus \pi_2^* V_2)^*$$

with the sheaf

$$\mathcal{O}_{S_1 \times S_2} : U \mapsto \mathcal{O}_{M_1 \times M_2}(U)[\theta^1, ..., \theta^n, \chi^1, ..., \chi^{n'}]$$

where $\theta^1, ..., \theta^n$ are local fiber coordinates on V_1^* and $\chi^1, ..., \chi^{n'}$ are local fiber coordinates on V_2^*. $\square$

Lemma 1.14. *Let S be a supermanifold and $|S| = M$. Let $f : M' \to M$ be a map between manifold (which is also a morphism between locally ringed space). Then there exists a pullback supermanifold $S' = f^*S$ and an induced morphism $f : S' \to S$ of locally ringed space. Moreover, $S' = f^*S$ is split (resp. projected) if S is split (resp. projected).*

Proof. We first make sense of $S' = f^*S$ as a locally ringed space. Clearly $|S'| = M'$. To define the structure sheaf $\mathcal{O}_{S'}$, note that $f : M' \to M$ defines a pullback map $f^* : \mathcal{O}_M \to f_*\mathcal{O}_{M'}$. Moreover, we have local isomorphisms $S \cong S(M,V)$ where $V \to M$ is a vector bundle, hence we may consider the pullback bundle $f^*V \to M'$. This allows us to define the structure sheaf as

$$\mathcal{O}_{S'} : U \mapsto \mathcal{O}_{M'}(f^{-1}(U))[\theta^1, ..., \theta^n]$$

where θ^i's are local fiber coordinates on $f^*V \to M'$. In other words, $\mathcal{O}_{S'} = f^*\mathcal{O}_S$.

To show the claim, note that if S is split, i.e., if there is a global isomorphism $S \cong (M, \mathcal{O}_M \otimes \bigwedge^\bullet V^*)$, then $S' = f^*S \cong (M', \mathcal{O}_{M'} \otimes \bigwedge^\bullet f^*V^*)$ since pullback commutes with wedge product. Now suppose S is projected, i.e., there is a projection $\pi : S \to M$ such that $i^*\pi^* = \mathrm{id}_{\mathcal{O}_M}$, then the pullback functor f^* takes the split exact sequence

$$0 \to J \to \mathcal{O}_S \xrightarrow{i^*} \mathcal{O}_M \to 0$$

to a split exact sequence

$$0 \to f^*J \to \mathcal{O}_{S'} \to \mathcal{O}_{M'} \to 0.$$

Hence S' is also split. □

Remark 1.15. Note that however, a submanifold of a split supermanifold need not in general be split as a supermanifold.

Now we introduce sheaves on supermanifolds.

Definition 1.16. A ***sheaf*** $\mathcal{F}$ on a supermanifold S is simply a sheaf on the reduced space M, and sheaf cohomology $H^*(S, \mathcal{F})$ of $\mathcal{F}$ on S is simply the sheaf cohomology $H^*(M, \mathcal{F})$ on the reduced space.

Remark 1.17. Despite the fact that sheaves and cohomology are defined to be the same as the reduced space, it is usually more illuminating to denote the cohomology as $H^*(S, \mathcal{F})$ instead of $H^*(M, \mathcal{F})$. For example, it is much more natural to think of a sheaf of $\mathcal{O}_S$-modules as a sheaf on S rather than a sheaf on M.

Example 1.18 (tangent bundle)**.** The ***tangent bundle*** T_S of supermanifold S is an example of a sheaf of $\mathcal{O}_S$-modules. It can be defined in terms of derivations, i.e., as sheaves

$$T_S := \mathrm{Der}(\mathcal{O}_S).$$

It is a $\mathbb{Z}_2$-graded vector bundle, or a locally free sheaf of $\mathcal{O}_S$-modules.

In the simplest example where $S = \mathbb{A}^{m|n}$, TS is the free $\mathcal{O}_S$-module generated by even tangent vectors $\frac{\partial}{\partial x^\mu}$ for $\mu = 1, ..., m$ and odd tangent vectors $\frac{\partial}{\partial \theta^\nu}$ for $\nu = 1, ..., n$.

Remark 1.19. In general, for a split supermanifold $S = S(M, V)$, the tangent bundle T_S need not have distinguished complementary even and odd subbundles: The even and odd parts are not sheaves of $\mathcal{O}_S$ modules. For example, $\frac{\partial}{\partial \theta^1}$ is an odd vector field, but multiplication by an element of $\mathcal{O}_S$ might change parity: Consider $\theta^1 \frac{\partial}{\partial \theta^1}$. This is now an even vector field. However, we note that since elements in $\mathcal{O}_M$ are even, if we restrict ourselves to multiplying only elements in $\mathcal{O}_M$ then parity is preserved. More specifically, what we are saying is that the restriction $T_S|_M$ of T_S to the reduced space M does split (By restriction to M we mean pullback, under the natural inclusion $i : M \to S$, of a sheaf of $\mathcal{O}_S$-modules to a sheaf of $\mathcal{O}_M$-modules. In this case, this is accomplished by setting all odd functions to 0). Explicitly, the splitting is given by

$$T_{S,+} : = T_M,$$
$$T_{S,-} : = V.$$

For a general supermanifold S, we only have local isomorphisms between S and $S(M, V)$ for some model space M and vector bundle V, but there is no canonical identification $T_{S,-} = V$. However, there is still a way to define $T_{S,-}$. We consider the natural embedding $M \to S$, where we view M as the subspace that is locally given by $\theta^\alpha = 0$ for all α. Then then we have a normal bundle sequence

$$0 \to T_M \to T_S|_M \to N_{M/S} \to 0.$$

Now $N_{M/S}$ is a purely odd bundle. Therefore, we may define $T_{S,+} = T_M$ and $T_{S,-} = N_{M/S}$.

1.2 Examples of Supermanifolds

Example 1.20. The simplest example is the affine superspace $\mathbb{A}^{m|n}$, which we already saw. In the algebraic case, the global function ring is

$$\mathcal{O}_{\mathbb{A}^{m|n}}(\mathbb{A}^{m|n}) = K[x_1, ..., x_m, \theta_1, ..., \theta_n]$$

is just the polynomial ring in m commuting variables and n anticommuting variables. In the C^0, C^1, C^∞ or analytic case we allow the even part of the function to be C^0, C^1, C^∞ or analytic, but the anticommuting part is unchange, still the polynomial ring in θ_j's.

Example 1.21. The complex ***projective superspace*** $\mathbb{P}^{m|n}$ is defined by the quotient

$$\mathbb{P}^{m|n} = \{(x^0, x^1, ..., x^m, \theta^1, ..., \theta^n) \in \mathbb{A}^{m+1|n} : x^\mu \neq 0 \text{ for some } \mu\}/\sim_{\mathbb{C}^*}$$

where $(x^\mu, \theta^\nu) \sim_{\mathbb{C}^*} (x^{\mu'}, \theta^{\nu'})$ iff $x^\mu = \lambda x^{\mu'}$ and $\theta^\nu = \lambda\theta^{\nu'}$ for some $\lambda \in \mathbb{C}^*$. Note that the group $\mathbb{C}^*$ is purely even, i.e., λx^μ is even and $\lambda\theta^\nu$ is odd for all $\lambda \in \mathbb{C}^*$.

The complex projective superspace are covered by local charts of open sets

$$U_\alpha = \{(x^0, ..., x^m, \theta^1, ..., \theta^n) \in \mathbb{P}^{m|n} : x^\alpha \neq 0\}$$

for $\alpha = 0, ..., m$. Each U_α can be identified with affine superspace $\mathbb{A}^{m|n}$ via the ratios $x^\beta/x^\alpha, \beta \neq \alpha$, and θ^ν/x^α, for $\nu = 1, ..., n$.

Note that for $m = 0$ there is a unique open set U_α, so $\mathbb{P}^{0|n} \cong \mathbb{A}^{0|n}$.

Example 1.22 (submanifolds)**.** Given a supermanifold, one can construct super submanifolds by imposing equations. For example in $\mathbb{P}^{m|n}$ we can impose the equation $f(z^0, ..., z^m, \theta^1, ..., \theta^n) = 0$ where f is a homogeneous polynomial in the homogeneous coordinate of $\mathbb{P}^{m|n}$ that is either even or odd. If f is even and sufficiently generic, this will give a complex supermanifold of dimension $m - 1|n$. For suitable odd f, this gives a supermanifold of dimension $m|n - 1$.

A ***divisor*** is a super submanifold of *codimension* $1|0$, i.e., when the defining polynomial is even.

Example 1.23 (covering of supermanifolds)**.** Let $S = (M, \mathcal{O}_S)$ be a supermanifold, and $f : \tilde{M} \to M$ a covering map of the underlying reduced space. Then there is a way to define a covering supermanifold $F : \tilde{S} \to S$.

Of course $|\tilde{S}| = \tilde{M}$. So it remains to construct the structure sheaf $\mathcal{O}_{\tilde{S}}$. We define $\mathcal{O}_{\tilde{S}} := f^{-1}\mathcal{O}_S$ to be the pullback sheaf of $\mathcal{O}_S$ under f. Recall that this is defined to be

$$f^{-1}\mathcal{O}_S : U \mapsto \varinjlim \mathcal{O}_S(V)$$

where the direct limit is taken over all open sets $f(U) \subset V \subset M$, for $U \subset \tilde{M}$ open. We claim that this gives $\tilde{S} = (\tilde{M}, \mathcal{O}_{\tilde{S}})$ the structure of a supermanifold, i.e., it is locally isomorphic to some model space. To see this, note that we can choose an open cover $(U_\alpha)_{\alpha\in A}$ of $\tilde{S}$ such that

- $f|_{U_\alpha} : U_\alpha \to V_\alpha := f(U_\alpha)$ is a isomorphism of underlying space. So this implies that V_α is open in M, in particular.
- For each α, there is an isomorphism $\varphi_\alpha : V_\alpha \to \varphi_\alpha(V_\alpha) \subset \mathbb{A}^{m|n}$ of locally ringed spaces, since S is a supermanifold to start with.

Now we simply consider $\varphi_\alpha f : U_\alpha \to \varphi_\alpha(V_\alpha)$, clearly this is an isomorphism of the underlying space. So it remains to show that this induces isomorphism on structure sheaves for all α. Since φ_α^* is an isomorphism for every α to begin with, it suffices to show that $f^* : \mathcal{O}_S \to f_*\mathcal{O}_{\tilde{S}}$ restricts to isomorphism on each U_α. Since f^* is defined to be precomposition with f, which restricts to local isomorphisms of the underlying space, we conclude that $f^* : \mathcal{O}_S|_{V_\alpha} \to f_*\mathcal{O}_{\tilde{S}}|_{U_\alpha}$ is an isomorphism of structure sheaves.

We now look at a more complicated example, which is a variant of the covering supermanifold. We first need to introduce the notion of a branched cover. This might seem a little bit abstract at this stage, but branched cover will not play a huge roll until later part of the book. We will give a more thorough treatment of this topic in Section 5.4.

Definition 1.24. A map $f : \tilde{M} \to M$ between complex manifolds is called a ***branched covering*** if there exists a divisor B on M such that f is a covering map on $M - B$, and for every $p \in B$, there is a local coordinate $(z_1, z_2, ..., z_m)$ around p, such that in this local coordinate B is given by the equation $z_1 = 0$, and f is of the from

$$f : (z_1, z_2, ..., z_m) \to (z_1^k, z_2, ..., z_m).$$

Example 1.25 (branched covering supermanifold)**.** Suppose we are given the following data:

- A supermanifold $S = (M, \mathcal{O}_S)$;
- A branched covering $f : \tilde{M} \to M$ of the reduced space, with a smooth branch divisor $B \subset M$;
- A divisor $D \subset S$ whose intersection with M is B. Note that here we view M as a subset of B via the natural inclusion $i : M \to S$ by setting all the odd coordinates to zero. Hence this means that D becomes B when we set all $\theta^\nu = 0$ in D.

We will now construct a supermanifold $\tilde{S}$, with a morphism $F : \tilde{S} \to S$ which is a branched covering with branch divisor D, and such that on the reduced space F is simply $f : \tilde{M} \to M$.

We will do the construction for the case where we can find global coordinates on S, as we can also glue local pieces together. Now let $(z_1, ..., z_m, \theta_1, ..., \theta_n)$ be global coordinates on S, where $(z_1, ..., z_m)$ are global coordinates on M. Without loss of generality we can assume the branch divisor B is given by $z_1 = 0$. Then the corresponding coordinate on $\tilde{M}$ are $(w_1, z_2, ..., z_m)$. Then locally f is of the form $(w_1, z_2, ..., z_m) \mapsto (w_1^k, z_2, ..., z_m)$. Hence we see that $f^* : \mathcal{O}_M \to \mathcal{O}_{\tilde{M}}$ takes z_1 to $(w_1)^k$.

Since the divisor D is given by vanishing of some even function $z' = z'(z_\mu, \theta_\nu)$ whose image is z_1 when we set $\theta_\nu = 0$ for all $\nu = 1, ..., n$, this implies that the coordinate ring $\mathcal{O}_S(S)$ which is generated by $z_1, ..., z_m, \theta_1, ..., \theta_n$ is also generated by $z', z_2, ..., z_m, \theta_1, ..., \theta_n$.

Now we construct $\mathcal{O}_{\tilde{S}}$. Let $w' = (z')^{1/k}$ be a k-th root of z'. Now we define

$$\mathcal{O}_{\tilde{S}} : U \mapsto f^{-1}\mathcal{O}_S(U)[w'].$$

Note that this makes $\mathcal{O}_{\tilde{S}}$ into a sheaf of $\mathcal{O}_S$-algebra. To check that this definition makes sense, we first look at the coordinate ring. This is given by

$$\begin{aligned}\mathcal{O}_{\tilde{S}}(\tilde{S}) = f^{-1}\mathcal{O}_S(\tilde{S})[w'] = \mathcal{O}_S(S)[w'] &= K[w', z', z_2, ..., z_m, \theta_1, ..., \theta_n] \\ &= K[w', z_2, ..., z_m, \theta_1, ..., \theta_n]\end{aligned}$$

since $(w')^k = z'$. Thus the coordinate functions on $\tilde{S}$ are $w', z_2, ..., z_m, \theta_1, ..., \theta_n$. We then see that there is a natural identification $\tilde{S} \cong \tilde{M}$, by $w' \mapsto w_1, z_i \mapsto z_i, \theta_j \mapsto 0$. We now construct a map $F : \tilde{S} \to S$ of locally ringed space. By definition, this comes with a morphism of reduced space, which is required to be $f : \tilde{M} \to M$. Thus we need to construct $F^* : \mathcal{O}_S \to F_*\mathcal{O}_{\tilde{S}}$ accordingly. It suffices to specify F^* on generators of $\mathcal{O}_S$, namely on $z', z_2, ..., z_m, \theta_1, ..., \theta_n$. This is done by sending $z_i \mapsto z_i, \theta_j \mapsto \theta_j$. What about z'? Since we discussed that on the level of reduced spaces $f^*(z_1) = (w_1)^k$, and w_1 is now identified with w', z' is now replacing the role of z_1, it only makes sense to define $F^* : z' \mapsto (w')^k = z'$.

This completes the construction.

Remark 1.26. We have constructed a branched covering $\tilde{S} \to S$. There is a notion of branched covering of a super Riemann surface, which is not quite the same as the branched covering of a supermanifold in general. A branched covering of a super Riemann surface is not a super Riemann surface, but can be modified into a super Riemann surface by blowing up along the divisor.

Example 1.27 (Parity Reversed Tangent Bundle ΠTM)**.** Let M be a manifold of dimension p and $E \to M$ a vector bundle of rank q. Then there is an associated supermanifold ΠE of dimension $p|q$ with $|\Pi E| = M$, defined by revering the parity of the fiber coordinates. Namely, for every local chart $U \subset M$, we note that the space $C^\infty(U, E)$ of smooth sections of E over U is a free $C^\infty(U)$-module generated by q sections $\sigma_1, ..., \sigma_q$. We now identify these sections with grassmann variables $\theta^1, ..., \theta^q$, and define the structure sheaf

$$\mathcal{O}_{\Pi E}(U) = C^\infty(U)[\theta^1, ..., \theta^n].$$

In the particular case where $E = TM$, we have the ***parity reversed tangent bundle*** ΠTM. Let $C^\infty(\Pi TM)$ be the smooth global sections of the structure sheaf $\mathcal{O}_{\Pi TM}$, namely the superalgebra of smooth functions on ΠTM. Then there is an isomorphism

$$\rho : \mathcal{A}^\bullet(M) \xrightarrow{\cong} C^\infty(\Pi TM) \tag{1.1}$$

between smooth functions on ΠTM and differential forms on M, where we identify

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} a_{i_1 \ldots i_p}(x) \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \in \mathcal{A}^p(M)$$

with

$$\rho(\omega) = \sum_{1 \le i_1 < \cdots < i_p \le n} a_{i_1 \ldots i_p}(x) \theta^{i_1} \cdots \theta^{i_p} \in C^\infty(\Pi TM).$$

Using the identification (1.1), we can integrate smooth functions on ΠTM by defining

$$\int_{\Pi TM} \rho(\omega) \mathrm{d}x^p \cdots \mathrm{d}x^p \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^p := \int_M \omega$$

where $\mathrm{d}x^p \cdots \mathrm{d}x^p \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^p$ is the canonical volume form on ΠTM, which we will denote as $\mathrm{d}x\mathrm{d}\theta$ for short.

Chapter 2

Obstruction Theory on Supermanifolds

The goal of this chapter is to classify up to isomorphism supermanifolds S which have a given underlying manifold M and a given odd tangent bundle V. For this we will use sheaf cohomology. We assume some familiarity with this machinery. We can ask a very general question: given a model object S, can we classify objects which are locally isomorphic to it? There is a general framework for approaching this question (refer to Chapter A for details).

Theorem 2.1 (Deformation Theory Meta-Theorem)**.** *Let S be a geometric object on a manifold M, in our case a supermanifold. Let $\mathcal{A}$ be the sheaf of automorphisms of S. That is, to each open set $U \subset M$ we associate $\mathrm{Aut}(S|_U)$. Then the first sheaf cohomology group $H^1(M, \mathcal{A})$ parametrizes isomorphism classes of objects which are locally isomorphic to S.*

Remark 2.2 (recovering automorphism of spaces)**.** In fact, the collection of automorphisms of locally ringed spaces can be recovered from only the collection of automorphisms of structure sheaves in the case where the space is a manifold. To illustrate this, let $f : (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$ be an automorphism of manifold as locally ringed spaces. If we only have informations of $f^* : \mathcal{O}_M \to \mathcal{O}_M$, then let x^μ be a set of coordinates of $U \subset M$ and consider $f^* : \mathcal{O}_M(U) \to \mathcal{O}_M(U)$. Then $f^*(x^\mu) = x^\mu \circ f$ is precisely the μ-th component of $f|_U : U \to f(U)$. Since we know these data for every μ and for any arbitrary open set $U \subset M$, this completely and uniquely determines $f : M \to M$ as an automorphism of the underlying space.

2.1 Obstruction to Splitting

Applying Theorem 2.1 to the local model $S = S(M, V)$, we see that the sheaf of nonabelian group of interest is $\mathrm{Aut}(S(M, V))$. By Remark 2.2, we see that we only need to concern

$$\mathrm{Aut}(\mathcal{O}_{S(M,V)}) = \mathrm{Aut}(\textstyle\bigwedge^\bullet V^*),$$

where we view V as a sheaf of $\mathcal{O}_M$-module (recall that we identified $V = T_-S$). The sheaf of automorphisms of the $\mathbb{Z}_2$-graded sheaf of algebras $\bigwedge^\bullet V^*$. Note that these automorphisms send J to itself, which implies they send J^2 to itself as well since they are homomorphisms in particular. Hence by Remark 1.9, they send $V^* = J/J^2$ to itself. In other words, we have a map $\mathrm{Aut}(\bigwedge^\bullet V^*) \to \mathrm{Aut}(V^*)$ by restriction, or equivalently (by transpose inverse) we have a map $\mathrm{Aut}(\bigwedge^\bullet V^*) \to \mathrm{Aut}(V)$. Let G be the kernel of this map. So G is the group of automorphisms of the split model $S(M,V)$ that preserves both M and V (Note that this does not mean that these automorphisms are literally identity, because for example we can have an automorphism described in Remark 1.8: it preserves the reduced space because the reduced map is obtained when setting odd coordinates to zero, and it fixes the odd direction by our construction).

Thus we have a short exact sequence of nonabelian sheaves

$$1 \to G \to \operatorname{Aut}(\textstyle\bigwedge^\bullet V^*) \to \operatorname{Aut}(V) \to 1 \tag{*}$$

which will induce a long exact sequence in sheaf cohomology:

$$\begin{aligned} 1 &\to H^0(M,G) \to H^0(M,\operatorname{Aut}(\textstyle\bigwedge^\bullet V^*)) \to H^0(M,\operatorname{Aut}(V)) \\ &\to H^1(M,G) \to H^1(M,\operatorname{Aut}(\textstyle\bigwedge^\bullet V^*)) \to H^1(M,\operatorname{Aut}(V)) \to \cdots \end{aligned}$$

Note that since our sheaves are nonabelian sheaves, the sheaf cohomology need not be groups in general, but only pointed sets. By Theorem 2.1, the term in the long exact sequence that parameterizes isomorphism classes of supermanifolds with reduced space M and odd dimension n is

$$H^1(M,\operatorname{Aut}(\mathcal{O}_{S(M,V)})) = H^1(M,\operatorname{Aut}(\textstyle\bigwedge^\bullet V^*)).$$

The distinguished point in this cohomology set corresponds to the isomorphism class $S(M,V)$. But since any two vector bundles over M of the same rank are locally isomorphic, we could have used another vector bundle V' of the same rank instead of V. This would have yielded another description of the same set of isomorphism classes of supermanifolds S with given reduced space M and given odd dimension, and the only difference being that the base point would now be $S(M,V')$.

$H^1(M,\operatorname{Aut}(\bigwedge^\bullet V^*))$ is certainly an object of our interest. However, the object of most of our concern is

$$H^1(M,G),$$

which consists of isomorphism classes of pairs (S,ρ) where S is a supermanifold with reduced space M and $\rho : V \cong T_-S$ is an isomorphism. The advantage of studying $H^1(M,G)$ instead of $H^1(M,\operatorname{Aut}(\bigwedge^\bullet V^*))$ is that the former parameterizes isomorphism classes of supermanifold that has a particular prescribed odd tangent bundle V, whereas in the latter case the isomorphism classes can be supermanifolds whose odd tangent bundle is merely of rank n, as any two rank n vector bundles are locally isomorphic.

Remark 2.3. The inclusion $G \to \operatorname{Aut}(\bigwedge^\bullet V^*)$ induces a map

$$H^1(M,G) \to H^1(M,\operatorname{Aut}(\textstyle\bigwedge^\bullet V^*))$$

that is given by $(S,\rho) \mapsto S$, which forgets the isomorphism $V \cong T_-S$ and just consider S as a supermanifold with $|S| = M$ and odd tangent bundle of rank n.

Remark 2.4 (vanishing of $H^1(M,G)$ and implications)**.** Recall that a supermanifold S is said to be split if it is globally isomorphic to some model $S(M,V)$. Let us consider the exact sequence (*) with this particular M and V. Clearly the isomorphism class of $(S(M,V),\mathrm{id})$ is an element of $H^1(M,G)$. If $H^1(M,G)$ vanishes, that means there is only one isomorphism class whose base space is M and odd tangent bundle is V. This shows that any supermanifolds satisfying such conditions must split.

The group G has a descending filtration by normal subgroups G^i for $i \geq 2$:

$$G = G^2 \supset G^3 \supset G^4 \supset \cdots$$

We give three description of these subgroups: algebraic, geometric, and analytic. We then use these groups to describe obstruction theory for splitting of a supermanifold.

Algebraically, we recall the filtration J^i that appeared in Remark 1.11, and we define

$$G^i := \{g \in G : g(x) - x \in J^i, \forall x \in \bigwedge^\bullet V^*\}.$$

First, observe that $G^2 = G$, since elements in G fixes V, and G^i is trivial if i exceeds the odd dimension of S.

Geometrically, we interpret G^i as a filtration of S itself. Given a supermanifold $S = (M, \mathcal{O}_S)$ and a nilpotent subsheaf $J \subset \mathcal{O}_S$, we introduce the locally ringed space

$$S^i := (M, \mathcal{O}_{S^i} = \mathcal{O}_S / J^{i+1}).$$

Note that S^i need not be supermanifolds except for extremes $i = 0, n$. They form an increasing filtration of S given by

$$|S| = S^0 \subset S^1 \subset \cdots \subset S^{i-1} \subset S^i \subset \cdots \subset S^n = S.$$

Recall that $\mathrm{Aut}(\bigwedge^\bullet V^*)$ preserves J, hence they preserve J^i, which means they also preserve the filtration of S by the S^i. If we take $S = S(M, V)$ to be the split model, an equivalent definition of G^i is as those automorphisms of $S(M, V)$ that act as the identity on $S(M, V)^{i-1}$.

Now we come to the (interesting) analytic perspective. Maybe it is helpful to first review the story for ordinary manifolds. Let M be a smooth manifold and $G = \mathrm{Diff}(M)$ is the group of diffeomorphisms on M. Then its Lie algebra $\mathfrak{g}$ can be interpreted as vector fields on M. Somewhat informally, this is because there is a correspondence between infinitesimal diffeomorphisms

$$x^\mu \mapsto x^\mu + \epsilon h^\mu(x)$$

with the vector field

$$h^\mu \frac{\partial}{\partial x^\mu}.$$

The same story holds for the case of supermanifolds. Concretely, the Lie algebra $\mathfrak{g}$ of the group G of automorphisms on the split model $S(M, V)$ that preserves M and V is generated by vector fields on $S(M, V)$ that are roughly of the form $\theta^{2k}\partial_x$ or $\theta^{2k+1}\partial_\theta$. More specifically,

Proposition 2.5. In local coordinates x^μ, θ^ν, these vector fields takes the form

$$f^\mu_{\nu_1,\ldots,\nu_{2k}}(x)\theta^{\nu_1}\cdots\theta^{\nu_{2k}}\frac{\partial}{\partial x^\mu} \tag{2.1}$$

or

$$f^\nu_{\nu_1,\ldots,\nu_{2k+1}}(x)\theta^{\nu_1}\cdots\theta^{\nu_{2k+1}}\frac{\partial}{\partial \theta^\nu}. \tag{2.2}$$

Proof. Recall that G is the kernel of the restriction map $\mathrm{Aut}(\bigwedge^\bullet V^*) \to \mathrm{Aut}(V^*)$, where we should keep in mind that $\mathrm{Aut}(V^*)$ is obtained from restricting the automorphism to J and quotient by J^2, since $V^* \cong J/J^2$ in our identification.

Similar to what we discussed above, an infinitesimal automorphism of $S(M,V)$ parameterized by ϵ is given by

$$\begin{aligned} x^\mu &\mapsto x^\mu + \epsilon f^\mu_{\nu_1,\ldots,\nu_{2k}}(x)\theta^{\nu_1}\cdots\theta^{\nu_{2k}} \\ \theta^\nu &\mapsto \theta^\nu + \epsilon f^\nu_{\nu_1,\ldots,\nu_{2k+1}}(x)\theta^{\nu_1}\cdots\theta^{\nu_{2k+1}}. \end{aligned}$$

Note that these are indeed automorphisms on $S(M,V)$ fixing M and $V^* \cong J^2/J$, as the first equation preserves x^μ on the level of reduced spaces when we set $\theta^{\nu_i} = 0$, and the second line preserves θ^ν modulo J^2. It is not hard to see that these are all the automorphisms that preserves M and V. Hence we are done. □

In these terms, the Lie algebra $\mathfrak{g}^i$ of the subgroup G^i is generated by vector fields on $S(M,V)$ are that schematically of the form $\theta^j\partial_x$ or $\theta^j\partial_\theta$, depending on the parity of j, for $j \geq i$. Using this language, we get an concrete description of the consecutive quotients of these filtration subgroups:

Theorem 2.6. *There is an isomorphism of sheaves of abelian groups*

$$G^i/G^{i+1} \cong T_{(-1)^i} \otimes \bigwedge^i V^*.$$

Here $T_{(-1)^i}$ denotes the odd or even part of the tangent bundle of S, i.e., V in the odd case and TM in the even case.

Proof. The filtration by the degree of θ shows that $\mathfrak{g}$ is a nilpotent Lie algebra, hence the exponential map $\exp : \mathfrak{g} \to G$ is an isomorphism of manifolds, which also induces isomorphism $\exp : \mathfrak{g}^i \to G^i$. Then the statement of the theorem follows from the description of $\mathfrak{g}^i$ above. □

The obstruction theory for splitting of the supermanifold S is based on filtering $H^1(M,G)$ by the images of $H^1(M,G^i)$:

$$0 = H^1(M,G^n) \to H^1(M,G^{n-1}) \to \cdots \to H^1(M,G^2) = H^1(M,G).$$

The geometric interpretation of $H^1(M,G^i)$ is as the set of isomorphism classes of pairs $\varphi_{i-1} = (S,\rho_{i-1})$, where S is a supermanifold with $|S| = M$, and ρ_{i-1} is an isomorphism between $S(M,V)^{i-1}$ and S^{i-1}.

Proposition 2.7. In order for a class $\varphi = (S,\rho) \in H^1(M,G)$ to vanish, it is necessary and sufficient that, for each $i \geq 2$, this class is the image of some $\varphi_{i-1} \in H^1(M,G^i)$.

Proof. Suppose for each $i \geq 2$, the class $\varphi = (S,\rho) \in H^1(M,G)$ is the image of some $\varphi_{i-1} \in H^1(M,G^i)$. Then clearly φ vanishes because $H^1(M,G^n) = 0$. Conversely, if φ vanishes, we just take $\varphi_{i-1} \in H^1(M,G^i)$ to be trivial for every i. □

There is nothing to check for $i = 2$, since $G^2 = G, \rho_1 = \rho, \varphi_1 = \varphi$. If a given class (S,ρ) is in the image for some i, then to decide whether it is in the image for $i+1$, we note that by Theorem 2.6 we have a short exact sequence

$$0 \to G^{i+1} \to G^i \to T_{(-1)^i}M \otimes \bigwedge^i V^* \to 0$$

which induces a long exact sequence of sheaf cohomology, which in part reads

$$H^1(M,G^{i+1}) \to H^1(M,G^i) \xrightarrow{\omega} H^1(M,T_{(-1)^i}M \otimes \bigwedge^i V^*).$$

The obstruction for a class $\varphi_{i-1} \in H^1(M,G^i)$ to come from some $\varphi_i \in H^1(M,G^{i+1})$ is that its image

$$\omega_i = \omega(\varphi_{i-1}) \in H^1(M,T_{(-1)^i}M \otimes \bigwedge^i V^*) \tag{2.3}$$

must vanish. This class is called the i-th ***obstruction class for splitting*** of S.

2.2 Obstruction to Projection

So far, we discussed obstructions for splitting. We now briefly discuss obstruction to projections. The group G has a subgroup G_- generated by vector fields $\theta^{2k+1}\partial_\theta$. This is the subgroup of G that preserves a projection. Accordingly, a supermanifold S that are projected but not necessarily split are labeled by a class in $H^1(M, G_-)$.

In general, a supermanifold S modeled on M, V is projected iff the class $\omega(S) \in H^1(M, G)$ that represents it is in the image of $H^1(M, G_-)$. If the subgroup G_- were normal in G, then we would have an exact sequence

$$H^1(M, G_-) \to H^1(M, G) \to H^1(M, G/G_-)$$

and then S would be projected iff $\omega(S)$ maps to zero in $H^1(M, G/G_-)$. But G_- is not normal, and it does not appear to be possible to express the obstruction to a projection in such simple terms.

Instead, we can define G_-^i to be the subgroup of G generated by G^i and G_-. This gives a sequence of subgroups

$$G = G_-^2 \supset G_-^3 = G_-^4 \supset G_-^5 = G_-^6 \supset G_-^7 = \cdots$$

Similarly, in order for a class $\omega(S) \in H^1(M, G)$ to come from $H^1(M, G_-)$, it is necessary and sufficient that, for each (odd) $i \geq 2$, this class is in the image of $H^1(M, G_-^i)$. To convert this to the vanishing of a series of obstructions, we need G_-^i to be a normal subgroup of G_-^{i-2}. This turns out to be true. We sketch the proof below:

Lemma 2.8. *For each i, the group G_-^i is normal in its predecessor G_-^{i-2}.*

Proof. The condition for a subgroup to be normal is that its Lie algebra should be an ideal in the ambient Lie algebra. Recall that the Lie algebra $\mathfrak{g}$ of G is generated by vector fields on the split model $S(M, V)$ that are schematically of the form $\theta^{2k}\partial_x$ or $\theta^{2k+1}\partial_\theta$, for $k \geq 1$, as in (2.1) and (2.2). The Lie algebra $\mathfrak{g}_-$ of G_- is the subalgebra generated by the vector fields $\theta^{2k+1}\partial_\theta$. The Lie algebra $\mathfrak{g}_-^i$ of G_-^i for odd i is generated by all the above vector fields except $\theta^2\partial_x, ..., \theta^{i-1}\partial_x$. Similarly, $\mathfrak{g}_-^{i-2}$ is generated by all the above vector fields except $\theta^2\partial_x, ..., \theta^{i-3}\partial_x$. Then the rest follows from a simple Lie bracket computation, which we refer interested readers to [2, p.18]. □

Hence similar to the obstruction theory for splitting, given a class $\omega(S)$ in the image of $H^1(M, G_-^{2k})$ for some k, then to determine whether it is in the image of $H^1(M, G_-^{2k+1})$, we look at the exact sequence

$$H^1(M, G_-^{2k+1}) \to H^1(M, G_-^{2k}) \xrightarrow{\omega^-} H^1(M, T_+M \otimes \textstyle\bigwedge^{2k} V^*).$$

The obstruction for a class $\varphi_{2k-1}^- \in H^1(M, G_-^{2k})$ to come from $H^1(M, G_-^{2k+1})$ is that its image

$$\omega_{2k}^- = \omega^-(\varphi_{2k-1}^-) \in H^1(M, T_+M \otimes \textstyle\bigwedge^{2k} V^*)$$

must vanish. This class ω_{2k}^- is called the $2k$-th ***obstruction class for projectedness*** of S.

Note that in particular $\omega_2^- = \omega_2$. Hence we have the following

Corollary 2.9. *The vanishing of $\omega_2 \in H^1(M, T_+ \otimes \bigwedge^2 V^*)$ is a necessary condition for a supermanifold S modeled on M, V to be projected.*

In fact, this is the only necessary criterion for projectedness that [2] used in their paper, hence we will not use higher obstruction classes either in the following part of the book.

2.3 Applications of Obstruction Theory

In this section we give some immediate applications to the obstruction theory developed above.

Corollary 2.10. *Any C^∞ supermanifold S is split.*

Proof. A C^∞ locally-free sheaf is fine because of the existence of a partition of unity, hence its H^1 vanishes. □

Corollary 2.11. *Any supermanifold of dimension $m|1$ is split.*

Proof. Indeed, since $\bigwedge^i V^* = 0$ for $i \geq 2$. □

Corollary 2.12. *A supermanifold of dimension $m|2$ is determined by the triple (M, V, ω), where $M = |S|$, $V = T_-S$, and*

$$\omega = \omega_2 \in H^1(M, \mathrm{Hom}(\textstyle\bigwedge^2 T_-S, T_+S)).$$

Moreover, any such triple arises from some S. A supermanifold of dimension $m|2$ is projected iff it is split.

Note that in the above statement we used the identification

$$T_+S \otimes \textstyle\bigwedge^2 V^* = \mathrm{Hom}((\bigwedge^2 V^*)^*, T_+S) = \mathrm{Hom}(\bigwedge^2 V, T_+S) = \mathrm{Hom}(\bigwedge^2 T_-S, T_+S),$$

where the isomorphism $(\bigwedge^2 V^*)^* \cong \bigwedge^2 V^*$ is obtained by choosing a basis e_μ of V and its dual basis ϵ^μ of V^* and identifying $\epsilon^\mu \wedge \epsilon^\nu$ with $e_\mu \wedge e_\nu$ and extend linearly.

A result that we will use later concerns the projectedness of the covering supermanifold:

Corollary 2.13. *Let $\pi : \tilde{S} \to S$ be a finite covering map of supermanifolds. If $\omega_2(S) \neq 0$ then $\omega_2(\tilde{S}) \neq 0$, so $\tilde{S}$ is not projected.*

Proof. For a local isomorphism $\pi : \tilde{S} \to S$, the structure sheaf $\mathcal{O}_{\tilde{S}}$ is the pullback of $\mathcal{O}_S$. It follows that $\omega_2(\tilde{S}) = \pi^*\omega_2(S)$. But the composition π^* with the trace map is multiplication by degree of π, hence π^* is injective and the result follows. For a more elementary proof, see [2, p.22-23]. □

We conclude this section by a discussion of submanifolds. Suppose S is a supermanifold modeled on M, V. Let M' be a submanifold of M and V' be a subbundle of $V|_{M'}$. We ask the following question: whether there exists a submanifold S' of S with $|S'| = M$ and $T_-S' = V'$.

The answer in general is no. To analyze the obstruction, we define the subgroup sheaf G_* of G consisting of automorphisms of $S(M, V)$ that restricts to automorphisms of $S(M', V')$. The condition that S contains S' is that the class in $H^1(M, G)$ that represents S should be the image of a class in $H^1(M, G_*)$. As in the discussion of projections, there would be an easy criterion for this if G_* were normal, but it is not.

However, we can partially reduce to the normal case if we replace S with S^2 and ask whether S^2 contains a subspace S'^2 modeled on $S(M', V')^2$. For this, we may replace G by the sheaf of abelian groups

$$G/G^3 = \mathrm{Hom}(\textstyle\bigwedge^2 T_-M, T_+M)$$

and G_* by G_*/G_*^3, which is the subsheaf of $\operatorname{Hom}(\bigwedge^2 T_- M, T_+ M)$ consistsing of homomorphisms that, when restricted to M', map $\bigwedge^2 T_- M'$ to $T_+ M'$. We call this sheaf $\operatorname{Hom}^*(\bigwedge^2 T_- M, T_+ M)$. The quotient of these two is given by the sheaf

$$\operatorname{Hom}(\textstyle\bigwedge^2 T_- M, T_+ M)/\operatorname{Hom}^*(\bigwedge^2 T_- M, T_+ M) = \operatorname{Hom}(\bigwedge^2 T_- M', N)$$

supported on M', where N is the normal bundle to M' in M. So we have an exact sequence in cohomology

$$\begin{aligned} H^1(M, \operatorname{Hom}^*(\textstyle\bigwedge^2 T_- M, T_+ M)) &\to H^1(M, \operatorname{Hom}(\textstyle\bigwedge^2 T_- M, T_+ M)) \\ &\xrightarrow{b} H^1(M', \operatorname{Hom}(\textstyle\bigwedge^2 T_- M', N)), \end{aligned}$$

which gives

Corollary 2.14. *A necessary condition for a supermanifold S modeled on M, V to contain a submanifold S' modeled on M', V' is that*

$$b(\omega_2) \in H^1(M', \operatorname{Hom}(\textstyle\bigwedge^2 T_- M', N))$$

vanishes.

Since G_* is defined to be automorphisms of $S(M, V)$ that restrict to automorphisms of $S(M', V')$, there is a tautological map from G_* to the sheaf G' of automorphisms of $S(M', V')$ that acts trivially on M' and its normal bundle. Hence G_* maps to both G and G',

$$\begin{tikzcd} G_* \arrow[r, "u"] \arrow[d, "r"'] & G \\ G' & \end{tikzcd}$$

which induces maps in cohomology

$$\begin{tikzcd} H^1(M, G_*) \arrow[r, "u"] \arrow[d, "r"'] & H^1(M, G) \\ H^1(M, G') & \end{tikzcd}$$

If we abelianize by replacing G_*, G, G' by their quotients by G_*^3, G^3, G'^3, then we can complete the diagram above to a commutating square:

$$\begin{tikzcd} \operatorname{Hom}^*(\bigwedge^2 T_- M, T_+ M) \arrow[r, "u"] \arrow[d, "r"'] & \operatorname{Hom}(\bigwedge^2 T_- M, T_+ M) \arrow[d, "\iota"] \\ \operatorname{Hom}(\bigwedge^2 T_- M', T_+ M') \arrow[r, "j"'] & \operatorname{Hom}(\bigwedge^2 T_- M', T_+ M|_{M'}) \end{tikzcd}$$

where u, r are the linearizations of the corresponding maps above. Note that the top row are sheaves on M and bottom row are sheaves on M'. The verticle maps are therefore restrictions from M to M', followed by restrictions from $\bigwedge^2 T_- M$ to $\bigwedge^2 T_- M'$. Finally, j comes from the inclusion $T_+ M' \to T_+ M|_{M'}$. Hence this induces a corresponding commutative square in cohomology:

$$\begin{tikzcd} H^1(M, \operatorname{Hom}^*(\bigwedge^2 T_- M, T_+ M)) \arrow[r, "u"] \arrow[d, "r"'] & H^1(M, \operatorname{Hom}(\bigwedge^2 T_- M, T_+ M)) \arrow[d, "\iota"] \\ H^1(M', \operatorname{Hom}(\bigwedge^2 T_- M', T_+ M')) \arrow[r, "j"'] & H^1(M', \operatorname{Hom}(\bigwedge^2 T_- M', T_+ M|_{M'})) \end{tikzcd}$$

If S' exists, the class $\omega_2(S) \in H^1(M, G)$ should be the image of some class $\omega_2^*(S) \in H^1(M, G_*)$ under u. The map $r : G_* \to G$ just restricts an element of G_* to its action on $S(M', V')$, so the image $\omega_2^*(S)$ in $H^1(M', G')$ is just the class $\omega_2(S')$ that represents S'. Hence the commutativity of the square implies the following compatibility condition:

Corollary 2.15 (Compatibility Lemma)**.** *If S' is a submanifold of a supermanifold S, then $\omega_2(S')$ and $\omega_2(S)$ are compatible in the sense that $j(\omega_2(S')) = \iota(\omega_2(S))$.*

Particularly important is the case that the normal sequence decomposes:

Corollary 2.16. *Let S be a supermanifold and $S' \subset S$ a submanifold, with reduced space $M' \subset M$, such that the normal sequence of M' decomposes: $TM|_{M'} = T_{M'} \oplus N$, where N is the (even) normal bundle. If $\omega_2(S') \neq 0$, then also $\omega_2(S) \neq 0$ and S is not projected.*

Proof. This follows from Corollary 2.15 and the fact that, when the normal sequence decomposes, the map j is injective. □

Chapter 3

Super Atiyah Classes

In this chapter we give a more geometric perspective of obstruction theory on supermanifolds, via super Atiyah classes. We will first give a sheaf theoretic treatment of obstruction classes. Then we will introduce obstruction theory using differential operators on supermanifolds. After that, we introduce the ordinary Atiyah classes, which obstructs existence of holomorphic connections on a vector bundle. Finally, we introduce the super Atiyah class.

3.1 Obstruction Theory Revisited

Throughout this section, let $S = (M, \mathcal{O}_S)$ be a supermanifold with reduced space M of dimension m and modeled on the vector bundle V over M of rank n. So $\dim S = (m|n)$. Let J denote the nilpotent ideal of $\mathcal{O}_S$. So $V = J/J^2$, and $\mathcal{O}_M = \mathcal{O}_S/J$.

We define, for every $i \geq 0$, the ringed space

$$S^{(i)} = (M, \mathcal{O}_S/J^{i+1}).$$

These need not be supermanifolds, except for the extreme cases $S^{(0)} = (M, \mathcal{O}_M) = M$ and $S^{(n)} = (M, \mathcal{O}_S) = S$. Nevertheless, the form an increasing filtration

$$M = S^{(0)} \subset S^{(1)} \subset \cdots \subset S^{(i)} \subset S^{(i+1)} \subset \cdots \subset S^{(n)} = S.$$

This is the geometric filtration, defined by Manin [25]. A supermanifold S splits iff all of its filtrations $S^{(i)}$ split, and similarly S is projected iff all of its filtrations $S^{(i)}$ are projected.

3.1.1 Obstructions to Projection

A projection is a morphism $\phi : S \to M$ of supermanifolds which is the identity on M. Such a morphism must induce a morphism of ringed space $\phi^{(i)} : S^{(i)} \to M$ that is identity on M for each i.

When $i = 0$, this is just a map between ringed spaces $M \to M$, which we obviously take to be the identity. This can always be done. Hence the obstruction to splitting boils down to, inductively, given $\phi_i : S^{(i)} \to M$, can we extend it to the next level of filtration $\phi_{i+1} : S^{(i+1)} \to M$.

Theorem 3.1. *The obstruction of extending ϕ_i one filtration level up to ϕ_{i+1} is characterized by a cohomology class*

$$\omega_i^- \in H^1(M, T_+ \otimes \textstyle\bigwedge^i V^*).$$

In other words, such an extension is possible provided this cohomology class vanishes.

Proof. On the level of topological space, of course we can (and we need to) set ϕ_{i+1} to be the identity on M. Hence the problem lies in extending the morphism on structure sheaves.

We choose a good cover $M = \bigcup_\alpha U_\alpha$, such that on each open set the supermanifolds S and $S(M,V)$ are isomorphic.

Now suppose $\phi_i : S^{(i)} \to M$ is given. Note that we have an exact sequence

$$0 \to J^{i+1}/J^{i+2} \to \mathcal{O}_{S^{(i+1)}} \to \mathcal{O}_{S^{(i)}} \to 0.$$

By an extension of ϕ_i on U_α we mean a morphism (identity on the base) ϕ_{i+1} such that the diagram

$$\begin{tikzcd} \Gamma(U_\alpha, \mathcal{O}_M) \arrow[r, "\phi^*_{i+1,\alpha}"] \arrow[d, equal] & \Gamma(U_\alpha, \mathcal{O}_S/J^{i+2}) \arrow[d] \\ \Gamma(U_\alpha, \mathcal{O}_M) \arrow[r, "\phi^*_{i,\alpha}"'] & \Gamma(U_\alpha, \mathcal{O}_S/J^{i+1}) \end{tikzcd}$$

commutes, where the vertical map on the right is induced by the quotient of the exact sequence above. That is, $\phi^*_{i+1,\alpha}$ agrees with $\phi^*_{i,\alpha}$ mod J^{i+1}. This is achievable on U_α, because by our choice of U_α there are local isomorphisms

$$\Gamma(U_\alpha, \mathcal{O}_S) \cong \Gamma(U_\alpha, \mathcal{O}_{S(M,V)}) = \Gamma(U_\alpha, \mathcal{O}_M \otimes \textstyle\bigwedge^\bullet V^*),$$

which gives a natural inclusion map

$$\iota_\alpha : \Gamma(U_\alpha, \mathcal{O}_M) \to \Gamma(U_\alpha, \mathcal{O}_S).$$

We claim that we can construct the required extension $\phi^*_{i+1,\alpha}$ by defining it to be the composition of the ι_α and the quotient by J^{i+2}:

$$\phi^*_{i+1,\alpha} : \Gamma(U_\alpha, \mathcal{O}_M) \to \Gamma(U_\alpha, \mathcal{O}_S) \to \Gamma(U_\alpha, \mathcal{O}_S/J^{i+2}).$$

One can check that the construction indeed satisfies the commutative diagram above and moreover they are extensions of $\phi^*_{0,\alpha} = \mathrm{id}_{\mathcal{O}_M(U_\alpha)}$.

We further note that by our choice of U_α we have the isomorphism

$$(\mathcal{O}_S/J^{i+2})/(\mathcal{O}_S/J^{i+1}) \cong J^{i+1}/J^{i+2} \cong \textstyle\bigwedge^{i+1} V^*$$

on U_α. Now define, on the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the quantity

$$\omega^-_{\alpha\beta} \equiv \phi^*_{i+1,\alpha}|_{U_{\alpha\beta}} - \phi^*_{i+1,\beta}|_{U_{\alpha\beta}} : \Gamma(U_{\alpha\beta}, \mathcal{O}_M) \to \Gamma(U_{\alpha\beta}, \mathcal{O}_S/J^{i+2}).$$

But note that since ϕ^*_{i+1}'s are extensions of ϕ^*_i's, the difference $\omega^-_{\alpha\beta}$ lives entirely in $J^{i+1}/J^{i+2} \cong \bigwedge^{i+1} V^*$. Hence $\omega^-_{\alpha\beta}$ is actually $(\bigwedge^{i+1} V^*)$-valued.

Now we show that $\omega^-_{\alpha\beta}$ is an even derivation of $\mathcal{O}_M(U_{\alpha\beta})$. To see this, we take two sections $f, g \in \Gamma(U_{\alpha\beta}, \mathcal{O}_M)$. Then

$$\begin{aligned} \omega^-_{\alpha\beta}(fg) &= \phi^*_\alpha(fg) - \phi^*_\beta(fg) \\ &= \phi^*_\alpha(f)\phi^*_\alpha(g) - \phi^*_\beta(f)\phi^*_\beta(g) \\ &= \phi^*_\alpha(f)\phi^*_\alpha(g) - \phi^*_\alpha(f)\phi^*_\beta(g) + \phi^*_\alpha(f)\phi^*_\beta(g) - \phi^*_\beta(f)\phi^*_\beta(g) \\ &= \phi^*_\alpha(f)\omega^-_{\alpha\beta}(g) + \omega^-_{\alpha\beta}(f)\phi^*_\beta(g). \end{aligned}$$

Now note that $\omega^-_{\alpha\beta}(g)$ is already valued in $\bigwedge^{i+1} V^* \cong J^{i+1}/J^{i+2}$. Moreover, since $\phi^*_\alpha = \phi^*_{i+1,\alpha}$ is an extension of $\phi^*_{0,\alpha}$ which is the identity, they must agree mod J. Hence $\phi^*_\alpha(f) = f + J$, which means the last expression above becomes

$$\omega^-_{\alpha\beta}(fg) = f\omega^-_{\alpha\beta}(g) + \omega^-_{\alpha\beta}(f)g.$$

This shows that $\omega^-_{\alpha\beta}$ is an even derivation of $\mathcal{O}_M(U_{\alpha\beta})$, hence valued in $T_+ \otimes \bigwedge^{i+1} V^*$. Moreover, one can check that $(\omega^-_{\alpha\beta})$ defines a Čech cocycle, hence gives rise to a cohomology class

$$\omega^- \in H^1(M, T_+ \otimes \bigwedge^{i+1} V^*),$$

which is the right place for obstruction class according to the statement of the theorem.

Finally, we show that why vanishing of ω^- implies the ability to extend ϕ_i to ϕ_{i+1}. Suppose ω^- vanishes, then it is a coboundary, so there must exists $\sigma \in H^0(M, T_+ \otimes \bigwedge^{i+1} V^*) \cong \Gamma(M, T_+ \otimes \bigwedge^{i+1} V^*)$ such that on the set $U_{\alpha\beta}$ we have

$$(\delta^0 \sigma)_{\alpha\beta} = \sigma_\alpha - \sigma_\beta = \phi_\alpha - \phi_\beta = \omega_{\alpha\beta}.$$

So $\phi^*_{i+1,\alpha} - \sigma_\alpha = \phi^*_{i+1,\beta} - \sigma_\beta$. Then define $\psi_\alpha = \phi^*_{i+1,\alpha} - \sigma_\alpha$ on U_α. One can check that ψ_α extends $\phi^*_{i,\alpha}$ on U_α because σ_α is valued in $\bigwedge^{i+1} V^*$, which means it vanishes mod J^{i+1} in the commutative diagram above, and $\phi^*_{i+1,\alpha}$ is an extension of $\phi^*_{i,\alpha}$. This concludes the proof. □

This beautiful geometric proof was essentially given by Manin in [25]. Instead of using filtration by $S^{(i)}$, Donagi and Witten [2] used filtration of automorphism groups of splitting for the split model $S(M, V)$ and gave a more algebraic proof. Hence it is not surprising that their results are slightly stronger than what we obtain above. In particular, they showed that the odd obstructions to projection ω^-_{2k+1} always vanish, and only the even obstructions ω^-_{2k} survive. Moreover, they showed that the second obstruction ω^-_2, is actually the same class as the second obstruction to splitting ω_2, which we will study next.

3.1.2 Obstructions to Splitting

A splitting of supermanifold S is a global isomorphism $\phi : S \to S(M, V)$ that is identity on the reduced space M, together with an isomorphism of structure sheaves. Therefore, it must induce isomorphisms on every level of filtration

$$\phi_i : S^{(i)} \xrightarrow{\cong} S(M, V)^{(i)}.$$

When $i = 0$, this is just a map between ringed spaces $M \to M$, which we obviously take to be the identity. This can always be done. Hence the obstruction to splitting boils down to, inductively, given $\phi_i : S^{(i)} \to S(M, V)^{(i)}$ an isomorphism, can we extend it to an isomorphism $\phi_{i+1} : S^{(i+1)} \to S(M, V)^{(i+1)}$.

Theorem 3.2. *The obstruction of extending ϕ_i one filtration level up to ϕ_{i+1} is characterized by a cohomology class*

$$\omega_i \in H^1(M, T_{(-1)^i} \otimes \bigwedge^i V^*).$$

In other words, such an extension is possible provided this cohomology class vanishes.

Proof. On the level of topological space, of course we can (and we need to) set ϕ_{i+1} to be the identity on M. Hence the problem lies in extending the morphism on structure sheaves.

We choose a good cover $M = \bigcup_\alpha U_\alpha$, such that on each U_α the supermanifolds S and $S(M,V)$ are isomorphic. On each U_α, we are given the pullback isomorphisms

$$\phi_{i,\alpha}^* : \mathcal{O}_{S(M,V)^{(i)}}(U_\alpha) \to \phi_{i*}\mathcal{O}_{S^{(i)}}(U_\alpha) = \mathcal{O}_{S^{(i)}}(U_\alpha).$$

By definition of structure sheaves of both supermanifolds, this is an isomorphism

$$\phi_{i,\alpha}^* : \Gamma(U_\alpha, \textstyle\bigwedge^{\leq i} V^*) \to \Gamma(U_\alpha, \mathcal{O}_S/J^{i+1}),$$

where by convention $\bigwedge^0 V^* = \mathcal{O}_M$. Now the question is whether we can extend $\phi_{i,\alpha}^*$. That is, whether we can find $\phi_{i+1,\alpha}^*$ such that the diagram

$$\begin{tikzcd}
\Gamma(U_\alpha, \Lambda^{\leq i+1} V^*) \arrow[r, "\phi_{i+1,\alpha}^*"] \arrow[d] & \Gamma(U_\alpha, \mathcal{O}_S/J^{i+2}) \arrow[d] \\
\Gamma(U_\alpha, \Lambda^{\leq i} V^*) \arrow[r, "\phi_{i,\alpha}^*"'] & \Gamma(U_\alpha, \mathcal{O}_S/J^{i+1})
\end{tikzcd}$$

commutes, where the vertical maps are induced by quotient. This is of course achievable provided U_α is sufficiently small, since we have local the isomorphism

$$\textstyle\bigwedge^{\leq i+1} V^* / \bigwedge^{\leq i} V^* \cong J^{i+1}/J^{i+2}$$

on U_α. Hence we conclude that the extension problem is locally solvable, and the obstruction lies in our ability to patch these local solutions glocally. Namely, for two such open covers U_α, U_β, define on the intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$

$$\omega_{\alpha\beta} \equiv \phi_{i+1,\alpha}^*|_{U_{\alpha\beta}} - \phi_{i+1,\beta}^*|_{U_{\alpha\beta}} : \Gamma(U_{\alpha\beta}, \mathcal{O}_{S(M,V)^{i+1}}) \to \Gamma(U_{\alpha\beta}, \mathcal{O}_S/J^{i+2}).$$

Note that on $U_{\alpha\beta}$, we have the isomorphism

$$\begin{aligned}
\mathcal{O}_S/J^{i+2} &\cong \mathcal{O}_M \oplus J/J^2 \oplus J^2/J^3 \oplus \cdots \oplus J^i/J^{i+1} \oplus J^{i+1}/J^{i+2} \\
&\cong \mathcal{O}_M \oplus \textstyle\bigwedge^1 V^* \oplus \bigwedge^2 V^* \oplus \cdots \oplus \bigwedge^i V^* \oplus \bigwedge^{i+1} V^* \\
&\cong \textstyle\bigwedge^{\leq i+1} V^*.
\end{aligned}$$

So $\omega_{\alpha\beta}$ is valued in the truncated exterior power $\bigwedge^{\leq i+1} V^*$. But in fact it is more constrained: Because both $\phi_{i+1,\alpha}^*$ and $\phi_{i+1,\beta}^*$ are extensions of the same morphism $\phi_{i,\alpha}^* = \phi_{i,\beta}^*$ when restricted on the intersection, they agree on elements of $\Lambda^{\leq i} V^*$. Therefore, $\omega_{\alpha\beta}$ is actually is valued in the sheaf

$$\textstyle\bigwedge^{\leq i+1} V^* / \bigwedge^{\leq i} V^* \cong \bigwedge^{i+1} V^*.$$

We claim that that $\omega_{\alpha\beta}$ is an even derivation of $\mathcal{O}_{S(M,V)^{i+1}}(U_{\alpha\beta})$, and the proof is exactly the same calculation as we did in Theorem 3.1. There is one subtle point though: we follow the same calculation until we reach the last step, which reads

$$\omega_{\alpha\beta}(fg) = \phi_\alpha^*(f)\omega_{\alpha\beta}(g) + \omega_{\alpha\beta}(f)\phi_\beta^*(g),$$

where this time $f, g \in \Gamma(U_\alpha, \mathcal{O}_{S(M,V)^{(i+1)}}) = \Gamma(U_\alpha, \bigwedge^{\leq i+1} V^*)$. We use the same argument that $\phi_\alpha^* = \phi_{i+1,\alpha}^*$ is an extension of $\phi_{0,\alpha}^* = \mathrm{id}_{\mathcal{O}_M}$, so the following diagram must commute:

$$\begin{tikzcd}
\Gamma(U_\alpha, \Lambda^{\leq i+1} V^*) \arrow[r, "\phi_\alpha^*"] \arrow[d, "\text{mod } \Lambda^{>0} V^*"'] & \Gamma(U_\alpha, \mathcal{O}_S/J^{i+2}) \arrow[d, "\text{mod } J"] \\
\Gamma(U_\alpha, \mathcal{O}_M) \arrow[r, "\phi_0^* = \mathrm{id}"'] & \Gamma(U_\alpha, \mathcal{O}_S/J)
\end{tikzcd}$$

and similarly for β. Therefore, if we decompose $f = f_0 + f_n$ where f_0 is the part in $\mathcal{O}_M$ and f_n is the nilpotent part, then the commutativity of the diagram actually says

$$\phi_\alpha^*(f) \equiv \phi_0^*(f_0) = f_0 \mod J.$$

Hence the expression of Libniz rule before actually becomes

$$\omega_{\alpha\beta}(fg) = f_0 \omega_{\alpha\beta}(g) + \omega_{\alpha\beta}(f) g_0.$$

But this is in fact the desired Libniz rule in this case. Because to say that $\phi_{i+1,\alpha}^*$ is a derivation from

$$\Gamma(U_\alpha, \mathcal{O}_{S(M,V)^{(i+1)}}) = \Gamma(U_\alpha, \bigwedge\nolimits^{\leq i+1} V^*)$$

to $\Gamma(U_\alpha, \mathcal{O}_S/J^{i+2})$, we need to specify the action from the domain (a supercommutative ring) to the codomain (a module over that ring). Here, after we identify $\Gamma(U_\alpha, \mathcal{O}_S/J^{i+2}) \cong \Gamma(U_\alpha, \bigwedge^{\leq i+1} V^*)$, the action simply becomes multiplication modulo $\bigwedge^{>i+1} V^*$, which means the action of f on $\Gamma(U_\alpha, \mathcal{O}_S/J^{i+2})$ is literally multiplication by f_0. The same is true when we restrict to $U_{\alpha\beta}$, and for the right action. We refer the readers to p.156 and p.158-159 of [25] for detailed definitions of superalgebra. Therefore, we may as well write

$$\omega_{\alpha\beta}(fg) = f \cdot \omega_{\alpha\beta}(g) + \omega_{\alpha\beta}(f) \cdot g.$$

This shows that $\omega_{\alpha\beta}$ is an even derivation of $\mathcal{O}_{S(M,V)^{(i+1)}}(U_{\alpha\beta})$, as previously claimed.

Since $\omega_{\alpha\beta}$ is even, by parity it must follows that

$$\omega_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, T_{(-1)^{i+1}} \otimes \bigwedge\nolimits^{i+1} V^*).$$

Finally, one can check that the family $(\omega_{\alpha\beta})$ forms a Čech cocycle, and descends to a well-defined cohomology class

$$\omega_{i+1} \in H^1(M, T_{(-1)^{i+1}} \otimes \bigwedge\nolimits^{i+1} V^*),$$

as desired. The final statement can be proved precisely in the same way we proved the analogous statement in Theorem 3.1. □

Remark 3.3. One should note that the definition of ω_i depends on the choice of ϕ_{i-1}, as is evident from our construction. Indeed, in section 2.2.2 of [2], the authors showed that even for the split model $S(M,V)$, nonstandard choice of ϕ_2 gives $\omega_3 \neq 0$. So non-vanishing of ω_i for a particular choice of ϕ_{i-1} is not sufficient to show that a supermanifold does not split. Same problem occurs for the obstruction class ω_i^- to projection.

However, none of these affects the obstruction class ω_2, which is an invariant of any supermanifold S and obstructs splitting or projection (recall that $\omega_2 = \omega_2^-$ as we previously mentioned). Hence nonvanishing of ω_2 is sufficient to deduce the failure of a supermanifold to be split or projected. In fact, this is the only obstruction class that we will use in the remaining part of this writing.

3.2 Obstruction Theory via Differential Operators

Let $S = (M, \mathcal{O}_S)$ be a supermanifold. Recall that the second obstruction class ω_2 (for simplicity we denote by ω), is given by

$$\omega \in H^1(M, T_+ \otimes \textstyle\bigwedge^2 T_-^*) = \mathrm{Ext}^1(\textstyle\bigwedge^2 T_-, T_+)$$

using the identification $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) \cong H^1(M, \mathrm{Hom}(\mathcal{F}, \mathcal{G}))$ and the isomorphism $V^* \otimes W \cong \mathrm{Hom}(V, W)$.

Definition 3.4. An ***extension*** of $\mathcal{F}$ by $\mathcal{G}$ (where $\mathcal{F}, \mathcal{G}$ are sheaves or vector bundles on M) is an exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

of sheaves on M. Two extensions $\mathcal{E}$ and $\mathcal{E}'$ of $\mathcal{F}$ by G are said to be equivalent if there exists an isomorphism $\mathcal{E} \to \mathcal{E}'$ making the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Theorem 3.5. *The equivalent classes of extension of $\mathcal{F}$ by $\mathcal{G}$ are in bijection with elements in* $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$.

Proof. This is a standard result in homological algebra. See Theorem 7.30 of [26], for example. ☐

Therefore, the class ω can be interpreted as the extension class $[D_\omega]$ of vector bundles

$$0 \to T_+ \to D_\omega \to \textstyle\bigwedge^2 T_- \to 0 \tag{3.1}$$

on M. We claim that D_ω can be realized as a certain sheaf of differential operators along S. We first introduce several sheafs of differential operators on S or M:

- $\mathcal{D}_S$: sheaf of differential operators on S.
- $\mathcal{D}_S|_M$: the restriction of $\mathcal{D}_S$ to M. This is achieved by setting all the Grassmann coordinates to zero. Algebraically speaking, this is achieved by tensoring the $\mathcal{O}_S$-module $\mathcal{D}_S$ with $\mathcal{O}_M = \mathcal{O}_S/J$ over M.
- $\mathcal{D}_M$: sheaf of differential operators on M. Note that we can identify $\mathcal{D}_M$ as a subsheaf of $\mathcal{D}_S|_M$ because if A is a differential operator on M, we can define its action on a function f on S by $A(f|_M)$.

  Note that when $S = S(M, V) = (M, \mathcal{O}_M \otimes \bigwedge^\bullet V^*)$ is split, we can view $\bigwedge^\bullet V$ as a subsheaf of $\mathcal{D}_S|_M$, since we can identify

  $$\frac{\partial}{\partial \theta^\mu} \wedge \frac{\partial}{\partial \theta^\nu} \equiv \frac{\partial}{\partial \theta^\mu}\frac{\partial}{\partial \theta^\nu}.$$

  Note that both sides are antisymmetric. Using the fact that for a split supermanifold S the most general differential operator in $\mathcal{D}_S|_M$ takes the form

  $$A = \sum_{I=(i_1,\dots,i_l), J=(j_1,\dots,j_s)} a^{IJ}(x^\mu, \theta^\nu = 0)\frac{\partial}{\partial \theta^{i_1}} \cdots \frac{\partial}{\partial \theta^{i_l}}\frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_s}},$$

  we can identify

  $$\mathcal{D}_{S(M,V)}|_M = \textstyle\bigwedge^\bullet V \otimes_{\mathcal{O}_M} \mathcal{D}_M. \tag{3.2}$$

- $\mathcal{D}_S^i$: subsheaf of $\mathcal{D}_S$ consisting of differential operator of order (maximum number of differentiation involved, with respect to both even and odd variables) $\leq i$.

- $D^i = D_S^i$: the restriction of $\mathcal{D}_S^i$ to M. This is a subsheaf of $\mathcal{D}_S|_M$.

- $D_-^i = D_{S,-}^i$: the subsheaf of D^i whose symbol, i.e., the i-th order term, is purely odd. That is, it is the sum of terms of the form

$$A = f(x^\mu)\frac{\partial}{\partial\theta^{k_1}}\cdots\frac{\partial}{\partial\theta^{k_i}} + \text{lower order terms.}$$

The precise claim is that

Theorem 3.6. *The vector bundle D_ω in (3.1) corresponding to the obstruction class $\omega \in H^1(M, T_+ \otimes \bigwedge^2 T_-^*) = \operatorname{Ext}^1(\bigwedge^2 T_-, T_+)$ can be identified as the sheaf*

$$\bar{D} := D_-^2/D_-^1$$

on M. In local coordinates, $D_-^2 = \mathcal{O}_M\langle 1, \partial_x, \partial_\theta, \partial_\theta^2\rangle$ and $D_-^1 = \mathcal{O}_M\langle 1, \partial_\theta\rangle$. Hence $\bar{D} = \mathcal{O}_M\langle 1, \partial_\theta\rangle$.

This is the key theorem of the section. The proof is quite technical, and requires some preparation. We first collect some useful results and notions in algebra.

3.2.1 Algebra Preliminaries

Lemma 3.7. *Let V be a vector space over a field F. Then we have the isomorphism*

$$\operatorname{End}(V) \cong F \oplus \operatorname{End}_0(V)$$

where $\operatorname{End}_0(V)$ denote traceless endomorphisms.

Proof. For any linear map $T : V \to V$, let $\operatorname{Tr}(T) = t$ be its trace. Then we have the decomposition $T = tI + (T - tI)$, where the first part is clearly isomorphic to the base field F, while the second part is in $\operatorname{End}_0(V)$. □

Lemma 3.8. *Let V, W be F-vector spaces. Then we have the isomorphism*

$$V \otimes W^* \cong \operatorname{Hom}(W, V).$$

Proof. Let $v \in V$ and $\epsilon_W \in W^*$. Then the isomorphism maps $v \otimes \epsilon_W$ to the map $w \mapsto \epsilon_W(w)v$ for all $w \in W$. □

Lemma 3.9. *Let V be a F-vector space, we have the isomorphism*

$$\textstyle\bigwedge^k V^* \cong (\bigwedge^k V)^*.$$

Proof. We need to exhibit a pairing $(-,-) : \bigwedge^k V^* \times \bigwedge^k V \to F$. The pairing is

$$(\epsilon_1 \wedge \cdots \wedge \epsilon_k, v_1 \wedge \cdots \wedge v_k) = \det\big(\epsilon^i(v_j)\big).$$

The rest of the detail is omitted. □

Lemma 3.10. *Suppose we are given F-vector spaces A, B, C. Suppose we have X and X' that fits into exact sequences $0 \to A \otimes C \to X \to B \to 0$ and $0 \to A \to X' \to B \otimes C^* \to 0$. Then we can recover X from X', and vice versa. More specifically, we have*

$$X' = (X \otimes C^*)/(A \otimes \mathrm{End}_0(C))$$

and

$$X = \ker(X' \otimes C \to B \otimes \mathrm{End}_0(C)).$$

Proof. Let us start with the exact sequence $0 \to A \otimes C \to X \to B \to 0$. Then we can tensor everything by C^*, and using Lemma 3.8 we have $C \otimes C^* \cong \mathrm{End}(C)$, hence the sequence reads $0 \to A \otimes \mathrm{End}(C) \to X \otimes C^* \to B \otimes C^* \to 0$. Now using Lemma 3.2.1, we see that $A \otimes \mathrm{End}(C) \cong A \oplus (A \otimes \mathrm{End}_0(C))$. The map $A \otimes \mathrm{End}(C) \to X \otimes C^*$ thus factors through $A \otimes \mathrm{End}_0(C)$ and descends to a well-defined exact sequence

$$0 \to A \to (X \otimes C^*)/(A \otimes \mathrm{End}_0(C)) \to B \otimes C^* \to 0.$$

The other direction is similar, starting by tensoring with second sequence by C. We leave the rest of the proof to interested readers. □

The above lemmas on vector spaces generalizes directly to vector bundles or sheaves of vector spaces. Next, we introduce the notion of torsors.

Definition 3.11. Let G be an abelian group[1]. By a ***G-torsor*** or a ***G-principal homogeneous space*** we mean a set X, together with a group action $G \times X \to X$, such that the map

$$\rho : G \times X \to X \times X \qquad (g, x) \mapsto (x, g.x)$$

is a bijection.

Remark 3.12. Note that ρ defined above being bijection implies two things: First, injectivity of ρ implies only identity element $e_G \in G$ can fix every element of X. In other words, the action is free. Surjectivity of ρ implies that the action of G on X is transitive. Hence, equivalently, one can say that a G-torsor is a set with an action of G on it that is freely transitive.

We now introduce the most relevant example of a G-torsor that we will use in this writting:

Definition 3.13. Let V be a vector space, viewed as an abelian group under addition. An ***affine space*** A modeled on V is simply a V-torsor A.

Remark 3.14. Let $P, Q \in A$ be two points in the affine space A modeled on V. By definition, there exists $v \in V$ such that $P = v.Q$. If A is a subset of the vector space V itself, then the action of V on A is by translation, hence we can switch to the additive notation $P = v + Q$. Thus $P - Q = v$.

This might sound familiar. An affine space is often described as a vector space that has "forgotten its origin." You can't add two points P, Q together, but you can subtract two points to get a vector $P - Q = v$ (the displacement vector).

The most useful example of affine space to think about is plane P_λ defined by $z = \lambda$ for some $\lambda \neq 0$ in $\mathbb{R}^3$. If you add two such points in the plane, the resulting z coordinate would be $2\lambda \neq \lambda$, so is no longer in P_λ. But the difference of two points in P_λ would still be a vector, hence in $\mathbb{R}^3$.

Again, the above notion generalizes to vector bundles.

[1] If G is nonabelian then one must distinguish between left and right torsors according to whether the action is on the left or right.

3.2.2 Proof of Theorem 3.6

Now we start on our proof of Theorem 3.3.

Recall that the family of all splittings of S is parameterized by the group of isomorphisms from $S(M,V)$ to S that fixes M and V. So

$$\mathrm{Splt}(S) := \mathrm{Isom}_{M,V}(S(M,V),S).$$

Note that $\mathrm{Splt}(S)$ is a $\mathrm{Splt}(S(M,V))$-torsor. A bit more generally, we can define

$$\mathrm{Splt}(S^{(i)}) := \mathrm{Isom}_{M,V}(S(M,V)^{(i)},S^{(i)}).$$

Similarly, this is a $\mathrm{Splt}(S(M,V)^{(i)})$-torsor. Working locally over M, we can also define *sheaves* of groups

$$\mathbf{Splt}(S^{(i)}) : U \mapsto \mathrm{Isom}_{U,V}(S(M,V)^{(i)}|_U, S^{(i)}|_U)$$

and similarly $\mathbf{Splt}(S(M,V)^{(i)})$. And we note that $\mathbf{Splt}(S^{(i)})$ is a $\mathbf{Splt}(S(M,V)^{(i)})$-torsor. In particular when $i=2$, by [2] just before their equation (2.7), using $G=G^2$ (defined in the paper) we have the identification

$$\mathrm{Splt}(S(M,V)^{(2)}) \cong H^0(G/G^3) = H^0(G^2/G^3) \cong H^0(M, T_+ \otimes \textstyle\bigwedge^2 T_-^*),$$

and similarly when we restrict to U we get the global section of $T_+ \otimes \bigwedge^2 T_-^*$ over U. This tells us that the sheaves of groups $\mathbf{Splt}(S(M,V)^2)$ can be identified with the vector bundle $T_+ \otimes \bigwedge^2 T_-^*$. In general (also discussed in the same place in [2]), the torsor $\mathbf{Splt}(S^{(2)})$ over it becomes an affine bundle modeled on $T_+ \otimes \bigwedge^2 T_-^*$, the structure group being the group of translations.

Now we need a core lemma. The proof is technical and not so obvious, so we omit it. But it follows directly from our treatment of obstruction classes from the previous section.

Lemma 3.15. *The first obstruction class defined in [2] can be identified with the class of this affine bundle discussed above. Equivalently, it is identified with the extension class of the sequence of vector bundles*

$$0 \to T_+ \otimes \textstyle\bigwedge^2 T_-^* \to D_\omega \xrightarrow{\pi} \mathcal{O}_M \to 0,$$

from which $\mathbf{Splt}(S^{(2)})$ *is recovered as* $\pi^{-1}(1)$.

Now we begin the proof of Theorem 3.6.

Proof of Theorem 3.6. First, note that by the definition of $\bar{D}$ from the statement of Theorem 3.6, it fits into an exact sequence

$$0 \to T_+ \to \bar{D} \to \textstyle\bigwedge^2 T_- \to 0 \tag{*}$$

or in local coordinates this is $0 \to \langle \partial_x \rangle \to \langle \partial_x, \partial_\theta^2 \rangle \to \langle \partial_\theta^2 \rangle \to 0$. To compare it with the extension class D_ω, we first message the sequence above into the sequence in Lemma 3.15 by tensoring with $\bigwedge^2 T_-^*$. Using the conversion recipe from Lemma 3.10, with $A = T_+, B = \mathcal{O}_M, C = \bigwedge^2 T_-^*, X' = \bar{D}$, we see that we need to make sense of identifying D_ω with

$$K := \ker(\alpha_0 : \bar{D} \otimes \textstyle\bigwedge^2 T_-^* \to \mathrm{End}_0(\bigwedge^2 T_-^*)).$$

Let us recall what this α_0 is by tracing through the construction. We started by tensoring $(*)$ with $\bigwedge^2 T_-^*$, from which we get a map

$$\alpha : \bar{D} \otimes \bigwedge^2 T_-^* \to \bigwedge^2 T_- \otimes \bigwedge^2 T_-^* \cong \mathrm{End}(\bigwedge^2 T_-^*),$$

where if we trace through the isomorphisms Lemma 3.8, this is the map

$$\alpha : \delta \otimes (\mathrm{d}\theta^i \wedge \mathrm{d}\theta^j) \mapsto \bar{\delta} \otimes (\mathrm{d}\theta^i \wedge \mathrm{d}\theta^j) \mapsto (\sigma \mapsto \bar{\delta}(\sigma)\mathrm{d}\theta^i \wedge \mathrm{d}\theta^j),$$

where $\delta \in \bar{D}$, and $\bar{\delta}$ is its image mod T_+, and $\sigma \in \bigwedge^2 T_-^*$. Finally, for $\sigma = \mathrm{d}\theta^p \wedge \mathrm{d}\theta^q$, and $\bar{\delta} = \frac{\partial}{\partial\theta^m}\frac{\partial}{\partial\theta^n}$, the pairing is $\bar{\delta}(\sigma) = \det\big(\frac{\partial}{\partial\theta^\mu}(\mathrm{d}\theta^\nu)\big) := \det\big(\mathrm{d}\theta^\nu(\frac{\partial}{\partial\theta^\mu})\big)$, using Lemma 3.9. The point is, by Lemma 3.2.1, we have a decomposition $\mathrm{End}(\bigwedge^2 T_-^*) \cong \mathcal{O}_M \oplus \mathrm{End}_0(\bigwedge^2 T_-^*)$. Therefore, α decomposes into two maps α_0 that maps to $\mathrm{End}_0(\bigwedge^2 T_-^*)$, and α_1 that maps to $\mathcal{O}_M$. Recalling the proof of Lemma 3.2.1, we see that to say $\varphi \in \bar{D} \otimes \bigwedge^2 T_-^*$ is in $\ker \alpha_0$ is to say that $\alpha(\varphi)$ is a multiple of $\mathrm{id}_{\bigwedge^2 T_-^*}$, hence its data must be entirely captured by its image under α_1, and we note that $\alpha_1(\varphi) = \mathrm{Tr}\,\alpha(\varphi)$.

Because we are asked to identify D_ω with K, we must compare the exact sequence in Lemma 3.15 with the following exact sequence:

$$0 \to T_+ \otimes \bigwedge^2 T_-^* \to K \xrightarrow{\alpha_1} \mathcal{O}_M \to 0$$

and show that these are two equivalent extensions. Since D_ω is characterized by the property that the affine bundle $\mathbf{Splt}(S^{(2)})$ is recovered by $\pi^{-1}(1)$, it suffices to identify $\mathbf{Splt}(S^{(2)})$ with $\alpha_1^{-1}(1)$, the elements $\tau \in \bar{D} \otimes \bigwedge^2 T_-^*$ such that $\alpha(\tau)$ is a multiple of identity on $\bigwedge^2 T_-^*$ with trace 1.

To this end, suppose we are given a splitting $s : S(M,V)^{(2)} \to S^{(2)}$. The equality (2.5) in [31], namely $D^i_{S^{(i)}} = D^i_S$ gives us an identification of $D^i_{S(M,V)}$ with D^i_S for $i = 1, 2$. Thus we get a corresponding identification of $\bar{D}_{S(M,V)}$ with $\bar{D}_S$. But by (3.2), the inclusion $\bigwedge^\bullet V \to D_{S(M,V)}$ naturally gives rise to a map

$$\bigwedge^2 T_- \to \bar{D}_S$$

which in local coordinate reads

$$\frac{\partial}{\partial\theta^m} \wedge \frac{\partial}{\partial\theta^n} \mapsto \frac{\partial}{\partial\theta^m}\frac{\partial}{\partial\theta^n}.$$

This map gives an element of $\mathrm{Hom}(\bigwedge^2 T_-, \bar{D}) \cong \bar{D} \otimes \bigwedge^2 T_-^*$, which, after tracing through the isomorphism of Lemma 3.8, reads

$$\tau = \frac{1}{2}(\mathrm{d}\theta^\mu \wedge \mathrm{d}\theta^\nu)\frac{\partial}{\partial\theta^\mu}\frac{\partial}{\partial\theta^\nu}.$$

Indeed, as one can check, using the determinant pairing in Lemma 3.9, one obtains

$$\tau\left(\frac{\partial}{\partial\theta^m} \wedge \frac{\partial}{\partial\theta^n}\right) = \frac{1}{2}(\delta^\mu_m\delta^\nu_n - \delta^\mu_n\delta^\nu_m)\frac{\partial}{\partial\theta^\mu}\frac{\partial}{\partial\theta^\nu} = \frac{1}{2}\left(\frac{\partial}{\partial\theta^m}\frac{\partial}{\partial\theta^n} - \frac{\partial}{\partial\theta^n}\frac{\partial}{\partial\theta^m}\right) = \frac{\partial}{\partial\theta^m}\frac{\partial}{\partial\theta^n},$$

which is indeed the map of interest. Since α is just modulo T_+ and our τ is built up purely of odd elements,

$$\alpha(\tau) = \frac{1}{2}(\mathrm{d}\theta^\mu \wedge \mathrm{d}\theta^\nu)\frac{\partial}{\partial\theta^\mu}\frac{\partial}{\partial\theta^\nu}.$$

Doing exactly the same calculation shows that $\alpha(\tau)$ is actually identity on $\bigwedge^2 T_-^*$. Normalizing τ so that $\mathrm{Tr}\,\alpha(\tau) = 1$, in other words replacing $\tau \to \tau/(\dim \bigwedge^2 T_-^*)$, we see that the splitting s indeed gives an element τ in $\alpha_1^{-1}(1)$. Finally, note that $\tau = \tau(s)$ depends on the splitting s, since the inclusion $\bigwedge^2 T_- \to \bar{D}_S$ is defined by the following commutative diagram

$$\begin{tikzcd}
\Lambda^2 V \arrow[r] \arrow[d, "\cong_s"'] & \bar{D}_{S(M,V)} \arrow[d, "\cong_s"] \\
\Lambda^2 T_- \arrow[r] & \bar{D}_S
\end{tikzcd}$$

whose vertical isomorphisms are induced by s, hence is splitting dependent. Thus our identification is given by

$$\mathbf{Splt}(S^{(2)}) \xrightarrow{\cong} \alpha_1^{-1}(1) \qquad s \mapsto \tau(s).$$

This concludes the proof. □

3.3 Obstruction Class via Map Spaces

In this section, we give another equivalent characterization of the obstruction class. Let

$$B_k = \mathbb{C}[\eta_1, ..., \eta_k]/(\eta_i\eta_j = -\eta_j\eta_i)$$

be the polynomial ring of k independent odd variables η_m for $1 \leq m \leq k$. Note that $\operatorname{Spec} B_k = \mathbb{A}^{0|k}$. In fact, for any supermanifold S and a $\mathbb{Z}_2$-graded Artinian $\mathbb{C}$-algebra B, we can define the set of ***B-points*** of S

$$S(B) := \operatorname{Maps}(\operatorname{Spec} B, S),$$

which is the set of all maps from $\operatorname{Spec} B$ to S. This set $S(B)$ is naturally a manifold: it is finite dimensional, smooth, and purely even. Indeed, the reduced space of $\operatorname{Spec} B$ is just a finite collection of points, hence this set of maps has a natural structure of a finite dimensional manifold.

Example 3.16. We first look at the example $S(B_1) = \operatorname{Maps}(\operatorname{Spec} B_1, S)$. So let $\varphi \in S(B_1)$ be such a map. Since $\operatorname{Spec} B_1 = \mathbb{A}^{0|1}$ is topologically just a point, the map on the level of topological space is not interesting, and in fact the information of where φ maps the point to can be fully recovered from the information of $\varphi^* : \mathcal{O}_S \to \varphi_*\mathcal{O}_{\mathbb{A}^{0|1}}$.

Since $\mathcal{O}_{\mathbb{A}^{0|1}} = \mathbb{C}[\eta_1]$, and φ^* is a $\mathbb{Z}_2$-homomorphism, we see that

$$\varphi^* x^a = \lambda^a \qquad \varphi^*\theta^b = v^b\eta_1,$$

where λ^a, v^b are even parameters, for $1 \leq a \leq m, 1 \leq b \leq n$. Using an abuse of notation for simplicity, we write

$$x^a = \lambda^a \qquad \theta^b = v^b\eta_1.$$

In fact, λ^a's describe the coordinates of a point on the reduced space M of S, which is exactly the coordinate of the point that φ maps to in M.

On the other hand, v^b's are the coefficients of an odd tangent vector. The easiest way to see this is to imagine if we change the coordinate $x^a \to y^a(x, \theta)$ and $\theta^b \to \psi^b(x, \theta)$. Say with respect to this new coordinate the map φ^* on the odd direction reads

$$\psi^b = w^b\eta_1.$$

Then Taylor expanding ψ^b near $\theta = 0$ gives

$$\psi^b(x, \theta) = \psi_0^b(x) + \theta^j \frac{\partial \psi^b}{\partial \theta^j}(x).$$

Plugging in $\theta^j = v^j \eta_1$, and collecting only the first order term in η_1, we see that

$$w^b = v^j \frac{\partial \psi^b}{\partial \theta^j}(x),$$

which is the transformation rule for an odd vector restricted on M (note that since it is restricted on M, any functions on M cannot have θ dependence, so the above transformation rule is indeed correct). So λ^a's describe a point on M, while v^b's describe an odd tangent vector, and these information completelet determines φ. Therefore, we conclude that $S(B_1)$ can be identified with the total space of $T_- M$.

Next we come to the core example of this section, $S(B_2)$. Using the same local coordinate analyze as the previous example, we see that a map $\mathrm{Spec}(B_2) \to S$ is determined by the even parameters λ^a, h^a, v^b, w^b, where $1 \le a \le m, 1 \le b \le n$, given by

$$x^a = \lambda^a + \eta_1 \eta_2 h^a \qquad \theta^b = v^b \eta_1 + w^b \eta_2.$$

Now we change coordinates to

$$y^a(x, \theta) = f^a(x) + \theta^i \theta^j \frac{\partial^2 y^a}{\partial \theta^i \partial \theta^j}(x) + \cdots \qquad \psi^b(x, \theta) = \theta^i \frac{\partial \psi^b}{\partial \theta^i}(x) + \cdots .$$

We will get a new set of rules defining φ in this coordinate, namely

$$y^a = \Lambda^a + \eta_1 \eta_2 H^a \qquad \psi^b = V^b \eta_1 + W^b \eta_2.$$

We do the same computation as before, collecting and comparing terms of $\mathbb{C}[\eta_1, \eta_2]$, we see that

$$\begin{aligned}
\Lambda &= f(\lambda) \\
V^b &= v^i \frac{\partial \psi^b}{\partial \theta^i}(\lambda) \\
W^b &= w^i \frac{\partial \psi^b}{\partial \theta^i}(\lambda) \\
H^a &= h^b \frac{\partial y^a}{\partial x^b}(\lambda) + (v^i w^j - w^i v^j) \frac{\partial^2 y^a}{\partial \theta^i \partial \theta^j}(\lambda), \qquad (**)
\end{aligned}$$

where $f(\lambda)$ is a point whose component is given by $f^a(\lambda)$ in the Taylor expansion above. Hence we see that $\lambda = (\lambda^a)$ describes a point in M, while $v = (v^b)$ and $w = (w^b)$ describes two odd tangent vectors. So λ together with v, w describe a point of the total space of the bundle $T_- \oplus T_-$ over M. The more interesting piece is $h = (h^a)$. For each fixed λ and v, w, the transformation rule (**) is a linear transformation $f'(\lambda)$, which is the transition functions for the even tangent space T_+, plus a constant shift $(v^i w^j - w^i v^j) \frac{\partial^2 y^a}{\partial \theta^i \partial \theta^j}(\lambda)$, which is an affine transformation. This shows that h is an affine bundle over the total space of $Y = T_- \oplus T_-$, modeled on $\pi^* T_+$, where $\pi : Y \to M$ is the projection map. Note that if we insist that the affine bundle is over Y, then it has to be modeled on a vector bundle over Y, so it is modeled on $\pi^* T_+$, not on T_+. But we can also view it as an affine bundle modeled on T_+ for every given parameters λ, v, w. In fact, the bilinear dependence of (**) shows that a point $S(B_2)$ is a family of affine bundle modeled on T_+, parameterized linearly by $\bigwedge^2 T_-$.

Proposition 3.17. The family of affine bundles over a model space F over M that is parameterized linearly by another vector bundle E over M, is classified by the cohomology group $H^1(M, \mathrm{Hom}(E,F))$.

Proof. We choose a good cover $M = \bigcup_\alpha U_\alpha$ by open sets. Over each open set U_α, say the section of our affine bundle over that is given by A_α. Then on the overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the transition rule is that of an affine transition

$$A_\beta = M_{\alpha\beta} A_\alpha + s_{\alpha\beta},$$

where $M_{\alpha\beta}$ is a section of $\mathrm{End}(F)$, which are transition functions that are already specified when we construct the vector bundle F, while $s_{\alpha\beta}$ is a section of F that depends on the parameter in E, so in other words a section of $\mathrm{Hom}(E,F)$. Moreover, compatibility asserts that they satisfy the cocycle condition

$$s_{\alpha\gamma} = s_{\alpha\beta} + s_{\beta\gamma}.$$

Hence $s = (s_{\alpha\beta})$ defines an element in $H^1(M, \mathrm{Hom}(E,F))$. Moreover, this family modeled on F that is parameterized by E is completely determined by the affine shift $s = (s_{\alpha\beta})$ on the overlap. Therefore, $H^1(M, \mathrm{Hom}(E,F))$ classifies such family, as desired. □

By the previous Proposition, we see that $S(B_2)$, which is an affine bundle modeled on T_+ parameterized linearly by $\bigwedge^2 T_-$, corresponds to an element in $H^1(M, \mathrm{Hom}(\bigwedge^2 T_-, T_+))$, which is the right place for the obstruction class. We claim that these two classes in fact coincide. It suffices to prove, because of the previous section, that the class of $S(B_2)$ is the same class as the extension class of $\bar{D}$ defined previously. But this is easy: In local coordinates, an operator in $\bar{D}$ takes the form

$$L = h^a \frac{\partial}{\partial x^a} + \xi^{ij} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j}.$$

Now we change coordinates $x^a \to y^a(x,\theta)$ and $\theta^b \to \psi^b(x,\theta)$. Then the differential operator in $\bar{D}$ takes the form

$$L = H^a \frac{\partial}{\partial y^a} + \Xi^{ij} \frac{\partial}{\partial \psi^i} \frac{\partial}{\partial \psi^j}.$$

To figure out how these coefficients relate, we use the chain rule modulo D_-^1:

$$\frac{\partial}{\partial x^a} = \frac{\partial y^k}{\partial x^a} \frac{\partial}{\partial y^k} + \frac{\partial \psi^l}{\partial x^a} \frac{\partial}{\partial \psi^l} = \frac{\partial y^j}{\partial x^a} \frac{\partial}{\partial y^j}$$

and

$$\frac{\partial}{\partial \theta^i} = \frac{\partial y^k}{\partial \theta^i} \frac{\partial}{\partial y^k} + \frac{\partial \psi^l}{\partial \theta^i} \frac{\partial}{\partial \psi^l}.$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} &= \left(\frac{\partial y^k}{\partial \theta^i} \frac{\partial}{\partial y^k} + \frac{\partial \psi^l}{\partial \theta^i} \frac{\partial}{\partial \psi^l} \right) \left(\frac{\partial y^p}{\partial \theta^j} \frac{\partial}{\partial y^p} + \frac{\partial \psi^q}{\partial \theta^j} \frac{\partial}{\partial \psi^q} \right) \\
&= \frac{\partial \psi^l}{\partial \theta^i} \frac{\partial \psi^q}{\partial \theta^j} \frac{\partial}{\partial \psi^l} \frac{\partial}{\partial \psi^q} + \frac{\partial y^k}{\partial \theta^i} \frac{\partial^2 y^p}{\partial y^k \partial \theta^j} \frac{\partial}{\partial y^p} + \cdots \\
&\equiv \frac{\partial \psi^l}{\partial \theta^i} \frac{\partial \psi^q}{\partial \theta^j} \frac{\partial}{\partial \psi^l} \frac{\partial}{\partial \psi^q} + \frac{\partial y^k}{\partial \theta^i} \frac{\partial^2 y^p}{\partial y^k \partial \theta^j} \frac{\partial}{\partial y^p} \quad \mathrm{mod}\ D_-^1.
\end{aligned}$$

Hence we see that the transformation rule is

$$H^p = h^a \frac{\partial y^p}{\partial x^a} + \xi^{ij} \frac{\partial^2 y^p}{\partial \theta^i \partial \theta^j}$$
$$\Xi^{lq} = \xi^{ij} \frac{\partial \psi^l}{\partial \theta^i} \frac{\partial \psi^q}{\partial \theta^j}.$$

We see that if we identify ξ with $v \wedge w$ in $S(B_2)$, then the first transformation equation above is precisely (**), and the second transformation equation is the combination of that of v and w.

But what we have established, $S(B_2)$ as an affine T_+-bundle over the total space Y of $T_- \oplus T_-$ over M corresponds to an element

$$[S(B_2)] \in H^1(M, \mathrm{Hom}(\bigwedge^2 T_-, T_+)) = H^1(M, T_+ \otimes \bigwedge^2 T_-^*) = \mathrm{Ext}^1(\bigwedge^2 T_-, T_+).$$

Hence $S(B_2)$ corresponds to a vector bundle that is an extension of $\bigwedge^2 T_-$ by T_+. Moreover, the computation above shows that the transition functions of this affine bundle is the same as that of the transition functions of the vector bundle $\bar{D}$, hence we conclude that $[S(B_2)]$ and $[\bar{D}]$ are the same class. Summarizing what we have established in the last two sections, we come to the first crutial theorem of this chapter:

Theorem 3.18. *For any supermanifold S with reduced space M, the following three classes in $H^1(M, \mathrm{Hom}(\bigwedge^2 T_-, T_+))$ are equal:*

1. *The first obstruction class $\omega = \omega_2$ to the splitting of S;*
2. *The extension class $[\bar{D}]$ of the ring of the differential operators $\bar{D} := D_-^2 / D_-^1$;*
3. *The class $[S(B_2)]$ as an affine T_+-bundle over the total space Y of $T_- \oplus T_-$ over M.*

3.4 Atiyah Classes

Atiyah class was first proposed by M. Atiyah in his paper [27], and is briefly reviewed in [28]. We will give an overview of Atiyah class, presenting several equivalent definitions, but not worried about showing these definitions being equivalent.

3.4.1 Jet Bundle and Extension Class

Let E be a vector bundle over a manifold X of dimension m. Let $p \in X$ be a point. We denote $\Gamma(E, p)$ the set of all local sections of E whose domain contains p. Let $\sigma, \eta \in \Gamma(E, p)$ be two such local sections. Near p, we can choose a local frame $e_\alpha \in \Gamma(E, p)$ for $1 \leq \alpha \leq \mathrm{rank}\, E$. Then we can write $\sigma = \sigma^\alpha e_\alpha$ and $\eta = \eta^\alpha e_\alpha$ where $\sigma^\alpha, \eta^\alpha \in \Gamma(\mathcal{O}_X, p)$. For multi-index $I = (I(1), ..., I(m))$, where $I(k)$ is to be thought of the order of differentiation with respect to the k-th variable, and $|I| = \sum_k I(k)$, we define the derivative operator

$$\partial^I = \frac{\partial^{|I|}}{\partial x^I} := \prod_{k=1}^m \left(\frac{\partial}{\partial x^k} \right)^{I(k)}$$

in a local chart of M near p. We say that σ, η have the same r***-jet*** at p if

$$\partial^I \sigma^\alpha(p) = \partial^I \eta^\alpha(p)$$

for all $0 \leq |I| \leq r$ and for all p. An r-jet is an equivalence class under this relation, and the equivalence class of r-jet with representative σ is denoted by $j_p^r \sigma$. We now define the set

$$J^r(E) = \{j_p^r \sigma : p \in X, \sigma \in \Gamma(E, p)\}.$$

There is an obvious projection $\pi : J^r(E) \to X$, by $\pi(j_p^r \sigma) = p$. A coordinate system on E will induce a set of coordinates on $J^r(E)$ as follows: If (x^i, u^α) is a set of coordinate systems on E, where x^i is the coordinate on an open set $U \subset X$ and u^a the fiber coordinates, then a coordinate system on $J^r(E)$ is given by $(x^i, u^\alpha, u_I^\alpha)$, where $x^i(j_p^r \sigma) = x^i(p)$, $u^\alpha(j_p^r \sigma) = u^\alpha(\sigma(p))$, and $u_I^\alpha(j_p^r \sigma) = \partial^I \sigma^\alpha$, for $0 \leq |I| \leq r$. These coordinate systems makes $J^r(E)$ into a vector bundle over X called the ***r-th jet bundle*** of E.

In particular, the first jet bundle $J^1(E)$ of E is the vector bundle whose fiber at $p \in X$ is the equivalence class of sections of E with the same first order derivatives at p.

Proposition 3.19 (Atiyah Sequence)**.** *Let X be a complex manifold and E a holomorphic vector bundle on X, then $J^1(E)$ fits into the following exact sequence called the **Atiyah sequence**.*

$$0 \to \Omega_X \otimes E \to J^1(E) \to E \to 0. \tag{3.3}$$

Proof. The map $J^1(E) \to E$ is defined fiberwise by $j_p^r \sigma \mapsto \sigma(p)$. Hence its kernel consists of $j_p^r \sigma$ such that $\sigma(p) = 0$. Hence it suffices to show $K \cong \Omega_X \otimes E$ holds fiberwise. We may choose local holomorphic coordinates around p, say (z^i) on an open set of $U \subset X$ containing p. We may choose a local frame e_α near p on E for $1 \leq \alpha \leq \operatorname{rank} E$. Then a section $\sigma = \sigma^\alpha e_\alpha$. Since for $j_p^r \sigma \in K_p$, all the information are contained in the first order derivatives, we define the isomorphism $\Phi : K_p \to \Omega_X|_p \otimes E|_p$ is given in this local coordinates by sending

$$\Phi(j_p^r \sigma) = \frac{\partial \sigma^\alpha}{\partial z^i}(p) \mathrm{d} z^i \otimes e_\alpha|_p.$$

It is easy to check that this is a linear isomorphism. Hence we get the desired claim. □

The sequence (3.3) therefore gives rise to the extension class known as the ***Atiyah class***:

$$\alpha_E \in \operatorname{Ext}^1(E, \Omega_X \otimes E) = H^1(X, \Omega_X \otimes \operatorname{End}(E)).$$

Let D_X be the sheaf of differential operators on X, and D^p be the subsheaf of operators of order $\leq p$. Then we have the following:

Lemma 3.20. *$J^1(E^*)^* \cong D^1 \otimes_{\mathcal{O}_X} E$.*

Proof. We work near $p \in X$ and choose a local coordinate z^μ around it on X once and for all, and choose a dual frame ϵ^α for $1 \leq \alpha \leq \operatorname{rank} E^*$ of E^* to that of e_α of E once and for all. Then $j_p^1 \varphi$ where $\varphi \in \Gamma(E^*, p)$ is given by the tuple $(\varphi(p), \partial_\mu \varphi_\alpha)$ where $\varphi = \varphi_\alpha \epsilon^\alpha$ near p. Then contraction gives a map $T_p M \to E_p^*$:

$$v^\mu \mapsto v^\mu \partial_\mu \varphi_\alpha.$$

Hence we see that the derivative part of $j_p^1 \varphi$ can be identified with an element in

$$\operatorname{Hom}(T_p M, E_p^*) \cong T_p^* X \otimes E_p^*.$$

Now, a section L of $D^1 \otimes E$ can be decomposed into the zeroth and first order part $L = L_0 + L_1$, each is a E-valued differential operator with the given order. We define a bilinear pairing from $D^1 \otimes_{\mathcal{O}_X} E$ and $J^1(E)$ to $\mathbb{C}$ by

$$\langle L, j_p^1 \varphi \rangle = \langle L_0, \varphi \rangle + \langle L_1, \partial_\mu \varphi_\alpha \rangle.$$

One can check that this pairing is nondegenerate (left as exercise), hence the statement immediately follows. □

By Lemma 3.20, we can dualize (3.3) with E replaced by E^* and get an exact sequence

$$0 \to E \to D^1 \otimes_{\mathcal{O}_X} E \to T_X \otimes E \to 0.$$

By properties of Ext^1 from homological algebra, the extension class corresponding to this sequence is $-\alpha_E$. Now replacing E by E^* again gives the representative of α_E. Hence we get the following.

Theorem 3.21. *The following two classes in $H^1(X, \Omega_X \otimes \mathrm{End}(E))$ are equal, for any complex manifold X and holomorphic vector bundle E on it:*

1. *The Atiyah class α_E;*
2. *The extension class $[D^1 \otimes_{\mathcal{O}_X} E^*]$ of $0 \to E^* \to D^1 \otimes_{\mathcal{O}_X} E^* \to T_X \otimes E^* \to 0$ given by the sheaf of first order differential operators from E to $\mathcal{O}_X$.*

3.4.2 Obstruction to Existence of Holomorphic Connections

Next, we introduce Atiyah class as the obstruction of finding a holomorphic connection on a vector bundle. Our treatment follows [29]. The advantage of this formulation is that we have a very concrete discription of the Atiyah class as a Čech 1-cocycle.

Definition 3.22. Let $E \to X$ be a holomorphic vector bundle over a complex manifold. A ***holomorphic connection*** on E is a complex-linear map $\nabla : E \to \Omega_X \otimes E$ such that the Libniz rule holds

$$\nabla(f \cdot \sigma) = \partial(f) \otimes \sigma + f \cdot \nabla\sigma,$$

for any local holomorphic function f on X and any local holomorphic section σ of E. Here, $\partial f = (\partial f/\partial z^i)\mathrm{d}z^i$ is the holomorphic differential.

Proposition 3.23. Locally, any holomorphic connection is of the form $\partial + A$ where A is a holomorphic section of $\Omega^1_X \otimes \mathrm{End}(E)$. Given $\sigma = \sigma^\mu e_\mu$ for some local frame e_α, the map ∂ is defined by $\partial(\sigma^\mu e_\mu) = (\partial\sigma^\mu)e_\mu \in \Omega^1_X \otimes E$.

Proof. This can be proved by using applying the Libniz rule above to the local expression $\sigma = \sigma^\mu e_\mu$. Then we have

$$\nabla\sigma = (\partial\sigma^\mu)e_\mu + \sigma^\mu \nabla e_\mu.$$

We write $\nabla e_\mu = A^\nu_\mu e_\nu$, for some section A^ν_μ of Ω_X. Then the equation above becomes

$$\nabla\sigma = (\partial\sigma^\mu + A^\mu_\nu \sigma^\nu)e_\mu.$$

Using matrix notation $\sigma = (\sigma^1, ..., \sigma^r)$, the equation becomes $\nabla = \partial + A$, as desired. □

Now let E be a holomorphic vector bundle and $X = \bigcup_i U_i$ be an open covering with holomorphic trivilizations $\Psi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r$. On the intersection $U_{ij} = U_i \cap U_j$, if a point on $E|_{U_{ij}}$ is given by (p, v_i) under Ψ_j and (p, v_j) as Ψ_i, then we define the transition map $\Psi_{ij} : U_i \to GL(r, \mathbb{C})$ by $v_i = \Psi_{ij}(p)v_j$. Note that we have the cocycle condition of the transition map: $\Psi_{ij}\Psi_{jk}\Psi_{ki} = \mathrm{id}$. Now we introduce the Atiyah class as follows:

Definition 3.24. The ***Atiyah class*** $\alpha_E \in H^1(X, \Omega^1_X \otimes \mathrm{End}(E))$ of the holomorphic vector bundle E is given by the Čech 1-cocycle

$$(\alpha_E)_{ij} := \Psi_j^{-1} \circ (\Psi_{ij}^{-1} \mathrm{d}\Psi_{ij}) \circ \Psi_j$$

on U_{ij}. Here, by $\Psi_{ij}^{-1}\mathrm{d}\Psi_{ij}$ we mean taking the exterior derivative of the components of the matrix Ψ_{ij}, then multiply by the inverse matrix Ψ_{ij}^{-1} of Ψ_{ij}. This gives an element $\Psi_{ij}^{-1}\mathrm{d}\Psi_{ij} \in \Gamma(U_{ij}, \Omega_X \otimes \mathrm{End}(\mathbb{C}^r))$. Then by conjugation of Ψ_j we get an element in $\Gamma(U_{ij}, \Omega_X \otimes \mathrm{End}(E))$.

Proposition 3.25. A holomorphic vector bundle E admits a holomorphic connection iff its Atiyah class $\alpha_E \in H^1(\Omega_X \otimes \mathrm{End}(E))$ vanishes.

Proof. Since Ψ_{ij} are holomorphic, we have that $\mathrm{d}\Psi_{ij} = \partial\Psi_{ij}$. Local holomorphic connections on $U_i \times \mathbb{C}^r$ are of the form $\partial + A_i$ by Proposition 3.23. These can be glued to a connection globally on E iff on U_{ij} we have

$$\Psi_i^{-1} \circ (\partial + A_i) \circ \Psi_i = \Psi_j^{-1} \circ (\partial + A_j) \circ \Psi_j.$$

Equivalently this is

$$\Psi_i^{-1} \circ \partial \circ \Psi_i - \Psi_j^{-1} \circ \partial \circ \Psi_j = \Psi_j^{-1} A_j \Psi_j - \Psi_i^{-1} A_i \Psi_i.$$

The left hand side of the equation above is

$$\begin{aligned}
\Psi_i^{-1} \circ \partial \circ \Psi_i - \Psi_j^{-1} \circ \partial \circ \Psi_j &= \Psi_j^{-1} \circ (\Psi_{ij}^{-1} \circ \partial \circ \Psi_i \circ \Psi_j^{-1}) \circ \Psi_j - \Psi_j^{-1} \circ \partial \circ \Psi_j \\
&= \Psi_j^{-1} \circ (\Psi_{ij}^{-1} \circ \partial \circ \Psi_{ij} - \partial) \circ \Psi_j \\
&= \Psi_j^{-1} \circ (\Psi_{ij}^{-1} \partial(\Psi_{ij})) \circ \Psi_j,
\end{aligned}$$

where the last equality $\Psi_{ij}^{-1} \circ \partial \circ \Psi_{ij} - \partial = \Psi_{ij}^{-1}\partial(\Psi_{ij})$ can be proved easily by applying both sides to a holomorphic section. The right hand side $\Psi_j^{-1} A_j \Psi_j - \Psi_i^{-1} A_i \Psi_i$ is the boundary of the 0-cocycle $B_i = \Psi_i^{-1} A_i \Psi_i$. Hence $\alpha_E = 0$ iff there exists local connections on E that can be glued to a global one. $\square$

3.4.3 Atiyah Class of Tangent Bundle

Now we specializes to the case where $E = T_X$ is the tangent bundle. In this case we use shorthand notation α_X to denote α_{T_X}. The Atiyah class α_X lives in $H^1(X, \Omega_X \otimes \mathrm{End}(T_X))$. We know

$$\Omega_X \otimes \mathrm{End}(T_X) = T_X^* \otimes T_X^* \otimes T_X$$

contains a piece $\mathrm{Sym}^2 T_X^* \otimes T_X = \mathrm{Hom}(\mathrm{Sym}^2 T_X, T_X)$. In fact, in this case the Atiyah class α_X lies completely in this summand, i.e., $\alpha_X \in H^1(\mathrm{Hom}(\mathrm{Sym}^2 T_X, T_X))$. For a proof, see discussion in [31] and (2.1.1) of [28].

We now present the bosonic analogue of Theorem 3.18 on a complex manifold X. Recall that D^i is the sheaf of differential operators of order $\le i$ on X. We can identify $D^1/D^0 \cong T_X$ and $D^2/D^1 \cong \mathrm{Sym}^2 T_X$. Then we have a short exact sequence of locally free sheaves on X:

$$0 \to T_X \to D^2/D^0 \to \mathrm{Sym}^2 T_X \to 0, \tag{3.4}$$

whose extension class we shall denote by $[D^2]$. Also, let $A_i = \mathbb{C}[\epsilon_1, ..., \epsilon_i]/(\epsilon_1^2, ..., \epsilon_i^2)$. Define

$$X(A_i) := \mathrm{Maps}(\mathrm{Spec}\, A_i, X).$$

By similar argument as we did in the supermanifold case, $X(A_0) = X, X(A_1) = T_X$, and $X(A_2)$ is the bosonic analogue of $S(B_2)$. It is an affine T_X-bundle over the total space of $T_X \oplus T_X$. Then Theorem 3.18 has a straightforward analogue:

Theorem 3.26. *The following three classes in $H^1(X, \mathrm{Hom}(\mathrm{Sym}^2 T_X, T_X))$ are equal for any complex manifold X:*

1. *The Atiyah class α_X;*
2. *The extension class $[D^2]$ of (3.4);*
3. *The class of $X(A_2)$ as an affine T_X-bundle over the total space of $T_X \oplus T_X$ over X.*

3.5 Super Atiyah Classes

We defined the Atiyah class α_E as an obstruction to admitting a holomorphic connection on E in Proposition 3.25. The notion of an affine connection can be extended to supermanifolds quite easily. See [30] section 4 for example. Therefore, we can extend the notion of super Atiyah class: Let S be a supermanifold. The ***super Atiyah class*** $\alpha_V \in H^1(S, T_S^* \otimes \mathrm{End}(V))$ of a (super) vector bundle V on S is defined as the obstruction class to the existence of an affine connection on V [30, Theorem 4.3], represented by a Čech 1-cocycle exactly like 3.25, namely $(\alpha_V)_{ij} = \nabla_i - \nabla_j$ is the difference between two affine connections on the intersection of two fine covers.

We aim to get a straightforward analogue of Theorem 3.26 for supermanifolds. The second equivalence is easy to generalize: We define the sheaf $\mathcal{D}^i$ of differential operators of order $\leq i$ on S, and the extension class $[\mathcal{D}^2]$ corresponding to that of (3.4).

Now we generalize the third equivalence, which is more tricky. Let A be a graded Artinian algebra. We wish to define $\mathbf{S}(A)$ of A-valued points of S. The first guess of defining it by $\mathbf{S}(A) := \mathrm{Maps}(\mathrm{Spec}\, A, S)$ is wrong because it is only a manifold, not a supermanifold. Hence $S(A) := \mathrm{Maps}(\mathrm{Spec}\, A, S)$ is the reduced space of the desired $\mathbf{S}(A)$.

The correct definition requires us to consider the internal-Hom in the category of supermanifolds. That is, we require

$$\mathbf{S}(A) = \mathbf{Maps}(\mathrm{Spec}\, A, S)$$

to be defined by the property that for every supermanifold S' there should be a natural identification

$$\mathrm{Maps}(S' \times \mathrm{Spec}\, A, S) = \mathrm{Maps}(S', \mathbf{Maps}(\mathrm{Spec}\, A, S)).$$

This property is strong enough so that any two candidates for $\mathbf{S}(A)$ are naturally identified. So if $S = S_1 \cup S_2$ is the union of two open subsets S_1, S_2 which are also supermanifolds, then we can build $\mathbf{S}(A)$ from $\mathbf{S_1}(A)$ and $\mathbf{S_2}(A)$. Hence it suffices to consider the case where S is affine. When $S = W$ is a super vector space, we define $\mathbf{Maps}(\mathrm{Spec}\, A, W) := A \otimes W$, where the tensor product is taken as $\mathbb{Z}_2$-graded vector spaces. In general, if S is affine, then it is defined by an ideal I in some super vector space W, in which case we take $\mathbf{Maps}(\mathrm{Spec}\, A, S)$ to be the subspace of $\mathbf{Maps}(\mathrm{Spec}\, A, W)$ Defined by the ideal

$$\mathbf{I} := \{\varphi \otimes r : \varphi \in A^*, r \in I\}$$

where A^* is the super vector space of linear functions on A.

As before, we consider the Artinian ring $A_i = \mathbb{C}[\epsilon_1, ..., \epsilon_i]/(\epsilon_1^2, ..., \epsilon_i^2)$. Then similar to the previous discussions for the bosonic case, we have $\mathbf{S}(A_1) = T_S$, and $\mathbf{S}(A_2)$ is our super version of $X(A_2)$ in Theorem 3.26. Then Theorem 3.26 has a straightforward generalization:

Theorem 3.27. *The following three classes in $H^1(S, \mathrm{Hom}(\mathrm{Sym}^2 T_S, T_S))$ are equal for any complex supermanifold S:*

1. *The super Atiyah class $\alpha_S = \alpha_{T_S}$;*

2. *The extension class $[\mathcal{D}^2]$ of the sheaf of differential operators $\mathcal{D}^2/\mathcal{D}^0$ on S;*

3. *The class of $\mathbf{S}(A_2)$ as an affine T_S-bundle over the total space Y of $T_S \oplus T_S$ over S.*

Now comes the key point. When we restrict the supermanifold S to the reduced space M, the tangent bundle decomposes:

$$T_S|_M = T_+ \oplus T_-.$$

Therefore, $\mathrm{Sym}^2 T_S$ also decomposes:

$$\mathrm{Sym}^2 T_S = \mathrm{Sym}^2 T_+ \oplus \mathrm{Sym}^2 T_- \oplus (T_+ \otimes T_-),$$

where we note that the symmetric product of a graded vector bundle T is defined in the graded sense: $ab = (-1)^{|b||a|}ba$. Since T_- is purely odd, we have that $\mathrm{Sym}^2 T_- = \bigwedge^2 T_-$.

By parity, the Hom group also splits into direct summands:

$$\mathrm{Hom}(\mathrm{Sym}^2 T_S, T_S) = \mathrm{Hom}(\mathrm{Sym}^2 T_+, T_+) \oplus \mathrm{Hom}(\textstyle\bigwedge^2 T_-, T_+) \oplus \mathrm{Hom}(T_+ \otimes T_-, T_-),$$

where the second arguments of Hom in the decomposition are given by above because these are $\mathbb{Z}_2$-homomorphisms, so it preserves gradings. This means the $H^1(M, \mathrm{Hom}(\mathrm{Sym}^2 T_S, T_S))$ also splits into three pieces, which are $H^1(M, -)$ where $-$ are the three pieces in the Hom decomposition above. From an equivalent perspective: Since Ext^1 is additive in both arguments, this means that we have a corresponding decomposition

$$\mathrm{Ext}^1(\mathrm{Sym}^2 T_S, T_S) = \mathrm{Ext}^1(\mathrm{Sym}^2 T_+, T_+) \oplus \mathrm{Ext}^1(\textstyle\bigwedge^2 T_-, T_+) \oplus \mathrm{Ext}^1(T_+ \otimes T_-, T_-). \quad (3.5)$$

Theorem 3.28. *Under the decomposition (3.5), the three components of the super Atiyah class (when restricted to M) $\alpha_S \in H^1(M, \mathrm{Hom}(\mathrm{Sym}^2 T_S, T_S)) = \mathrm{Ext}^1(\mathrm{Sym}^2 T_S|_M, T_S|_M)$ are:*

1. *The Atiyah class α_M of the reduced manifold M;*

2. *The obstruction class $\omega(S)$ to the splitting of S;*

3. *The Atiyah class α_{T_-} of the vector bundle T_- on M.*

Proof. We use (2) of Theorem 3.27 and consider the extension class of $\mathcal{D}^2/\mathcal{D}^0$ on S instead. The corresponding extension of (3.5) splits into three extensions, each involving three submodules $\mathcal{D}^2_+, \mathcal{D}^2_-, \mathcal{D}^2_\pm$ of $\mathcal{D}^2$, which we will prove to be invariant under choice of coordinates:

$$\mathcal{D}^2_+ = \mathcal{O}_M\langle \partial_x^2, \partial_x \rangle \qquad \mathcal{D}^2_- = \mathcal{O}_M\langle \partial_\theta^2, \partial_x \rangle \qquad \mathcal{D}^2_\pm = \mathcal{O}_M\langle \partial_x\partial_\theta, \partial_\theta \rangle,$$

with the three obvious extensions

$$0 \to T_+ \to \mathcal{D}^2_+ \to \mathrm{Sym}^2 T_+ \to 0$$
$$0 \to T_+ \to \mathcal{D}^2_- \to \textstyle\bigwedge^2 T_- \to 0$$
$$0 \to T_- \to \mathcal{D}^2_\pm \to T_+ \otimes T_- \to 0$$

corresponding to (3.5). Then the claim immediately follows from Theorem 3.26, Theorem 3.18, and Theorem 3.21 with $E = T_{-}$. Hence it remains to show that the three submodules of $\mathcal{D}^2$ above are indeed invariant under change of coordinates.

Suppose we change coordinates $x^\mu \to y^\mu(x, \theta)$ and $\theta^\nu \to \psi^\nu(x, \theta)$. Then by chain rule, we have

$$\begin{aligned}
\frac{\partial}{\partial x^\mu} &= \frac{\partial y^a}{\partial x^\mu}\frac{\partial}{\partial y^a} + \frac{\partial \psi^b}{\partial x^\mu}\frac{\partial}{\partial \psi^b} \\
\frac{\partial}{\partial \theta^\nu} &= \frac{\partial y^a}{\partial \theta^\nu}\frac{\partial}{\partial y^a} + \frac{\partial \psi^b}{\partial \theta^\nu}\frac{\partial}{\partial \psi^b}.
\end{aligned}$$

Since we are on M where the odd coordinates are zero, we see that

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^a}{\partial x^\mu}\frac{\partial}{\partial y^a} \qquad \frac{\partial}{\partial \theta^\nu} = \frac{\partial \psi^b}{\partial \theta^\nu}\frac{\partial}{\partial \psi^b}.$$

Hence $\mathcal{D}^2_+, \mathcal{D}^2_-, \mathcal{D}^2_\pm$ are indeed invariant submodules of $\mathcal{D}^2$ under a change of coordinates, hence are well-defined. □

Chapter 4

Forms and Dualities on Supermanifolds

4.1 Berezinian and Integration on Supermanifolds

On a purely bosonic manifold M^p, the natural object that we can integrate is a top dimensional differential form, i.e., a section of the canonical bundle $\bigwedge^p \Omega_M$. On a supermanifold, however, the differential of the odd variable commutes. Hence the complex of differential forms is not bounded from above. That is, there is no top degree differential form. The Berezinian will be the analogue of the canonical bundle in the supermanifold case, and we can naturally integrate a section of the Berezinian (We already mentioned it a little bit in Example **??**).

Let us first define the ***Berezin integral*** on $\mathbb{R}^{p|q}$. Let the coordinate of $\mathbb{R}^{p|q}$ be $t^1, ..., t^p|\theta^1, ..., \theta^q$. Then a general function $g(t^1...|...\theta^q)$ can be expanded as a polynomial in θ's:

$$g(t^1...|...\theta^q) = \sum_I g_I(t)\theta^I = g_0(t^1, ..., t^p) + \cdots + \theta^q\theta^{q-1}\cdots\theta^1 g_q(t^1, ..., t^p).$$

Then we define the integral of g on $\mathbb{R}^{p|q}$ as

$$\int_{\mathbb{R}^{p|q}} [\mathrm{d}t^1...|...\mathrm{d}\theta^q] g(t^1...|...\theta^q) = \int_{\mathbb{R}^p} \mathrm{d}t^1...\mathrm{d}t^p g_p(t^1, ..., t^p).$$

The left hand side is just formal notation, and the right hand side is to be taken as the definition. But to generalize integration to arbitrary supermanifolds, we need to make sense of the object $[\mathrm{d}t^1...|...\mathrm{d}\theta^q]$. We know it should be a section of some vector bundle over $\mathbb{R}^{p|q}$, hence is completely determined by its transition functions.

Definition 4.1 (Berezinian)**.** Let V be a $\mathbb{Z}_2$-graded vector space of dimension $p|q$. Let $V = V^0 \oplus V^1$ be a decomposition of V into even and odd subspaces, then any linear map $T : V \to V$ decomposes as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We define the ***Berezinian*** of T to be

$$\operatorname{Ber} T = \det\bigl(A - BD^{-1}C\bigr) \det(D)^{-1}.$$

One can check that the definition of $\mathrm{Ber}\, T$ does not depend on the choice of the decomposition $V = V^0 \oplus V^1$. This is not trivial, but several references can be found in [5].

Now let M be a compact supermanifold of dimension $p|q$. Let $\xi = (t^1...|...\theta^q)$ and $\xi' = (t^{1'}...|...\theta^{q'})$ be two local coordinates on M. Then one can show that

$$[\mathrm{d}t^{1'}...|...\mathrm{d}\theta^{q'}] = \mathrm{Ber}\left(\frac{\partial \xi'}{\partial \xi}\right)[\mathrm{d}t^1...|...\mathrm{d}\theta^q].$$

Therefore, the object $[\mathrm{d}t^1...|...\mathrm{d}\theta^q]$ can be interpreted as a section of a vector bundle over M which we define as follows:

Definition 4.2. Let $M^{p|q}$ be a supermanifold. For two local charts $\varphi_i : U_i \to \mathbb{R}^{p|q}$ and $\varphi_j : U_j \to \mathbb{R}^{p|q}$ of M, let $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$ be the change of coordinate morphism. The ***Berezinian*** $\mathcal{B}\mathrm{er}(M)$ of M is the line bundle of rank $1|0$ whose transition functions on $U_i \cap U_j$ is given by $\mathrm{Ber}\, \varphi_{ij}$.

Then, what can be naturally integrated over M is a section of $\mathcal{B}\mathrm{er}(M)$. For more details, see [5]. Now we define integration on a supermanifold $M^{p|q}$. First, take $\sigma \in \Gamma(M, \mathcal{B}\mathrm{er}(M))$ a section of Berezinian that is supported on some open set $U \subset M$ such that $U \cong \mathbb{R}^{p|q}$. Then since this is a line bundle, locally

$$\sigma = [\mathrm{d}t^1...|...\mathrm{d}\theta^q] g(t^1...|...\theta^q).$$

So we define

$$\int_M \sigma = \int_{\mathbb{R}^{p|q}} [\mathrm{d}t^1...|...\mathrm{d}\theta^q] g(t^1...|...\theta^q).$$

For a general section $\sigma \in \Gamma(M, \mathcal{B}\mathrm{er}(M))$, we choose an atlas $M = \cup_\alpha U_\alpha$ of M, such that $U_\alpha \cong \mathbb{R}^{p|q}$, and a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. Then we define

$$\int_M \sigma = \sum_\alpha \int_{U_\alpha} \rho_\alpha \sigma,$$

and now each term in the summand is defined above.

A particularly common situation is when we have fiber bundle maps of supermanifolds. Then one can show that the Berezinian is functorial in the following sense:

Proposition 4.3. Let $\pi : E \to B$ be a locally trivial fibration of supermanifolds with fiber F. Then the integration along fiber map $\pi_* : \mathcal{B}\mathrm{er}(E) \to \mathcal{B}\mathrm{er}(B)$ is functorial in the sense that

$$\int_E \sigma = \int_N \pi_* \sigma$$

for any section σ of $\mathcal{B}\mathrm{er}(E)$.

Proof. By a partition of unity argument, it suffices to check the equality in the case that the fibration is given by $\pi : \mathbb{R}^{p'|q'} \to \mathbb{R}^{p|q}$. Then this is just a simple application of Fubini's theorem. ☐

4.2 Differential and Integral Forms

Let $M^{p|q}$ be a supermanifold, with local coordinates $t^1...|...\theta^q$. Then we introduce the corresponding 1-forms $\mathrm{d}t^1, ..., \mathrm{d}t^p, \mathrm{d}\theta^1, ..., \mathrm{d}\theta^q$. As established in Example **??**, forms on M will corresponds to functions

$$\omega(t^1, ..., \mathrm{d}\theta^q | \theta^1, ..., \mathrm{d}t^p)$$

on ΠTM.

Definition 4.4. If the function $\omega(t^1, ..., \mathrm{d}\theta^q|\theta^1, ..., \mathrm{d}t^p) \in C^\infty(\Pi TM)$ has polynomial dependence on the even 1-forms $\mathrm{d}\theta^i$, then it is called a ***differential form*** on M. If in its dependence on $\mathrm{d}\theta^1, ..., \mathrm{d}\theta^q$, the function ω is a distribution supported at the origin (i.e., δ functions and its derivatives), then we call it an ***integral form*** on M.

A wedge product of two differential forms or a differential form and an integral form is defined by multiplying corresponding functions. The differential forms $\mathcal{A}^\bullet(M)$ are naturally graded by the number of differentials. The ***exterior derivative*** is defined in the obvious way as a vector field on ΠTM:

$$\mathrm{d} = \sum_i \mathrm{d}t^i \frac{\partial}{\partial t^i} + \sum_j \mathrm{d}\theta^j \frac{\partial}{\partial \theta^j}. \tag{4.1}$$

Proposition 4.5. $\mathrm{d}^2 = 0$.

Proof. The proof is similar to the bosonic case. □

Therefore, we have a chain complex $\mathcal{A}^0(M) \xrightarrow{\mathrm{d}} \mathcal{A}^1(M) \xrightarrow{\mathrm{d}} \cdots$ of differential forms. The cohomology $H^*_{\mathrm{dR}}(M) := H^*(\mathcal{A}^\bullet(M), \mathrm{d})$ of this chain complex is called the ***de Rham cohomology*** of M.

We will now introduce a grading on integral forms. A top degree integral form is given by element of the form

$$f(t^1 ...|...\theta^q)\mathrm{d}t^1...\mathrm{d}t^p\delta(\mathrm{d}\theta^1)...\delta(\mathrm{d}\theta^q).$$

Therefore, a top degree integral form has degree p which is the even dimension of the supermanifold M. We denote this space by $\Sigma^p(M)$. Note that such a form is annihilated by multiplication by $\mathrm{d}t^i$ (anticommutativity of $\mathrm{d}t$'s) or $\mathrm{d}\theta^i$ (because this will give a term proportional to $\mathrm{d}\theta^i\delta(\mathrm{d}\theta^i)$, and we know $x\delta(x) = 0$ in the distributional sense). An integral form of ***codimension*** i can be obtained from a section of $\Sigma^p(M)$ by applying i derivatives with respect to the variables $\mathrm{d}t^i$ or $\mathrm{d}\theta^j$. That is, a general element of an integral form of codimension i is given by

$$\frac{\partial}{\partial(\mathrm{d}t)^I \partial(\mathrm{d}\theta)^J} f(t^1 ...|...\theta^q)\mathrm{d}t^1...\mathrm{d}t^p\delta(\mathrm{d}\theta^1)...\delta(\mathrm{d}\theta^q),$$

where $|I| + |J| = i$. Since we are viewing $\mathrm{d}t^i$ and $\mathrm{d}\theta^j$ as independent variables in the coordinate ring of ΠTM, differentiation by $\mathrm{d}t^i$ needs no further explanation. Since the only expression above that depends on $\mathrm{d}\theta^j$'s are the delta functions, we explain a bit more about how the differentiation works. Clearly, if $i \neq j$ then

$$\frac{\partial}{\partial(\mathrm{d}\theta^j)}\delta(\mathrm{d}\theta^i) = 0.$$

If $i = j$, then we formally write

$$\frac{\partial}{\partial(\mathrm{d}\theta^i)}\delta(\mathrm{d}\theta^i) = \delta'(\mathrm{d}\theta^i),$$

and it formally becomes a derivative of the delta function. You can think of this as distributional derivative if you want. Similarly, taking k derivatives gives $\delta^{(k)}(\mathrm{d}\theta^i)$. We said before that $\mathrm{d}\theta^j\delta(\mathrm{d}\theta^j) = 0$. More generally, one can verify that in the sense of distributions

$$x \cdot \delta^{(k)}(x) = -k\delta^{(k-1)}(x). \tag{4.2}$$

Therefore, it follows that

$$\mathrm{d}\theta^j\delta^{(k)}(\mathrm{d}\theta^j) = -k\delta^{(k-1)}(\mathrm{d}\theta^j).$$

Let us make the above discussion into a formal definition:

Definition 4.6 (Integral forms)**.** Let $M^{p|q}$ be a supermanifold with local coordinates $t^1...|...\theta^q$. Locally, any integral form can be written as an $\mathcal{O}_M$-linear combination of elements of the form

$$\omega = f(t^1...|...\theta^q)\mathrm{d}t^{i_1}\dots\mathrm{d}t^{i_r}\delta^{(j_1)}(\mathrm{d}\theta^1)\dots\delta^{(j_q)}(\mathrm{d}\theta^q).$$

The ***degree*** $|\omega|$ of the integral form ω above is defined as

$$|\omega| = r - j_1 - \dots - j_q.$$

We denote the space of integral forms of degree m as $\Sigma^m(M)$. By default we set $\Sigma^m(M) = 0$ for all $m > p$, and we refer to $\Sigma^{p-i}(M)$ as the space of integral forms of codimension i.

Proposition 4.7. The exterior derivative increases the degree by 1. That is, for each m we have a map $\mathrm{d} : \Sigma^m(M) \to \Sigma^{m+1}(M)$.

Proof. We decompose the exterior derivative (4.1) into the even and odd part $\mathrm{d} = \mathrm{d}^0 + \mathrm{d}^1$, where $\mathrm{d}^0 = \sum_i \mathrm{d}t^i \frac{\partial}{\partial t^i}$ and $\mathrm{d}^1 = \sum_j \mathrm{d}\theta^j \frac{\partial}{\partial \theta^j}$. Let

$$\omega = f(t^1...|...\theta^q)\mathrm{d}t^{i_1}\dots\mathrm{d}t^{i_r}\delta^{(j_1)}(\mathrm{d}\theta^1)\dots\delta^{(j_q)}(\mathrm{d}\theta^q)$$

where $|\omega| = m$. Then we have

$$\mathrm{d}^0\omega = \sum_{i=1}^{p}\left(\frac{\partial f}{\partial t^i}\right)\mathrm{d}t^i \wedge \mathrm{d}t^{i_1}\dots\mathrm{d}t^{i_r}\delta^{(j_1)}(\mathrm{d}\theta^1)\dots\delta^{(j_q)}(\mathrm{d}\theta^q)$$

which increases the degree by 1 since we are wedging an extra $\mathrm{d}t^i$. On the other hand,

$$\mathrm{d}^1\omega = \sum_{\beta=1}^{q}\left(\frac{\partial f}{\partial \theta^\beta}\right)\mathrm{d}\theta^\beta \wedge \mathrm{d}t^{i_1}\dots\mathrm{d}t^{i_r}\delta^{(j_1)}(\mathrm{d}\theta^1)\dots\delta^{(j_q)}(\mathrm{d}\theta^q).$$

Now we apply (4.2) adn we see that each summand gives a term $\mathrm{d}\theta^\beta \cdot \delta^{(j_\beta)}(\mathrm{d}\theta^\beta) = -j_\beta\delta^{(j_\beta-1)}(\mathrm{d}\theta^\beta)$. Hence each term in the summand now has one degree higher. Hence we conclude that $\mathrm{d}\omega \in \Sigma^{m+1}(M)$, as desired. □

Since $\mathrm{d}^2 = 0$, it follows that we have another chain complex

$$\dots \to \Sigma^{-1}(M) \xrightarrow{\mathrm{d}} \Sigma^0(M) \xrightarrow{\mathrm{d}} \Sigma^1(M) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Sigma^{p-1}(M) \xrightarrow{\mathrm{d}} \Sigma^p(M) \xrightarrow{\mathrm{d}} 0.$$

We define the ***Spencer cohomology*** (or integral cohomology) to be the cohomology $H^*_{\mathrm{Sp}}(M) = H^*(\Sigma^\bullet(M), \mathrm{d})$ of this chain complex.

We will now give a different description of the space of integral forms.

Definition 4.8. Let V be a $\mathbb{Z}_2$-graded vector space and $V = V_0 \oplus V_1$ be its even and odd decomposition. The k-th ***supersymmetric power*** S^kV of V is defined as

$$S^kV = \bigoplus_{r+s=k} \mathrm{Sym}^r V_0 \oplus \bigwedge^s V_1.$$

If $\mathcal{E}$ is a $\mathbb{Z}_2$ graded vector bundle, then $S^k\mathcal{E}$ is defined similarly.

Theorem 4.9. $\Sigma^k(M) \cong \mathcal{B}\mathrm{er}(M) \otimes S^{p-k}\Pi TM$.

Proof. See Theorem 9.24 of [40]. □

Remark 4.10. The significance of introducing integral forms on supermanifold is that these are the object that we can integrate on. More specifically, we can integrate a section of $\Sigma^p(M) \cong \mathcal{B}\mathrm{er}(M)$ on M, and we can integrate a section of $\Sigma^{p-i}(M)$ over a supermanifold of codimension i inside M. The reason is that as we discussed in Example **??**, integrating a form ω on M is the same as integrating the function ω on ΠTM. That is,

$$\int_M \omega = \int_{\Pi TM} \beta(\Pi TM) \cdot \omega, \tag{4.3}$$

where $\beta(\Pi TM)$ is the volume form on ΠTM as introduced in Example **??**. That is, $\beta(\Pi TM)$ is a normalized section of $\mathcal{B}\mathrm{er}(\Pi TM)$. In fact, in the previous section we only introduced how to integrate a function on a supermanifold, so (4.3) is actually the definition of $\int_M \omega$.

Now, if ω has polynomial dependence on $\mathrm{d}\theta^1, ..., \mathrm{d}\theta^q$, i.e., if ω is a differential form on M, then the right hand side of (4.3) does not converge. Conclusion: integrating a differential form over a supermanifold does not make sense in general due to convergence issues, and we can only integrate differential forms over a purely bosonic manifold. However, if ω has distributional dependence on $\mathrm{d}\theta^1, .., \mathrm{d}\theta^q$, then the right hand side of (4.3) does make sense, and we can integrate ω.

Integration of an integral form on M and Berezin integration is equivalent in the following sense:

Proposition 4.11. Let $\pi : \Pi TM \to M$ be the parity-reversed tangent bundle and let $\omega \in \Sigma^p(M)$ be an integral form of top degree. Let $\sigma = \pi_*(\beta(\Pi TM)\omega) \in \mathcal{B}\mathrm{er}(M)$. Then

$$\int_M \sigma = \int_M \omega.$$

Proof. Apply Proposition 4.3 to the fibration $\pi : \Pi TM \to M$. □

4.3 Stokes's Theorem

There is a super analogue of Stokes's theorem for integral forms. Similar to the ordinary Stokes's theorem for bosonic manifold, the first step of the proof is to check the theorem holds in $\mathbb{R}^{p|q}$.

Proposition 4.12 (Stokes's Theorem, local version)**.** Let ν be a compactly supported integral form on $\mathbb{R}^{p|q}$ of codimension 1. Then

$$\int_{\mathbb{R}^{p|q}} \mathrm{d}\nu = 0.$$

Proof. Again, We decompose the exterior derivative (4.1) into the even and odd part $\mathrm{d} = \mathrm{d}^0 + \mathrm{d}^1$, where $\mathrm{d}^0 = \sum_i \mathrm{d}t^i \frac{\partial}{\partial t^i}$ and $\mathrm{d}^1 = \sum_j \mathrm{d}\theta^j \frac{\partial}{\partial \theta^j}$. Then we claim that

$$\int_{\mathbb{R}^{p|q}} \mathrm{d}^0\nu = \int_{\mathbb{R}^{p|q}} \mathrm{d}^1\nu = 0.$$

The integral of $\mathrm{d}^0\nu$ over the even variables vanishes by the ordinary bosonic version of Stokes's theorem. The integral of $\mathrm{d}^1\nu$ over the odd variables vanish because $\mathrm{d}^1\nu$ is a sum of terms none of which are proportional to the product $\theta^1...\theta^q$ of all the odd variables. □

We can immediately extend this to a general supermanifold:

Theorem 4.13 (Stokes's Theorem, version I)**.** *Let M be a supermanifold and ν be a compactly supported integral form on M of codimension 1. Then*

$$\int_M \mathrm{d}\nu = 0.$$

Proof. By a partition of unity argument, this follows immediately from the local version of the Stokes's theorem. ☐

Finally, we give the familiar version of Stokes's theorem for supermanifolds with boundary. We first must define what it means for a supermanifold to have boundary.

Definition 4.14. Let M be a supermanifold such that the reduced space $|M|$ is an ordinary manifold with boundary. We say M is a ***supermanifold with boundary*** of dimension $p|q$ if there are open covers $M = (\cup_\alpha U_\alpha) \cup (\cup_a V_a)$ such that for each α we have isomorphisms $\varphi_\alpha : U_\alpha \cong \mathbb{R}^{p|q}$ and isomorphisms

$$\psi_a : V_a \xrightarrow{\cong} \{(t_a^1 ... t_a^p|\theta_a^1 ... \theta_a^q) \in \mathbb{R}^{p|q} : t_a^p \geq 0\}.$$

As usual, we have the gluing functions in the intersections $U_\alpha \cap U_\beta, U_\alpha \cap V_b, V_a \cap V_b$. As for the gluing function ψ_{ab} on $V_a \cap V_b$, we require that

$$t_a^p = \exp\big(\psi_{ab}(t_b^1 ...|...\theta_b^q)\big)t_b^p.$$

The ***boundary*** ∂M of M is the subset defined by the equations $t_a^p = 0$.

Theorem 4.15 (Stokes's Theorem, Version II)**.** *Let M be a supermanifold with boundary and ν is a compactly supported integral form on M of codimension 1, then*

$$\int_M \mathrm{d}\nu = \int_{\partial M} \nu.$$

Proof. As usual, one first proves it by reduction to the ordinary form of Stokes's theorem for the case that ν is compactly supported in just one of the U_α or V_a. The general case follows by using a partition of unity. ☐

4.4 Poincaré Duality

Recall that for a supermanifold $M^{p|q}$, we have two chain complexes over M: The de Rham complex $(\mathcal{A}^\bullet, \mathrm{d})$ and the integral complex or Spencer complex $(\Sigma^\bullet, \mathrm{d})$. The corresponding cohomologies of these two complexes are the de Rham cohomology $H^*_{\mathrm{dR}}(M)$ and Spencer cohomology $H^*_{\mathrm{Sp}}(M)$. We will show that Berezin integral defines a perfect pairing of these two cohomology groups in the correct dimension. First, just as in the bosonic case, we need to show every closed form is locally exact:

Lemma 4.16 (Poincaré lemma)**.** *Let ω be a differential or integral form on a supermanifold M. If ω is closed, then locally there exists a form η such that $\mathrm{d}\eta = \omega$.*

Proof. Let the local coordinates of U be $Z^A = (x^1, \ldots, x^p|\theta^1, \ldots, \theta^q)$. We define a radial contraction flow $h_t : U \to U$ for $t \in [0, 1]$ by scaling all coordinates equally:

$$h_t(Z) = tZ.$$

Notice that $h_1 = \mathrm{id}_U$ (the identity map) and $h_0 = 0$ (projection to the origin). Next, we define the Euler vector field E on U, which generates this scaling:

$$E = \sum_{i=1}^{p} x^i \frac{\partial}{\partial x^i} + \sum_{a=1}^{q} \theta^a \frac{\partial}{\partial \theta^a} = \sum_A Z^A \frac{\partial}{\partial Z^A}.$$

We construct a homotopy operator $K : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$, and respectively $K : \Sigma^k(M) \to \Sigma^{k-1}(M)$, as follows: For every k-form ω, we define

$$K\omega = \int_0^1 \frac{1}{t} i_E(h_t^* \omega) \mathrm{d}t,$$

where i_E is the contraction with the Euler vector field. Now by Cartan's formula we have

$$\mathrm{d}(K\omega) + K(\mathrm{d}\omega) = \int_0^1 \frac{1}{t} \left(\mathrm{d} i_E(h_t^*\omega) + i_E\, \mathrm{d}(h_t^*\omega)\right) \mathrm{d}t = \int_0^1 \frac{1}{t} \mathcal{L}_E(h_t^*\omega) \mathrm{d}t.$$

A standard property of the radial scaling map is that the Lie derivative with respect to the Euler vector field matches the formal time derivative scaled by t:

$$\frac{1}{t} \mathcal{L}_E(h_t^*\omega) = \frac{\mathrm{d}}{\mathrm{d}t}(h_t^*\omega).$$

Substituting this identity into our integral reduces the entire expression to a total derivative:

$$\mathrm{d}(K\omega) + K(\mathrm{d}\omega) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(h_t^*\omega) \mathrm{d}t = h_1^*\omega - h_0^*\omega = \omega,$$

since $h_1^* = \mathrm{id}$ and $h_0^* = 0$. Now if ω is closed, the equation above reads $\mathrm{d}(K\omega) = \omega$, which means we can take $\eta = K\omega$. ☐

Next, we show that the de Rham and Spencer cohomology of M are both isomorphic to ordinary de Rham cohomology of the reduced space $|M|$. This should make sense intuitively: since as a topological space a supermanifold M is just identical to its reduced space $|M|$, we should not be able to get new topological invariants.

Theorem 4.17 (Quasi-Isomorphism)**.** *Let M be a supermanifold. Then for any k, we have isomorphisms*

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(|M|) \cong H^k_{\mathrm{Sp}}(M).$$

Proof. By Poincaré lemma, every d-closed form is locally d-exact. Then it follows from the existence of partitions of unity on M that the sheaves $\mathcal{A}^k$ and Σ^k are acyclic, meaning $H^i(M, \mathcal{A}^k) = H^i(M, \Sigma^k) = 0$ for all $i > 0$. Therefore, we have an acyclic resolution

$$0 \to \underline{\mathbb{R}}_{|M|} \hookrightarrow \mathcal{A}^1 \xrightarrow{\mathrm{d}} \mathcal{A}^2 \xrightarrow{\mathrm{d}} \mathcal{A}^3 \xrightarrow{\mathrm{d}} \ldots$$

where we recall that a sheaf on M is simply a sheaf on the reduced space $|M|$. Therefore, we have

$$H^k_{\mathrm{dR}}(M) = R^k\Gamma(M, \mathcal{A}^\bullet) =: H^k(|M|, \underline{\mathbb{R}}_{|M|}) \cong H^k(|M|; \mathbb{R}) \cong H^k_{\mathrm{dR}}(|M|).$$

Now we look at the Spencer resolution. The isomorphism $H^k_{\mathrm{dR}}(|M|) \cong H^k_{\mathrm{Sp}}(M)$ is a consequence of the Čech-to-de Rham spectral sequence for the double complex $(\Sigma^\bullet, \delta, \mathrm{d})$, where δ is the Čech differential and d the exterior differential of integral forms. On one hand, we still have the de Rham isomorphism $H^k_{\mathrm{dR}}(|M|) \cong H^k(|M|, \mathbb{R}_{|M|})$. On the other hand, we have the Spencer sheaf

$$H^k_\delta(\Sigma^\bullet) := \frac{\ker(\mathrm{d} : \Sigma^k \to \Sigma^{k+1})}{\mathrm{im}(\mathrm{d} : \Sigma^{k-1} \to \Sigma^k)},$$

which is the quotient of the kernel sheaf of d by the image sheaf of d (it is defined to be the sheafification of the image presheaf of d). The Poincaré lemma implies that

$$H^k_\delta(\Sigma^\bullet) = \begin{cases} \mathbb{R}_M & k = 0 \\ 0 & k > 0. \end{cases}$$

Hence it follows from the spectral sequence that $H^k_{\mathrm{Sp}}(M) \cong H^k(|M|, \mathbb{R}_{|M|}) \cong H^k_{\mathrm{dR}}(|M|)$. This concludes the proof. □

For details of this proof can be found in [40]. The quasi-isomorphism theorem above immediately gives the following:

Corollary 4.18.

$$H^k_{\mathrm{dR}}(\mathbb{R}^{p|q}) \cong H^k_{\mathrm{Sp}}(\mathbb{R}^{p|q}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0. \end{cases}$$

Now we present the Poincaré duality theorem on supermanifolds. We first remark that one can define compactly supported cohomology $H^*_{\mathrm{dR},c}(M)$ and $H^*_{\mathrm{Sp},c}(M)$ exactly as we defined de Rham and Spencer cohomology on M, except that we obviously require the forms in the cochain groups to have compact support. When M is compact, then these are just ordinary de Rham and Spencer cohomologies.

Theorem 4.19 (Poincaré Duality for Supermanifolds)**.** *Let M be a real supermanifold of dimension $p|q$ with $|M|$ oriented. Then, for any $n \geq 0$, the Berezin integral defines a perfect pairing in cohomology*

$$\langle -, - \rangle : H^{p-n}_{\mathrm{Sp},c}(M) \times H^n_{\mathrm{dR}}(M) \xrightarrow{\sim} \mathbb{R} \qquad \langle [\omega], [\eta] \rangle = \int_M \omega \wedge \eta.$$

Hence we have an induced isomorphism $\alpha_M : H^k_{\mathrm{Sp},c}(M) \xrightarrow{\sim} H^{p-k}_{\mathrm{dR}}(M)^*$.

Notation 4.20. In the proof below, we will commit to a slight abuse of notation and write $\langle [\omega], [\eta] \rangle$ as $\langle \omega, \eta \rangle$.

Proof. The pairing is well-defined by Stokes's theorem. We first assume that $M \cong \mathbb{R}^{p|q}$. by the previous corollary, the only nontrivial pairing occurs when $n = 0$, and this is a pairing between one dimensional vector spaces. There exists a section of compactly supported Berezinian β such that $\int_{\mathbb{R}^{p|q}} \beta = 1$, which shows that $\alpha : H^p_{\mathrm{Sp},c}(\mathbb{R}^{p|q}) \to H^0_{\mathrm{dR}}(\mathbb{R}^{p|q})$ is surjective, hence is an isomorphism. This proves the Poincaré duality in the case of $M = \mathbb{R}^{p|q}$.

The remainder of the proof is a classic Mayer-Vietoris argument and induction on cardinality of good covers, See [41],§5 for details. We consider the case where $M = U \cup V$, where the two open sets $\{U, V\}$ are each isomorphic to $\mathbb{R}^{p|q}$ and all possible intersections are contractible (this is called a ***good cover***). We consider the short exact sequences of complexes. For de Rham forms, we have the restriction sequence:

$$0 \to \mathcal{A}^n(U \cup V) \xrightarrow{r} \mathcal{A}^n(U) \oplus \mathcal{A}^n(V) \xrightarrow{\delta} \mathcal{A}^n(U \cap V) \to 0$$

where $r(\eta) = (\eta|_U, \eta|_V)$ and $\delta(\eta_U, \eta_V) = \eta_U|_{U\cap V} - \eta_V|_{U\cap V}$. For compactly supported integral forms, we have the extension-by-zero sequence:

$$0 \to \Sigma_c^k(U \cap V) \xrightarrow{e} \Sigma_c^k(U) \oplus \Sigma_c^k(V) \xrightarrow{s} \Sigma_c^k(U \cup V) \to 0$$

where $e(\omega) = (\omega, -\omega)$ and $s(\omega_U, \omega_V) = \omega_U + \omega_V$, which extends the forms ω_U, ω_V by zero and then adds them. Taking the long exact sequences of these two complexes and taking dual of the second one gives a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^n_{\mathrm{Sp},c}(U\cap V) \arrow[r, "e_*"] \arrow[d, "\alpha_{U\cap V}"'] & H^n_{\mathrm{Sp},c}(U) \oplus H^n_{\mathrm{Sp},c}(V) \arrow[r, "s_*"] \arrow[d, "\alpha_U \oplus \alpha_V"'] & H^n_{\mathrm{Sp},c}(M) \arrow[r, "d_*"] \arrow[d, "\alpha_M"'] & H^{n+1}_{\mathrm{Sp},c}(U\cap V) \arrow[r, "e_*"] \arrow[d, "\alpha_{U\cap V}"'] & \cdots \\
\cdots \arrow[r] & H^{p-n}_{\mathrm{dR}}(U\cap V)^* \arrow[r, "\delta^*"] & H^{p-n}_{\mathrm{dR}}(U)^* \oplus H^{p-n}_{\mathrm{dR}}(V)^* \arrow[r, "r^*"] & H^{p-n}_{\mathrm{dR}}(M)^* \arrow[r, "d^*"] & H^{p-n-1}_{\mathrm{dR}}(U\cap V)^* \arrow[r, "\delta^*"] & \cdots
\end{tikzcd}$$

We check that all the squares are commutative up to a sign. Then since the rows are exact, by five lemma we can deduce α_M is an isomorphism, proving Poincaré duality for the case where $M = U \cup V$ is the union of two split superspace isomorphic to $\mathbb{R}^{p|q}$.

Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. Checking commutativity of the first two squares is straightforward. For the map e_* and δ^*:

$$\langle e_*\omega, (\eta_U, \eta_V)\rangle = \int_U \omega \wedge \eta_U + \int_V (-\omega) \wedge \eta_V = \int_{U\cap V} \omega \wedge (\eta_U - \eta_V) = \langle \omega, \delta_*(\eta_U, \eta_V)\rangle.$$

This commutes exactly. Similarly, for s_* and r^*:

$$\langle s_*(\omega_U, \omega_V), \eta\rangle = \int_{U\cup V} (\omega_U + \omega_V) \wedge \eta = \int_U \omega_U \wedge \eta|_U + \int_V \omega_V \wedge \eta|_V = \langle (\omega_U, \omega_V), r_*(\eta)\rangle.$$

It remains to check the third square that involves the connecting homomorphisms. We recall the definitions from p.24,45 of [41]: Let $[\eta] \in H^n_{\mathrm{dR}}(U \cap V)$. Its image under the de Rham connecting homomorphism $d^* : H^n_{\mathrm{dR}}(U \cap V) \to H^{n+1}_{\mathrm{dR}}(U \cup V)$ is represented by $\mathrm{d}(\rho_U \eta) = \mathrm{d}\rho_U \wedge \eta$. Let $[\omega] \in H^{p-n-1}_{\mathrm{Sp},c}(U \cup V)$. Its image under the connecting homomorphism $d_* : H^{p-n-1}_{\mathrm{Sp},c}(U \cup V) \to H^{p-n}_{\mathrm{Sp},c}(U \cap V)$ is represented by $\mathrm{d}(\rho_U \omega) = \mathrm{d}\rho_U \wedge \omega$. Hence we have

$$\langle d_*\omega, \eta\rangle = \int_{U\cap V} (\mathrm{d}\rho_U \wedge \omega) \wedge \eta = \pm \int_{U\cap V} \omega \wedge (\mathrm{d}\rho_U \wedge \eta) = \pm\langle \omega, d^*\eta\rangle.$$

Note that in the third equality we used the fact that $d^*\omega$ is supported in $U \cap V$, see p.24 of [41]. Hence, by five lemma, α_M is an isomorphism.

So we have proven that if the Poincaré duality holds on good covers U and V, then it holds on $U \cup V$. The proof is concluded by induction on the number of good covers. □

We know that a compactly supported integral form on $M^{p|q}$ of codimension zero can be naturally integrated over M. Now we claim that an integral form of codimension r can be naturally integrated over a submanifold of codimension $r|0$.

Indeed, for every submanifold $N^{p-r|q} \subset M^{p|q}$ of codimension $r|0$, there is a ***Poincaré dual*** r-form δ_N. If N is locally defined by the vanishing of even functions $f_1, ..., f_r$, which are real valued when the odd variables θ^s all vanish, then locally

$$\delta_N = \delta(f_1)...\delta(f_r)\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_r,$$

where one orders the factors so as to agree with the orientation of the normal bundle $\mathcal{N}_{N/M}$ to N. One can check that changing the defining equations does not change δ_N, so the definition make sense globally.

Remark 4.21. Several remarks are in order: (1) The Poincaré dual δ_N, as the name suggests, arises naturally from Poincaré duality as follows: Poincaré duality gives an isomorphism (after taking duals) $H^{p-r}_{\mathrm{Sp},c}(M)^* \cong H^r_{\mathrm{dR}}(M)$. Now if N is a submanifold of M of codimension r, then integration over N is an element of $H^{p-r}_{\mathrm{Sp},c}(M)^*$. Hence by Poincaré duality, this corresponds to a unique element $\delta_N \in H^r_{\mathrm{dR}}(M)$, which is what we call the Poincaré dual of N. By construction, its defining property is that for any μ representing a class in $H^{p-r}_{\mathrm{Sp},c}(M)$, we have

$$\int_M \delta_N \wedge \mu = \int_N \mu.$$

This explains the notation δ_N: It is the form analogue of a delta function over N.

(2) If N is locally the zero set of $f_1, ..., f_r$, then the functions $f_1, ..., f_r$ provides the local fiber coordinates of the normal bundle $\mathcal{N}_{N/M}$. Therefore, by the expression of δ_N given above, if $\pi : \mathcal{N}_{N/M} \to N$ is the fibration map, then fiber integrating the Poincaré dual δ_N of N gives:

$$\pi_* \delta_N = 1.$$

In fact, this can be proven in general: The Poincaré dual δ_N of N can be chosen to be the Thom class of the normal bundle $\mathcal{N}_{N/M} \to N$, which is characterized by exactly the property that $\pi_*\delta_N = 1$. See §6 of [41].

(3) Technically, since $\delta_N \in H^r_{\mathrm{dR}}(M)$ is a differential form, we should demand the coefficient function of δ_N to be smooth functions on M, but in our expression we have $\delta(f_1)...\delta(f_r)$ which are distributions on M. This formal but clever definition is given by Witten in [5]. There are two ways to remedy this: First is to note that by remark (2) and uniqueness of Poincaré dual, we can take the Poincaré dual of N to be any form that is supported in an arbitrarily small neighborhood of the zero section of $\mathcal{N}_{N/M}$, that fiber integrates to 1. Hence we can explicitly construct a family of forms cohomologous to the Poincaré dual of N by taking a sequence of smooth bump functions φ_ϵ with total integral 1 and shrinking support around 0. Then the family is given by

$$\Phi_\epsilon = \varphi_\epsilon(f_1) \dots \varphi_\epsilon(f_r) \mathrm{d}f_1 \wedge \dots \wedge \mathrm{d}f_r.$$

Then as $\epsilon \to 0$, the forms Φ_ϵ converge to δ_N in the sense of distribution. Another way to interpret this is that we can actually allow distributional coefficients, and instead of a differential form, we will get an object known as the ***current***. There is a rich theory of cohomology of currents, and it turns out that they are isomorphic to the de Rham cohomology [42]. Hence we can actually allow forms to have distributional coefficients.

If we perturb N slightly to a homologous submanifold N, then Poincaré dual allows us to prove that the integral of any closed integral form of correct codimension over N and N' are the same:

Proposition 4.22. If $\mu \in \Sigma^{p-r}(M)$ is a closed integral form of codimension r, and $N \sim N'$ (which means N and N' are homologous submanifold in M), then

$$\int_N \mu = \int_{N'} \mu.$$

Proof. Since $N \sim N'$, we have $\delta_N - \delta_{N'} = \mathrm{d}\tau$ for some compactly supported $(r-1)$-form τ. Since μ is closed, we have $\mathrm{d}(\tau \wedge \mu) = \mathrm{d}\tau \wedge \mu$. Therefore,

$$\int_N \mu - \int_{N'} \mu = \int_M \mathrm{d}\tau \wedge \mu = \int_M \mathrm{d}(\tau \wedge \mu) = 0$$

by Stokes's theorem. □

Chapter 5

Super Riemann Surfaces and Supermoduli

5.1 Super Riemann Surfaces

Definition 5.1. Let S be a supermanifold of dimension $1|1$. A distribution (i.e., a subsheaf of the tangent sheaf) $\mathcal{D} \subset TS$ is called a ***superconformal structure*** if it is

- an odd distribution, i.e., a subbundle of rank $0|1$.
- ***everywhere non-integrable***. Recall that by Frobenius theorem [14, p.48], a distribution is integrable if it is closed under Lie bracket. Since $\mathcal{D}$ is an odd distribution, its Lie bracket will be anticommutator. By assumption $\mathcal{D}$ is of rank $0|1$, so it will be generated by a single odd vector field v. Since $v^2 := \frac{1}{2}\{v, v\}$, we define $\mathcal{D}$ to be ***integrable*** if $v^2 \in \mathcal{D}$, and define it to be ***everywhere non-integrable*** if v^2 is everywhere independent of v over $\mathcal{O}_S$.

Definition 5.2. A ***Super Riemann Surface*** (SRS) is a pair $(S, \mathcal{D})$, where $S = (C, \mathcal{O}_S)$ is a complex supermanifold of dimension $1|1$, and $\mathcal{D}$ is a superconformal structure.

Remark 5.3. We know from above that $\mathcal{D}$ is everywhere non-integrable if $v^2 \notin \mathcal{D}$. But since v is an odd vector field, its square is an even vector field, so you might wonder how could v^2 possibly be in $\mathcal{D}$, which is an odd distribution. But remember that unlike in the ordinary manifold situation where we can multiply vectors only by even functions $\mathcal{O}_M$, now on supermanifolds we can multiply by element in $\mathcal{O}_S$, which could be odd functions. For example, it could happen that $\theta v = v^2$, as both sides are even. But in this case $\theta v - v^2 = 0$ so v, v^2 are not linearly independent over $\mathcal{O}_S$.

Remark 5.4. Since S is of dimension $1|1$, TS has rank $1|1$. Since $\mathcal{D}$ has rank $0|1$ and $\mathcal{D}^2$ is even hence has rank $1|0$, together they generate the entire tangent bundle TS. Moreover, we have an isomorphism $TS/\mathcal{D} \cong \mathcal{D}^2$. Therefore we actually have an exact sequence of sheaves

$$0 \to \mathcal{D} \to TS \to \mathcal{D}^2 \to 0.$$

Lemma 5.5. *Locally on a SRS S one can choose coordinates x and θ, refered to as* ***superconformal coordinates****, such that $\mathcal{D}$ is generated by the vector field*

$$v := \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}.$$

Proof. We first choose any local coordinate system $y|\eta$ on S. Let $\mathcal{D}$ be generated by the odd vector field v, which we can always locally write as

$$v = F(y,\eta)\partial_\eta + G(y,\eta)\partial_y.$$

Taylor expand $G(y,\eta)$ with respect to η, and since η is fermionic, η^2 and higher order term vanishes, so $G(y,\eta) = G_0(y) + \eta G_1(y)$, where $G_0(y)$ is an odd function that depends only on y and not η, and $G_1(y)$ an even function that depends only on y. The condition that v, v^2 generate TS implies that $G_1 \neq 0$ everywhere. Given this assumption, it becomes a standard exercise in differential geometry to find a coordinate transformation $x(y,\eta), \theta(y,\eta)$ that puts v into the desired form above. □

Corollary 5.6. *Assume set up in Lemma 5.5, we have $v^2 = \partial_x$. In some sense, we found a square root of the vector ∂_x.*

Proof. We apply the vector v to a function $h(x,\theta) = f(x) + \theta g(x)$, where the right hand side is Taylor expansion of h with respect to θ, and again since $\theta^2 = 0$ the Taylor series terminate at linear order. Using the expression in Lemma 5.5 we see that

$$v(h) = g(x) + \theta f'(x).$$

Doing this once again we see that

$$v^2(h) = f'(x) + \theta g'(x) = \partial_x(h).$$

This proves the claim. □

Recall that a divisor on a supermanifold has codimension $1|0$. Hence it is natural to define

Definition 5.7. A divisor on a SRS is a dimension $0|1$ submanifold. Hence it is a collection of points, with some odd dimension.

Example 5.8. One way to specify a subvariety (or a divisor) of dimension $0|1$ in any complex supermanifold is to give a point p and an odd tangent vector at p. There is then a unique subvariety of dimension $0|1$ that passes through p with the given tangent vector.

For the case of a SRS, let p be the point defined by equations $x = x_0, \theta = \theta_0$. In this case, a subvariety D passing through p has local equation $x = x_0 + \alpha\theta, \theta = \theta_0 + \alpha$, where α is an odd parameter that parameterizes D. Hence D is given by the equation

$$x = x_0 - \theta_0\theta.$$

In general, on a supersubmanifold of dimension $1|1$, a divisor that in some local coordinate system defined by the above equation is called a ***minimal divisor*** or a ***prime divisor***. Since coordinates $p = (x_0, \theta_0)$ are arbitrary, there is a 1-1 correspondence between points and minimal divisors on a SSR. On a general supermanifold of dimension $1|1$, there is no such natural correspondence between points and minimal divisors, which makes sense since a SRS is a more specialized classes of dimension $1|1$ supermanifolds (with an odd distribution $\mathcal{D}$).

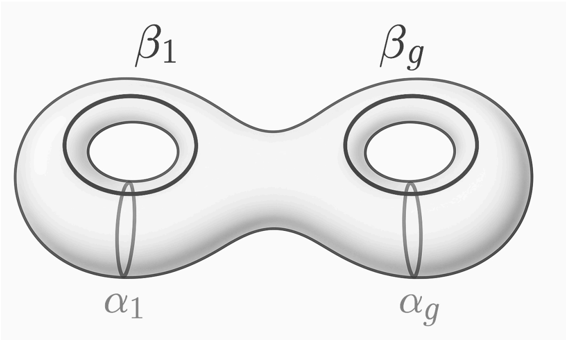


Figure 5.1: A genus g Riemann surface Σ (Here $g = 2$). The loops α_i, β_i is a set of generators of $H_1(\Sigma) = \mathbb{Z}^6$.

5.2 Moduli and Supermoduli

Although moduli space and supermoduli space is the main subject of study in this book, going through all details on the fundamentals of moduli space will take a whole book itself. Therefore, we content ourselves with the intuitive picture of moduli space, and refer interested reader to any textbooks on moduli spaces, and many good notes on the subject, for example [15].

In a word, a moduli space is a parameter space that parameterizes isomorphism classes of objects of our interest. In our situation, the object of interests are ***Riemann surfaces*** or ***algebraic curves*** (or simply curves), which are compact complex manifolds with complex dimension 1. Such a Riemann surface Σ will always satisfy $H_1(\Sigma) = \mathbb{Z}^{2g}$, we we define the ***genus*** of Σ to be g. See Figure 5.1 for an example.

We can consider the moduli space $\mathcal{M}_g$ of curves of genus g. A point in $\mathcal{M}_g$ will correspond to an isomorphism class of curves. Its boundary corresponds to singular curves. Inside the curve, the moduli space $\mathcal{M}_g$ is smooth away from the curves that has nontrivial automorphisms. In other words, automorphism of curves actually prevents the moduli space from being smooth. This is because if a curve Σ has nontrivial automorphism group $\mathrm{Aut}(\Sigma)$, then locally around the point Σ, the moduli space looks like a quotient of a smooth space by a finite group, in other words there is a quotient singularity.

Hence a more abstract but somehow better way, in some sense, to describe moduli space is to consider the moduli stack M_g, which involves the language of algebraic stacks. We will not use this machinery in our book, because in our cases it is enough to think about $\mathcal{M}_g$ as a manifold with possibly some singularities. We refer the reader to [15, Chap.5] for an introduction to the theory of algebraic stacks.

Similarly, one can consider the supermoduli space $\mathfrak{M}_g$ of super Riemann surface (SRS) with genus g, which we mean the genus of the reduced space. However, since every split SRS has a nontrivial automorphism $(x, \theta) \mapsto (x, -\theta)$, which is identity on the reduced space, we conclude that SRS with nontrivial automorphisms are actually dense in any family of SRS. This means the stacky nature of the moduli problem is more essential for SRS than the ordinary ones, and what we call supermoduli space $\mathfrak{M}_g$ must be properly understood not as a supermanifold but as the supermanifold analogue of a stack.

Recall from Corollary 2.11, a $1|1$ supermanifold S defined over a field is split, so it is specified by a pair (C, V), where C is a Riemann surface and V is the analytic line bundle on C underlying $\mathcal{D}$. Recall from Corollary 5.6, we actually have $V^{\otimes 2} \cong T_C$. Any V that satisfies this condition is called a ***spin structure*** on C. We will thus denote V by $T_C^{1/2}$, and similarly we shall denote its n-th power by by $T_C^{n/2}$ or equivalently $K_C^{-n/2}$

where $K_C = T_C^* = T_C^{-1}$ is the canonical bundle on C.

An ordinary Riemann surface with a spin structure is often called a ***spin curve***. Hence the reduced space of C of a super Riemann surface S is a spin curve. It follows that the reduced space of the supermoduli $\mathfrak{M}_g$ is the moduli space $\mathcal{SM}_g$ of spin curves.

Now we can do deformation theory on SRS. Thus we must consider automorphisms on the SRS. Locally, the infinitesimal automorphisms are generated by ***superconformal vector fields***, which preserves the distribution $\mathcal{D}$. In superconformal coordinates, a short calculation shows that the even superconformal vector field takes the form

$$f(x)\partial_x + \frac{1}{2}f'(x)\theta\partial_\theta, \tag{5.1}$$

while the odd one takes the form

$$-g(x)(\partial_\theta - \theta\partial_x). \tag{5.2}$$

One define a subsheaf W of TS whose local sections are of the form (5.1) and (5.2). But note that W is not a locally-free sheaf, since multiplying a superconformal vector field by a function of x and θ, or even a function of x, does not in general give a new superconformal vector field. In fact W is only a sheaf of $\mathbb{Z}_2$-graded Lie algebras.

However, we can give W the structure of a locally free sheaf because of the following

Theorem 5.9. *Under the projection $TS/\mathcal{D} = \mathcal{D}^2$, we can project W to the sheaf of sections $\mathcal{D}^2$ via the inclusion $W \to TS$. This projection gives an isomorphism $W \cong \mathcal{D}^2$.*

Proof. We recall that a general section of $\mathcal{D}^2$ is of the form $A(x,\theta)\partial_x$ for some function A, we must show that $A(x,\theta)\partial_x$ can in a unique fashion be written as a section of W modulo $\mathcal{D}$. But note that since we can add vector of the form in Lemma 5.5 since we are computing modulo $\mathcal{D}$, we have

$$\begin{aligned}
f(x)\partial_x + \frac{f'(x)}{2}\theta\partial_\theta - g(x)(\partial_\theta - \theta\partial_x) &= f(x)\partial_x + g(x)\theta\partial_x + \frac{f'(x)}{2}\theta\partial_\theta - g(x)\theta \\
&\sim f(x)\partial_x + 2g(x)\theta\partial_x + \frac{f'(x)}{2}\theta\partial_\theta \\
&\sim f(x)\partial_x + 2g(x)\theta\partial_x \frac{f'(x)}{2}\theta(-\theta\partial_x) \\
&= f'(x)\partial_x + 2g(x)\theta\partial_x
\end{aligned}$$

since $\theta^2 = 0$, we can choose f, g such that $A(x,\theta) = f(x) + 2\theta g(x)$. In fact the left hand side is in 1-1 correspondence with the right hand side by Taylor expanding A with respect to θ, and since $\theta^2 = 0$ there are precisely two terms, corresponding to $f(x)$ and $2g(x)$. This shows that the map in question is indeed an isomorphism. $\square$

Hence by the theorem above, we have $H^1(S, W) = H^1(S, \mathcal{D}^2)$. In fact, this gives the tangent space to the supermoduli $\mathfrak{M}_g$ at the point S:

$$T_S\mathfrak{M}_g = H^1(S, W) = H^1(S, \mathcal{D}^2).$$

This should not come as a surprise, since in ordinary algebraic geometry, the tangent of moduli space $\mathfrak{M}_g$ at a point X is given by $T_X\mathfrak{M}_g = H^1(X, T_X)$. Note that under the correspondence of infinitesimal automorphism $x^\mu \mapsto x^\mu + \epsilon h^\mu$ and the vector field $h^\mu\partial/\partial x^\mu$, the automorphism of X is identified with T_X (See Chapter A for more details).

When S is split, the sheaf W of supercomformal vector fields decomposes into $W = W_+ \oplus W_-$ where $W_+ = T_C$ and $W_- = T_C^{1/2}$. Hence we may identify the even and odd tangent spaces $T_\pm \mathfrak{M}_g$ via

$$\begin{aligned} T_{+,S}\mathfrak{M}_g &= H^1(C, T_C) \\ T_{-,S}\mathfrak{M}_g &= H^1(C, T_C^{1/2}). \end{aligned}$$

5.3 Punctures

In the construction of moduli space $\mathcal{M}_g$ of genus g curves, we identified two curves Σ, Σ' in the moduli space $\mathcal{M}_g$ if there is an isomorphism $\varphi : \Sigma \to \Sigma'$. We can also consider $\mathcal{M}_{g,n}$, the moduli space of curves with n marked point. This space consists of equivalence classes of $(\Sigma, p_1, ..., p_n)$, where Σ is a genus g curve, and $p_1, ..., p_n \in \Sigma$ are just n points on the curve. We identify $(\Sigma, p_1, ..., p_n)$ and $(\Sigma', q_1, ..., q_n)$ if there exists an isomorphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi(p_i) = q_i$ for $i = 1, ..., n$.

The notion of marked point, or a ***puncture*** on an ordinary Riemann surface has two analogs on a super Riemann surface. In string theory, they are known as the ***Neveu-Schwarz puncture*** (NS puncture) and ***Ramond puncture*** (R puncture).

An NS puncture on a SRS S is the obvious analog of a puncture on an ordinary Riemann surface: it is simply the choice of a point in S, given by $x = x_0, \theta = \theta_0$. From Example 5.8, we learned that an NS puncture determines and is determined by a minimal divisor on S. In deformation theory, NS puncture at a point p is obtained by restricting the sheaf W of superconformal vector fields to its subsheaf W_p consisting of superconformal vector fields that leave the point p fixed. We have the following result:

Proposition 5.10. Let F be the minimal divisor that corresponds to the point p. Then W_p can be given a natural structure of a locally-free sheaf. In fact,

$$W_p \cong \mathcal{D}^2(-F),$$

where the right hand side is a short hand notation for $\mathcal{D}^2 \otimes \mathcal{O}_S(-F)$, where we used the identification between the divisor class group $\mathrm{Div}(S)$ and line bundles $\mathrm{Pic}(S)$, given by $D \mapsto \mathcal{O}_S(D)$, as in classical algebraic geometry.

Proof. We take superconformal coordinate x, θ, and without loss of generality we set p to be the point where $x = \theta = 0$. The condition that the superconformal vector field vanishes at p, from (5.1) and (5.2), is that $f(0) = g(0) = 0$. From the proof of Theorem 5.9, this means that $A(x, \theta) = f(x) + 2\theta g(x)$ vanishes at $x = 0$. But the divisor corresponding to the point p is precisely defined by $x = 0$ by Example 5.8, see also [2, Remark 3.2]. So the condition $A(0, \theta) = 0$ precisely mean that $A(x, \theta)\partial_x$ is a section of $\mathcal{D}^2(-F)$. □

Let us now consider the supermoduli $\mathfrak{M}_{g,1}$ with just a single NS puncture. Then its reduced space is $\mathcal{SM}_{g,1}$, which parameterizes spin curves S of genus g with a single puncture p. $\mathcal{SM}_{g,n}$ has two components $\mathcal{SM}_{g,n}^\pm$, corresponding to even and odd spin structures. Similarly, $\mathfrak{M}_g$ has two components $\mathfrak{M}_g^\pm$, corresponding to even and odd spin structure.

And we have the analogus tangent space identification:

$$\begin{aligned} T_{+,S}\mathfrak{M}_{g,1} &= H^1(C, T_C(-p)) \\ T_{-,S}\mathfrak{M}_g &= H^1(C, T_C^{1/2}(-p)). \end{aligned}$$

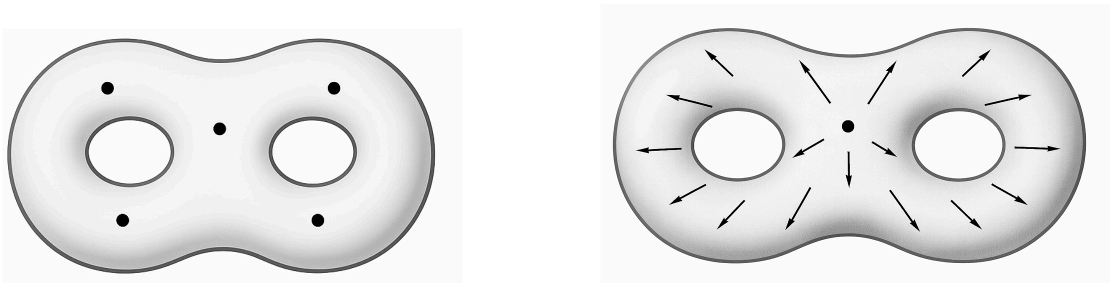
Now we introduce the second kind of puncture: ***Ramond puncture***. In this situation, we assume the underlying supermanifold $(C, \mathcal{O}_S)$ is still smooth, but the odd distribution $\mathcal{D} \subset T_{-}S$ is no longer everywhere non-integrable. The generator v in the local form in Theorem 5.5 is replaced by

$$v := \frac{\partial}{\partial \theta} + x\theta \frac{\partial}{\partial x}.$$

In other words, $\mathcal{D}$ fails to be a distribution along the divisor $x = 0$. It can be shown that in the presence of a Ramond puncture, the sheaf of superconformal vector fields is still isomorphic to $\mathcal{D}^2$, but $\mathcal{D}^2$ is no longer isomorphic to $TS/\mathcal{D}$. Instead, we have

$$\mathcal{D}^2 \cong (TS/\mathcal{D}) \otimes \mathcal{O}_S(-F).$$

We can also have multiple Ramond punctures, by which we mean $\mathcal{D}$ fails to be distribution along multiple divisors, and near each divisor we can find local coordinates as described above. See Fig 5.2 for an illustration of two kinds of punctures.



(a) **(b)**

Figure 5.2: Two kinds of punctures on SRS. (a): A $g = 2$ SRS with a series of marked points, i.e., NS punctures; (b): A $g = 2$ SRS with a Ramond punctures marked in black, i.e., points where the distribution $\mathcal{D}$ degenerates (We surpress θ by looking at the reduced space).

5.4 Branched Covering

We have already seen branched covering of complex manifolds in Definition 1.24. For our purposes, we will focus on branched covering of supercurves. However, it is helpful to first get a feeling of this construction in the ordinary (bosonic case), which we will discuss below.

Suppose we have a smooth map $f : X \to Y$ between manifolds. We know what it means for f to be a covering map. A branched covering f is something that is almost a covering map, except at a bad subset $B \subset Y$ of codimension one. Recall that a codimension one submanifold is called a divisor, hence the bad subset B on which f fails to be a covering map is called the ***branch divisor***. Intuitively, B is the set of points where different sheets of the cover merges together, or equivalently the set of points $p \in Y$ where $f^{-1}(p)$ has less points than excepted compared to points outside of B, where f is a normal covering map. The points $R = f^{-1}(B)$ are the points on which f fails to be a covering, and we say f is ***ramified*** on R. The points in R are called the ***ramification points*** of f. See Fig.5.3 for an illustration.

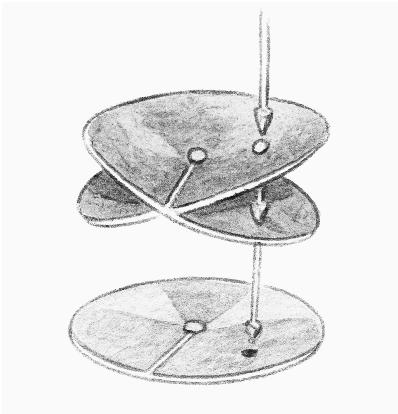


Figure 5.3: A 2-sheeted branched cover X of a disk Y. The branched divisor $B \subset Y$ in this case is a line radiating from the origin, on which X fails to be a 2-sheeted cover.

Example 5.11. The canonical example of branched cover we should keep in mind is the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^n$ for some n. The map f is clearly a covering map of degree n away from $z = 0$, but is ramified at the point $z = 0$. Hence in this case, the branch divisor is simply given by the point $z = 0$.

Now we give formal definition of branched cover between complex curves (Riemann surfaces), and we will see that Example 5.11 fully characterizes branched covers, at least locally in this case.

Definition 5.12. Let $f : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. A point $x \in X$ is called a ***branch point*** or ***ramification points*** of f, if there is no neighborhood U of x such that $f|_U$ is injective. The map f is called an ***unbranched holomorphic map*** if f has no branched points.

Hence, a ***branched cover*** $f : X \to Y$ of Riemann surfaces is specified by a divisor $B \subset Y$ of branched points of f, which is called the ***branch divisor***, and is a covering map restricted to $X - f^{-1}(B) \to Y - B$.

Theorem 5.13. *Let $f : X \to Y$ be a holomorphic branched cover of Riemann surfaces with branch divisor $B \subset Y$. Then for any $p \in f^{-1}(B)$, we can always find local holomorphic charts $p \in U \subset X$, $\varphi : U \to \varphi(U) \subset \mathbb{C}$, and $f(U) \subset V \subset Y$, $\psi : V \to \psi(V) \subset \mathbb{C}$, such that*

- $\varphi(p) = 0, \psi(f(p)) = 0.$
- *The local representation $\hat{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is given by $\hat{f}(z) = z^k$ for some $k \geq 2$ and for all $z \in \varphi(U)$.*

Proof. The statement just follows from the classification theorem for local behavior of holomorphic mappings, which is proven in [17, p.10]. Note that the assumption that f is holomorphic is essential for the theorem to apply. □

Next, we define branched covers of supercurves. But in order for us to formulate the definition, we need a little bit of algebra:

Definition 5.14 (Finite presentation)**.** We say $R \to S$ is of ***finite presentation*** if there exist integers $n, m \in \mathbb{N}$ and polynomials $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ and an isomorphism of R-algebras

$$R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \cong S.$$

Definition 5.15 (étale morphism)**.** We say that a ring homomorphism $\varphi : A \to B$ is ***étale*** (resp. ***smooth***, ***unramified***), or that B is étale (resp. smooth, unramified) over A if the following two conditions are satisfied:

- $A \to B$ is ***formally étale*** (resp. ***formally smooth***, ***formally unramified***): for every square-zero extension of A-algebras $R' \to R$ (meaning that the kernel I satisfies $I^2 = 0$), the natural map

$$\mathrm{Hom}_A(B, R') \to \mathrm{Hom}_A(B, R)$$

  is bijective (resp. surjective, injective).

- B is ***essentially of finite presentation*** over A: $A \to B$ factors as $A \to C \to B$, where $A \to C$ is of finite presentation and $C \to B$ is C-isomorphic to a localization morphism $C \to S^{-1}C$ for some multiplicatively closed subset $S \subset C$.

The second condition is just a finiteness condition; the meat of the concept is in the first one. Formal smoothness is often referred to as the infinitesimal lifting property. Geometrically speaking, it says that if the affine scheme $\operatorname{Spec} B$ is smooth over $\operatorname{Spec} A$, then any map from $\operatorname{Spec} R$ to $\operatorname{Spec} B$ lifts to any square-zero (and hence any infinitesimal) deformation $\operatorname{Spec} R'$. Moreover, if $\operatorname{Spec} B$ is étale over $\operatorname{Spec} A$ this lifting is unique.

Differential-geometrically, unramifiedness, smoothness and étaleness correspond to the tangent map of $\operatorname{Spec} \varphi$ being injective, surjective and bijective, respectively. In particular, étale is the generalization to the algebraic case of the concept of local isomorphism.

Now we are ready to define branched cover of supercurves.

Definition 5.16. Let S be a supercurve, i.e., a dimension $1|1$ supermanifold, and let $B \subset S$ be a closed subscheme of codimension $1|0$, and let $U = S - \mathcal{B}$.

A ***branched cover*** of S with ***branch locus*** $\mathcal{B}$ is a finite, surjective morphism $\pi : \tilde{S} \to S$ such that the local morphisms $\pi_p^* : \mathcal{O}_{S,\pi(p)} \to \mathcal{O}_{\tilde{S},p}$ are étale for all $p \in U$.

The cover π may be ramified along a closed subscheme $\mathcal{R} \subset \pi^{-1}(\mathcal{B}) \subset \tilde{S}$ of codimension $1|0$ called the ***ramification divisor*** of π. The bosonic reduction $\pi_{\mathrm{bos}} : \tilde{C} \to C$ of $\pi : \tilde{S} \to S$ is ramified along a divisor $R = |\mathcal{R}|$ with branch locus $B = |\mathcal{B}|$.

An ***isomorphism of branched covers*** is a pair of isomorphisms of supercurves $\phi : S \to S'$ and $\tilde{\phi} : \tilde{S} \to \tilde{S}'$ such that the following diagram commutes

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi"'] & \tilde{S}' \arrow[d, "\pi'"] \\
S \arrow[r, "\phi"'] & S'
\end{tikzcd}$$

and such that $\phi(\mathcal{B}) = \mathcal{B}'$ and $\tilde{\phi}(\mathcal{R}) = \mathcal{R}'$.

Remark 5.17 (local description)**.** Let $\pi : \tilde{S} \to S$ be a branched cover, and let $\mathcal{P} \subset \mathcal{R}$ be a connected component of the ramification divisor of π. Near $\mathcal{P}$, we can find (local) coordinates (w, θ') and (z, θ) on $\tilde{S}$ and S, respectively, such that $\mathcal{B} \subset S$ is defined by the equation $z = 0$ and π is locally of the form

$$z = w^k, \quad \theta = w^\ell \theta'$$

for some $k > 0$, $\ell \geq 0$.

Branched covering has intimate relations with NS and R punctures. In fact, we have the following theorem:

Theorem 5.18. *Let S be a SRS, $\tilde{S}$ a supermanifold and $\pi : \tilde{S} \to S$ is a branched cover. We can blow up $\tilde{S}$ at an ideal to get a new SRS Q with punctures, such that the composition $\Pi : Q \to S$ is still a branched cover, and the NS punctures of Q are precisely the ramification points of π with odd local degree, and R punctures of Q are precisely the ramification points of π with even local degree.*

Proof. We choose superconformal coordinates x, θ on S. Recall that from Example 1.25 that we can find local coordinates z, θ on $\tilde{S}$ such that locally $w^k = z$ near a branch point (so k is the local degree).

Now by Lemma 5.5, the superconformal vector field v on S is given by $v = \partial_\theta + \theta\partial_z$. But now using $\partial_z = (kw^{k-1})^{-1}\partial_w$, we see that the lifted vector field is given by

$$\tilde{v} = \frac{\partial}{\partial\theta} + \frac{1}{kw^{k-1}}\theta\frac{\partial}{\partial w},$$

which now has a parabolic structure. What we can do, is to blow up $\tilde{S}$ at the point $w = \theta = 0$. This will give a new set of coordinates y, ψ on the blow up Q with $y = w$ and $\psi = \theta/w$. Now using chain rule:

$$\begin{aligned}\frac{\partial}{\partial\theta} &= \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\psi} = 0 + \frac{1}{w}\frac{\partial}{\partial\psi} = \frac{1}{y}\frac{\partial}{\partial\psi},\\ \frac{\partial}{\partial w} &= \frac{\partial y}{\partial w}\frac{\partial}{\partial y} + \frac{\partial\psi}{\partial w}\frac{\partial}{\partial\psi} = \frac{\partial}{\partial y} - \frac{\psi}{y}\frac{\partial}{\partial\psi}.\end{aligned}$$

Therefore, the vector $\tilde{v}$ on Q now becomes

$$\chi = \frac{1}{y}\frac{\partial}{\partial\psi} + \frac{y\psi}{ky^{k-1}}\left(\frac{\partial}{\partial y} - \frac{\psi}{w}\frac{\partial}{\partial\psi}\right) = \frac{1}{y}\frac{\partial}{\partial\psi} + \frac{1}{ky^{k-2}}\frac{\partial}{\partial y}.$$

Away from $y = 0$, the superconformal structure on Q is generated by

$$y\chi = \frac{\partial}{\partial\psi} + \frac{1}{ky^{k-3}}\frac{\partial}{\partial y}.$$

Therefore, we see that blowing up once decreases the parabolic structure from order $\sim w^{-(k-1)}$ to order $\sim w^{-(k-3)} = w^{-(k-1)+2}$ by 2.

In general, we can consider the ideal on $\tilde{S}$ given by $I_l = (w^l, w^k, \theta)$, and we blow up $\tilde{S}$ along this ideal to obtain Q. This will decrease the parabolic structure form $\sim w^{-(k-1)}$ to $\sim w^{-(k-1)+2l}$ by $2l$.

We see that if k is odd, then $k-1$ is even, so we can blow up along some ideal I_l so that Q reduces the parabolic structure to just NS punctures (marked points). However, if k is even, then $k-1$ is odd, so after blow up along some ideal will this form will decrease the parabolic structure at most to R punctures. ☐

Part II

A Crash Course in Quantum Field Theory

Chapter 6

First Installment

6.1 Introduction

Quantum field theory (QFT) is the universal language for many phenomena. Some examples includes particle physics (LHC collisions, ν scattering...), condensed matter physics, statistical physics (phase transitions, superconductivity, quantum Hall effect...), cosmology (perturbations from inflation, freeze in and freeze out), and also in mathematics (topology, differential geometry, algebraic geometry, representation theory...). We will mostly focus on particle physics side, but many of QFT can be transported in many different settings mentioned above.

QFT is helpful whenever lots of states are involved, and symmetries and locality are in play. Historically, physicists were forced to do two things at once: combining quantum mechanics and special relativity.

Now that I have told you how wonderful QFT is, I should tell you what QFT actually is. What is a definition of a QFT? Nobody knows. What we know are many examples. So QFT is a work in progress, and to define it abstractly to the level where mathematicians are satisfied is still an open problem.

We will mainly work in flat space in 4 dimension, $\mathbb{R}^{3,1}$. We will write $x^\mu = (\mathbf{x}, t)$ as spacetime coordinates. A field is an assignment of a scalar (spinor, vector...) operator to each point in spacetime. Very abstractly, we write $\hat{\mathcal{O}}(x^\mu)$ where $\hat{\mathcal{O}}$ is a quantum mechanical operator. These could be scalar fields (Higgs bosons, for example), electromagnetic fields...

Classically, we write down a Lagrangian or a Hamiltonian, and we derive the equations of motion. What we do in quantum mechanics? We quantize the system. So we turn fields into operators: $\varphi \mapsto \hat{\varphi}$.

6.2 Harmonic Oscillators

Example 6.1. We look at an example of QFT in $D = 0 + 1$ dimensions, no one time dimension, and no space dimension. So we will have a field $\varphi(t)$ that depends only on t. I will take φ to be my favourite example: the harmonic oscillator, whose Hamiltonian is

$$H = \frac{P_\varphi^2}{2m} + \frac{1}{2}k\varphi^2.$$

And its Lagrangian is

$$L = \frac{1}{2}m\dot{\varphi}^2 - \frac{1}{2}k\varphi^2.$$

In QM class, they might write φ as the position x in space, but here we will have no space dimension, φ is just a field. Now I quantize the system but putting hats on everything. We universally set $\hbar = 1$. We require

$$[\hat{\varphi}, \hat{P}_\varphi] = i.$$

We can write them in terms of the raising and lowering operator

$$\hat{\varphi} = \sqrt{\frac{1}{2m\omega}}(a^\dagger + a)$$

and

$$\hat{P}_\varphi = i\sqrt{\frac{m\omega}{2}}(a^\dagger - a)$$

where $\omega = \sqrt{k/m}$, and $[a, a^\dagger] = 1$, $a\,|0\rangle = 0$ and $a^\dagger\,|0\rangle = \sqrt{n+1}\,|n+1\rangle$. The field evolution is

$$\hat{\varphi}(t) = \mathrm{e}^{i\hat{H}t}\hat{\varphi}(0)\mathrm{e}^{-i\hat{H}t}.$$

Example 6.2 (Coupled Harmonic Oscillators)**.** This is the model of acoustic phons on a lattice. The Lagrangian is

$$L = \sum_{i=1}^{N} \frac{1}{2}m\dot{\phi}_i^2 - \frac{mc^2}{2}\left(\frac{\phi_i - \phi_{i+1}}{\ell}\right)^2$$

where we set $\phi_1 = \phi_{N+1}$. So there are only nearest neighbor interactions, and ℓ is the spacing of the lattice. See Fig.6.1 for an illustration.

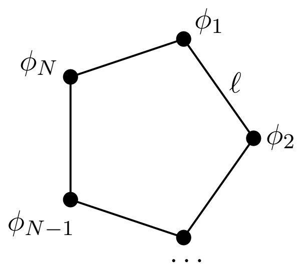


Figure 6.1: Acoustic phons on a lattice

Let us view this model as the discretized approximation to a (1+1)-dimensional QFT. Allowing continuous variables for i and setting $\ell \to 0$, the Lagrangian becomes

$$L = \left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2. \tag{6.1}$$

In the above we set $m = c = 1$. What about Hamiltonian? The discretized system has Hamiltonian

$$H = \sum_{i=1}^{N} \frac{1}{2}\frac{P_i^2}{m} + \frac{mc^2}{2}\left(\frac{\phi_i - \phi_{i+1}}{\ell}\right)^2.$$

Quantizing, we require $[\hat{\phi}_i, \hat{P}_j] = i\delta_{ij}$. We want to diagonalize H, just like where in SHO we switch from position and momentum to raising and lowering operator where the Hamiltonian is diagonalized. So we change the basis.

Let us first look at classical equation of motions for the discrete version. The Euler-Lagrange equations for L reads

$$m\ddot{\phi}_i - \left(-\frac{mc^2}{\ell^2}(2\phi_i - \phi_{i+1} - \phi_{i-1})\right) = 0$$

for each $i = 1, ..., N$. This equation is solved by the technique of (discretized) Fourier transform. We introduce the Fourier modes

$$\tilde{\phi}_k(t) = \frac{1}{\sqrt{N}} \sum_n \mathrm{e}^{ik\ell n} \phi_n(t)$$

and the inverse Fourier transform

$$\phi_n(t) = \frac{1}{\sqrt{N}} \sum_{k>0}^{2\pi/\ell} \mathrm{e}^{-ik\ell n} \tilde{\phi}_k(t)$$

where $k = 2\pi s/N\ell$ with $s \in \mathbb{Z}$. Plug this into EOM gives

$$\frac{\mathrm{d}^2\tilde{\phi}_k}{\mathrm{d}t^2} = -\frac{c^2}{\ell^2}(2 - 2\cos k\ell)\tilde{\phi}_k.$$

Using half-angle formula, this is

$$\frac{\mathrm{d}^2\tilde{\phi}_k}{\mathrm{d}t^2} = -\frac{4c^2}{\ell^2}\sin^2\frac{k\ell}{2}\tilde{\phi}_k.$$

Thus we see that in the frequency space, the EOM decouples. We now define

$$\omega_k = \frac{2c}{\ell}|\sin(k\ell/2)|$$

so that

$$\frac{\mathrm{d}^2\tilde{\phi}_k}{\mathrm{d}t^2} = -\omega_k^2\tilde{\phi}_k.$$

Now that I have noticed that the expression of ϕ_n is a good guess for the classical solution, we rewrite L in terms of the Fourier modes. Note that

$$\begin{aligned}\sum_n \dot{\phi}_n^2 &= \sum_n \frac{1}{N} \sum_{r,s} \mathrm{e}^{-2\pi i(sn+rn)/N} \frac{\mathrm{d}\tilde{\phi}_s}{\mathrm{d}t}\frac{\mathrm{d}\tilde{\phi}_r}{\mathrm{d}t} \\ &= \frac{1}{N}\sum_{r,s} N\delta_{r,-s}\frac{\mathrm{d}\tilde{\phi}_s}{\mathrm{d}t}\frac{\mathrm{d}\tilde{\phi}_r}{\mathrm{d}t} = \sum_s \frac{\mathrm{d}\tilde{\phi}_s}{\mathrm{d}t}\frac{\mathrm{d}\tilde{\phi}_{-s}}{\mathrm{d}t}.\end{aligned}$$

Hence the expression of L becomes

$$L = \frac{1}{2}\sum_k \left(\frac{\mathrm{d}\tilde{\phi}_k}{\mathrm{d}t}\frac{\mathrm{d}\tilde{\phi}_{-k}}{\mathrm{d}t} - \omega_k^2\tilde{\phi}_k\tilde{\phi}_{-k}\right).$$

Let's start quantizing! We promote everything into operators and we introduce the conjugate momenta:

$$\frac{\partial L}{\partial\dot{\phi}_n} = m\dot{\phi}_n \equiv \Pi_n.$$

We require

$$[\phi_n, \Pi_{n'}] = i\delta_{n,n'}.$$

In the frequency space, we have

$$[\tilde{\phi}_k, \tilde{\Pi}_{k'}] = \sum_{n,n'} U_{kn} U_{k'n'} [\phi_n, \Pi_{n'}] = i \sum_n U_{kn} U_{k'n} = i\delta_{k,-k'},$$

where we define

$$U_{kn} = \frac{1}{\sqrt{N}} \mathrm{e}^{-ik\ell n}.$$

We write

$$\tilde{\phi}_k = \sqrt{\frac{1}{2\omega_k}} (a_k + a^\dagger_{-k}) \tag{6.2}$$

and

$$\tilde{\Pi}_k = -i\sqrt{\frac{\omega_k}{2}} (a_k - a^\dagger_{-k})$$

where $[a_k, a^\dagger_{k'}] = \delta_{k,k'}$. The Hamiltonian now reads

$$H = \sum_{k=-\pi/\ell}^{\pi/\ell} \hbar\omega_k \left(a^\dagger_k a_k + \frac{1}{2} \right) + \frac{\tilde{P}_0^2}{2}$$

where the last term is the zero mode energy. Of course, there is a ground state $|0\rangle$ which is annihilated $a_k |0\rangle = 0$. And we have $a^\dagger_k |0\rangle = |k\rangle$. The interpretation is that the raising operator created a particle on the lattice on the site k. We can also have $a^\dagger_k a^\dagger_{k'} |0\rangle = |k, k'\rangle$, which means now we have two particles at site k, k' respectively.

We define the Hilbert space $\mathbf{H}_M$ of M particles, which is spanned by all elements of the form

$$(a^\dagger_{k_1})^{N_1} \cdots (a^\dagger_{k_m})^{N_m} |0\rangle$$

such that $\sum_i N_i = M$. We can also introduce the ***Fock space***, which is defined by

$$\mathbf{F} = \bigoplus_M \mathbf{H}_M.$$

The next approximation will be to take continuim limit. We take $N \to \infty$, $\ell \to 0$, so $k\ell \to 0$ in the above example. So that

$$\omega_k = \frac{2c}{\ell} \sin(k\ell/2) \simeq \frac{2c}{\ell}\frac{k\ell}{2} = ck.$$

Then the L above becomes (6.1) in the continuous limit.

So what is the moral of the story? The moral is that we should write scalar quantum field $\phi(x^\mu)$ as a sum of annihilation and creation operators a_k and $a^\dagger_k$, because when we Fourier transform and quantize, the Fourier modes (6.2) will be a sum of these operators. When we study spin fields as summations of these operators with a spin index (creates/annihilates spin up and down particles). In the bosonic case we introduce commutation relations to these operators, but fermionic case will also require anticommutation relation.

6.3 Symmetries

We will be studying fields of the form $\phi(x^\mu)$, where $x^\mu \in \mathbb{R}^{3,1}$. Compared to discrete field theory, our setting has a lot more symmetries, which is one reason why we are interested with continuum field theory.

Our goal is to set up the path integral for continuum QFT's. We will balance two competing considerations. One one hand, we will be interested in spacetime symmetries, but we will also pay attention to theory that are unitary (the consideration is quantum mechanical). The first consideration prefers Lagrangians, while the second prefers Hamiltonians.

So, what symmetries do we want? Obviously, we have rotations. The rotation group is

$$O(N) = \{R \in GL_n(\mathbb{R}) : R^T R = 1\}$$

There are two connected components of $O(N)$. The component in which $\det R = +1$ is called $SO(N)$. It is clear that rolation preserves length of a vector. But in special relativity, we instead want to preserve the relativity norm

$$g_{\mu\nu} x^\mu x^\nu = c^2 t^2 - x^2 - y^2 - z^2$$

where $x^\mu = (t, x, y, z)$. Here, we choose the convention

$$g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1).$$

We will be interested in ***Lorentz transformations***. These are transformation of the form

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$$

that preserves relativity norm. The Lie group is called $SO(3, 1)$.

Example 6.3. Ordinary rotations in $\mathbb{R}^3$ (the spacial direction) is a Lorentz transformation. Another example is a boost in the t and x direction:

$$\begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and we allow $-\infty < \beta < \infty$, where β is called ***rapidity***. To see how this matches the boost we are used to in special relativity, recall that those boosts are given by

$$x \mapsto \frac{x + vt}{\sqrt{1 - v^2}} \qquad t \mapsto \frac{t + vx}{\sqrt{1 - v^2}}.$$

Here, we just set

$$\cosh\beta = \frac{1}{\sqrt{1 - v^2}} \qquad \sinh\beta = \frac{v}{\sqrt{1 - v^2}}$$

and we recover the form of boost above.

What happens to a vector V^μ or a tensor $T^{\mu\nu}$ under Lorentz transformations?

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$$

Also recall the 4-momentum is $p = (E, p^1, p^2, p^3)$, which transforms exactly the same:

$$p^\mu \mapsto \Lambda^\mu{}_\nu p^\nu.$$

And tensor transforms as
$$T^{\mu\nu} \mapsto \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}.$$
In general,
$$W^{\mu_1\cdots\mu_m} \mapsto \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_m}{}_{\nu_m}W^{\nu_1\cdots\nu_m}.$$
We call vectors with $V^\mu V_\mu > 0$ ***timelike***, vectors with $V^\mu V_\mu < 0$ ***spacelike***, vectors with $V^\mu V_\mu = 0$ ***null***. We stress that discrete transformations are not in $SO(3,1)$. For example, the transformation
$$\mathrm{diag}(1,-1.-1,-1)$$
flips the space direction, so is called a ***parity*** transformation. Similarly
$$\mathrm{diag}(-1,1,1,1)$$
is called ***time-reversal***. Finally,
$$\mathrm{diag}(1,-1,1,1)$$
is called ***reflection***. How about fields? What happens to these fields when I apply a Lorentz transformation? The transformation rule, for example for a vector field, transforms like
$$A^\mu(x) \mapsto \Lambda^{\mu}{}_{\nu}A^\nu(\Lambda x).$$

Another interesting question is how a field changes under translation. Doing a Taylor expansion gives
$$\phi(x^\mu) \mapsto \phi(x^\mu + \epsilon^\mu) \simeq \phi(x) + \epsilon^\mu\partial_\mu\phi(x)$$
which shows that ∂_μ is the infinitesimal generator for translations. If we do similar analysis for Lorentz transformation, one can check that to first order,
$$\Lambda_{\mu\nu} \simeq g_{\mu\nu} + L_{\mu\nu}$$
where
$$L_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu.$$

6.4 Quantization of Free Field

Let us consider the Lagrangian
$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$
The EL equation reads
$$g^{\mu\nu}\partial_\mu\partial_\nu\phi = 0.$$
Note that expressions above are clearly Lorentz invariant because all the indices are contracted and by our previous discussion on how Lorentz transformation acts on tensors. The equation reads
$$\partial_t^2\phi = \nabla^2\phi.$$
It is not hard to check that the solution to this equation is
$$\phi(x^\mu) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}[a_\mathbf{p}(t)\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}} + a^*_\mathbf{p}(t)\mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}}],$$

where $\omega_{\mathbf{p}}^2 = \mathbf{p}^2$, and such that $(\partial_t^2 + \mathbf{p}^2)a_p(t) = 0$. Equivalently, we can absorb the time dependence into the exponent and get a cleaner solution

$$\phi(x^\mu) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}[a_{\mathbf{p}}\mathrm{e}^{-ip_\mu x^\mu} + \hat{a}_{\mathbf{p}}^*\mathrm{e}^{ip_\mu x^\mu}]$$

such that $p_\mu p^\mu = 0$. The expression above now looks more relativistic. In the future, we are not even going to write $p_\mu x^\mu$, we will simply write $p \cdot x$ or even just px.

Now we are ready to quantize. We replace the amplitudes by creation and annihilation operators. Hence the quantized free field is

$$\hat{\phi}(x^\mu) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}[\hat{a}_{\mathbf{p}}\mathrm{e}^{-ipx} + \hat{a}_{\mathbf{p}}^\dagger\mathrm{e}^{ipx}]$$

where we require (ignoring the hat)

$$[a_{\mathbf{p}}, a_{\mathbf{k}}^\dagger] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{k}).$$

What is the Hilbert space? We need a ground state $|0\rangle$. Of course we have $a_{\mathbf{p}}|0\rangle = 0$. Moreover, we require

$$a_{\mathbf{p}}^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}|\mathbf{p}\rangle,$$

where $|\mathbf{p}\rangle$ is a state representing a particle with momentum $\mathbf{p}$, and again $\omega_{\mathbf{p}}^2 = \mathbf{p}^2$.

Let us check this normalization is correct. Using the commutation relation, we have

$$\langle\mathbf{p}|\mathbf{k}\rangle = \sqrt{4\omega_{\mathbf{p}}\omega_{\mathbf{k}}}\langle 0|a_{\mathbf{p}}a_{\mathbf{k}}^\dagger|0\rangle = \sqrt{4\omega_{\mathbf{p}}\omega_{\mathbf{k}}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k}) = 2\omega_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k}).$$

Let us check causality, Say x^μ is a point inside the lightcone, and y^μ outside. So $(x-y)^2 < 0$, they are spacelike separated. A natural question to ask is whether $\hat{\phi}(x^\mu)$ affects $\hat{\phi}(y^\mu)$. Causality says this must not be the case. How do we see this? We calculate $[\phi(x), \phi(y)]$. Because by Hisenberg's uncertainty, if the commutator vanishes, then they do not affect each other. We have

$$\begin{aligned}[\phi(x), \phi(y)] &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}[a_{\mathbf{p}}\mathrm{e}^{-ipx} + a_{\mathbf{p}}^\dagger\mathrm{e}^{ipx}, a_{\mathbf{q}}\mathrm{e}^{-iqy} + a_{\mathbf{q}}^\dagger\mathrm{e}^{iqy}] \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(\mathrm{e}^{-ip(x-y)} - \mathrm{e}^{ip(x-y)}).\end{aligned}$$

Can we sent $x-y$ to $y-x$? Hence set $z = x-y$, we are asking does there exist a Lorentz transformation such that $z \mapsto -z$? If so, then by a change of coordinate, the two integrals above cancels each other, and we are done. Indeed, because x, y are spacelike separated, there is indeed such a Lorentz transformation. Hence we conclude that $[\phi(x), \phi(y)] = 0$.

6.5 Euler-Lagrange Equation for Fields, Symmetries

In this section we will review classical field theory, and their symmetries. The Lagrangian L of a field theory usually involves fields and their derivatives, so $L = L[\phi, \dot{\phi}]$. An equivalent formulation is achieved by using Hamiltonian $H = H[\phi, \Pi]$ where Π is the ***conjugate momentum*** of the field ϕ. The conversion between these two formulations are achieved by Legendre transform

$$\Pi = \frac{\partial L}{\partial \dot{\phi}} \qquad H = \Pi\dot{\phi} - L.$$

Similarly the discussion carries over to Lagrangian densities and Hamiltonian densities. A typical example is the theory (now using Lagrangian density)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi).$$

The Hamiltonian density is therefore

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi).$$

The equation of motion is obtained by variation of the action

$$S[\phi] = \int \mathrm{d}^4x\mathcal{L}[\phi, \partial_\mu\phi].$$

The variation reads

$$\begin{aligned}
S[\phi+\delta\phi] &= \int \mathrm{d}^4x\mathcal{L}[\phi+\delta\phi, \partial_\mu(\phi+\delta\phi)] \\
&= \int \mathrm{d}^4x\left(\mathcal{L}[\phi,\partial_\mu\phi] + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right) \\
&= S[\phi] + \int \mathrm{d}^4x\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi
\end{aligned}$$

after integration by parts. Note that we used $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$ because ∂_μ, δ are both derivatives, hence they commute. The variation $S[\phi+\delta\phi] - S[\phi]$ must be zero, hence we recovered the ***Euler-Lagrange equation*** (EL for short):

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

Example 6.4. Consider the theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

Since $\partial\mathcal{L}/\partial\phi = -m^2\phi$ and $\partial\mathcal{L}/\partial(\partial_\mu\phi) = \partial^\mu\phi$, the EL equation reads

$$-\partial_\mu\partial^\mu\phi - m^2\phi = 0.$$

Or equivalently, introducing the operator $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$, we have

$$(\Box + m^2)\phi = 0.$$

The field ϕ satisfying the above equation is called the ***Klein-Gordan field*** of mass m.

Example 6.5. Now suppose we have a theory with two Klein-Gordan fields;

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}\omega\psi^2.$$

The EL equations are just

$$(\Box + m^2)\phi = 0 \qquad (\Box + \omega^2)\psi = 0.$$

Assuming for the moment that $m = \omega$, then let us define the complex fields

$$\varphi = \frac{1}{\sqrt{2}}(\phi + i\psi) \qquad \varphi^* = \frac{1}{\sqrt{2}}(\phi - i\psi).$$

Then using these new fields the Lagrangian becomes

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi.$$

Note that the expression above is still real, because it is invariant under conjugation. This new expression is completely equivalent to the Lagrangian $\mathcal{L}$ we started with, but now it has a new interesting symmetry: it is invariant under the symmetry

$$\varphi \mapsto \mathrm{e}^{-i\alpha}\varphi.$$

This is a global $U(1)$-symmetry, where α is independent of x^μ. More generally, if the Lagrangian is invariant under the symmetry $\varphi \mapsto \mathrm{e}^{-i\alpha(x)}\varphi$, then we say that the theory has a ***gauge symmetry***.

Since the Lagrangian $\mathcal{L}[\varphi]$ is the same as the Lagrangian $\mathcal{L}[\phi, \psi]$, the latter should also have this symmetry. Indeed, one can check that the original Lagrangian is invariant under the transformation

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Hence equivalently, the theory has a $SO(2)$-rotation symmetry. It is very easy to generalize to the theory

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2}(\partial_\mu \phi_{(i)})^2 - \frac{1}{2}m^2 \phi_{(i)}^2.$$

And it is immediate just from the expression that the theory has a $SO(N)$-symmetry.

6.6 Noether's Theorem

The reason that symmetry is so good is that they implies that there are conserved quantities. More concretely, each symmetry gives a conserved current J^μ, conserved in the sense that $\partial_\mu J^\mu = 0$.

This gives the conserved charge

$$Q = \int_\Sigma \mathrm{d}^3 x J^0,$$

where Σ is a connected smooth submanifold. Physically, if we require that nothing can flow out of Σ, i.e., the surface integral of Σ dotted with any vector is zero (which is a physical requirement), then the charge Q is conserved by Stoke's theorem:

$$\partial_t Q = \int_\Sigma \mathrm{d}^3 x \partial_0 J^0 = \int_\Sigma \mathrm{d}^3 x (\nabla \cdot \mathbf{J}) = \int_{\partial\Sigma} \mathbf{J} \cdot \mathbf{dA} = 0.$$

Theorem 6.6 (Noether)**.** *Every continuous symmetry gives rise to a conserved current.*

Proof. Suppose we have a continuous symmetry parameterized by α, such that infinitesimally $\phi_a \to \phi_a + \delta\phi_a = \phi_a + R^b{}_a \delta\phi_b$, where ϕ_a's are all the fields that appears in the

Lagrangian $\mathcal{L}$, for $a = 1, ..., N$. Then symmetry of $\mathcal{L}$ under α implies

$$\begin{aligned} 0 = \frac{\delta \mathcal{L}}{\delta \alpha} &= \frac{\partial \mathcal{L}}{\partial \phi_a}\frac{\delta \phi_a}{\delta \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{\delta(\partial_\mu \phi_a)}{\delta \alpha} \\ &= \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\frac{\delta \phi_a}{\delta \alpha} + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{\delta \phi_a}{\delta \alpha}\right). \end{aligned}$$

The first term vanishes by EL, hence if we define

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{\delta \phi_a}{\delta \alpha} \tag{6.3}$$

then clearly we have $\partial_\mu J^\mu = 0$, as desired. Note that the expression is summed over all fields, i.e., summed over a. □

Example 6.7. In the example where

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi,$$

we have

$$\frac{\delta \varphi}{\delta \alpha} = -i\varphi \qquad \frac{\delta \varphi^*}{\delta \alpha} = i\varphi^*.$$

Then (6.3) reads

$$J^\mu = \partial^\mu \varphi^*(-i\varphi) + \partial^\mu \varphi(i\varphi^*).$$

If we lower the index $J_\mu = g_{\mu\nu} J^\nu$ we have

$$J_\mu = -i(\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi).$$

It is not explicitly obvious that $\partial_\mu J^\mu = 0$. But if we use the classical equations of motion, it follows that $\partial_\mu J^\mu = 0$. It is not surprising that equations of motion are needed, since in the proof of Noether's theorem, we used the EL equation.

Example 6.8. We can also talk about spacetime symmetries. We have translation symmetry, for example. Under infinitesimal translation

$$\phi(x) \mapsto \phi(x + \xi) \simeq \phi(x) + \xi^\nu \partial_\nu \phi(x).$$

In this case

$$\frac{\delta \phi}{\delta \xi^\nu} = \partial_\nu \phi$$

and

$$\partial_\nu \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \xi^\nu} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{\delta \phi_a}{\delta \xi^\nu}\right)$$

where the second equality is obtained by the same variational procedure. Hence moving the lefthand side from the left, we get

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{\delta \phi_a}{\delta \xi^\nu} - g_{\mu\nu}\mathcal{L}\right) = 0.$$

Hence in this case, we have four Noether currents, one for each ν:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{\delta \phi_a}{\delta \xi^\nu} - g_{\mu\nu}\mathcal{L}.$$

These are called ***energy-momentum tensor***. Note that the component $T^0{}_0$ is the energy density.

Example 6.9 (gauge theories)**.** Let's go back to the $U(1)$-example. The Lagrangian is

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi - V(|\varphi|).$$

What if we consider the symmetry $\varphi \mapsto \mathrm{e}^{-i\alpha(x)}\varphi$? This will give a change to the Lagrangian $\mathcal{L}[\varphi] \mapsto \mathcal{L}[\mathrm{e}^{-i\alpha(x)}\varphi]$. Let us introduce

$$\mathcal{L}' = \mathcal{L}[\phi] - A_\mu J^\mu$$

where the transformation of A_μ is given by $A_\mu \to A_\mu + \partial_\mu \alpha$, then the same variational Lagrangian $\mathcal{L}'$ is invariant under $\mathcal{L}'[\varphi] \mapsto \mathcal{L}'[\mathrm{e}^{-i\alpha(x)}\varphi]$ by simple calculation. These A_μ's are called ***gauge fields*** or ***vector potentials***. If a Lagrangian contains a vector field A_μ and is invariant under the transformation $A_\mu \to A_\mu + \partial_\mu \alpha$ for all $\alpha(x)$, then the theory is called a ***gauge theory***.

Maxwell's theory (setting $e = 1$) in vaccum is the most important example of a gauge theory:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

One can check that it is indeed invariant under $A_\mu \to A_\mu + \partial_\mu \alpha$. There is a mathematical explanation to this. What we call A_μ is actually the component of a differential 1-form

$$A = A_\mu \mathrm{d}x^\mu.$$

What we defined to be $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is actually the component of a differential 2-form $F = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, which is the exterior derivative of A:

$$F = \mathrm{d}A.$$

Our manifold $X = \mathbb{R}^{1,3}$ is 4-dimensional. The Hodge star operator $\star : \Omega^2(X) \to \Omega^2(X)$ maps a 2-form to a form with complementary dimension, which is again a 2-form. The Lagrangian $\mathcal{L}$ in this language is simply given by

$$\mathcal{L} = F \wedge \star F = \mathrm{d}A \wedge \star(\mathrm{d}A).$$

A gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$ is simply a transformation $A \to A + \mathrm{d}\alpha$, where $\alpha \in C^\infty(X)$, hence $\mathrm{d}\alpha$ is a 1-form. Since the action is given by

$$S = \int_X F \wedge \star F = \int_X \mathrm{d}A \wedge \star(\mathrm{d}A),$$

we see that under such a transformation

$$\mathrm{d}A \to \mathrm{d}(A + \mathrm{d}\alpha) = \mathrm{d}A + \mathrm{d}^2\alpha = \mathrm{d}A$$

using the fact that de Rham differential satisfies $\mathrm{d}^2 = 0$. Another example of a gauge theory is

$$\mathcal{L}_{\mathrm{SQED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |\partial_\mu \varphi + ieA_\mu \varphi|^2 - m^2 |\varphi|^2.$$

where again $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. As we will see, this theory, called ***scalar quantum electrodynamics*** (SQED), is an important example of gauge theory.

6.7 Maxwell Equations and Tree Level Diagrams

Following the previous example, let us look at the most famous example of a $U(1)$-gauge theory, the Maxwell theory:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu.$$

where again $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $F_{0i} = E_i$ is the component of electric field, and $F_{ij} = \epsilon_{ijk}B_k$ gives the component of magnetic field. We have

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -eJ^\mu \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}(F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu}.$$

Hence the EL equations read

$$\partial_\mu F^{\mu\nu} = eJ^\nu. \tag{6.4}$$

This is the famous ***Maxwell equations*** in component form. Actually in components we recover two of them:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \partial_t \mathbf{E}.$$

The other two is just intrinsically encoded in the Bianchi identity:

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0.$$

Now we will focus on solving A_μ, given the information of J^μ. The Maxwell equation reads

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = eJ^\nu.$$

Or equivalently

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = eJ^\nu. \tag{6.5}$$

We will solve (6.5) by the method of Green's function. We will find an operator $\Pi_{\mu\nu}$ such that

$$\Pi_{\mu\nu}(eJ^\mu) = A_\mu,$$

which will yield the solution A_μ. We will refer to $\Pi_{\mu\nu}$ a ***propagator***. Since this is a differential equation, it is often more convenient to Fourier transform the equation into momentum space. Let us instead look for a Fourier transformed counterpart to this relation. Fourier transforming gives

$$A_\mu(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{ikx}\tilde{A}_\mu(k).$$

Let us look for

$$\tilde{A}_\mu(k) = \tilde{\Pi}_{\mu\nu}\tilde{J}^\nu(k). \tag{6.6}$$

Now, in equation (6.5), the notation $\Pi_{\mu\nu}(eJ^\nu)$ means $\Pi_{\mu\nu}$ acts on eJ^μ in someway. In momentum space we will require something stronger, namely the right hand side of (6.6) simply is given by multiplication by $\tilde{\Pi}_{\mu\nu}$. After Fourier transform, (6.4) reads

$$-k^2\tilde{A}^\nu + k^\nu k_\mu \tilde{A}^\mu = e\tilde{J}^\nu. \tag{6.7}$$

We will use the best method to solve equation (6.6): Guessing! Here is my guess:

$$\tilde{\Pi}_{\mu\nu} = -\frac{g_{\mu\nu} - (1-\xi)k_\mu k_\nu/k^2}{k^2}. \tag{6.8}$$

Now I am going to simply check if my guess works. Let us check it: Substitute (6.8) into the Fourier transformed Maxwell equation gives

$$
\begin{aligned}
-k^2\tilde{A}^\nu + k^\nu k_\mu \tilde{A}^\mu &= -k^2\tilde{\Pi}^{\nu\rho}\tilde{J}_\rho + k^\nu k_\mu \tilde{\Pi}^{\mu\rho}\tilde{J}_\rho \\
&= \left(g^{\nu\rho} - (1-\xi)\frac{k^\nu k^\rho}{k^2} - k^\mu k_\mu \frac{g_{\mu\nu} - (1-\xi)k^\mu k^\nu/k^2}{k^2}\right)\tilde{J}^\rho \\
&= \left(g^{\nu\rho} - \frac{k^\nu k^\rho}{k^2}\right)\tilde{J}_\rho.
\end{aligned}
$$

So we need to show $k^\nu k^\rho \tilde{J}_\rho/k^2 = 0$. But note that $k^\rho \tilde{J}_\rho = 0$ because there is a physical requirement $\partial_\mu J^\mu = 0$, i.e., conservation of current. And note that $k^\rho \tilde{J}_\rho = 0$ is just the Fourier transform of the current conservation equation. Therefore, we conclude that (6.8) is indeed a good guess that solves (6.7).

Note that in (6.8) there is a unphysical parameter ξ. This is just a choice of gauge. In other words, we can choose ξ to be anything we want (most popular choice is Feynman gauge $\xi = 1$, there is also Landau gauge $\xi = 0$), and any physical observable will not depend on it (which we will show later).

6.8 Columb's Law

Let us derive the ***Columb's law*** using these results. We take

$$
J^\mu = (\delta^3(\mathbf{x}), 0, 0, 0).
$$

Since the current J^μ is time independent, so must our vector potential A_μ. Hence in (6.8) we may set $\xi = 1$ and drop all the time dependence. Equivalently, this can be obtained by going back to the original equation (6.4), choosing a gauge such that $\partial_\mu A^\mu = 0$. Hence the Maxwell equation becomes $\Box A^\nu = 0$. By time independence this is just $\nabla^2 A^\nu = eJ^\nu$. Then

$$
A_0(\mathbf{x}) = \Pi_{00}J_0(\mathbf{x}) = -\frac{e}{\nabla^2}\delta^3(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e}{|\mathbf{k}|^2}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}.
$$

In the second equality, we used $\nabla^2 A^\nu = J^\nu$ (setting $e = 1$), and $1/\nabla^2$ is a formal symbol that means we Fourier transform and multiply by $1/|\mathbf{k}|^2$ in momentum space. In spherical coordinate, this integral becomes

$$
\begin{aligned}
A_0(\mathbf{x}) &= \frac{e}{(2\pi)^3}\int_0^\infty k^2\mathrm{d}k \int_{-1}^1 \mathrm{d}(\cos\theta)\int_0^{2\pi}\mathrm{d}\phi \frac{1}{k^2}\mathrm{e}^{ikr\cos\theta} \\
&= \frac{e}{(2\pi)^2}\int_0^\infty k^2\mathrm{d}k \int_{-1}^1 \mathrm{d}(\cos\theta)\,\frac{\mathrm{e}^{ikr\cos\theta}}{k^2} \\
&= \frac{e}{(2\pi)^2}\frac{1}{ir}\int_0^\infty \mathrm{d}k\,\frac{\mathrm{e}^{ikr} - \mathrm{e}^{-ikr}}{k}.
\end{aligned}
$$

The last integral can be evaluted using Residue theorem. This is a standard computation in complex analysis. The result is

$$
A_0(\mathbf{x}) = \frac{e}{4\pi}\frac{1}{|\mathbf{x}|}.
$$

This is exactly Columb's law.

6.9 Green's Function

Consider $G(x,y)$ such that
$$\Box_x G(x,y) = -\delta^4(x-y).$$
Written formally, G is the solution to
$$G = \frac{1}{\Box}.$$
Again, the expression makes sense in momentum space. Using Fourier transform, one can find solution
$$G(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2} \mathrm{e}^{ik(x-y)}.$$
One can check easily that
$$\Box_x G(x,y) = -\delta^4(x,y).$$
We call $G(x,y)$ the ***Green's function*** in a field theory. This is really useful in field theory. For example, let us consider the following theory of a scalar field:
$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{\lambda}{4!}h^4 - Jh.$$
The EL reads
$$-\Box h - \frac{\lambda h^3}{3!} - J = 0.$$
When $\lambda = 0$, the solution is simply the Green's function
$$h_0 = -\frac{1}{\Box}J.$$
More specifically, we have
$$h_0(x) = \int \mathrm{d}^4y \; G(x,y)J(y). \tag{6.9}$$
When $\lambda \neq 0$, there is no closed solution. We perturbatively expand h in orders of λ:
$$h = h_0 + \lambda h_1 + \lambda^2 h_2 + \cdots$$
Then the EL equation reads
$$-\Box(h_0 + \lambda h_1 + \lambda^2 h_2 + \cdots) - \frac{\lambda}{3!}(h_0 + \lambda h_1 + \lambda^2 h_2 + \cdots)^3 - J = 0.$$
Using $h_0 = J/\Box$, to the first order in λ we must have
$$-\Box \lambda h_1 - \frac{\lambda}{3!}h_0^3 = 0.$$
Thus
$$h_1(x) = -\frac{1}{\Box_x}\left(\frac{1}{3!}h_0^3\right) = \frac{1}{3!}\int \mathrm{d}^4y G(x,y)h_0^3(y).$$
Using
$$\begin{aligned} h_0^3(y) &= h_0(y)h_0(y)h_0(y) \\ &= \int \mathrm{d}^4y_1 G(y,y_1)J(y_1) \int \mathrm{d}^4y_2 G(y,y_2)J(y_2) \int \mathrm{d}^4y_3 G(y,y_3)J(y_3), \end{aligned}$$

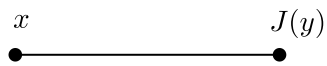
we get

$$h_1(x) = \frac{1}{3!} \int \mathrm{d}^4y \mathrm{d}^4y_1 \mathrm{d}^4y_2 \mathrm{d}^4y_3 G(x,y)G(y,y_1)G(y,y_2)G(y,y_3)J(y_1)J(y_2)J(y_3). \quad (6.10)$$

There is a nice diagrammatic interpretation the calculation above. The perturbative solution h, up to first order in λ, is given by

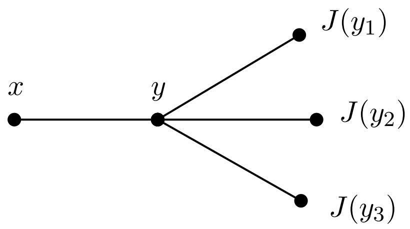
$$h(x) = h_0(x) + \lambda h_1(x),$$

which is the sum of expressions (6.9) and (6.10). The order λ^0 approximation, which is h_0, is just the solution to the free theory where $\lambda = 0$ (λ is usually a measure of how strong the interaction is, when $\lambda = 0$ the theory has no interaction, which we call free). By looking at the expression (6.9), we can represent it by the following diagram:

$h_0(x) =$ 

The line segment with endpoint x and $J(x)$ is simply a pictorial instruction for us to compute the integral $\int \mathrm{d}^4yG(x,y)J(y)$. So we can think of the line segment with vertices $x, J(y)$ as the propagator $G(x,y)$, and we need to truncate the end by multiplying $J(y)$.

Now we look at $h_1(x)$. This is (6.10), which is much more interesting. We can represent the expression (6.10) pictorially by:

$\lambda h_1(x) =$ 

Again, using the previously established rule for h_0, we replace each line segment with a propagator. So for example the line segment with endpoints x, y corresponds to $G(x,y)$, and the line segments with endpoints $y, J(y_i)$ is replaced by $G(y, y_i)$, for $i = 1, 2, 3$, and we truncate by $J(y_1)J(y_2)J(y_3)$. We add two new rule: Any internal vertices (in this case y) is an free variable, that we need to integrate over. And for every internal vertex, we need to multiply by a factor of λ. Integrating over all the variables other than x so that the result will be a function of x only, we see that we exactly recover (6.10) multiplied by λ, which is what we want in the perturbative expansion, except we are off by a factor of $1/3!$.

This mysterious factor $1/3!$ can in fact be covered as the order of the automorphism group of the graph above. These are called ***Feynman diagrams*** for this theory, and the rules associating each diagram to a term of mathematical expression in the perturbative expansion of h are called ***Feynman rules***. In the case of this simple theory, the Feynman rules can be summarized as follows:

1. Draw a point x and a line from x to a new point x_i.

2. Either truncate a line at a source J or let the line branch into two lines adding a new point and a factor of λ.

3. Repeat previous step.

4. The final value for $h(x)$ is given by graphs up to some order in λ with the ends capped by currents $J(x_i)$, the lines replaced by propagators $G(x_i, x_j)$, and all internal points integrated over, and divide the value of each diagram by the symmetry factor (i.e., order of the automorphism group of the graph).

It is worth remarking that the diagram corresponding of h_1 has 4 edges coming out of each internal vertex y is due to our theory having a interaction term $-\lambda h^4/4!$ in the Lagrangian. In general, if the interaction term is given by $-\lambda h^n/n!$, then each internal vertex will have n edges coming out of it.

So the upshot is: Up to a certain order λ^n, the perturbative expression of h can be recovered by summing over all allowed Feynman diagrams with no more than n internal vertices using the above Feynman rules. This is just a pictorial (but much faster) way of doing the perturbative computation that we could have done without using Feynman diagrams and just grind the math.[1] Of course, that way you are much easier to make a mistake, and you need to write down very long expressions.

Feynman diagram is the key to perturbative QFT (i.e., QFT whose Lagrangian contains a nonzero interaction term). As we will see later, for each theory there is an associated set of Feynman rules that allow us to do perturbation theory.

[1] Feynman diagrams were invented by Richard Feynman. Julian Schwinger, who won the Nobel Prize in Physics together with Feynman, disliked him so much that he forbade his students to do any perturbative computation of QFT using Feynman diagrams!

Chapter 7

Path Integral Formalism

Thirty-one years ago Richard Feynman told me about his sum over histories version of quantum mechanics. The electron does anything it likes, he said. It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function. I said to him, Youre crazy. But he wasnt.
– F. J. Dyson

Perhaps the best way to introduce the path integral formalism is by starting with the double slit experiment. A particle emitted from a source at time $t = 0$ passes through one or the other of two holes drilled in a board and is detected at time $t = T$ by a detector located on the screen, as shown in Fig.7.1. The amplitude for detection is given by a fundamental postulate of quantum mechanics, the superposition principle, as the sum of the amplitude for the particle to propagate from the source through one hole and then onward to the detector and the amplitude for the particle to propagate from the source through the other hole and then onward to the detector.

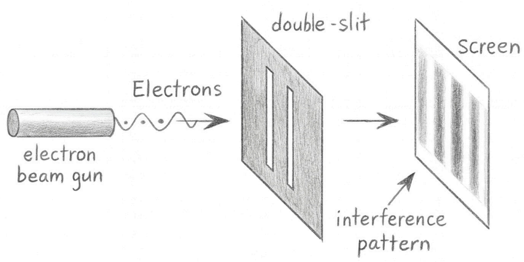


Figure 7.1: The double slit experiment

There is an obvious generalization to this situation: If we drill more than two holes, say $A_1, ..., A_n$ of the holes, and we want to measure the amplitude $\mathcal{A}_{S,O}$ for the particle to propagate from the source S to the detector O, it should be given by

$$\mathcal{A}_{S,O} = \mathcal{A}_{S,A_1,O} + \mathcal{A}_{S,A_2,O} + \cdots + \mathcal{A}_{S,A_n,O},$$

where $\mathcal{A}_{S,A_j,O}$ is the amplitude of the particle starts from S, passes A_j, then ending up at O. If we drill so many holes on the board that it feels like the screen is not really there, then we should get the contribution $\mathcal{A}_{S,P,O}$ from every point P on the board. Similarly, if we have a sequence of boards, say $B_1, ..., B_m$ of them, then the amplitude $\mathcal{A}_{S,O}$ should recieve contributions from every

$$\mathcal{A}_{S,P_1,P_2,...,P_m,O}$$

where P_j is an arbitrary point on B_j. Taking $n, m \to \infty$, the amplitude $\mathcal{A}_{S,O}$ should recive contribution from amplitude of every possible path from S to O. This clever thought experiment was due to Richard Feynman, who later formulated his deep insights to the path integral formalism.

To understand the path integral formalism, we have to be more precise about how do we interpret $\mathcal{A}_{S,O}$. The symbol is not totally precise because we didn't consider the time in which detection is made. Suppose we want to express the amplitude of a particle started at an initial position q_I at time t_I, and ended up at a final position q_F at time t_F. Let $|q_I\rangle$ denote the position eigenstate at q_I, and similarly define $|q_F\rangle$. Suppose the particle is governed by the Hamiltonian H. Then if we denote $T = t_F - t_I$, after we waited T units of time, the particle started off as $|q_I\rangle$ would have been in the state $\mathrm{e}^{-iHT}|q_I\rangle$ by Schrödinger equation, and the amplitude for it to end up at q_F at time t_F is given by the inner product

$$\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle,$$

which is the quantity we are interested in evaluating. Now we use Feynman's idea described above to slice up the path into small intervals. Namely, we divide T into N segments by defining $\delta t = T/N$. Thus

$$\langle q_F|\,\mathrm{e}^{-iHT}\,|q_I\rangle = \langle q_F|\,\mathrm{e}^{-iH\delta t}\mathrm{e}^{-iH\delta t}\cdots\mathrm{e}^{-iH\delta t}\,|q_I\rangle\,.$$

Since the position eigenstates are complete, we insert a bunch of identity operators of the form $\int \mathrm{d}q_j|q_j\rangle\langle q_j| = 1$. So now we have that

$$\begin{aligned}&\langle q_F|\mathrm{e}^{-iHT}|q_i\rangle\\ &= \left(\prod_{j=1}^{N-1}\int \mathrm{d}q_j\right)\langle q_F|\,\mathrm{e}^{-iH\delta t}\,|q_{N-1}\rangle\,\langle q_{N-1}|\,\mathrm{e}^{-iH\delta t}\,|q_{N-2}\rangle\cdots\langle q_1|\,\mathrm{e}^{-iH\delta t}\,|q_I\rangle\,.\end{aligned}$$

Let us focus on one factor $\langle q_{j+1}|\mathrm{e}^{-iH\delta t}|q_j\rangle$. Suppose[1] first that our Hamiltonian is just given by the Hamiltonian of a free particle $H = \hat{p}^2/2m$. Inserting the identity using momentum eigenstates $\int \frac{\mathrm{d}p}{2\pi}|p\rangle\langle p| = 1$ and using $\langle q|p\rangle = \mathrm{e}^{ipq}$ gives

$$\begin{aligned}\langle q_{j+1}|\,\mathrm{e}^{-i\delta t(\hat{p}^2/2m)}\,|q_j\rangle &= \int\frac{\mathrm{d}p}{2\pi}\,\langle q_{j+1}|\,\mathrm{e}^{-i\delta t(\hat{p}^2/2m)}|p\rangle\,\langle p|q_j\rangle\\ &= \int\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-i\delta t(p^2/2m)}\,\langle q_{j+1}|p\rangle\,\langle p|q_j\rangle\\ &= \int\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-i\delta t(p^2/2m)}\mathrm{e}^{ip(q_{j+1}-q_j)}.\end{aligned}$$

The integral over p is an Gaussian integral, which can be evaluated. Doing this integral yields

$$\begin{aligned}\langle q_{j+1}|\,\mathrm{e}^{-i\delta t(\hat{p}^2/2m)}\,|q_j\rangle &= \left(\frac{-im}{2\pi\delta t}\right)^{1/2}\mathrm{e}^{[im(q_{j+1}-q_j)^2]/2\delta t}\\ &= \left(\frac{-im}{2\pi\delta t}\right)^{1/2}\mathrm{e}^{i\delta t(m/2)[(q_{j+1}-q_j)/\delta t]^2}.\end{aligned}$$

[1] The general case $H = \hat{p}^2/2m + V(\hat{x})$ with potential is similar.

Putting all these pieces back we get

$$\langle q_F| \, \mathrm{e}^{-iHT} \, |q_I\rangle = \left(\frac{-im}{2\pi\delta t}\right)^{N/2} \left(\prod_{k=1}^{N-1} \int \mathrm{d}q_k\right) \mathrm{e}^{i\delta t(m/2)\sum_{j=0}^{N-1}[(q_{j+1}-q_j)/\delta t]^2},$$

where $q_0 = q_I, q_N = q_F$. Now we take the limit where $N \to \infty$, or equivalently where $\delta t \to 0$. Then the exponent in exponential above becomes $\int_0^T \mathrm{d}t \frac{1}{2}m\dot{q}^2$. Thus the formula above becomes

$$\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle = \int \mathcal{D}q \mathrm{e}^{i\int_0^T \mathrm{d}t \frac{1}{2}m\dot{q}^2}$$

where we define

$$\int \mathcal{D}q = \lim_{N\to\infty} \left(\frac{-im}{2\pi\delta t}\right)^{N/2} \prod_{k=1}^{N-1} \int \mathrm{d}q_k = \lim_{N\to\infty} \left(\frac{-imN}{2\pi T}\right)^{N/2} \prod_{k=1}^{N-1} \int \mathrm{d}q_k.$$

The integral with respect to dq_k above should be interpreted as integrating over all points on a screen in which the particle passes at time $t = k\delta t$. So what we are doing is assigning a weight of $\mathrm{e}^{i\int_0^T \mathrm{d}t\frac{1}{2}m\dot{q}^2}$ to each path $q(t)$ that the particle can possibly take and integrate over all such paths. However, everything we have done here is at a formal level, and immediately we see two concerning things from the formula above:

The first thing is that the formula above is just nonsense. There is a constant in front that diverges at $N \to \infty$. However, what physicists usually do is to calculate time ordered correlation function (as we will see later), which is the ratio of two path integrals. Hence, this infinite constant will be canceled because both the numerator and the denominator will contribute the same diverging constant.

The second thing is that $\mathcal{D}q$ can not be interpreted as a measure on the space of paths. This is a more mathematical statement, but it is known that in many cases that such measure cannot exist. Therefore, we should interpret the path integral as intergrating over all possible paths only at a heuristic level, while in reality it always means we do the procedure above and consider the limit of infinite product of integrals in small time interval. In fact, the only path integral that can be evaluated exactly are those that are Gaussian, and for non-Gaussian integral we will need to use perturbation theory (as we will see later).

As a second step, repeat the derivation above with the Hamiltonian now replaced with its most general form: $\hat{H} = \hat{p}^2/2m + V(\hat{q})$. I claim that the result we will get is

$$\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle = \int \mathcal{D}q \mathrm{e}^{i\int_0^T \mathrm{d}t\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)}. \tag{7.1}$$

We recognize that what is inside the exponential is just i times the action $S = \int_0^T \mathrm{d}tL(q,\dot{q})$. This is the path integral formalism: The amplitude $\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle = \langle q_F, t_F|q_I, t_I\rangle$ recieves contribution from all possible path $q(t)$ from q_I to q_F, each weighed by a factor of $\mathrm{e}^{iS(q,\dot{q})}$.

We will prove (7.1) shortly, but let us discuss first why the path integral formalism is more favourable compared to the Schrödinger picture that we have developed. This formalism is highly advantages because classical solutions emerges naturally in the limit $\hbar \to 0$. Note that in the derivation above (in fact in pretty much every other places of this writing), we assumed $\hbar = 1$. If we restore the unit in (7.1), the actually formula should read

$$\langle q_F|\mathrm{e}^{-iHT/\hbar}|q_I\rangle = \int \mathcal{D}q \mathrm{e}^{iS(q,\dot{q})/\hbar}.$$

When $\hbar \to 0$, the integrand $\mathrm{e}^{iS(q,\dot{q})/\hbar}$ will oscillate rapidly. For any q, if q and its nearby path does not minimize the action $S(q,\dot{q})$, then they will destructively interfere and the contribution from them cancels out with each other. Therefore, the only significant contribution to the amplitude come from those paths that minimizes the action, which are the classical paths by the principle of least action.

To prove (7.1), note that each term in the insertion will be of the form

$$\begin{aligned}
\langle q_{j+1}|\mathrm{e}^{-i\delta t(\hat{p}^2/2m)}\mathrm{e}^{-i\delta t V(\hat{q})}|q_j\rangle &= \int \frac{\mathrm{d}p}{2\pi}\langle q_{j+1}|\mathrm{e}^{-i\delta t(\hat{p}^2/2m)}|p\rangle\langle p|\mathrm{e}^{-i\delta t V(\hat{q})}|q_j\rangle \\
&= \int \frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-i\delta t(p^2/2m)}\langle q_{j+1}|p\rangle\langle p|q_j\rangle\mathrm{e}^{-i\delta t V(q_j)} \\
&= \mathrm{e}^{-i\delta t V(q_j)}\left(\frac{-im}{2\pi\delta t}\right)^{1/2}\mathrm{e}^{i\delta t(m/2)[(q_{j+1}-q_j)/\delta t]^2} \\
&= \left(\frac{-im}{2\pi\delta t}\right)^{1/2}\mathrm{e}^{i\delta t\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V(q_j)\right)}.
\end{aligned}$$

So it follows that when we multiply all the insertion, and take limit as $N \to \infty$, we would get (7.1).

Example 7.1 (Gaussian integral in one variable)**.** As a warm up, in the finite dimensional case, we consider the integral

$$\int \mathrm{d}p\ \mathrm{e}^{-\alpha p^2 + Jp}.$$

Recalling that $\int \mathrm{d}p\ \mathrm{e}^{-\alpha p^2} = \sqrt{\pi/\alpha}$, we just need to complete the square, and the above expression becomes

$$\int \mathrm{d}p\ \mathrm{e}^{-\alpha\left(p-\frac{J}{2\alpha}\right)^2 + \frac{J^2}{4a}} = \sqrt{\frac{\pi}{\alpha}}\mathrm{e}^{J^2/4\alpha}.$$

A somewhat equivalent method is called the method of stationary phase. Taking the derivative of the exponent to find the critical point:

$$0 = \frac{\partial}{\partial p}(-\alpha p^2 + Jp) = 2\alpha p + J$$

gives the critical point

$$p_* = \frac{J}{2\alpha}.$$

Then up to the constant $\sqrt{\pi/\alpha}$, we see that

$$\int \mathrm{d}p\ \mathrm{e}^{-\alpha p^2 + Jp} \sim \mathrm{e}^{-\alpha p_*^2 + Jp_*} = \mathrm{e}^{J^2/4\alpha}.$$

Example 7.2 (Gaussian integral in finite dimension)**.** Suppose now J be a n-dimensional vector, and A an $n \times n$ matrix. We now consider the n-dimensional integral

$$\int \mathrm{d}^n p \mathrm{e}^{-\frac{1}{2}p^\dagger A p + J^\dagger p}.$$

After diagonalizing A the integral becomes just a product of integrals over the p_i, and the result is the product of one-dimensional Gaussian integrals, with a being replaced by an eigenvalue of A. That is,

$$\int \mathrm{d}^n p \mathrm{e}^{-\frac{1}{2}p^\dagger A p + J^\dagger p} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}\mathrm{e}^{\frac{1}{2}J^\dagger A^{-1} J}. \tag{7.2}$$

7.1 Time Ordered Correlators and Partition Functions

Now suppose I have some operator of interest. In the Hisenberg picture, the operator $\mathcal{O}(t)$ evolves with time t. What we will be interested in calculating is calculating expectation values of the form, say

$$\frac{\langle q_b, t_b | \mathcal{O}(t)\mathcal{O}'(t') | q_a, t_a\rangle}{\langle q_b, t_b | q_a, t_a\rangle}.$$

The denominator is the path integral that we already did. So let us redo the calculation of the numerator. Now assuming that $\mathcal{O}, \mathcal{O}'$ is just built up from the position operator Q (example, $\mathcal{O} = Q^2$). Now assuming $t_a < t < t' < t_b$. Then going through the computation. Routine calculation shows that the expression above becomes

$$\frac{\int \mathcal{D}q\ \mathrm{e}^{iS}\mathcal{O}(t)\mathcal{O}'(t')}{\int \mathcal{D}q\ \mathrm{e}^{iS}}$$

It is important to stress that inside the integral, $\mathcal{O}(t), \mathcal{O}'(t')$ now become classical functions of position and momentum, instead of being operators. We now introduce the ***time ordering operator***, which is defined by

$$T\{\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\} = \begin{cases} \mathcal{O}_1(t_1)\mathcal{O}_2(t_2) & t_1 \geq t_2 \\ \mathcal{O}_2(t_2)\mathcal{O}_1(t_1) & t_1 < t_2. \end{cases}$$

Then in fact we have that

$$\langle q_b, t_b | T\{\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\} | q_a, t_a\rangle = \int \mathcal{D}q\ \mathrm{e}^{iS[q]}\mathcal{O}_1(t_1)\mathcal{O}_2(t_2).$$

This is not surprising, since the right hand side are all classical objects, hence they must be independent of orderings. Therefore, the left hand side should also be something that does not depend on the ordering of the operators. Indeed, if we go back into the derivation of the path integral, we recall that we slice up time in order, so the only way that the expectation on the left agrees with the path integral on the right, is that the left hand side is already time ordered.

The formal object we obtain is called functional integral. We now introduce ***functional derivatives***. The rule is defined as follows: We have

$$\frac{\delta}{\delta q(t)} q(t_a) = \delta(t - t_a).$$

This is the analogue of ordinary derivative $\partial q_a / \partial q_j = \delta_{ij}$. Now suppose I want to look at the following functional, called the ***partition function***:

$$Z[J] = \int \mathcal{D}q \exp\left(i\int \mathrm{d}t\ L + i\int \mathrm{d}t\ J(t)q(t)\right).$$

Here, J is called the ***current***, and $Z[J]$ is commonly referred to as the ***partition function*** of the theory. Suppose we know the answer to this expression, then essentially we know everything there is to know about the theory defined by the Lagrangian L. To see this, consider calculating

$$(-i)\frac{\delta}{\delta J(t')} Z[J].$$

Note that

$$(-i)\frac{\delta}{\delta J(t')}\mathrm{e}^{i\int \mathrm{d}t J(t)q(t)} = \mathrm{e}^{i\int Jq\mathrm{d}t}\int \delta(t - t')q(t)\mathrm{d}t = \mathrm{e}^{\int Jq\mathrm{d}t} q(t').$$

Hence

$$(-i)\frac{\delta}{\delta J(t')}Z[J]\bigg|_{J=0} = \langle q_b, t_b|Q(t')|q_a, t_a\rangle = \langle Q(t')\rangle.$$

Generalizing, we see that

$$(-i)^n \frac{1}{Z[0]}\frac{\delta}{\delta J(t_1)}\cdots\frac{\delta}{\delta J(t_m)}Z[J]\bigg|_{J=0} = \langle T\{Q(t_1)\cdots Q(t_m)\}\rangle.$$

Similarly, changing the appropriate expression of the integral involving J, we can recover all time ordered correlators by taking functional derivative to path integrals of the theory.

Example 7.3 (Harmonic Oscillators)**.** We will next show how this works for a concrete example: harmonic oscillator. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}m^2q^2.$$

This is slightly different from the usual expression you see in classical mechanics because the mass is not in the kinetic term but in the potential term now. We can translate in-between by a rescaling of x, but this expression is more common in QFT so we will study the expression above instead. We would like to calculate

$$\langle 0|T\hat{q}(t_1)\hat{q}(t_2)|0\rangle.$$

We see that the potential term will have an integrand which is a Gaussian. Let's compute

$$\begin{aligned} Z[J] &= \int \mathcal{D}q\ \mathrm{e}^{iS[q]+i\int \mathrm{d}t\ Jq} \\ &= \int \mathcal{D}q\ \exp\left(\int -\frac{1}{2}q\partial_t^2 q - \frac{1}{2}m^2\dot{q}^2 + Jq\right). \end{aligned}$$

The integral is quadratic in q, hence is just an infinite dimensional Gaussian. The answer should be the infinite-dimensional analogue of (7.2). Using stationary phase method, we take derivative of the exponent with respect to q and set it to zero shows the extremum appears at the solution to the differential equation

$$-\partial_t^2 q - m^2 q = -J.$$

Note that instead of $-\partial_t^2 q/2$ we get $-\partial_t^2 q$ in the first term, because the derivative can hit both q's in $q\partial_t^2 q$. To solve the equation above, we formally write

$$q = -\frac{1}{-\partial_t^2 - m^2}J.$$

Formally, we are taking Fourier transform and invert the operator on the left hand side of the differential equation. Now, seperating the part that depends on J, we get:

$$Z[J] = Z_0 \exp\left(i\int \mathrm{d}t\mathrm{d}t'\ \frac{1}{2}J(t')\frac{1}{-\partial_t^2 - m^2}J(t)\right).$$

Note that this is precisely what we expected: This is nothing but an infinite dimensional analogue of (7.2). Now let us take functional derivatives and calculate some correlation functions. We have

$$
\begin{aligned}
\langle 0|T\hat{q}(t_1)\hat{q}(t_2)|0\rangle &= \frac{1}{Z_0}(-i)^2\frac{\delta}{\delta J(t_2)}\frac{\delta}{\delta J(t_1)}Z[J]\bigg|_{J=0} \\
&= (-i)\frac{\delta}{\delta J(t_2)}\exp\left(i\int \mathrm{d}t\mathrm{d}t'\ \frac{1}{2}J(t')\frac{1}{-\partial_t^2-m^2}J(t)\right) \\
&\quad\times\frac{1}{2}\left(\int \mathrm{d}t\mathrm{d}t'\ \delta(t'-t_1)\frac{1}{-\partial_t^2-m^2}J(t)+\int \mathrm{d}t\mathrm{d}t' J(t')\frac{1}{-\partial_t^2-m^2}\delta(t-t_1)\right)\bigg|_{J=0}.
\end{aligned}
$$

Now we take functional derivative with respect to $J(t_2)$ again. If the derivative hit the J in the exponential, we will get some integral with J in it, which vanishes when we set $J=0$ in the end. Hence we only need to worry about the situation where the derivative hits the J in the integral. Thus the expression above reads

$$
\langle T\hat{q}(t_1)\hat{q}(t_2)\rangle = \mathrm{e}^{i\int \mathrm{d}t\mathrm{d}t'\ J(t')\frac{1}{-\partial_t^2-m^2}J(t)}\int \mathrm{d}t\mathrm{d}t'\ \delta(t'-t_1)\frac{1}{-\partial_t^2-m^2}\delta(t-t_2)\bigg|_{J=0}.
$$

Recall that the Green's function satisfies

$$
G(t_1,t_2) = -\frac{1}{\partial_t^2+m^2}\delta(t_1-t_2).
$$

Here, the derivative ∂_t^2 is taken with respect to t_1. The function $\delta(t_1-t_2)$ acts like identity matrix on the space of functions, because the expression $\int \mathrm{d}t_1\mathrm{d}t_2\ J(t_1)\delta(t_1-t_2)J(t_2)$ is just the infinite dimensional generalization of the vector contraction expression $J^\dagger AJ$ for some matrix A (which is identity in our case) and vector J. Compare this with Example 7.2. Hence $G(t_1,t_2)$ is the inverse matrix of the operator $-(\partial_t^2+m^2)$. Hence the above expression becomes

$$
\langle T\hat{q}(t_1)\hat{q}(t_2)\rangle = \mathrm{e}^{i\int \mathrm{d}t\mathrm{d}t'\ J(t')\frac{1}{-\partial_t^2-m^2}J(t)}\int \mathrm{d}t\mathrm{d}t'\ \delta(t'-t_1)G(t',t)\delta(t-t_2)\bigg|_{J=0} = G(t_1,t_2).
$$

Thus we have shown that the two point time ordered correlator is exactly the Green's function. Let's see what this looks like in frequency space. In frequency space (with variable ω), the derivative ∂_t becomes $-i\omega$. Doing Fourier transform, we see that

$$
\langle T\hat{q}(t_1)\hat{q}(t_2)\rangle = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{i\omega(t_1-t_2)}}{-\omega^2+m^2}.
$$

The integral is the same as

$$
\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{i\omega(t_1-t_2)}}{-\omega^2+m^2-i\epsilon}
$$

and send $\epsilon\to 0$. The above integral can be evaluated using residue technique from complex analysis. The poles (Fig.7.2) are located at the solutions $\omega^2=m^2-i\epsilon$. In other words,

$$
\omega = \pm\sqrt{m^2-i\epsilon} = \pm m\sqrt{1-i\epsilon/m^2} \simeq \pm m\big(1-i\epsilon/2m^2\big).
$$

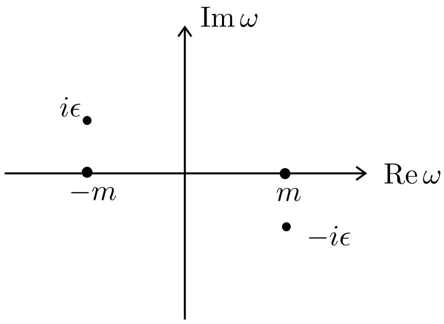


Figure 7.2: Pole Location.

The interested readers are encouraged to complete the computation using residue theory and contour integral techniques. The result is

$$G(t_1,t_2) = \frac{i}{2\omega}\exp(-i\omega|t_1-t_2|).$$

7.2 Path Integrals in QFT

Next, we will generalize path integral to QFT setting. Let us consider the path integral whose Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi).$$

In the above, $\Pi = \partial_t\phi$ is the canonical momentum. The Langrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi).$$

In QM, the eigenstates of the position are $|q\rangle$ which satisfies $Q\,|q\rangle = q\,|q\rangle$. In QFT, the field $\hat{\phi}(\mathbf{x})$ will become an operator valued function. The eigenstates are $|\phi(\mathbf{x})\rangle$ such that $\hat{\phi}(\mathbf{x})\,|\phi(\mathbf{x})\rangle = \phi(\mathbf{x})\,|\phi(\mathbf{x})\rangle$, where $\phi(\mathbf{x})$ is a number, namely the eigenvalue, which depends on the location $\mathbf{x}$. Similarly, we have momentum eigenstates $|\Pi(\mathbf{y})\rangle$ such that $\hat{\Pi}(\mathbf{y})\,|\Pi(\mathbf{y})\rangle = \Pi(\mathbf{y})\,|\Pi(\mathbf{y})\rangle$. We also require the commutation relation

$$[\hat{\phi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}).$$

The functional integral formalism derived in the last chapter should hold for all quantum mechanical systems. In particular it should hold for quantum field theories. In the scalar QFT theory above, it is merely a quantum mechanical systems with infinite degree of freedoms, where each degree of freedom is simply the value of the field $\phi(\mathbf{x})$ instead of position q. Suppose now at time t_a the field configuration is $\phi_a(\mathbf{x})$, and at time t_b the field configuration is $\phi_b(\mathbf{x})$, then the transition amplitude $\mathcal{A}$ must be of the form

$$\begin{aligned}\mathcal{A} &\equiv \langle\phi_b(\mathbf{x})|\mathrm{e}^{-iH(t_b-t_a)}|\phi_a(\mathbf{x})\rangle \\ &= \int\mathcal{D}\phi\mathcal{D}\Pi\,\exp\left(i\int_{t_a\le t\le t_b}\mathrm{d}^4x\left(\Pi\dot{\phi} - \frac{1}{2}\Pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right).\end{aligned} \tag{7.3}$$

where the $\mathcal{D}\phi$ integral are constrained to the specific configurations $\phi_a(\mathbf{x})$ at t_a and $\phi_b(\mathbf{x})$ at t_b. Note that (7.3) is simply infinite dimensional generalization of (**??**). Note that the exponents in (7.3) is quadratic in Π, so we can first do the Gaussian integral involving $\mathcal{D}\Pi$. Using the stationary phase method, we obtain

$$\mathcal{A} = \int\mathcal{D}\phi\exp\left(i\int_{t_a\le t\le t_b}\mathrm{d}^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right)\right) = \int\mathcal{D}\phi\,\mathrm{e}^{iS[\phi,\partial_\mu\phi]}.$$

7.3 Free Theory and Wick's Theorem

Let us work out the path integral formalism for the free theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

This should be familiar as it is just the 4D generalization of Example 7.3. So we already know the answer to the time ordered product in some sense from that example. We know we should get something like

$$\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}\mathrm{e}^{ik(x_1-x_2)}. \tag{7.4}$$

Indeed, we carry over the exact same steps as in Example 7.3, with $\partial_t^2 + m^2$ becomes $\Box - m^2$, which leads to (7.4). Note that (7.4) is manifestly Lorentz invariant.

Now that we are masters in two-point functions, what about three-point functions? Let us calculate now

$$\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)|0\rangle.$$

To do this, we calculate the functional derivative as expected. The partition function for the free field theory reads

$$Z[J] = \int \mathcal{D}\phi \exp\left(i\int \mathrm{d}^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi(x)\right)\right).$$

This expression is quadratic in ϕ. Hence the path integral can be computed directly using the method of stationary phase. Direct generalization of (7.2) leads us to the result

$$Z[J] = Z_0 \exp\left(i\int \mathrm{d}^4x\mathrm{d}^4y\; J(x)\frac{1}{\Box_x - m^2}J(y)\right),$$

where Z_0 is independent of J. We can define the Green's function $G(x,y)$ to satisfies the relation

$$(\Box_x - m^2)G(x_1,x_2) = -\delta(x_1 - x_2).$$

One can check that $G(x_1,x_2)$ is precisely given by the right hand side of (7.4). Hence

$$\begin{aligned}(-i)\frac{1}{Z_0}\frac{\delta Z[J]}{\delta J(x_1)}\bigg|_{J=0} = \exp\left(i\int \mathrm{d}^4x\mathrm{d}^4y\; J(x)G(x,y)J(y)\right)\times \\ \left(\int \mathrm{d}^4yG(x_i,y)J(y) + \int \mathrm{d}^4xJ(x)G(x,x_i)\right),\end{aligned}$$

which we see is zero when we set $J=0$. When we take the second derivative, the nonzero contribution comes from $\delta/\delta J$ hits the J in the integrand of the integral on the right. But when we take third derivative again we have similar expression above, hence becomes zero. Therefore, we conclude that all the correlator for an odd number of insertions vanishes.

Hence very generally, we wish to compute correlators with an even number of insertions:

$$\langle T\hat{\phi}(x_1)\cdots\hat{\phi}(x_{2m})\rangle = \frac{1}{Z_0}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{2m})}Z[J]\bigg|_{J=0}. \tag{7.5}$$

The computation in Example 7.3 already showed that taking two functional derivatives gives the propagator:

$$\frac{1}{Z_0}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}Z[J] = G(x_1,x_2),$$

where $G(x_1,x_2)$ is defined by the integral on the right hand side of (7.4). In general, the answer is given by the famous ***Wick's Theorem***:

Theorem 7.4 (Wick's Theorem)**.** *The value of the expression* (7.5) *is given by the sum of products of propagators valued at all possible pairings of the insertion points* $x_1, \ldots, x_{2m}$*. That is,*

$$\langle T\hat{\phi}(x_1)\cdots\hat{\phi}(x_{2m})\rangle = \sum_{\textit{all pairings}} G(x_{i_1}, x_{i_2})\cdots G(x_{i_{2m-1}}, x_{i_{2m}}). \tag{7.6}$$

We will use ***Wick contractions*** to indicate the pairing, as shown below. For example:

$$\langle T\overbrace{\phi(x_1)\phi(x_2)}\cdots\overbrace{\phi(x_{2m-1})\phi(x_{2m})}\rangle \equiv G(x_1, x_2)\cdots G(x_{2m-1}, x_{2m}).$$

The proof of Wick's theorem is not very illuminating, so we omit it for now. What's more important is convincing oneself that the theorem is right by working through examples. For instance, the 4-point function will be given by the sum of three terms:

$$\langle T\hat{\phi}(x_1)\cdots\hat{\phi}(x_4)\rangle = G(x_1, x_2)G(x_3, x_4) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3).$$

The expression above has a nice pictorial representation given by ***Feynman diagrams***. It is given by the sum of the three diagram in Fig.7.3, where each line from x_i to x_j is the instruction for us to multiply by a Green's function $G(x_i, x_j)$:

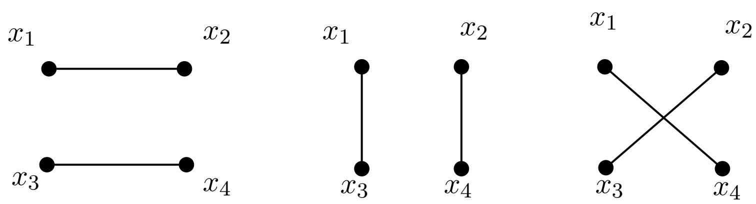


Figure 7.3: 4-point correlator function

7.4 Interacting Quantum Fields and Feynman Diagrams

The Lagrangian in section 7.3 only has a quadratic term in ϕ. In general, we may have some higher order polynomials in ϕ. So the theory might have some interaction terms. So the Lagrangian will be the same Lagrangian in section 7.3, called $\mathcal{L}_{\text{free}}$, plus some higher order interaction terms:

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}.$$

For example, we might have something like

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi - \frac{\lambda}{3!}\phi^3. \tag{7.7}$$

In this case, $\mathcal{L}_{\text{int}} = \lambda\phi^3/3!$. Theory like this cannot be solved exacly like what we did in Section 7.3. But QFT like this is of great importance because most practical quantum field theories have higher order interaction terms in the Lagrangian. Therefore, it is still of great interest to calculate the time ordered correlators with arbitrary number of insertions. To this end, we need to compute the correlators of theory (7.7) perturbatively: That is, we assume λ is a small number and do Taylor expansion. Assuming the quantity $\mathcal{Q}$ is what we want to compute for the theory (7.7). This could be a correlator with 4 insertion points. Let $\mathcal{Q}_0$ be the corresponding quantity we would get if $\lambda = 0$ (i.e., in free theory). Then, we Taylor expand $\mathcal{Q}$ in terms of λ:

$$\mathcal{Q} = \mathcal{Q}_0 + \lambda\mathcal{Q}_1 + \lambda^2\mathcal{Q}_2 + \cdots.$$

Feynman diagram is a powerful tool that tells us how to determine the coefficients $\mathcal{Q}_i$ when $\mathcal{Q}$ are time ordered correlators with arbitrary number of insertion points. We will formulate this idea for the remaining of this chapter. The partition function for the theory (7.7) is given by

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\phi \exp\left(i \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi + J\phi - \frac{\lambda}{3!}\phi^3\right)\right) \\
&= \int \mathcal{D}\phi\, \mathrm{e}^{i\int \mathrm{d}^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi + J\phi\right)} \exp\left(i \int \mathrm{d}^4x\, \frac{-\lambda}{3!}\phi^3\right) \\
&= \int \mathcal{D}\phi\, \mathrm{e}^{i\int \mathrm{d}^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi + J\phi\right)} \\
&\quad \times \left(1 + \frac{-i\lambda}{3!}\int \mathrm{d}^4z \phi^3(z) + \left(\frac{-i\lambda}{3!}\right)^2 \frac{1}{2}\int \mathrm{d}^4z \int \mathrm{d}^4w\, \phi^3(z)\phi^3(w) + \cdots\right)
\end{aligned}$$

Each term in this expansion can be viewed as a path integral in the free theory. Using

$$Z_0[J] = \int \mathcal{D}\phi \exp\left(i \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi + J\phi\right)\right)$$

to denote the partition function of the free theory, we can write the Taylor expansion of $Z[J]$ above as

$$\begin{aligned}
Z[J] &= Z_0[J] + \frac{-i\lambda}{3!}\int \mathrm{d}^4z(-i)^3\frac{\delta^3 Z_0[J]}{\delta J(z)^3} + \left(\frac{-i\lambda}{3!}\right)^2\frac{1}{2}\int \mathrm{d}^4z\int \mathrm{d}^4w(-i)^6\frac{\delta^6 Z_0[J]}{(\delta J(z))^3(\delta J(w))^3} + \cdots \qquad (7.8) \\
&= \exp\left(\frac{-i\lambda}{3!}(-i)^3\int \mathrm{d}^4z\frac{\delta^3}{\delta J(z)^3}\right)Z_0[J].
\end{aligned}$$

Let us denote the ground state of the theory (7.7) by $|\Omega\rangle$. We avoided using $|0\rangle$ because we want to reserve it for the ground state of the free theory, which may be different from $|\Omega\rangle$. We are interested to compute the time ordered correlator

$$\langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle = \frac{(-i)^n}{Z[0]}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}Z[J]\bigg|_{J=0}. \qquad (7.9)$$

From (7.8) we know that $Z[J]$ can be Taylor expanded in λ, with the constant term being $Z_0[J]$. Hence the correlator also has a Taylor expansion in λ, whose constant term is the free theory correlator function:

$$\langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle = \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle + \lambda\mathcal{Q}_1 + \lambda^2\mathcal{Q}_2 + \cdots.$$

Now we discuss how to determine the coefficients $\mathcal{Q}_m$. These can be determined by Feynman diagrams and Feynman rules. We will construct a collection of diagrams called Feynman diagrams that are relevant (will be made explicit below), and using a set of rules called Feynman rules we can turn these diagrams into mathematical expressions. Then $\mathcal{Q}_m$ is simply the sum of all such relevant Feynman diagrams (namely their expressions obtained from applying Feynman rules to the diagrams).

The first step is to determine the number of insertion points $x_1, ..., x_n$. In this case we have n of them. This determines the external vertices of Feynman diagrams, which is equal to n.

The second step is to determine which coefficient we want to compute, in our case say it is $\mathcal{Q}_m$ for a fixed integer $m \geq 1$. This means we need to find terms in the Taylor expansion (7.9) that are proportional to λ^m. Using (7.8), we see that up to a constant,

$$\begin{aligned}\mathcal{Q}_m &\sim \frac{\delta^n}{\delta J_1 \cdots \delta J_n} \lambda^m \int \mathrm{d}^4 z_1 \cdots \mathrm{d}^4 z_m \frac{\delta^3}{\delta J(z_1)^3} \cdots \frac{\delta^3}{\delta J(z_m)^3} \exp\left(\frac{i}{2} \int \mathrm{d}^4 x \mathrm{d}^4 y J(x) G(x,y) J(y) \right) \\ &= \frac{\delta^n}{\delta J_1 \cdots \delta J_n} \lambda^m \int \mathrm{d}^4 z_1 \cdots \mathrm{d}^4 z_m \frac{\delta^3}{\delta J(z_1)^3} \cdots \frac{\delta^3}{\delta J(z_m)^3} \sum_{\ell=0}^{\infty} \frac{i^\ell}{2^\ell \ell!} \left(\int \mathrm{d}^4 x \mathrm{d}^4 y J(x) G(x,y) J(y) \right)^\ell .\end{aligned} \tag{7.10}$$

Note that in each summand of (7.10), the number of currents J is equal to 2ℓ, and the number of functional derivatives $\delta/\delta J$ is $n + 3\ell$. In this sum, if $2\ell > n + 3m$, then after we finish taking all derivatives, we will still have some current J left. Hence these terms will vanish when we evaluate at $J = 0$ in the end. On the other hand, if $2\ell < n + 3m$, then after taking some functional derivatives, we will run out of J's, which means taking functional derivatives again will give zero. Hence we conclude that

Proposition 7.5. The only terms in (7.10) that have nonzero contributions to $\langle \Omega | T \phi_1 \cdots \phi_n | \Omega \rangle$ are those that satisfy $n + 3m = 2\ell$.

Next, we note that for each fixed ℓ in the sum (7.10), we have terms proportional to

$$\frac{\delta^n}{\delta J_1 \cdots \delta J_n} \frac{\delta^3}{\delta J(z_1)^3} \cdots \frac{\delta^3}{\delta J(z_m)^3} \int \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_\ell \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_\ell \prod_{j=1}^{\ell} J(x_j) G(x_j, y_j) J(y_j).$$

By product rule, this is the sum of all possible cases where some $\delta/\delta J(p)$ hits some $J(q)$. Each time this happens, we will get a delta function $\delta(p-q)$. Since Proposition 7.5 tells us that $n + 3m = 2\ell$, that is, all of the vertices $p_1, ..., p_n$ (external vertices), and three copies of $z_1, ..., z_m$, must be paired with some current J. And the propagators G will record all possible pairings among these vertices. For example:

$$\int \mathrm{d}^4 x_j \mathrm{d}^4 y_j \frac{\delta J(x_j)}{\delta J(p_k)} G(x_j, y_j) \frac{\delta J(y_j)}{\delta J(z_s)} = G(p_k, z_s).$$

Now we are ready to build a bridge between the expressions in the perturbative series of the time ordered correlator and Feynman diagrams. We first need a definition

Definition 7.6. Let D be a (Feynman) diagram. The value $e(D)$ of the diagram D is given by

$$e(D) = \frac{1}{|\operatorname{Aut} D|} (-i\lambda)^{|V_0|} \prod_{(p_i, p_j) \in E} G(p_i, p_j),$$

where V_0 is the collection of all internal vertices of D, and $|V_0|$ the number of internal vertices, and E is the set of all edges of D. Finally, $\operatorname{Aut} D$ is the group of automorphisms of the diagram D fixing the external vertices (meaning these automorphisms cannot permute external vertices).

Therefore, we conclude that

Theorem 7.7 (Feynman Rule)**.** *The coefficient $\mathcal{Q}_m$ in the Taylor expansion of the time ordered correlator $\langle T \phi(p_1) \cdots \phi(p_n) \rangle$ is given by the sum over values of all possible diagrams with external vertices $p_1, ..., p_n$, and m internal vertices, where the sum is over all possible* *pairings between these vertices (every external vertices must be paired with some internal vertices). If we denote the set of all such graphs by $\mathcal{G}_{n,m}$, then the above statements simply becomes*

$$\mathcal{Q}_m = \sum_{D \in \mathcal{G}_{n,m}} e(D).$$

In the above statement, the only thing we did not justify is the coefficient $1/|\operatorname{Aut} D|$. This is just some combinatorial result which we will omit. However, we will give a sample calculation in the next section, showing that for a loop D we have $|\operatorname{Aut} D| = 2$. People usually denote $|\operatorname{Aut} D|$ by $s(D)$ and call it the ***symmetry factor***. Note that since we are pairing two verticles by an edge (propagator), of course we should get twice as many points as edges. That is,

$$n + dm = 2\ell,$$

where d is the degree of ϕ in the interaction piece $\mathcal{L}_{\text{int}}$. In our case $d = 3$. This is precisely Proposition 7.5.

Now, we have almost constructed a dictionary between physical quantities and geometric elements in Feynman diagrams:

Physical Quantities	**Elements in Feynman Diagram**
number of insertions in correlator	number of external vertices
order of λ in the Taylor expansion	number of internal vertex
propagators	edges of the diagram
degree of ϕ in the interaction term	valence of the diagram

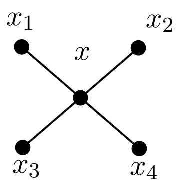
The only thing that is possibly not clear is why does valence of the diagram corresponds to the degree of the interaction. To see this, we go back to (7.10). Say we are looking at an internal vertex z_1. Then asking the valence of the graph at the vertex z_1 is the same as asking how many things can we pair z_1 into. Equivalently, this is asking how many $\delta/\delta J(z_1)$ we are taking in (7.10), which is evidently the degree of the interaction.

7.5 Momentum Space Feynman Rules

Usually, momentum space Feynman diagram computation is very messy. The calculation is greatly simplified if we do Fourier transforms and go to momentum space. Let us consider the example of a ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

One Feynman diagram in position space for the 4-point correlator is given by



In position space the Feynman rule gives the value of this diagram:

$$e(D) = -i\lambda \int \mathrm{d}^4x\, G(x, x_1)G(x, x_2)G(x, x_3)G(x, x_4).$$

Since the propagator reads

$$G(x,x') = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\mathrm{e}^{ik(x-x')},$$

we see that after Fourier transform, the diagram has value

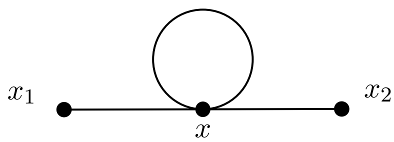
$$e(D) = -i\lambda\int \mathrm{d}^4x \prod_{j=1}^{4}\frac{\mathrm{d}^4k_j}{(2\pi)^4}\frac{i}{k_j^2-m^2}\mathrm{e}^{ik_j(x-x_j)}.$$

Note that the expression contains a term

$$\int \mathrm{d}^4x\ \mathrm{e}^{i(k_1+k_2+k_3+k_4)x} = \delta^4(k_1+k_2+k_3+k_4).$$

This enforces conservation of momentum. Therefore, in momentum space, we must always write down such a delta function associated to each diagram which enforces momentum conservation.

Let us give an example of how to compute the symmetry factor $|\operatorname{Aut}D|$ of a diagram. For example let us take this diagram in for the ϕ^4 theory:



The related term in the series expansion is given by

$$e(D) = \frac{-i\lambda}{4!}\int \mathrm{d}^4x\ \langle 0|\phi(x)\phi(x)\phi(x)\phi(x)\phi(x_1)\phi(x_2)|0\rangle.$$

Using Wick's theorem, $\phi(x_1)$ can hook onto 4 of the $\phi(x)$, and $\phi(x_2)$ can hook onto any of the remaining 3 of the $\phi(x_4)$, then the remaining 2 $\phi(x)$ must pair up. Hence the expression becomes

$$e(D) = -\frac{i\lambda}{4!}\int \mathrm{d}^4x\ 4G(x,x_1)3G(x,x_2), G(x,x) = -\frac{i\lambda}{2}\int \mathrm{d}^4x\ G(x,x_1)G(x,x_2)G(x,x).$$

Hence in this case we see that $|\operatorname{Aut}D| = 2$. Fourier transform to momentum space, this becomes

$$\frac{-i\lambda}{2}\frac{i}{k^2-m^2}\frac{i}{k^2-m^2}\int \mathrm{d}^4p\ \frac{i}{p^2-m^2}$$

which corresponds to the diagram in Fig.7.4.

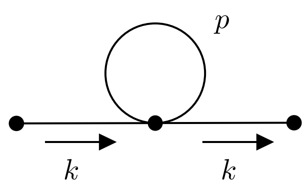


Figure 7.4: Momentum space loop diagram.

To get back to the position space, we simply inverse Fourier the expression above.

For now, it might not be obvious why we would prefer calculating Feynman diagrams in momentum space instead of in position space, except that it makes momentum conservation more obvious. This will come clear when we introduce the LSZ formula: What experimental physicists can measure is something called scattering amplitude, which is usually denoted by $\mathcal{M}$. The scattering amplitude is directly related to the time ordered correlator by the LSZ formula: In fact, it is just proportional to the sum of all Feynman diagrams, with external legs amputated (by this we mean when we evaluate Feynman diagrams, we just ignore all the propagators connecting to external vertices). In this case, to calculate $\mathcal{M}$ it is much easier to calculate the Feynman diagram in momentum space.

7.6 Example: Feynman Rules for Scalar QED

In the previous section, we have studied how we can generate Feynman diagrams given a theory with a specified Lagrangian. If the Lagrangian is free, then we can solve the theory exactly, otherwise if it includes a nontrivial interaction term, then we need to compute the correlator functions perturbatively by a systematic procedure of calculating Feynman diagrams. We have already seen how this works for ϕ^3 and ϕ^4 theory (and the story is pretty much the same for ϕ^n theory). In this chapter, we will work out another important example to illustrate this procedure: the theory of scalar quantum electrodynamics (SQED). We have already seen this theory in Example 6.9. What makes this example different from the ϕ^n-theory is that instead of just one real scalar field ϕ, we now have a complex scalar field ϕ (which gives two real scalar fields α, β where $\phi = \alpha + i\beta$, or equivalently we have two independent scalar fields ϕ, ϕ^* that we need to consider). Moreover, there is in addition the presence of a vector field A_μ.

Let us go back to the Lagrangian in Example 6.9. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi + ieA_\mu\phi)(\partial^\mu\phi^* - ieA^\mu\phi^*) - m^2|\phi|^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\Box + m^2)\phi - ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + \mathrm{e}^2 A_\mu A^\mu|\phi|^2. \end{aligned} \tag{7.11}$$

In the second equality we used integration by parts to replace $\partial_\mu\phi\partial^\mu\phi^*$ by $-\phi^*\partial^\mu\partial_\mu\phi = -\phi^*\Box\phi$. The resulting difference in action will be the integral of a total derivative, which under suitable boundary conditions will be zero by Stokes' theorem. Hence, the action is invariant under such replacement. In other words, in going from $\mathcal{L}$ to (7.11), we did not change any physics, since action is all that matters.

An easy way to remember this Lagrangian is by introducing the ***covariant derivative*** operator $D_\mu = \partial_\mu + ieA_\mu$. Then the SQED Lagrangian simply becomes

$$\mathcal{L} = -\frac{1}{2}F_{\mu\nu}^2 + |D_\mu\phi|^2 - m^2|\phi|^2.$$

Looking at (7.11), we must first determine what is the free piece and what is the interaction piece. Since the free piece should have at most quadratic order in fields, and interaction piece at least cubic order in fields, we conclude that

$$\mathcal{L}_{\text{free}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\Box + m^2)\phi, \tag{7.12}$$

$$\mathcal{L}_{\text{int}} = -ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + \mathrm{e}^2 A_\mu A^\mu \phi\phi^*. \tag{7.13}$$

As we have seen from the previous section, we must first solve the free theory, which will give us all the propagators in the theory. And then we will focus on the interaction piece, which will give us information of what our Feynman diagram beyond tree level will look like: coefficients for vertices, valence of the diagrams, etc.

As discussed, we will first look at the free theory (7.12). The free theory partition function is

$$
\begin{aligned}
&Z_0[J^\mu, J, J^*] \\
&\quad = \int \mathcal{D}A\mathcal{D}\phi\mathcal{D}\phi^* \exp\bigg(i\int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\Box + m^2)\phi + A_\mu J^\mu + J^*\phi + \phi^* J\right)\bigg).
\end{aligned}
\tag{7.14}
$$

In (7.14), there are three different currents, one is the vector current J^μ, and there are also J, J^* current for ϕ and ϕ^*. To calculate time ordered correlators, we take the corresponding functional derivatives of (7.14), and evaluate at $J^\mu = J = J^* = 0$.

7.6.1 Photon Propagator and Gauge Fixing

The first quantity we want to calculate is

$$
\langle TA_\mu(x_1)A_\nu(x_2)\rangle = (-i)^2\frac{1}{Z_0[0]}\frac{\delta^2 Z_0}{\delta J^\mu(x_1)\delta J^\nu(x_2)}\bigg|_{J=0}.
$$

Since anything in the exponent of (7.14) containing ϕ, ϕ^* does not couple with the vector current J^μ, we see that

$$
(-i)^2\frac{1}{Z_0[0]}\frac{\delta^2 Z_0}{\delta J^\mu(x_1)\delta J^\nu(x_2)}\bigg|_{J=0} = \frac{(-i)^2}{Z_0[0]}\int \mathcal{D}\phi\mathcal{D}\phi^*\ \mathrm{e}^{i\int \mathrm{d}^4x(-\phi^*(\Box+m^2)\phi)}\frac{\delta^2\mathcal{I}[J^\mu]}{\delta J^\mu(x_1)\delta J^\nu(x_2)}\bigg|_{J=0},
\tag{7.15}
$$

where

$$
\mathcal{I}[J^\mu] = \int \mathcal{D}A \exp\bigg(i\int \mathrm{d}^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\Big)\bigg).
\tag{7.16}
$$

Hence let us calculate $\mathcal{I}[J^\mu]$ first. Rewriting the first term in the exponent a little bit gives

$$
\begin{aligned}
-\frac{1}{4}\int \mathrm{d}^4x\ F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}\int \mathrm{d}^4x\ (\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu - \partial_\nu A_\mu\partial^\mu A^\nu + \partial_\nu A_\mu\partial^\nu A^\mu) \\
&= -\frac{1}{2}\int \mathrm{d}^4x\ (\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu) \\
&= \frac{1}{2}\int \mathrm{d}^4x\ (A_\nu\partial_\mu\partial^\mu A^\nu - A_\nu\partial^\mu\partial^\nu A_\mu) \\
&= \frac{1}{2}\int \mathrm{d}^4x\ A_\nu(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\mu.
\end{aligned}
$$

Hence $\mathcal{I}[J^\mu]$ now reads

$$
\mathcal{I}[J^\mu] = \int \mathcal{D}A \exp\bigg(i\int \mathrm{d}^4x\left(\frac{1}{2}A_\mu(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu + A_\mu J^\mu\right)\bigg).
\tag{7.17}
$$

This is a standard Gaussian integral. The integration yields

$$
\mathcal{I}[J^\mu] = \mathcal{N}\exp\bigg(\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ J^\mu(x)\frac{-i}{g^{\mu\nu}\Box - \partial^\mu\partial^\nu}J^\nu(y)\bigg),
$$

and the renormalization constant $\mathcal{N}$, multiplied together with the $\mathcal{D}\phi\mathcal{D}\phi^*$ integral on the right hand side of (7.15) would exactly cancel. Hence we would be able to conclude that the propagator $\langle TA_\mu(x_1)A_\nu(x_2)\rangle$ is proportional to the inverse of $g^{\mu\nu}\Box - \partial^\mu\partial^\nu$. Right?

Unfortunately, there is one very serious mistake in our argument. The operator

$$P^{\mu\nu} = g^{\mu\nu}\Box - \partial^\mu \partial^\nu$$

is not invertible. To see this, just consider any function of the form $\partial_\mu \alpha$, for any $\alpha(x)$, then we see that $P^{\mu\nu}\partial_\mu \alpha = 0$. Therefore, the operator $P^{\mu\nu}$ is not injective, hence cannot be inverted. Hence our argument above falls apart. If you remember Example 6.9, a gauge transformation is precisely given by $A_\mu \to A_\mu + \partial_\mu \alpha$ for some α. Indeed, the SQED Lagrangian is invariant under such a transformation. The fact that $P^{\mu\nu}$ fails to be injective for having kernels containing $\partial_\mu \alpha$'s is precisely the indication that we are working with a gauge theory. Looking at the integral (7.17) we see that when we are integrating over $A_\mu = \partial_\mu \alpha$, the exponents become zero when $J^\mu = 0$. Hence $\mathcal{I}[0]$ would receive infinite contribution equal to measure of all field configurations $A_\mu = \partial_\mu \alpha$, which means it is badly divergent. Since $\mathcal{I}[J^\mu]$ is a factor of $Z_0[J]$, this means that $Z_0[0]$ is also badly divergent, which is problematic. Physically, this means that we are counting contributions from the entire gauge orbits, when we should have just counted contribution from one representative from each gauge orbit, since fields differ by a gauge transformation are physically the same.

But we are really close. The solution is to add a gauge fixing term

$$\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

to our Lagrangian, which now reads

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu + \frac{1}{2\xi}(\partial_\mu A^\mu)^2.$$

This seems odd at first. But we claim that we can always choose a gauge such that $\partial_\mu A^\mu = 0$. To see this, we first note that our original Lagrangian is gauge invariant, i.e., invariant under any shift $A_\mu \to A_\mu + \partial_\mu \alpha$. Now under such a gauge transformation, we have

$$\partial_\mu A^\mu \to \partial_\mu A^\mu + \Box\alpha.$$

Hence, we may find an α such that $\Box\alpha = -\partial_\mu A^\mu$. Indeed, the solution is given formally by

$$\alpha = -\frac{1}{\Box}\partial_\mu A^\mu.$$

After adding such a term, performing integration by part, our operator $P^{\mu\nu}$ now becomes

$$P^{\mu\nu} = g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu.$$

Substituting this into the modified version of (7.17), and comparing with (7.2) in Example 7.2, we see that the propagator would be an operator $\Pi_{\mu\nu}$ such that

$$P^{\mu\nu}\Pi_{\nu\sigma} = \delta^\mu_\sigma.$$

Fourier transforming into frequency space, we see that we want an operator $\tilde{\Pi}_{\nu\sigma}$ such that

$$\tilde{P}^{\mu\nu}\tilde{\Pi}_{\nu\sigma} = \delta^\mu_\sigma.$$

Now the Fourier transform of $P^{\mu\nu}$ is

$$\tilde{P}^{\mu\nu} = -k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu.$$

We claim that

$$\tilde{\Pi}^{\mu\nu} = -\frac{1}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{7.18}$$

Compare (7.18) with (6.8), we see that this is exactly the Green's function obtained from solving the Maxwell equation. Indeed, the partition function $\mathcal{I}[J^\mu]$ we are working with in (7.16) is literally Maxwell theory! Let us verify that (7.18) is indeed the operator we want:

$$\begin{aligned}
\tilde{P}_{\alpha\mu}\tilde{\Pi}^{\mu\nu} &= -\frac{1}{k^2}\left(-k^2 g_{\alpha\mu}g^{\mu\nu} + k^2 g_{\alpha\mu}(1-\xi)\frac{k^\mu k^\nu}{k^2} + \left(1-\frac{1}{\xi}\right)k_\alpha k_\mu g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)k_\alpha k_\mu (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \\
&= -\frac{1}{k^2}\left(-k^2\delta^\nu_\alpha + (1-\xi)k_\alpha k^\nu + \left(1-\frac{1}{\xi}\right)k_\alpha k^\nu - \left(1-\frac{1}{\xi}\right)k_\alpha(1-\xi)k^\nu\right) \\
&= -\frac{1}{k^2}\left(-k^2\delta^\nu_\alpha + k_\alpha k^\nu\frac{1-\xi}{\xi} + \frac{\xi-1}{\xi}k_\alpha k^\nu\right) = \delta^\nu_\alpha.
\end{aligned}$$

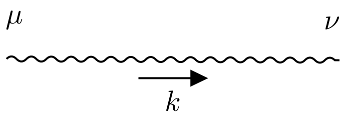
However, note that in (7.17), or rather the modified version after we add the gauge term, has quadratic fluctuation $iP^{\mu\nu}$ in the exponent inbetween A_μ and A_ν. Hence we really want to find the inverse of the operator $iP^{\mu\nu}$, which is $-i\Pi^{\mu\nu}$. But remember that in calculating the time ordered correlator (7.15) we need to multiply $-i\Pi^{\mu\nu}$ by $(-i)^2 = -1$, hence

$$\langle TA_\mu(x_1)A_\nu(x_2)\rangle = (-i)^2\frac{\delta\mathcal{I}}{\delta J^\mu \delta J^\nu} = \frac{i}{\Box g^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu} = i\Pi_{\mu\nu}(x_1, x_2).$$

Since our theory is translation invariant, propagator only depends on the distance between two points, i.e., $\Pi_{\mu\nu}(x_1, x_2) = \Pi_{\mu\nu}(x_1 - x_2)$. Hence it is related to the Fourier space propagator $\tilde{\Pi}_{\mu\nu}(k)$ by

$$\Pi_{\mu\nu}(x_1, x_2) = \Pi_{\mu\nu}(x_1 - x_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\tilde{\Pi}_{\mu\nu}(k)\mathrm{e}^{ik(x_1-x_2)}.$$

Therefore, in momentum space, the photon propagator $i\tilde{\Pi}^{\mu\nu}$ is simply



$$= \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right).$$

We will usually denote the photon propagator by a curly line. In the next section we will evaluate all other possible propagators in SQED, but the hard work is mostly done. The remaining calculation will be easier.

7.6.2 Complex Scalar Propagator and Others

We now want to evaluate the 2-point correlator $\langle T\phi^*(x)\phi(0)\rangle$. In fact, we could evaluate $\langle T\phi^*(x)\phi(y)\rangle$ for any arbitrary x, y. But as we discussed before, any reasonable theory should be translation invariant, hence without loss of generality we can set $y = 0$ and just focus on $\langle T\phi^*(x)\phi(0)\rangle$. Going back to (7.14), we see that

$$\langle T\phi^*(x)\phi(0)\rangle = \frac{(-i)^2}{Z_0[0]}\frac{\delta^2}{\delta J^*(x)\delta J(0)}Z_0[J^\mu, J, J^*]\bigg|_{J=0} = \frac{(-i)^2}{Z_0[0]}\mathcal{I}[J^\mu]\frac{\delta^2}{\delta J^*(x)\delta J(0)}\mathcal{Q}[J, J^*], \tag{7.19}$$

where

$$\mathcal{Q}[J, J^*] = \int \mathcal{D}\phi \mathcal{D}\phi^* \exp\left(i \int \mathrm{d}^4x(-\phi^*(\Box + m^2)\phi + J^*\phi + \phi^* J)\right). \tag{7.20}$$

We will directly evaluate $\mathcal{Q}[J, J^*]$. There is a version of Gaussian integral for complex scalars. In n-dimensional space, this is just the identity

$$\int \prod_{k=1}^{n} \mathrm{d}z_k \mathrm{d}z_k^* \exp\left(i \sum_{p,q} z_p^* M_{pq} z_q + i \sum_p (J_p^* z_p + z_p^* J_p)\right) = \frac{C}{\det M} \exp\left(-i \sum_{p,q} J_p^* (M^{-1})_{pq} J_q\right). \tag{7.21}$$

The proof of (7.21) is rather tedious and not very instructive. So we simply assume this result and omit the proof. The infinite dimensional generalization is immediate. Using this to evaluate (7.20) we get that

$$\mathcal{Q}[J, J^*] = \mathcal{N} \exp\left(-i \int \mathrm{d}^4x \mathrm{d}^4y \; J^*(x) \frac{1}{-\Box_x - m^2} J(y)\right). \tag{7.22}$$

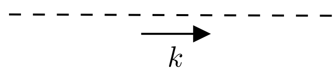
Taking functional derivative of the above expression, we see that

$$\frac{\delta^2}{\delta J^*(x) \delta J(0)} \mathcal{Q}[J, J^*] = \mathcal{N} \frac{-i}{-\Box - m^2} = \mathcal{N} \frac{i}{\Box + m^2}.$$

Substitute this into (7.19). We multiply the expression above by the overall coefficient $(-i)^2 = -1$, and again $\mathcal{N}$ and $\mathcal{I}[0]/Z_0[0]$ cancel out each other. Hence we conclude that

$$\langle T\phi^*(x)\phi(0)\rangle = \frac{i}{-\Box - m^2}.$$

Fourier transform into momentum space, the complex scalar propagator reads



$$= \frac{i}{k^2 - m^2 + i\epsilon}.$$

Finally, let us look at (7.14) again. Is there any other propagators that we haven't calculated yet? For example, what about $\langle T\phi(x)\phi(0)\rangle$ or $\langle TA_\mu(x)\phi(0)\rangle$? Let us discuss $\langle T\phi(x)\phi(0)\rangle$ first. We know that

$$\langle T\phi(x)\phi(0)\rangle \sim \left.\frac{\delta^2 Z_0}{\delta J^*(x) J^*(0)}\right|_{J^\mu = J = J^* = 0} \sim \left.\frac{\delta^2 \mathcal{Q}[J, J^*]}{\delta J^*(x) J^*(0)}\right|_{J = J^* = 0}.$$

Now using (7.22), it is not hard to convince oneself that the expression on the right hand side is zero: When we takes the first functional derivative with respect to J^*, it is the exponential time with derivative of its exponent, which is proportional to J (independent of J^*). Hence when we take functional derivative with respect to J^*, we pull down another derivative of the exponent, which is again proportional to J. All these factors multiplies, hence the expression on the right hand side is proportional to J, which vanishes when we set $J = 0$ in the end. Same argument also shows that $\langle T\phi^*(x)\phi^*(0)\rangle = 0$. The same argument also shows $\langle TA_\mu(x)\phi(0)\rangle = \langle TA_\mu(x)\phi^*(0)\rangle = 0$: When we take derivative of J^μ with respect to the gauge fixed $\mathcal{I}[J^\mu]$, we get a term proportional to J^μ, this is multiplied with $\delta\mathcal{Q}/\delta J(0)$ or $\delta\mathcal{Q}/\delta J^*(0)$, which is proportional to J or J^*. In the end when we set $J^\mu = J = J^* = 0$, the expression vanishes.

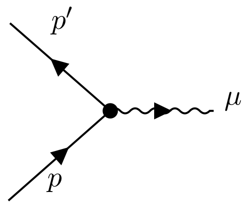
Hence, we conclude that in SQED, the only possible nonzero Wick contractions must pair (contract) A_μ with its own kind A_ν, whereas ϕ can only be contracted with ϕ^* and vice versa. All other Wick contraction will vanish. In other words, the photon propagator and the complex scalar propagator are all the propagators there are to calculate. Therefore, we have completely solved the free theory given by the partition function (7.14) at this point, and we can proceed to the full SQED partition function to study interactions of this theory.

7.6.3 Interactions

Now we can study interactions in SQED. Our goal will be modest: We want to determine what type of interaction vertices that we might have, and determine the coefficients for the interaction vertices. Recall that the interaction part of the SQED Lagrangian is

$$\mathcal{L}_{\text{int}} = -ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + \mathrm{e}^2 A_\mu A^\mu \phi\phi^*.$$

We see that there is a cubic term (in fields), and a quartic term. This means that we will get both 3-valent and 4-valent diagrams in SQED. Let us first look at the cubic terms. It contains 3 fields, hence in momentum space, we have a 3-valence diagram with vertex value



$$= -ie(p+p')^\mu.$$

Figure 7.5: Cubic vertex in SQED.

To see why the vertex coefficient is $-ie(p+p')^\mu$, we note that this cubic diagram is related to the three point function

$$(-ie)\int \mathrm{d}^4w\ \langle 0|\phi\phi^* A_\mu A_\alpha \phi^*\partial^\alpha\phi|0\rangle - \langle 0|\phi\phi^* A_\mu A_\alpha \phi\partial^\alpha\phi^*|0\rangle.$$

We are a bit careless in the above expression: The time ordered operator was omitted, also all position coordinates were omitted. The first three terms in front are external points, and the internal vertex is w.

Now, recall from the previous section that we expanded the interaction term $\exp\big(i\int \mathrm{d}^4x\mathcal{L}_{\text{int}}\big)$. Hence the associated coefficient in front of the vertex should be the coefficient in front of

$$i(-ie)\int \mathrm{d}^4w\ A^\mu(w)[\phi^*(w)\partial_\mu\phi(w) - \partial_\mu\phi^*(w)\phi(w)]. \tag{7.23}$$

For the sake of convenience, it is of our best interest to obtain the vertex coefficient in momentum space rather than position space. This means we need to Fourier transform (7.23). We have

$$A_\mu(w) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-ikw}\tilde{A}_\mu(k) \qquad \phi(w) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-ipw}\tilde{\phi}(p) \qquad \phi^*(w) = \int \frac{\mathrm{d}^4p'}{(2\pi)^4}\mathrm{e}^{ip'w}\tilde{\phi}^*(p').$$

Then (7.23) becomes

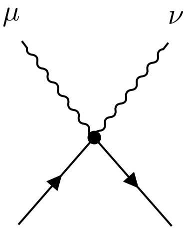
$$(-ie)\int \mathrm{d}^4w \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4p'}{(2\pi)^4}\tilde{A}^\mu(k)\mathrm{e}^{-ikw}[\tilde{\phi}^*(p')\mathrm{e}^{ip'w}p_\mu\tilde{\phi}(p)\mathrm{e}^{-ipw} + p'_\mu\tilde{\phi}^*(p')\mathrm{e}^{ip'w}\tilde{\phi}(p)\mathrm{e}^{-ipw}]$$

Note that the second term become a plus sign instead of minus becomes ∂_μ acting on $\mathrm{e}^{ip'w}$ produces a $+p'_\mu$. Hence the expression reads

$$(-ie)\int \mathrm{d}^4w\ \mathrm{e}^{-i(p-p'+k)w}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4p'}{(2\pi)^4}\tilde{A}^\mu(k)\tilde{\phi}^*(p')\tilde{\phi}(p)(p_\mu+p'_\mu).$$

The d^4w integral gives $(2\pi)^4\delta^4(p-p'+k)$, which is just a momentum conservation condition, which is automatically enforced by the momentum space Feynman rule. Hence in momentum space, with external legs amputated by LSZ formula. The cubic diagram has value just equal to its vertex value $-ie(p+p')^\mu$.

There is also an interaction term $\mathrm{e}^2A_\mu A^\mu\phi\phi^*$ in the Lagrangian. This gives a diagram with value



$$= 2i\mathrm{e}^2g^{\mu\nu}.$$

Figure 7.6: Quartic vertex in SQED.

To see why Fig.7.6 has value $2ieg^{\mu\nu}$, we first note that this diagram is associated with the 4-point correlator $\langle 0|A_\mu A_\nu\phi\phi^*A_\alpha A^\alpha\phi\phi^*|0\rangle$. Already, we know that there will be a factor of 2 coming from this diagram, because we have two different contractions

$$\langle 0|A_\mu A_\nu\phi\phi^*A_\alpha A^\alpha\phi\phi^*|0\rangle \qquad \langle 0|A_\mu A_\nu\phi\phi^*A_\alpha A^\alpha\phi\phi^*|0\rangle$$

that both give the same value by symmetry, hence the quartic diagram should include contributions from both contractions. Hence we conclude that the diagram Fig.7.6 should have value $2ie^2g^{\mu\nu}$.

Since one of the interaction term in (7.11) is of degree 3, another typical Feynman diagram for 4-point correlator is something like

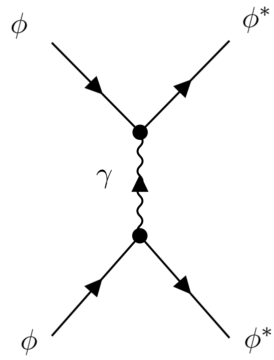


Figure 7.7: A typical Feynman diagram for SQED. The curly line is the vector boson propagator, and the straight line is the propagator for scalar boson.

We have completed our derivation of SQED Feynman rules in momentum space. We will come back to concrete SQED scattering amplitude calculations much later, but hopefully this chapter serves as a good exercise in applying the techniques we learned from the previous section to an interesting example.

7.7 Path Integrals and Partition Function

The Greens functions of quantum theory are related to time-evolution operators, which (roughly speaking) look like a complex exponential e^{-iEt}, with E an energy and t the time. The probabilities (and density matrices) of statistical physics are based on the Boltzmann factor, given (roughly) by the real exponential $\mathrm{e}^{-E/T}$, where T is the temperature and k_B is the Boltzmann constant. The fundamental elements of the two theories are therefore both exponentials, albeit one is real and the other is complex. This similarity can be exploited by making the identification $it = T^{-1}$. The claim is that this will map a quantum field theory on to a statistical field theory. The advantage of this mapping is that it allows us to transfer results from one subject directly to the other, and vice versa. Within the framework of the mapping, inverse temperature $\beta = 1/T$ in statistical physics behaves like imaginary time in a quantum field theory. It turns out that the concept of imaginary time is a remarkably powerful one with uses across quantum and statistical field theory. We examine the imaginary time formalism now. A ***Wick rotation*** is the substitution

$$\tau = it,$$

i.e., allowing imaginary time. With this substitution, we replace the Minkowski metric $(+,-,-,-)$ to the Euclidean metric $(+,+,+,+)$ up to a minus sign. We will see the benefit of this trick very soon.

Arguably the most important object in statistical physics is the ***partition function*** of a system. Suppose the system is governed by a Hamiltonian H, the partition function is given by

$$Z(\beta) = \operatorname{Tr} \mathrm{e}^{-\beta H}.$$

If we have an complete orthonormal basis of our Hilbert space, say $|\lambda\rangle$ for λ in some index set, then the partition function is expressed by

$$Z(\beta) = \sum_{\lambda} \langle \lambda | \mathrm{e}^{-\beta H} | \lambda \rangle.$$

In particular if these are chosen to be eigenstates of the Hamiltonian, $H\,|E_n\rangle = E_n\,|E_n\rangle$, then

$$Z(\beta) = \sum_{n} \langle E_n | \mathrm{e}^{-\beta H} | E_n \rangle = \sum_{n} \mathrm{e}^{-\beta E_n}.$$

Our next claim is that the partition function is really a path integral in disguise. Note that the expression $\langle \lambda | \mathrm{e}^{-\beta H} | \lambda \rangle$ can be interpreted as an amplitude $\langle q_F | \mathrm{e}^{-iHt} | q_I \rangle$ if we choose $|\lambda\rangle$ to be the position eigenstate at $q_I = q_F = \lambda$ (of course then instead of suming over λ in the $Z(\beta)$ expression we should integrate over all position λ), and make the replacement $\beta = it$. Then we have

$$Z(\beta) = \int \mathrm{d}\lambda \langle \lambda | \mathrm{e}^{-iHt} | \lambda \rangle = \int \mathrm{d}\lambda \int \mathcal{D}q \mathrm{e}^{i \int_0^t \mathrm{d}t L(q(t))}.$$

But we already made the replacement $t = -i\beta$. So our limit of integration goes from $t = 0$ to $t = -i\beta$. To make the expression nicer, we apply Wick rotation $t = -i\tau$. After performing change of variable, we end up with

$$Z(\beta) = \int \mathrm{d}\lambda \int \mathcal{D}q \mathrm{e}^{-\int_0^\beta \mathrm{d}\tau L_E(q(\tau))}, \tag{7.24}$$

where we denote the Lagrangian $L(q(t))$ under the change of variable by $L_E(q(\tau))$, the ***Euclidean Lagrangian***. For example, if the Lagrangian we started with is the standard Lagrangian

$$L(q(t)) = \frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - V(q),$$

then under the Wick rotation $\mathrm{d}q/\mathrm{d}t = i\mathrm{d}q/\mathrm{d}\tau$, thus

$$\begin{aligned} i\int_0^{-i\beta} \mathrm{d}t L(q(t)) &= i\int_0^{\beta} -i\mathrm{d}\tau\left[-\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 - V(q)\right] \\ &= -\int_0^{\beta} \mathrm{d}\tau\left[\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q)\right]. \end{aligned}$$

Hence we see that

$$L_E(q(\tau)) = \frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q).$$

Note that the sign of the kinetic term of L_E is not changed compared to L, but the sign of the potential term is reversed. But there is still more to (7.24). Note that in the formula above it we had to integrate $\langle\lambda|\mathrm{e}^{-iHt}|\lambda\rangle$ where t goes from 0 to $-i\beta$, so τ goes from 0 to β. Since $q_I = q_F = \lambda$, we have $q(t=0) = q(t=-i\beta) = \lambda$. Thus $q(\tau=0) = q(\tau=\beta) = \lambda$. But that's not all. If we imagine τ extending from $-\infty$ to ∞ then the boundary conditions are periodic: after a period β, the state returns to where it started. The integral over λ means that we integrate over all path subject to these periodic boundary conditions (PBC). Thus (7.24) becomes

$$Z(\beta) = \operatorname{Tr} \mathrm{e}^{-\beta H} = \int_{\mathrm{PBC}} \mathcal{D}q \mathrm{e}^{-\int_0^\beta \mathrm{d}\tau L_E(q(\tau))} \tag{7.25}$$

where PBC stands for the periodic boundary condition with period β. This result will be very useful, because as we will see later, we can write the index of a Dirac operator on spin manifold as a partition function, which can then be computed using the right hand side of (7.25) using path integral techniques, which will lead to the famous Atiyah-Singer index theorem.

Chapter 8

Vectors and Spinors

8.1 Canonical Quantization of Vector Bonsons

In this chapter we discuss the theory of vector-valued quantum fields. The most important example of a vector field theory is the Maxwell's theory $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ where as usual we have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Another example is the theory of massive vector bonson with Lagrangian given by $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$. Note that the signs of the coefficient is important, because in this way the term for the time derivative will be $\frac{1}{2}\dot{A}_i^2$. So far, these are classical fields, so we need to quantize A_μ, the vector boson. We will do something similar as the canonically quantized scalar field. We first present the expression for quantized vector bonson:

$$A_\mu(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \sum_j \epsilon_\mu^j(p) a_{p,j} \mathrm{e}^{-ipx} + \epsilon_\mu^{j*}(p) a_{p,j}^\dagger \mathrm{e}^{ipx}. \tag{8.1}$$

In the expression, p is a 4-vector with $p = (p^0, \mathbf{p})$. Note that we require the on shell condition $(p^0)^2 - |\mathbf{p}|^2 = m^2$ and $\omega_\mathbf{p} = \sqrt{|\mathbf{p}|^2 + m^2}$. In our case, $m = 0$ since the vector boson is assumed to be massless. The index j represents ***polarizations*** which we will get to shortly, and $a_{p,j}^\dagger$ is creation operator defined by

$$a_{p,j}^\dagger |0\rangle = \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left|p, \epsilon^j\right\rangle.$$

Now we need to determine the range of $j = 1, ..., n$ in the summand (8.1). For the massive boson, the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu.$$

The equation of motion is given by

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = 0.$$

The solution, as one can easily check, is given by the solution to the following two equations

$$\partial_\mu \partial^\mu A^\nu + m^2 A^\nu = 0, \tag{8.2}$$

$$\partial_\mu A^\mu = 0. \tag{8.3}$$

Using the expression (8.1), we find constraints on the vectors ϵ^μ. The first equation (8.2) applied to (8.1) gives

$$(p^0)^2 - |\mathbf{p}|^2 - m^2 = 0.$$

The second equation (8.3) applied to (8.1) gives

$$p_\mu \epsilon_j^\mu(p) = 0. \tag{8.4}$$

In addition, we also require the normalization condition

$$\epsilon_\mu^j(p)\epsilon^{\mu j *}(p) = -1.$$

Let us pick a reference frame such that the 4-momentum of our vector boson takes the form $p^\mu = (E, 0, 0, p_z)$. The condition (8.4) immediately gives 3 polarization vectors

$$\epsilon_{(1)}^\mu = (0,1,0,0) \qquad \epsilon_{(2)}^\mu = (0,0,1,0) \qquad \epsilon_{(3)}^\mu = (p_z/m, 0, 0, E/m).$$

These are the three physical polarizations. Thus in (8.1), if A_μ is massive, then the sum in j should go from 1 to 3. To get a more mathematically sophisticated explanation, we can look at Lorentz transformations which don't change p^μ. We can go to the rest frame of the particle, in which the 4-momentum is $p^\mu = (m, 0, 0, 0)$. Then clearly the Lorentz transformations which fixes p^μ is a copy of SO(3) which rotates the last three components. The group SO(3) fixing p^μ is called the ***little group***. Mathematically, spin 1 representation of little group tells us the propagating states (polarizations).

Now we can turn to the massless case. In this case we have a gauge redundancy (or a gauge symmetry), because now the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is invariant under the gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \alpha.$$

The equation of motion for the massless boson is simply $\partial_\mu F^{\mu\nu} = 0$. Expanding it out we get

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu).$$

As explained in the path integral derivation of SQED Feynman rules, we can choose a gauge where $\partial_\mu A^\mu = 0$. Then substitute (8.1) into the equation of motion and gauge condition yields same equations for ϵ^μ. But the gauge condition which says we identify A_μ with $A_\mu + \partial_\mu\alpha$ requires us to identify $|\epsilon_\mu\rangle \sim |\epsilon_\mu + p_\mu\rangle$. This is one more constraint, so we should expect to find less ϵ_μ's than the massive case (in which case there are 3). Let us now find ϵ_μ's. We still have the condition

$$p_\mu \epsilon^\mu = 0 \qquad \epsilon_\mu \epsilon^{\mu *} = -1.$$

We pick a frame where $p^\mu = (E, 0, 0, E)$. The above two conditions will give 3 polarizations

$$\epsilon_{(1)}^\mu = (0,1,0,0) \qquad \epsilon_{(2)}^\mu = (0,0,1,0) \qquad \epsilon_{(3)}^\mu = (E,0,0,E).$$

But the last polarization is not physical because $\epsilon_{(3)}^\mu$ is proportional to p^μ, hence is equivalent to zero because $|\epsilon_\mu\rangle \sim |\epsilon_\mu + p_\mu\rangle$. Therefore, we only get two polarizations.

8.2 Representations of the Lorentz Group

We will now give a precise definition of spin. Let's return to the group of Lorentz transformations. This is a six dimensional Lie group, with generators (basis of Lie algebra) three rotations $\mathbf{J}$ (rotation in the xy, xz, yz plane) and three boosts $\mathbf{K}$ (rotation in the tx, ty, tz plane). Hence the most general Lorentz transformation is obtained by exponentiating the basis:

$$\Lambda = \exp(i\boldsymbol{\theta} \cdot \mathbf{J} + i\boldsymbol{\beta} \cdot \mathbf{K}).$$

The commutation relations that the Lorentz algebra satisfies is

$$[J_i, J_j] = i\epsilon_{ijk}J_k \qquad [J_i, K_j] = i\epsilon_{ijk}K_k \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k.$$

We can define two operators

$$J_i^+ = \frac{1}{2}(J_i + iK_i) \qquad J_i^- = \frac{1}{2}(J_i - iK_i).$$

We see that $J_i^+ J_i^- = J_i$, and it is not hard to check that $[J_i^+, J_j^-] = 0$. Hence it is possible to diagonalize $J_i^\pm$ simultaneously. Moreover, $[J_i^+, J_j^+] = i\epsilon_{ijk}J_k^+$, and $[J_i^-, J_j^-] = i\epsilon_{ijk}J_k^-$. Hence we see that $J^\pm$ are two sets of Lie algebra satisfying the commutation relation of angular momentum operators, i.e., two copies of $\mathfrak{so}(3)$ algebra.

Let us look at finite dimensional representations of Lorentz group. By the discussion above, these are finite representations of $J^\pm$'s, hence are labeled by two indices (j^+, j^-), but the more common notation is (j_L, j_R), i.e., the left and right handed representations. The dimension of the representations will be $d_L = 2j_L + 1$ and $d_R = 2j_R + 1$. A possible list of representations could be $(j_L, j_R) = (0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (1,1)...$

If we have two irreducible representations j_L, j_R of $\mathfrak{so}(3)$, their tensor product representation $j_L \otimes j_R$ decomposes into direct sums of irreducible representations

$$j_L \otimes j_R = |j_L + j_R| \oplus \cdots \oplus |j_L - j_R|,$$

called the Clebsch-Gordan decomposition. Thus the list of $j_L \otimes j_R$ corresponding to the list above is $0, \frac{1}{2}, \frac{1}{2}, 1 \oplus 0, 2 \oplus 1 \oplus 0$. Thus there are two spin $\frac{1}{2}$ representations, associated to $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, which we refer to as the ***left-handed*** and ***right-handed*** Weyl fermion.

Let us find the representation of the Lorentz group corresponding to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. In the case $(\frac{1}{2}, 0)$ we take

$$J_i^+ = \sigma_i/2 \qquad J_i^- = 0.$$

Hence we see that

$$\mathbf{J} = \boldsymbol{\sigma}/2 \qquad \mathbf{K} = -i\boldsymbol{\sigma}/2.$$

Similarly, for the $(0, \frac{1}{2})$ representation, we take

$$J_i^+ = 0 \qquad J_i^- = \sigma_i/2.$$

Therefore

$$\mathbf{J} = \boldsymbol{\sigma}/2 \qquad \mathbf{K} = i\boldsymbol{\sigma}/2.$$

We denote an element of the left-handed representation $(\frac{1}{2}, 0)$ by

$$\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and an element of the right-handed representation $(0, \frac{1}{2})$ by

$$\psi_R = \begin{pmatrix} \psi^{\dot{1}} \\ \psi^{\dot{2}} \end{pmatrix}.$$

Then the corresponding Lorentz transformation is

$$\psi_L \to \exp\left(\frac{i}{2}\theta_j\sigma_j + \frac{1}{2}\beta_j\sigma_j\right)\psi_L$$

and

$$\psi_R \to \exp\left(\frac{i}{2}\theta_j\sigma_j - \frac{1}{2}\beta_j\sigma_j\right)\psi_R.$$

The ψ_L's are called the ***left-handed Weyl spinors***, and ψ_R's are called the ***right-handed Weyl spinors***. We also define the ***Dirac spinor*** as a 4-component object, which is the direct sum of the two representations $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, by

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi^{\dot{1}} \\ \psi^{\dot{2}} \end{pmatrix}.$$

8.3 Four Dimensional Representations of the Lorentz Group

We want our Lorentz algebra to act on Dirac spinors. There is another more covariant way to package the Lorentz algebra, whose commutation is given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}), \tag{8.5}$$

where $g^{\mu\nu}$ is the Minkowski metric, and $J^{\mu\nu}$ is a rotation in the $\mu\nu$ plane (if indices are spatial, this is a rotation, if it involves a time coordinate then it is a boost, since a boost is like a rotation between space and time). What we want are 4×4 matrices $(S^{\mu\nu})_{AB}$, where $\mu, \nu, A, B = 0, 1, 2, 3$, which acts on Dirac spinors, and satisfies (8.5). In fact, we also have a representation $(\mathcal{J}^{\mu\nu})_{\mu',\nu'}$ that acts on Lorentz 4-vectors, not on the spinors satisfying (8.5). In the above notation, $\mu\nu$ index tells us which generator $J^{\mu\nu}$ it corresponds to, and A, B, μ', ν' are matrix indices.

Theorem 8.1. *There exists 4×4 matrices $(\gamma^\mu)_{AB}$ with $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{I}$. Then*

$$S^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

will satisfies the Lorentz algebra (8.5) above.

Proof. The proof is constructive. We define

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{I}_{2\times 2} \\ \mathbf{I}_{2\times 2} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{8.6}$$

The more compact notation to denote this is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (\mathbf{I}_{2\times 2}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbf{I}_{2\times 2}, -\sigma^i)$. The proof that the given $S^{\mu\nu}$ satisfies (8.5) is just computational, so we omit it. $\square$

One can also check that

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu,$$

where

$$(\mathcal{J}^{\mu\nu})_{\mu'\nu'} = i(\delta^\mu_{\mu'}\delta^\nu_{\nu'} - i\delta^\nu_{\mu'}\delta^\mu_{\nu'}).$$

Moreover, one can check that if

$$\Lambda_D = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right),$$

then

$$\Lambda_D^{-1}\gamma^\mu\Lambda_D = \Lambda^\mu{}_\nu\gamma^\nu,$$

where $\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)$.

8.4 The Dirac Equation

Consider the quantity

$$(i\gamma^\mu\partial_\mu - m)\Psi(x).$$

If we do a Lorentz transform $\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)$, then the left hand side reads

$$\Lambda_D(i\gamma^\mu\partial_\mu - m)\Psi(\Lambda^{-1}\cdot x).$$

Therefore, under a Lorentz transformation, the left hand side transform like an honest spinor. But we are interested in building a Lorentz invariant action. Since $(i\gamma^\mu\partial_\mu - m)\Psi(x)$ is a column vector, we want to multiply it by a row vector and get a scalar that is Lorentz inavriant. Therefore, we want a row vector Φ such that under a Lorentz transformation $\Phi \to \Phi\Lambda_D^{-1}$.

The first guess is to simply take $\Phi = \Psi^\dagger$. Then since $\Psi \to \Lambda_D\Psi$, we have $\Psi^\dagger \to \Psi^\dagger\Lambda_D^\dagger$. Unfortunately, the representation is not unitary so $\Lambda_D^\dagger$ need not equal to Λ_D^{-1}. So we need to work harder. The correct answer turns out to be

$$\bar{\Psi} \equiv \Psi^\dagger\gamma^0,$$

where γ^0 is defined in (8.6). Then we see that if we have a Dirac spinor

$$\Psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

then

$$\bar{\Psi} = \begin{pmatrix}\psi_R^\dagger & \psi_L^\dagger\end{pmatrix}.$$

One can then check that the quantity

$$\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

is Lorentz invariant. A more common notation is $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$. More generally for a vector p_μ we define $p\!\!\!/ = \gamma^\mu p_\mu$. Since we have a Lorentz invariant quantity now, let us consider the ***Dirac action***

$$S = \int \mathrm{d}^4x\ \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi.$$

From this action, we can get the action for the left handed spinor ψ_α and the right handed spinors $\chi^{\dot{\alpha}}$ that together constitute Ψ. This is because we know all the ingredients in the action in terms of left handed and right handed spinors:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix} \qquad \bar{\Psi} = \begin{pmatrix} \chi^{\dagger\dot{\alpha}} & \psi^\dagger_\alpha \end{pmatrix}.$$

Plugging these into the Dirac action, we see that the action breaks into the left and right handed pieces:

$$S = \int \mathrm{d}^4x \, \Big(i\chi^\dagger \sigma^\mu \partial_\mu \chi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - m(\chi^\dagger \psi + \psi^\dagger \chi) \Big).$$

We see that the mass term is causing the left and right handed spinors to couple together. So, if there is no mass term, we could have introduced the left handed Weyl spinor action

$$S_L = \int \mathrm{d}^4x \; \psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi$$

and the right handed Weyl spinor action

$$S_R = \int \mathrm{d}^4x \; \chi^\dagger i\sigma^\mu \partial_\mu \chi.$$

The term $m(\chi^\dagger\psi + \psi^\dagger\chi)$ is called the ***Dirac mass term***.

8.5 Canonical Quantization of Spinors

Let us canonically quantize the spinor field $\Psi(x)$. The mode expansion reads

$$\Psi(x) = \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} (a^s_\mathbf{p} u^s_\mathbf{p} \mathrm{e}^{-ip\cdot x} + b^{s\dagger}_\mathbf{p} v^s_\mathbf{p} \mathrm{e}^{ip\cdot x})$$

and

$$\bar{\Psi}(x) = \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} (a^{s\dagger}_\mathbf{p} \bar{u}^s_\mathbf{p} \mathrm{e}^{-ip\cdot x} + b^s_\mathbf{p} \bar{v}^s_\mathbf{p} \mathrm{e}^{ip\cdot x}).$$

We require

$$\{a^s_\mathbf{p}, a^{s'\dagger}_{\mathbf{p}'}\} = \delta_{s,s'}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}') \qquad \{a, b^\dagger\} = 0.$$

Thus we see that the spinor component anticommutes: $\Psi_A\Psi_B = -\Psi_B\Psi_A$.

Recall that the Dirac equation reads

$$(i\not{\partial} - m)\Psi = 0,$$

and we have

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

Hence the Dirac equation of Ψ applied to the canonical quantization expression above, we see that

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u^s_\mathbf{p} = 0 \qquad \begin{pmatrix} -m & -p\cdot\sigma \\ -p\cdot\bar{\sigma} & -m \end{pmatrix} v^s_\mathbf{p} = 0.$$

Assume that $m \neq 0$ first. In the rest frame of the particle, we have that $p^\mu = (m, 0, 0, 0)$. Then the equations above becomes

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u_s = 0 \qquad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} v_s = 0.$$

For any nonzero ξ_s, η_s which are two component vectors (remeber each entry in the matrix is a 2×2 matrix) We have solutions

$$u_s = \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \qquad v_s = \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}.$$

Thus we have two linearly independent solutions, denoted by $s = \uparrow, \downarrow$, given by

$$u_\uparrow = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_\downarrow = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_\uparrow = \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_\downarrow = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Now we want to go to some general Lorentz frame, not just the rest frame of the particle. So we may apply a Lorentz transformation. We will discuss the frame where $p_\mu = (E, 0, 0, p_z)$,

$$u_s = \begin{pmatrix} \begin{pmatrix} \sqrt{E - p_z} & 0 \\ 0 & \sqrt{E + p_z} \end{pmatrix} \xi_s \\ \begin{pmatrix} \sqrt{E + p_z} & 0 \\ 0 & \sqrt{E - p_z} \end{pmatrix} \xi_s \end{pmatrix}.$$

Similarly,

$$v_s = \begin{pmatrix} \begin{pmatrix} \sqrt{E - p_z} & 0 \\ 0 & \sqrt{E + p_z} \end{pmatrix} \eta_s \\ \begin{pmatrix} -\sqrt{E + p_z} & 0 \\ 0 & -\sqrt{E - p_z} \end{pmatrix} \eta_s \end{pmatrix}.$$

Using

$$\sqrt{p \cdot \sigma} = \begin{pmatrix} \sqrt{E - p_z} & 0 \\ 0 & \sqrt{E + p_z} \end{pmatrix}, \quad \sqrt{p \cdot \bar{\sigma}} = \begin{pmatrix} \sqrt{E + p_z} & 0 \\ 0 & \sqrt{E - p_z} \end{pmatrix}$$

we can write

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \bar{\sigma}} \xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta_s \\ -\sqrt{p \cdot \bar{\sigma}} \eta_s \end{pmatrix}.$$

8.6 Path Integrals for Quantum Electrodynamics

Now we want to use path integral formalism to deduce Feynman rules of QED, and calculate scattering amplitudes. But we first need to define path integral for vector bosons, and path integral for fermions. The QED Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$$

So we denote the action by

$$S[A, \bar{\Psi}, \Psi] = \int \mathrm{d}^4 x \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \Big).$$

And we want to cinsider path integral

$$\int \mathcal{D}A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS[A,\bar{\Psi},\Psi]}.$$

We need to address the issue of defining path integral over fermionic fields because they anticommute, and we note that we have gauge redundancy $A_\mu \mapsto A_\mu + \partial_\mu \alpha$. So we also need to address gauge redundancy in the path integral.

8.7 Faddeev-Popov Procedure

To address the gauge redundancy, we choose a gauge $G(A) \equiv \partial_\mu A^\mu = 0$. Now consider a change of gauge

$$A_\mu^{(\alpha)} = A_\mu + \frac{i}{e}\partial_\mu \alpha.$$

Then $A_\mu^{(\alpha)}$ should be identified with A_μ because it is in the same gauge orbit of A_μ. The technique we will present is called the ***Faddeev-Popov procedure***. We use the identity

$$1 = \int \mathcal{D}\alpha\, \delta(G(A^{(\alpha)}))\det\left(\frac{\delta G(A^{(\alpha)})}{\delta \alpha}\right),$$

which is just an analogue of the identity in multivariable calculus

$$1 = \int \prod_{i=1}^{n} \mathrm{d}a_i\, \delta^n(\mathbf{g}(\mathbf{a}))\det\left(\frac{\partial g_i}{\partial a_j}\right).$$

This is just the standard change of variable expression of the identity $1 = \int \prod_i \mathrm{d}a_i\, \delta^n(\mathbf{a})$. Now the key point is that the analogue of the Jacobian

$$\det\left(\frac{\delta G(A^{(\alpha)})}{\delta \alpha}\right)$$

is a constant independent of A. We first pick the gauge $G(A) = \partial_\mu A^\mu + \omega(x)$. Then $G(A^\alpha) = \partial_\mu A^\mu - \omega(x) + \partial_\mu(\frac{i}{e}\partial^\mu \alpha)$. Note that the only α dependence is in the last term, so the functional Jacobian will just be a functional determinant

$$\det\left(\frac{\delta G(A^{(\alpha)})}{\delta \alpha}\right) = \det\left(\frac{i}{e}\partial_\mu \partial^\mu\right) \equiv C$$

where C will be a formally infinite constant. Thus, if we now look at the path integral

$$\int \mathcal{D}A e^{iS[A]}.$$

Inserting the identity above, we can write

$$\begin{aligned}\int \mathcal{D}A e^{iS[A]} &= C\int \mathcal{D}\alpha \mathcal{D}A e^{iS[A]}\delta(G(A^{(\alpha)})) \\ &= C\int \mathcal{D}\alpha \mathcal{D}A e^{iS[A]}\delta(G(A))\end{aligned}$$

because $S[A]$ is gauge invariant, so we can shift the integral by a gauge choice, which is fine since we are still integrating over all gauge choices and all vector bosons. Continuing, we have that

$$\int \mathcal{D}A\mathrm{e}^{iS[A]} = C\int \mathcal{D}\alpha\mathcal{D}A\mathrm{e}^{iS[A]}\delta(\partial_\mu A^\mu - \omega).$$

The last expression is proportional to

$$\int \mathcal{D}A\mathrm{e}^{iS[A]} \propto \det\left(\frac{i}{e}\partial^2\right)\int \mathcal{D}\alpha\mathcal{D}\omega\mathcal{D}A\mathrm{e}^{-\frac{i}{2\xi}\int \mathrm{d}^4x\omega^2(x)}\mathrm{e}^{iS[A]}\delta(\partial_\mu A^\mu - \omega).$$

The path integral is now quadratic in A:

$$\int \mathcal{D}A\mathrm{e}^{iS[A]} \propto \int \mathcal{D}A\mathrm{e}^{iS[A]-i\int \mathrm{d}^4x\frac{(\partial_\mu A^\mu)^2}{2\xi}}.$$

This is now quadratic in A. This yields the vector boson propagator

$$\Pi_{\mu\nu} = \frac{i}{p^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$$

as we did in section 7.6.

8.8 Fermionic Path Integrals

For fermionic fields we have $\psi(x_1)\psi(x_2) = -\psi(x_2)\psi(x_1)$. Thus we need to define integration over anticommuting variables. We start with some formal definitions. We introduce the ***Grassmann numbers*** $\theta_1, ..., \theta_m$, such that $\theta_i\theta_j = -\theta_j\theta_i$. A function $f(x,\theta)$ may depends on the position x^μ and Grassmann variables θ^α for $\alpha = 1, ..., m$. We have the nice property when we Taylor expand

$$f(x,\theta) = f_0(x) + \theta_\alpha f^\alpha(x) + \theta_\alpha\theta_\beta f^{\alpha\beta}(x) + \cdots + \theta_1\theta_2\cdots\theta_m f^{1\cdots m}(x).$$

The Taylor series terminates because $\theta_i^2 = 0$. We now introduce calculus of Grassmann numbers. The derivative operator satisfies the usual property:

$$\frac{\partial}{\partial\theta}(\theta) = 1 \qquad \frac{\partial}{\partial\theta}(1) = 0.$$

Integration is more interesting, it is exactly the same as derivatives:

$$\int \mathrm{d}\theta(\theta) = 1 \qquad \int \mathrm{d}\theta(1) = 0.$$

If we have multivariable integration, we require

$$\int \mathrm{d}\theta_1\cdots\mathrm{d}\theta_n(\theta_n\cdots\theta_1) = 1.$$

We require the measure $\mathrm{d}\theta_i$ and θ_j's to anticommute, so derivative and integration are order sensitive. We can also have conjugation variables $\bar{\theta}_j$'s, just like holomorphic and antiholomorphic coordinates $z_j, \bar{z}_j$. Let us now evaluate an important Grassmann integral:

$$\mathcal{I} = \int \mathrm{d}\bar{\theta}_1\mathrm{d}\theta_1\cdots\mathrm{d}\bar{\theta}_n\mathrm{d}\theta_n\exp\left(-\bar{\theta}_i A_{ij}\theta_j\right).$$

Recall that in the bosonic case, we have the Gaussian integral

$$\int \mathrm{d}\bar{z}_1 \mathrm{d}\bar{z}_1 \cdots \mathrm{d}\bar{z}_n \mathrm{d}z_n \exp(-\bar{z}_i A_{ij} z_j) \propto \frac{1}{\det A} \exp\bigl(-J^T A^{-1} J\bigr). \tag{8.7}$$

In fermionic case, we will show that actually $\mathcal{I} \propto \det A$. Our strategy of evaluating $\mathcal{I}$ is to do the Taylor expansion, since it always terminates. Let us expand $\exp\bigl(-\bar{\theta}_i A_{ij} \theta_j\bigr)$. We have

$$\exp\bigl(-\bar{\theta}_i A_{ij} \theta_j\bigr) = 1 - \bar{\theta}_i A_{ij} \theta_j + \cdots + \frac{1}{n!}(-\bar{\theta}_i A_{ij} \theta_j)^n + \mathcal{O}(\theta^{n+1}).$$

Note that the Grassmann integral only picks out the order n term in the Taylor expansion, hence

$$\mathcal{I} = \int \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_1 \cdots \mathrm{d}\bar{\theta}_n \mathrm{d}\theta_n \frac{1}{n!}(-\bar{\theta}_i A_{ij} \theta_j)^n = \sum_{\sigma \in S_n} \frac{1}{n!} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)} \propto \det A.$$

Thus, we conclude that

$$\int \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_1 \cdots \mathrm{d}\bar{\theta}_n \mathrm{d}\theta_n \exp\bigl(-\bar{\theta}_i A_{ij} \theta_j\bigr) \propto \det A. \tag{8.8}$$

Now we do the analogues source term in the bosonic case, by introducing additional Grassmannians $\bar{\eta}_i, \eta_i$'s. We consider the integral

$$\mathcal{I}' = \int \mathrm{d}\bar{\theta} \mathrm{d}\theta \exp\bigl(-\bar{\theta}_i A_{ij} \theta_j + \bar{\eta}_i \theta_i + \bar{\theta}_i \eta_i\bigr),$$

where $\mathrm{d}\bar{\theta}\mathrm{d}\theta$ is the shorthand for $\prod_i \mathrm{d}\bar{\theta}_i \mathrm{d}\bar{\theta}_i$, as before. Using matrix notation, we have

$$\mathcal{I}' = \int \mathrm{d}\bar{\theta} \mathrm{d}\theta \exp\bigl(-(\bar{\theta} - \bar{\eta} A^{-1}) A (\theta - A^{-1}\eta) + \bar{\eta} A^{-1} \eta\bigr) = \exp\bigl(\bar{\eta} A^{-1} \eta\bigr) \det A. \tag{8.9}$$

Compare (8.9) with (8.7). Now, (8.8) and (8.9) have direct generalizations to grassmannian path integrals. Now let us consider the action

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp\left(i \int \mathrm{d}^4 x \; \bar{\Psi}(i\partial\!\!\!/ - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta\right). \tag{8.10}$$

We are interested in two point functions $\langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle$. By (8.9), we already know the answer should be:

$$Z[\eta, \bar{\eta}] \propto \exp\bigl(\bar{\eta}(i\partial\!\!\!/ - m)^{-1} \eta\bigr) \det A.$$

Hence

$$\langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle = \frac{1}{Z[0]} \frac{\delta^2}{\delta \bar{\eta}(x) \delta \eta(y)} Z[\eta, \bar{\eta}] \bigg|_{\eta = \bar{\eta} = 0} \propto (i\partial\!\!\!/ - m)^{-1}.$$

Therefore, we would like to find $(i\partial\!\!\!/ - m)^{-1}$. In momentum space, we find

$$\langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle = \frac{i}{i\partial\!\!\!/ - m + i\epsilon} \delta^4(x - y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p\!\!\!/ - m + i\epsilon} \mathrm{e}^{-ip(x-y)}.$$

Therefore, the fermionic propagator Θ_{AB} is given by the spinor

$$\Theta_{AB} = \left(\frac{i}{p\!\!\!/ - m + i\epsilon}\right)_{AB}.$$

We want to simplify the expression a bit more. Using

$$(\not{p}+m)(\not{p}-m)=\not{p}\not{p}-m^2$$

and

$$\not{p}\not{p}=\gamma_\mu\gamma_\nu p^\mu p^\nu=\frac{1}{2}\{\gamma_\mu,\gamma_\nu\}p^\mu p^\nu=g_{\mu\nu}p^\mu p^\nu=p^2,$$

we can clear the denominator of Θ_{AB} because

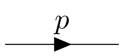
$$\frac{1}{\not{p}-m}=\frac{\not{p}+m}{p^2-m^2}.$$

8.9 Feynman Rules for QED

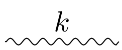
Recall that the QED Lagrangian is given by

$$S_{\text{QED}}=\int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\bar{\Psi}(i\not{D}-m)\Psi\right),$$

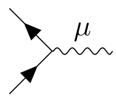
where $\not{D}=\gamma^\mu D_\mu$ with $D_\mu=\partial_\mu-ieA_\mu$. Now we write down the Feynman rules for QED. Previously we have computed the propagators for boson and fermions already. Similar to the SQED case, one can calculate that the vertex rule is given by $-ie\gamma^\mu$. Thus the Feynman rules for QED is given by:



$$=\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$



$$=\frac{-ig_{\mu\nu}}{k^2+i\epsilon}$$



$$=-ie\gamma^\mu$$

There is one more rule we need to discuss, that is the rule for fermionic loop. This involves a term like

$$\Psi(x)\bar{\Psi}(x)\Psi(y)\bar{\Psi}(y)$$

because if we expand out the path integral of the interaction terms, the Grassmann valued variables will appear exactly in this order. To make them into the correct position so that we can wick contract, we need to swing the first term all the way to the last, producing a minus sign:

$$\Psi(x)\bar{\Psi}(x)\Psi(y)\bar{\Psi}(y)=-\bar{\Psi}(x)\Psi(y)\bar{\Psi}(y)\Psi(x).$$

Thus a fermionic loop contributes to a minus sign to the overall result.

8.10 Fermionic Partition Functions

The fermionic harmonic oscillator is the direct analogue of the bosonic harmonic oscillator. The fermionic oscillator has Hamiltonian

$$H=\frac{1}{2}(c^\dagger c-cc^\dagger)\omega.$$

Recall that the bosonic oscillator hamiltonian is of the same form, with $c^\dagger, c$ replaced by the creation and annihilation operator $a^\dagger, a$, satisfying commutation relations $[a,a^\dagger]=1$ and $[a,a]=[a^\dagger,a^\dagger]=0$. Naturally, we would like to require the fermionic analogue of such relation:

$$\{c,c^\dagger\}=1\qquad\{c,c\}=\{c^\dagger,c^\dagger\}=0.$$

Similar to the bosonic case, we can define the ***number operator***

$$N = c^\dagger c,$$

which will put the Hamiltonian into simpler forms

$$H = \left(N - \frac{1}{2}\right)\omega.$$

Since $N^2 = c^\dagger c c^\dagger c = N$, the eigenvalue of N must be either 0 or 1. Let $|n\rangle$ denote the eigenvector of H with eigenvalue n, where $n = 0, 1$ as argued earlier. Then it is easy to verify that

$$c^\dagger |0\rangle = |1\rangle \qquad c\,|0\rangle = 0 \qquad c^\dagger |1\rangle = 0 \qquad c\,|1\rangle = |0\rangle\,.$$

Hence it follows that $c^\dagger c\,|n\rangle = n\,|n\rangle$, for $n = 0, 1$. Now consider hilbert space

$$\mathbf{H} = \mathrm{span}\,\{|0\rangle\,, |1\rangle\}.$$

Now we consider the states

$$|\theta\rangle = |0\rangle + |1\rangle\,\theta \qquad \langle\theta| = \langle 0| + \theta^*\,\langle 1|\,,$$

where θ and θ^* are Grassmann numbers. These states are called ***coherent states*** and are eigenstates of $c, c^\dagger$ respectively, since

$$c\,|\theta\rangle = |0\rangle\,\theta = |\theta\rangle\,\theta \qquad \langle\theta|\,c^\dagger = \theta^*\,\langle 0| = \theta^*\,\langle\theta|\,.$$

Now we have a very useful completeness relation:

Lemma 8.2. *Let $|\theta\rangle$ and $\langle\theta|$ be defined as above. Then we have*

$$\int \mathrm{d}\theta^*\mathrm{d}\theta\,|\theta\rangle\langle\theta|\,\mathrm{e}^{-\theta^*\theta} = 1. \tag{8.11}$$

Proof. This is just a straightforward computation:

$$\begin{aligned}\int \mathrm{d}\theta^*\mathrm{d}\theta\,|\theta\rangle\langle\theta|\,\mathrm{e}^{-\theta^*\theta} &= \int \mathrm{d}\theta^*\ \mathrm{d}\theta(|0\rangle + |1\rangle\theta)\,(\langle 0| + \theta^*\langle 1|)\,(1 - \theta^*\theta)\\ &= \int \mathrm{d}\theta^*\ \mathrm{d}\theta\,(|0\rangle\langle 0| + |1\rangle\theta\langle 0| + |0\rangle\theta^*\langle 1| + |1\rangle\theta\theta^*\langle 1|)\,(1 - \theta^*\theta)\\ &= |0\rangle\langle 0| + |1\rangle\langle 1|\\ &= 1.\end{aligned}$$

This completes the proof. □

Now we use the completeness relation to calculate the partition function of fermionic oscillator.

Lemma 8.3. *Let H be the Hamiltonian of a fermionic oscillator. Then the partition function of the system is given by*

$$\mathrm{Tr}\,\mathrm{e}^{-\beta H} = \int \mathrm{d}\theta^*\mathrm{d}\theta\langle -\theta|\mathrm{e}^{-\beta H}|\theta\rangle\mathrm{e}^{-\theta^*\theta}. \tag{8.12}$$

Proof. We insert the completeness relation (8.11) into the definition of a partition function to get

$$\begin{aligned}
Z(\beta) &= \sum_{n=0,1} \langle n|\mathrm{e}^{-\beta H}|n\rangle \\
&= \sum_n \int \mathrm{d}\theta^* \mathrm{d}\theta \mathrm{e}^{-\theta^*\theta} \langle n|\theta\rangle\langle\theta|\mathrm{e}^{-\beta H}|n\rangle \\
&= \sum_n \int \mathrm{d}\theta^* \mathrm{d}\theta (1-\theta^*\theta)(\langle n|0\rangle + \langle n|1\rangle\theta)(\langle 0|\mathrm{e}^{-\beta H}|n\rangle + \theta^*\langle 1|\mathrm{e}^{-\beta H}|n\rangle) \\
&= \sum_n \int \mathrm{d}\theta^* \mathrm{d}\theta (1-\theta^*\theta)[\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle - \theta^*\theta\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle \\
&\qquad + \theta\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle + \theta^*\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle].
\end{aligned}$$

The last term of the last line does not contribute to the integral, since $\langle n|0\rangle \neq 0$ only when $n = 0$, in which case $\langle 1|\mathrm{e}^{-\beta H}|0\rangle = 0$. Hence we may change θ^* to $-\theta^*$ in the last term. Therefore,

$$\begin{aligned}
Z(\beta) &= \sum_n \int \mathrm{d}\theta^* \mathrm{d}\theta (1-\theta^*\theta)[\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle - \theta^*\theta\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle \\
&\qquad + \theta\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle - \theta^*\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle] \\
&= \int \mathrm{d}\theta^* \mathrm{d}\theta \mathrm{e}^{-\theta^*\theta} \langle -\theta|\mathrm{e}^{-\beta H}|\theta\rangle.
\end{aligned}$$

This completes the proof. □

Part III

Supersymmetric Proof of the Atiyah-Singer Index Theorem

Chapter 9

Mathematical Aspects of the Atiyah-Singer Index Theorem

In this chapter we put together the mathematical backgrounds required to understand the statement of the Atiyah-Singer index theorem for Dirac operators on a spin manifold. The mathematical aspect involves a beautiful combination of analysis, algebraic topology, and Riemannian geometry, which demonstrates the richness of this theorem. Our treatment follows [32] quite closely.

9.1 Clifford Algebras and Spin Groups

The goal of this section is to construct, for every $n \geq 3$, the unique double cover of the rotation group $\mathrm{Spin}_n \to \mathrm{SO}_n$. When $n = 3$, using the standard quaternion argument we have a double cover $\mathrm{SU}(2) \to \mathrm{SO}_3$. This argument does not work for higher dimensions. To generalize this construction, we need to introduce the notion of Clifford algebra.

Definition 9.1. Let V be an (real) inner product space. The ***Clifford algebra*** $\mathrm{Cl}(V)$ of V is the quotient of the tensor algebra

$$\mathrm{Cl}(V) = \otimes^\bullet V / \sim$$

where the relation is $vw + wv = -2\langle v, w\rangle$ for all $v, w \in V$. Of course $\otimes^0 V \cong \mathbb{R}$.

Equivalently, we may choose an orthonormal basis $e_1, ..., e_n$ of V. Then $\mathrm{Cl}(V)$ is the algebra generated by $e_1, ..., e_n$ with the relation

$$e_i^2 = -1 \qquad e_i e_j = -e_j e_i$$

for $i \neq j$. Therefore, we see that as a vector space, we have $\mathrm{Cl}(V) \cong \Lambda^\bullet V$, where the isomorphism is given by $e_{i_1} \cdots e_{i_p} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_p}$. This identification gives $\mathrm{Cl}(V)$ a natural $\mathbb{Z}_2$-grading

$$\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V)$$

which is given by the parity of the number of vectors. Moreover, using the isomorphism with the exterior algebra, the number of vectors gives a grading

$$\mathrm{Cl}(V) = \bigoplus_{k=0}^{n} \mathrm{Cl}^{(k)}(V).$$

Now we study the case where $V = \mathbb{R}^n$ with the standard inner product and the standard orthonormal basis. Then we use shorthand Cl_n to denote $\mathrm{Cl}(\mathbb{R}^n)$. We first study its center $\mathscr{Z}(\mathrm{Cl}_n)$.

Proposition 9.2. If n is even, then $\mathscr{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)}$. If n is odd, then $\mathscr{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)} \oplus \mathrm{Cl}_n^{(n)}$. In any case, $\mathscr{Z}(\mathrm{Cl}_n) \cap \mathrm{Cl}_n^0 = \mathrm{Cl}_n^{(0)}$.

Proof. Clearly $\mathscr{Z}(\mathrm{Cl}_n) = \{a \in \mathrm{Cl}_n : av = va, \forall v \in \mathbb{R}^n\}$. Now let $e_I = e_{i_1} \cdots e_{i_p}$. For $j \neq I$ we have $e_I e_j = (-1)^p e_j e_I$, and if $j \in I$ then $e_I e_j = (-1)^{p-1} e_j e_I$.

Clearly $\mathrm{Cl}_n^{(0)} \subset \mathscr{Z}(\mathrm{Cl}_n)$ for any n. For $1 \leq p \leq n-1$, since if $e_I \in \mathscr{Z}(\mathrm{Cl}_n)$ we must have $e_I e_j = e_j e_I$ for all j, we must have that p is both even and odd, which is impossible. Now if n is even, and $e_I \in \mathrm{Cl}_n^{(n)}$, above calculation shows that $e_I e_j = -e_j e_I$ for all j, hence e_I cannot be in the center. If n is odd, then it is easy to see from above that $\mathrm{Cl}_n^{(n)} \subset \mathscr{Z}(\mathrm{Cl}_n)$. This concludes the proof. $\square$

Now we consider the group $\mathrm{Cl}_n^* \subset \mathrm{Cl}_n$, which is the group of multiplicative invertible elements in Cl_n. If $v \in \mathbb{R}^n$ is a nonzero vector then we have $v^{-1} = -v/\|v\|^2$. Therefore $\mathbb{R}^n \setminus \{0\} \subset \mathrm{Cl}_n^*$. In particular $S^{n-1} \subset \mathrm{Cl}_n^*$.

Definition 9.3. The ***pin group*** Pin_n is the subgroup of Cl_n^* generated by the unit sphere S^{n-1}. The ***spin group*** is the even part of the pin group: $\mathrm{Spin}_n := \mathrm{Pin}_n \cap \mathrm{Cl}_n^0$.

We now define an automorphism of Cl_n called ***transpose***, namely $(e_{i_1} \cdots e_{i_p})^t = e_{i_p} \cdots e_{i_1}$.

Proposition 9.4. We have that

$$\mathrm{Spin}_n = \{a \in \mathrm{Pin}_n : aa^t = 1\} = \{v_1 \cdots v_p : v_i \in S^{n-1},\ p \text{ even}\}.$$

In particular, $\mathrm{Spin}_n \subset \mathrm{Cl}_n^0$.

Proof. By definition, $a \in \mathrm{Pin}_n$ iff $a = v_1 \cdots v_p$ with $v_i \in S^{n-1}$. Thus $aa^t = (v_1 \cdots v_p)(v_p \cdots v_1) = (-1)^p$, and $aa^t = 1$ iff p is even. $\square$

Now we consider the group homomorphism

$$\rho : \mathrm{Pin}_n \to \mathrm{GL}_n(\mathbb{R}) \qquad \rho(a)v = ava^t.$$

In general, a is a product of vectors, and ava^t is an element in Cl_n. To show that it is an honest vector, we note that for a unit vector $u \in \mathbb{R}^n$, the Clifford relation gives

$$uvu^t = uvu = -v + 2\langle u, v\rangle u$$

which is a linear combination of vectors, hence is a vector. Now $a = u_1 \cdots u_p$ is a product of unit vectors by definition, hence ava^t is a successive conjugate of v by unit vectors, hence is a vector. Therefore $\rho : \mathrm{Pin}_n \to \mathrm{GL}_n(\mathbb{R})$ is indeed well-defined.

Proposition 9.5. $\rho(\mathrm{Pin}_n) = \mathrm{O}_n$.

Proof. We first look at the image of S^{n-1} under ρ. If $a \in S^{n-1}$ and $v \in \mathbb{R}^n$. By choosing a suitable orthonormal basis we may decompose $v = \lambda a + a'$ for some $\lambda \in \mathbb{R}$ and such that $\langle a, a'\rangle = 0$. Then

$$\rho(a)v = ava^t = a(\lambda a + a')a = \lambda a^3 + aa'a = -\lambda a + a'$$

since $aa' = -a'a - 2\langle a, a'\rangle = -a'a$. Therefore, $\rho(a)$ is the reflection with respect to the hyperplane perpendicular to a and the result follows from the classical fact that any element in O_n is a product of reflections. $\square$

Corollary 9.6. $\rho(\mathrm{Spin}_n) = \mathrm{SO}_n$.

Proof. This follows immediately from the previous results, since any element of SO_n is a product of an even number of reflections. □

Theorem 9.7. *If $n \geq 3$, then the map $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is the universal double covering map. In particular, Spin_n is compact.*

Proof. By looking at the long exact sequence of homotopy group induced by the fibration

$$\mathrm{SO}_n \to \mathrm{SO}_{n+1} \to S^n$$

one can compute that $\pi_1(\mathrm{SO}_n) \cong \pi_1(\mathrm{SO}_{n+1})$ for $n \geq 3$. Moreover, from the $n = 3$ long exact sequence, one computes $\pi_1(\mathrm{SO}_3) = \mathbb{Z}_2$. Hence we see that $\pi_1(\mathrm{SO}_n) = \mathbb{Z}_2$ for $n \geq 3$. Therefore, SO_n has a unique double cover. We claim that $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is that cover.

We have $a \in \ker \rho$ iff $va = av$ for any $v \in \mathbb{R}^n$ since $a^t = a^{-1}$ by Proposition 9.4. But this means by Proposition 9.2 that

$$a \in \mathrm{Spin}_n \cap \mathscr{Z}(\mathrm{Cl}_n) \subset \mathrm{Cl}_n^0 \cap \mathscr{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)} = \mathbb{R}$$

and hence $\ker \rho = \{\pm 1\}$. This already shows that $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is a double cover (local invertibility is obvious because we can conjugate by a^t). Since $\pi_1(\mathrm{SO}_n) = \mathbb{Z}_2$ for $n \geq 3$, and since the universal cover of SO_n must have degree $|\pi_1(\mathrm{SO}_n)| = 2$, we conclude that $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is the universal cover.

Finally, since Spin_n is a finite cover of a compact space SO_n, it is compact. □

9.2 Spinor Representation

Let us return to the general theory of Clifford algebra. Now our goal is to study representations of the Clifford algebra. As always, complex representations are much nicer to work with. Thus, let us complexify:

Definition 9.8. The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ is obtained after tensoring with $\mathbb{C}$ the real Clifford algebra Cl_n:

$$\mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}.$$

We have the natural inclusion $\mathrm{Cl}_n \subset \mathbb{C}\mathrm{l}_n$, and the gradings ($\mathbb{Z}$ or $\mathbb{Z}_2$) have direct counterparts in the complex case. In particular, with obvious notation we have inclusion $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n^0 \subset \mathbb{C}\mathrm{l}_n$. Now we study complex representations of $\mathbb{C}\mathrm{l}_n$:

Definition 9.9. A complex ***representation*** of a complex algebra $\mathcal{A}$ is an algebra homomorphism $\zeta : \mathcal{A} \to \mathrm{Hom}(V)$ where V is a finite dimensional vector space. We say that V is a (left) $\mathcal{A}$-module.

A representation ζ is ***irreducible*** if there exists no nontrivial proper subspace $V_0 \subset V$ such that $\zeta(a)(V_0) \subset V_0$ for any $a \in \mathcal{A}$. Otherwise, ζ is said to be ***reducible***.

A representation ζ is called ***completely reducible*** if there exists a direct sum decomposition $V = \bigoplus_{k=1}^r V_k$ of V into irreducible submodules.

There is a priori no reason why a representation of an arbitrary algebra should be completely reducible. But of course the situation for Clifford algebra is different.

Proposition 9.10. Any representation of $\mathbb{C}\mathrm{l}_n$ is completely reducible.

Proof. We consider the ***Clifford group*** Cliff_n formed by all elements of the type $\pm e_{i_1}\cdots e_{i_p}$ for $0 \le p \le n$. This is a finite multiplicative group of order 2^{n+1}. And clearly there exists a bijective correspondence between algebra representations $\zeta : \mathbb{C}\mathrm{l}_n \to \mathrm{Hom}(V)$ with group representations $\xi : \mathrm{Cliff}_n \to \mathrm{Aut}(V)$ with $\xi(-1) = -\mathrm{id}_V$. This means we just have to check any representation of Cliff_n is completely reducible. Indeed, let ξ be such a representation and let $\langle\cdot,\cdot\rangle$ be any hermitian product on V. Define

$$(v, w) = \frac{1}{2^{n+1}} \sum_{g \in \mathrm{Cliff}_n} \langle \xi(g)v, \xi(g)w\rangle$$

for all $v, w \in V$. It follows that ξ acts by isometries with respect to $(\cdot,\cdot)$. In particular, if V_o is a submodule then $V_o^\perp = \{v \in V : (v,w) = 0\ \forall w \in V_o\}$ is also a submodule. The result follows from splitting off one irreducible module at a time (such irreducible module exists by picking a representation with the least dimension). □

Remark 9.11. The averaging trick above shows that any representation of a finite group G is completely reducible. In the case where G is a compact Lie group, we integrate with respect to the ***Haar measure*** μ_G to get a continuous averaging expression

$$(v, w) = \int_G \langle gv, gw\rangle \mathrm{d}\mu_G(g)$$

for $v, w \in V$. This shows that if G is a compact Lie group, then any representation of G is completely reducible. In particular, every representation of Spin_n is completely reducible.

Now that we have established every representation of $\mathbb{C}\mathrm{l}_n$ is completely reducible, let's classify all irreducible representations of $\mathbb{C}\mathrm{l}_n$. The crucial ingredient to do this is the ***chirality operator***

$$\Gamma_n = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n.$$

Simple computation shows

Lemma 9.12. *The chirality operator Γ_n satisfies $\Gamma_n^2 = 1$, and $\Gamma_n v = (-1)^{n+1} v \Gamma_n$ for any $v \in \mathbb{R}^n$. In particular, $\Gamma_n \in \mathscr{Z}(\mathbb{C}\mathrm{l}_n)$ if n is odd and $\Gamma_n \in \mathscr{Z}(\mathbb{C}\mathrm{l}_n^0)$ if n is even.*

The main theorem of this section is:

Theorem 9.13. *If $n = 2k$ there exists a unique irreducible module of dimension 2^k over $\mathbb{C}\mathrm{l}_n$. If $n = 2k+1$, there exists exactly two inequivalent irreducible modules of dimension 2^k over $\mathbb{C}\mathrm{l}_n$.*

Proof. Since $\mathbb{C}\mathrm{l}_0 = \mathrm{Cl}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R} \otimes \mathbb{C} = \mathbb{C}$, the base case $n = 0$ follows immediately. Now assume the theorem holds for $n = 0, 1, ..., 2k-2$, and let V be a $\mathbb{C}\mathrm{l}_{2k-1}$-module. By Lemma 9.12, $\Gamma_{2k-1} = i^{k-1} e_1 ... e_{2k-1} \in \mathscr{Z}(\mathbb{C}\mathrm{l}_{2k-1})$ and $\Gamma_{2k-1}^2 = 1$. Thus we have a decomposition

$$V = V^+ \oplus V^-$$

into $\mathbb{C}\mathrm{l}_{2k-1}$-modules with $V^\pm = \{x \in V : \Gamma_{2k-1} x = \pm x\}$. Since $\mathbb{C}\mathrm{l}_{2k-2} \subset \mathbb{C}\mathrm{l}_{2k-1}$, both $V^\pm$ are $\mathbb{C}\mathrm{l}_{2k-2}$-modules and the inductive hypothesis gives the decomposition

$$V^\pm = \bigoplus_j W_j^\pm$$

into irreducible $\mathbb{C}\mathrm{l}_{2k-2}$-modules of dimension 2^{k-1}. But $x \in W_j^\pm$ implies

$$e_{2k-1} x = \pm e_{2k-1} \Gamma_{2k-1} x = \mp i^{k-1} e_1 ... e_{2k-2} x \in W_j^\pm.$$

Hence each $W_j^{\pm}$ is also a $\mathbb{C}_{2k-1}$-module. Moreover, as $\mathbb{C}\mathrm{l}_{2k-1}$-modules $W_j^+ \neq W_j^-$ because left multiplication by e_{2k-1} gives distinct results, and W_j^+'s (respectively W_j^-'s) are mutually equivalent since multiplication by e_{2k-1} gives the same result. This proves the theorem in this case.

Now assume the theorem holds for $n = 0, 1, ..., 2k-1$ and let V be a $\mathbb{C}\mathrm{l}_{2k}$-module. As before, $\mathbb{C}\mathrm{l}_{2k-1} \subset \mathbb{C}\mathrm{l}_{2k}$, V is a $\mathbb{C}\mathrm{l}_{2k-1}$-module and the induction hypothesis splits V into a sum of $\mathbb{C}\mathrm{l}_{2k-1}$-modules $V = V^+ \oplus V^-$, where $V^{\pm}$ is a direct sum of irreducible $\mathbb{C}\mathrm{l}_{2k-1}$-modules $W_j^{\pm}$ as above. Now since $\Gamma_{2k-1}e_{2k} = -e_{2k}\Gamma_{2k-1}$, multiplication μ by e_{2k} satisfies $\mu(V^{\pm}) \subset V^{\mp}$. Writing $W_j = W_j^+ \oplus \mu(W_j^+)$ we get a decomposition

$$V = \bigoplus_j W_j$$

into mutually equivalent $\mathbb{C}\mathrm{l}_{2k}$-modules of dimension 2^k. □

Next, we are ready of introduce the class of representations of Spin_n we have been searching for:

Definition 9.14. The ***spinor representation*** $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ is obtained by restricting any irreducible representation of $\mathbb{C}\mathrm{l}_n$ under the inclusion $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n$.

Since the only case we care is the even dimensional case, we have the following:

Corollary 9.15. *If $n = 2k$ is even, then the spin representation $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ splits as a sum*

$$\Psi = \Psi^+ \oplus \Psi^-$$

of two irreducible modules of dimension 2^{k-1}, where $\Psi^{\pm} = \{x \in \Psi : \Gamma_{2k}x = \pm x\}$ and we have the exchange relation

$$e_i(\Psi^{\pm}) \subset \Psi^{\mp} \qquad i = 1, ..., n.$$

Proof. The decomposition follows from Theorem 9.13 and the exchange relation follows from Lemma 9.12. □

Remark 9.16. Let V be a real vector space of dimension n with an inner product and an almost complex structure, i.e., a endomorphism $J : V \to V$ with $J^2 = -\mathrm{id}_V$. Then we can give an explicit description of the Ψ above. We have a decomposition

$$V \otimes \mathbb{C} = V^+ \oplus V^-$$

where $V^{\pm}$ is the $(\pm i)$-eigenspace of J. Then we can define

$$\Psi := \Lambda^{\bullet}(V^-)^*.$$

(Compare this with supermanifold case: The structure sheaf of a supermanifold is locally given by $\mathcal{O}_M \otimes \Lambda^{\bullet}V$ for a vector bundle $V \to M$, and odd sections of this sheaf is the fermionic functions, which are spinors) The action of $\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n(V)$ on Ψ is as follows: Any vector $a \in V$ has a unique decomposition $a = a^+ + a^-$ where $a^{\pm} \in V^{\pm}$. Now for any $\psi \in \Psi$, we define

$$a \cdot \psi = \sqrt{2}\Big((a^-)^{\flat} \wedge \psi - i_{a^+}\psi\Big)$$

where $\flat : V^- \to (V^-)^*$ is given by the inner product $(v^-)^{\flat} = \langle -, v^- \rangle$ and i_{a^+} is contraction by the vector a^+. Through tedious computation one can check that $(a^{\pm})^2 = 0$ and

$a^+a^- + a^-a^+ = -2\langle a^+, a^- \rangle$. Thus this indeed defines a representation of $\mathbb{C}\mathrm{l}_n$ on Ψ. One can think of wedging by a^+ as the creation operator (since it increases degree) and contracting by a^- as the annihilation operator (since it decreases degree). Now since $\dim(\Lambda^\bullet V^{-*}) = 2^k$ we conclude that it is the Ψ given in Theorem 9.13. Moreover, note that

$$\dim(\mathbb{C}\mathrm{l}_n) = 2^n = 2^{2k} = (2^k)^2 = \dim \mathrm{End}(\Lambda^\bullet V^{-*}) = \dim(\Psi).$$

Moreover, the map $\mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Lambda^\bullet V^{-*})$ given by this representation is surjective: One can reach any states $v_1 \wedge \cdots \wedge v_p$ from 1 by successive creations, and one can go back to 1 from any state by successive annihilation, it follows that this map is in fact an isomorphism. Therefore, we conclude that

$$\mathbb{C}\mathrm{l}_n = \mathrm{End}(\Psi). \tag{9.1}$$

In fact this last equation holds for the most general case, even when V is not an almost complex vector space.

9.3 Spin Bundles and Spin Manifolds

Throughout this section, let $n = 2k$ be an even number.

In the previous section, we constructed irreducible representation Ψ of $\mathbb{C}\mathrm{l}_n$, and we showed that $\mathbb{C}\mathrm{l}_n = \mathrm{End}(\Psi)$ when the vector space has an almost complex structure, but we remarked that this holds in general.

Now we want to make the pointwise vector space picture into family of vector spaces over a Riemannian manifold M of dimension n. Namely, we want to consider a bundle $\mathbb{C}\mathrm{l}(M)$ such that at every point $x \in M$ its fiber is $\mathbb{C}\mathrm{l}(T_xM)$. That is,

$$\mathbb{C}\mathrm{l}(M) = \coprod_{x \in M} \mathbb{C}\mathrm{l}(T_xM)$$

as a set. To make sure this construction works globally, we need the following construction:

Definition 9.17. Let $\pi : P \to M$ be a principal G-bundle and $\mu : G \to \mathrm{Aut}(V)$ a representation of G. Note that G acts on the product $P \times V$ on the right via

$$(p, v) \cdot g := (pg, \mu(g^{-1})v)$$

for $(p, v) \in P \times V$. The ***associated bundle*** $E := P \times_\mu V$ is the quotient $P \times V/\sim$ where $(p, v) \sim (p, v) \cdot g$ for all $g \in G$. Then it can be checked that with the obvious projection E is a bundle over M with fiber V.

We can consider the principal SO_n-bundle P_M^{SO} (the frame bundle), whose fiber over any $x \in M$ is the set of orthonormal frames of T_xM. Then we can define the ***Clifford algebra bundle*** to be the associated bundle

$$\mathbb{C}\mathrm{l}(M) := P_M^{\mathrm{SO}} \times_\mu \mathbb{C}\mathrm{l}_n$$

where $\mu : \mathrm{SO}_n \to \mathrm{Aut}(\mathbb{C}\mathrm{l}_n)$ is defined by

$$\mu(g)(v_1 \ldots v_p) = (gv_1) \ldots (gv_p).$$

Proposition 9.18. At any point $x \in M$, the fiber $\mathbb{C}\mathrm{l}(M)_x$ is canonically isomorphic to $\mathbb{C}\mathrm{l}(T_xM)$.

Proof. A point $p \in P_M^{\mathrm{SO}}$ is an orthonormal basis $(e_1, ..., e_n)$ of T_xM. This induces an isometry $i_p : \mathbb{R}^n \to T_xM$ that sends the standard basis of $\mathbb{R}^n$ to the basis $(e_1, ..., e_n)$ of T_xM. Therefore, this induces an isomorphism of Clifford algebras $i_p : \mathbb{C}\mathrm{l}_n \to \mathbb{C}\mathrm{l}(T_xM)$. Therefore we can define an map $\Phi_x : \mathbb{C}\mathrm{l}(M)_x \to \mathbb{C}\mathrm{l}(T_xM)$ by $\Phi_x([p, v]) = i_p(v)$. Note that there is an obvious inverse induced by $i_p^{-1} : T_xM \to \mathbb{R}^n$. To check that this is well-defined, let $[p', v'] = [p, v]$ in $\mathbb{C}\mathrm{l}(M)_x$. This means that $p' = pg$ and $v' = \mu(g^{-1})v$. Thus

$$i_{p'}(v') = i_{pg}(\mu(g^{-1})v) = i_p(\mu(g)\mu(g^{-1})v) = i_p(v).$$

Since Φ_x is an isomorphism, the proof is complete. □

We can ask whether there exists a bundle S over M such that for each $x \in M$, the fiber S_x is an irreducible module over $\mathbb{C}\mathrm{l}(M)_x$ and $\mathbb{C}\mathrm{l}(M) = \mathrm{End}(S)$. However, there exists topological obstructions to the existence of S, which we shall discuss now. We first need a description of space that parameterize the set of principal G-bundles over M:

Theorem 9.19. *The sheaf cohomology group $H^1(M, G)$ parameterizes the set of principal G-bundles over M.*

Proof. Let $M = \bigcup_\alpha U_\alpha$ be a good cover. Let P be a principal G-bundle. By shrinking the good cover we might as well assume that P trivializes over each U_α with transition functions $\xi_{\alpha\beta}$. The transition functions satisfy the cocycle condition

$$\xi_{\alpha\beta}\xi_{\beta\gamma} = \xi_{\alpha\gamma}.$$

Moreover, P' is another principal G-bundle that is isomorphic to P (by passing to common trivializing cover we might as well assume P' also trivializes over U_α) iff there exists a family of maps $\xi_\alpha : U_\alpha \to G$ such that

$$\xi'_{\alpha\beta} = \xi_\beta^{-1}\xi_{\alpha\beta}\xi_\alpha,$$

so they must differ by a coboundary. □

Now let $\mathcal{E}$ be an oriented vector bundle of rank r over the Riemannian manifold M. Thus we can consider the principal SO_r-bundle $P_{\mathcal{E}}^{\mathrm{SO}}$ which is its frame bundle. By Theorem 9.7, we have a short exact sequence of groups

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_r \xrightarrow{\rho} \mathrm{SO}_r \to 0.$$

This induces a long exact sequence in sheaf cohomology

$$\cdots \to H^0(M, \mathrm{SO}_r) \xrightarrow{\partial_*} H^1(M, \mathbb{Z}_2) \to H^1(M, \mathrm{Spin}_r) \xrightarrow{\rho_*} H^1(M, \mathrm{SO}_r) \xrightarrow{\partial_*} H^2(M, \mathbb{Z}_2) \to \cdots \tag{9.2}$$

The key observation is that the second Wtiefel-Whitney class $w_2(\mathcal{E})$ of $\mathcal{E}$ is involved here:

Proposition 9.20. $w_2(\mathcal{E}) = \partial_*(P_{\mathcal{E}}^{\mathrm{SO}})$.

Proof. This can be proven by looking at the axiomatic characterization of Stiefel-Whitney classes. □

From the exactness of (9.2), we see that $w_2(\mathcal{E}) = \partial_*(P_{\mathcal{E}}^{\mathrm{SO}}) = 0$ iff there exists a principal Spin_r-bundle $P_{\mathcal{E}}^{\mathrm{Spin}}$ over M such that

$$\rho_*(P_{\mathcal{E}}^{\mathrm{Spin}}) = P_{\mathcal{E}}^{\mathrm{SO}}.$$

This means that $P_{\mathcal{E}}^{\mathrm{SO}}$ is obtained from $P_{\mathcal{E}}^{\mathrm{Spin}}$ as the quotient by the $\mathbb{Z}_2$-action on the fibers. Therefore, we obtain

Proposition 9.21. Let $\mathcal{E}$ be a real oriented real vector bundle of rank r over M. The following statements are equivalent: (1) $w_2(\mathcal{E}) = 0$; (2) There exists a Spin_r-bundle $P_{\mathcal{E}}^{\mathrm{Spin}}$ over M and a nontrivial two-sheeted covering map $\gamma : P_{\mathcal{E}}^{\mathrm{Spin}} \to P_{\mathcal{E}}^{\mathrm{SO}}$ such that the diagram

$$\begin{tikzcd} P_{\mathcal{E}}^{\mathrm{Spin}} \arrow[rr, "\gamma"] \arrow[dr] & & P_{\mathcal{E}}^{\mathrm{SO}} \arrow[dl] \\ & M & \end{tikzcd}$$

commutes and $\gamma(pg) = \gamma(p)\rho(g)$, for $p \in P_{\mathcal{E}}^{\mathrm{Spin}}$ and $g \in \mathrm{Spin}_r$.

If either of the conditions above happens we say that $\mathcal{E}$ is a ***spin bundle***. An oriented Riemannian manifold of dimension $n \geq 3$ is said to be ***spin*** if its tangent bundle TM is spin in the above sense.

9.4 The Dirac Operator on Spin Manifolds

Now we turn our attention to geometry of spin manifolds. Let us now assume that M is a spin manifold of dimension $n = 2k$ with a Riemannian metric. We fix a spin structure, i.e, we fix a lift $\gamma : P_M^{\mathrm{Spin}} \to P_M^{\mathrm{SO}}$ of the frame bundle. If $A \in \mathcal{A}^1(P_M^{\mathrm{SO}}, \mathfrak{so}_n)$ is the connection 1-form on the frame bundle induced from the Levi-Civita connection ∇^{LC} on M, then $A^s := \gamma^* A \in \mathcal{A}^1(P_M^{\mathrm{Spin}}, \mathfrak{so}_n)$ is a connection 1-form on the spin bundle, and it defines a connection $\nabla^s := \mathrm{d} + A^s$ called the ***spin connection***. Recall that we have a spin representation $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$. Hence we can use the associated bundle construction to form the ***spinor bundle*** on M:

$$S(M) = P_M^{\mathrm{Spin}} \times_\tau \Psi.$$

This is a complex vector bundle of rank 2^k over M. Elements in $\Gamma(S(M))$ are referred to as ***spinors***. For each $x \in M$, the fiber $S(M)_x$ is an irreducible module over $\mathbb{C}\mathrm{l}(M)_x$ and in fact we have the global generalization of (9.1):

$$\mathbb{C}\mathrm{l}(M) = \mathrm{End}(S(M)). \tag{9.3}$$

Moreover, one can check that Ψ is unitary. In the case where Remark 9.16 applies, one can this direct. Therefore the Clifford action $c : \mathbb{C}\mathrm{l}_n \to \Psi$ given in Remark 9.16 satisfies $c(a)^\dagger = -c(a)$. Since $c^2(a) = c(a^2) = -1$, a skew-Hermitian operator that squares to minus one must be unitary.

Since Ψ is unitary, $S(M)$ can be equipped with a natural metric $\langle \cdot, \cdot \rangle$ that is compatible with the connection ∇^s. We point out two properties of these structures:

1. Clifford multiplication by unit tangent vectors is an isometry: For any $\varphi, \varphi' \in \Gamma(S(M))$ and any $u \in \Gamma(TM)$ with $\|u\| = 1$, we have

$$\langle u\varphi, u\varphi' \rangle = \langle \varphi, \varphi' \rangle. \tag{9.4}$$

2. Let ∇^c be the natural connections on $\mathbb{C}\mathrm{l}(M)$, which is the connection $\mathrm{End}(S(M))$ induced from the connection on $S(M)$, is compatible with the connection ∇^s on $S(M)$: For any $a \in \Gamma(\mathbb{C}\mathrm{l}(M))$ and $\varphi \in \Gamma(S(M))$ we have

$$\nabla^s(a\varphi) = \nabla^c a \cdot \varphi + a\nabla^s \varphi. \tag{9.5}$$

Now we can finally define the Dirac operator on a spin manifold.

Definition 9.22. In terms of a local orthonormal frame $e_1, ..., e_n$ of TM, the ***Dirac operator*** $\not{\partial} : \Gamma(S(M)) \to \Gamma(S(M))$ is defined by

$$\not{\partial} := \sum_{i=1}^{n} e_i \cdot \nabla^s_{e_i}.$$

We see that $\not{\partial}$ is a first order linear differential operator. Moreover, indeed we can view $\not{\partial}$ as an endomorphism on $\Gamma(S(M))$ because via the musical isomorphism $\sharp : T^*M \to TM$ given by the metric on M and the Clifford $\Gamma(TM \otimes S(M)) \to \Gamma(S(M))$, the sequence of arrows

$$\Gamma(S(M)) \xrightarrow{\nabla^s} \Gamma(T^*M \otimes S(M)) \xrightarrow{\sharp} \Gamma(TM \otimes S(M)) \xrightarrow{\cdot} \Gamma(S(M))$$

starts and ends at the correct spaces. With respect to the inner product on $\Gamma(S(M))$ defined by

$$(\varphi, \varphi') = \int_M \langle \varphi, \varphi' \rangle \mathrm{d}\Omega$$

where Ω is the Riemannian volume form on M, one can show that $\not{\partial}$ is self-adjoint:

Theorem 9.23. *With respect to the inner product on $\Gamma(S(M))$ defined above, $\not{\partial}$ is self-adjoint: For any $\varphi, \varphi' \in \Gamma(S(M))$ one has*

$$(\not{\partial}\varphi, \varphi') = (\varphi, \not{\partial}\varphi').$$

Proof. If u is a unit vector, since $u^2 = -1$, (9.4) implies that $\langle u\varphi, \varphi' \rangle = -\langle \varphi, u\varphi' \rangle$. Since the connection 1-form $A \in \mathcal{A}^1(P_M^{\mathrm{SO}}, \mathfrak{so}_n)$ is induced from the Levi-Civita connection on M, the connection ∇^c on $\mathbb{C}\mathrm{l}(M)$ coincides with the Levi-Civita connection on vectors. That is, at any point $x \in M$ we can choose local orthonormal frames $e_1, ..., e_n$ of TM such that $\nabla^c_{e_i} e_i|_x = 0$. Then at this point, with respect to this frame, using (9.4) and (9.5) we have

$$\begin{aligned}
\langle \not{\partial}\varphi, \varphi' \rangle - \langle \varphi, \not{\partial}\varphi' \rangle &= \sum_i \langle e_i \nabla^s_{e_i} \varphi, \varphi' \rangle - \langle \varphi, e_i \nabla^s_{e_i} \varphi' \rangle \\
&= \sum_i \langle e_i \nabla^s_{e_i} \varphi, \varphi' \rangle + \langle e_i \varphi, \nabla^s_{e_i} \varphi' \rangle \\
&= \sum_i \langle \nabla^s_{e_i}(e_i \varphi), \varphi' \rangle + \langle e_i \varphi, \nabla^s_{e_i} \varphi' \rangle \\
&= \sum_i e_i \langle e_i \varphi, \varphi' \rangle,
\end{aligned}$$

where the last equality is because ∇^s is compatible with the fiber metric on $S(M)$. Now we can define a vector field $X \in \Gamma(TM)$ by $\langle X, Y \rangle = \langle Y\varphi, \varphi' \rangle$, where the left hand side is the Riemannian inner product and on the right hand side the inner product is the fiberwise inner product on $S(M)$, where $Y\varphi$ denotes the Clifford multiplication. Therefore, we have that

$$\langle \not{\partial}\varphi, \varphi' \rangle - \langle \varphi, \not{\partial}\varphi' \rangle = \sum_i e_i \langle e_i \varphi, \varphi' \rangle = \sum_i e_i \langle X, e_i \rangle = \sum_i \langle \nabla_{e_i} X, e_i \rangle = \operatorname{div} X.$$

The rest is an easy application of Stokes' theorem. Since $\mathcal{L}_X \Omega = (\operatorname{div} X)\Omega$, and Cartan's magic formula gives $\mathcal{L}_X \Omega = \mathrm{d}(i_X \Omega)$ since $\mathrm{d}\Omega = 0$, one has

$$\int_M (\operatorname{div} X)\Omega = \int_M \mathrm{d}(i_X \Omega) = \int_{\partial M} i_X \Omega = 0$$

since $\partial M = \emptyset$. Therefore, we see that $(\not{\partial}\varphi, \varphi') - (\varphi, \not{\partial}\varphi') = 0$, as desired. $\square$

By Corollary 9.15, we have a decomposition

$$S(M) = S^+(M) \oplus S^-(M)$$

where $S^\pm(M) := P_M^{\text{Spin}} \times_{\tau^\pm} \Psi^\pm$. This decomposition is preserved by ∇^s and is orthogonal with respect to the fiber metric of $S(M)$. Now from the parity exchange relation in Corollary 9.15, we have $\not{\partial}(\Gamma(S^\pm(M))) \subset \Gamma(S^\mp(M))$. Therefore, in matrix form, there exists some differential operator $D : \Gamma(S^+(M)) \to \Gamma(S^-(M))$ such that

$$\not{\partial} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$$

since $\not{\partial}$ is self adjoint by Theorem 9.23. Using some technical analysis argument, one can show that $\not{\partial}$ is Fredhom, meaning both $\dim\ker D$ and $\dim\ker D^\dagger$ are finite. Hence we may define the ***index*** of the Dirac operator as

$$\operatorname{ind}(\not{\partial}) = \dim\ker D - \dim\ker D^\dagger.$$

The Atiyah-Singer index theorem is a striking result, which says the purely analytical quantity $\operatorname{ind}(\not{\partial})$ is already determined by the topology of M. Specifically, the curvature 2-form

$$F_{\mu\nu} = R_{\mu\nu\alpha\beta}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta.$$

of TM. The splitting principle for real vector bundle (See [32], Theorem 7.2.2) shows that if we want to define characteristic classes using the curvature 2-form F, we may assume that it splits into block diagonal form

$$F = \begin{pmatrix} 0 & r_1 & \cdots & 0 & 0 \\ -r_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & r_n \\ 0 & 0 & \cdots & -r_n & 0 \end{pmatrix}$$

Then the ***$\hat{A}$-class*** of M is defined by the Taylor expansion of

$$\hat{A}(M) := \prod_k \frac{r_k/2}{\sinh(r_k/2)}.$$

Extracting the top-dimensional piece of $\hat{A}(M)$ and integrate, the Atiyah-Singer index theorem in this case is

Theorem 9.24 (=Theorem 10.6)**.** *On a spin Riemannian manifold M of even dimension, we have*

$$\operatorname{ind}(\not{\partial}) = \int_M \hat{A}(M).$$

In the next chapter we will present a proof of this result on a heuristic level of rigor using path integral techniques.

9.5 General Atiyah-Singer Index Formula and Applications

In fact, there is a more general version of the index theorem without the need of a spin structure. The characteristics of this theory is very similar to the spin case discussed previously.

Throughout this section, let M be an oriented closed Riemannian manifold of dimension $n = 2k$.

Definition 9.25. A vector bundle $E \to M$ is called a ***Clifford bundle*** if the following holds:

1. E is a (complex) vector bundle whose typical fiber is a Clifford module. This means we have a Clifford action $c : \mathbb{C}\mathrm{l}(M) \to \mathrm{End}(E)$.
2. E has a fiber metric such that Clifford multiplication by unit tangent vectors is a unitary isomorphism. For any $\varphi, \varphi' \in \Gamma(E)$ and any $u \in \Gamma(TM)$ with $\|u\| = 1$, we have
$$\langle c(u)\varphi, c(u)\varphi' \rangle = \langle \varphi, \varphi' \rangle. \tag{9.6}$$
3. There is a connection ∇^E on E that is compatible with the fiber metric and with the Clifford connection ∇^c on $\mathbb{C}\mathrm{l}(M)$ (induced by the Levi-Civita connection). For any $a \in \Gamma(\mathbb{C}\mathrm{l}(M))$ and $\varphi \in \Gamma(E)$, we have the Leibniz rule:
$$\nabla^E(a \cdot \varphi) = (\nabla^c a) \cdot \varphi + a \cdot \nabla^E \varphi. \tag{9.7}$$

For a Clifford bundle E, we define the Dirac operator $\not{\partial} : \Gamma(E) \to \Gamma(E)$ in a local orthonormal frame $\{e_i\}$ of TM:
$$\not{\partial} = \sum_{i=1}^{n} e_i \cdot \nabla^E_{e_i}.$$

Definition 9.26. We say a Clifford bundle E is $\mathbb{Z}_2$-graded if there is a decomposition $E = E^+ \oplus E^-$ preserved by the connection, such that Clifford multiplication by a tangent vector v maps $E^\pm \to E^\mp$ (odd parity). The Dirac operator then takes the form:
$$\not{\partial} = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

The ***index*** of the Dirac operator is then defined as:
$$\mathrm{ind}(\not{\partial}) = \dim\ker D^+ - \dim\ker D^-.$$

Proposition 9.27. Let $S = S(M)$ be the spinor bundle as before. If E is a Clifford bundle over M, then E is always isomorphic to a twisted spinor bundle, in the sense that there exists a vector bundle Q and an isomorphism
$$E \cong S \otimes Q.$$

Explicitly, $Q = \mathrm{Hom}_{\mathbb{C}\mathrm{l}(M)}(S, E)$.

Proof. We claim that the map
$$\Upsilon : S \otimes \mathrm{Hom}_{\mathbb{C}\mathrm{l}(M)}(S, E) \to E \qquad \Upsilon(s \otimes \phi) = \phi(s)$$

is an isomorphism. Since this is a bundle map, we only need to check that it is an isomorphism on each fiber. At each $x \in M$, the map is $\Upsilon_x : S_x \otimes \mathrm{Hom}_{\mathbb{C}\mathrm{l}(M)}(S_x, E_x) \to E_x$. Since S_x is irreducible, we must have $E_x \cong S_x^{\oplus p}$ for some p. Under this identification we have $\Upsilon_x : S_x \otimes \mathrm{Hom}(S_x, S_x^{\oplus p}) \to S_x^{\oplus p}$. Now since S_x is irreducible, Schur's lemma says that every map in $\mathrm{Hom}(S_x, S_x)$ must be a constant multiple of identity, namely $\mathrm{Hom}(S_x, S_x) \cong \mathbb{C}$. Now clearly $\mathrm{Hom}(S_x, S_x^{\oplus p}) = \mathrm{Hom}(S_x, S_x)^{\oplus p} \cong \mathbb{C}^p$. Under this identification the we have

$$\Upsilon : S_x \otimes \mathbb{C}^p \to S_x^{\oplus p} \qquad \Upsilon(s \otimes (\lambda_1, ..., \lambda_p)) = (\lambda_1 s, ..., \lambda_p s).$$

Clearly this is a surjective map, hence it is an isomorphism since $\dim(S_x \otimes \mathbb{C}^p) = \dim(S_x^{\oplus p})$. □

Theorem 9.28 (Atiyah-Singer). *Let M be a compact, oriented, even-dimensional Spin manifold and let $\not{\partial}$ be the Dirac operator associated to the twisted Clifford bundle $E = S \otimes Q$. The analytical index is equal to the topological index:*

$$\mathrm{ind}(\not{\partial}) = \int_M \hat{A}(TM)\,\mathrm{ch}(Q).$$

This theorem is considered one of the greatest achievements of twentieth-century mathematics. We illustrate its power by sketching the proofs of several classical consequences.

Corollary 9.29 (Poincaré-Hopf-Gauss-Bonnet). *The integral of the Euler class of TM is the Euler characteristic of M:*

$$\chi(M) = \int_M e(TM).$$

Proof Sketch. Let $E = \Lambda^\bullet T^*M \otimes \mathbb{C}$. We define the Clifford action $c(v)$ on a form α by:

$$c(v)\alpha = v^\flat \wedge \alpha - i_v \alpha.$$

Here, $\flat : TM \to T^*M$ is the isomorphism $v^\flat = g(v, -)$ given by the metric. Then one can verify that $c(v)c(w) + c(w)c(v) = -2g(v, w)$. Hence this indeed defines a Clifford action. Now the bundle E is naturally $\mathbb{Z}_2$-graded by even or odd degree forms. Moreover, the metric on M gives E a fiberwise inner product by $\langle \mathrm{d}x^i, \mathrm{d}x^j \rangle = g^{ij}$ and can be extended to all p-forms by taking determinants, and integrating this over M gives an inner product on E, so we can make sense of the adjoint d^* of the exterior differentiation operator. The connection ∇^E is the Levi-Civita connection extended to differential forms: Namely,

$$(\nabla^E_X \alpha)(X_1, ..., X_p) = X\alpha(X_1, ..., X_p) - \sum_{i=1}^{p} \alpha(X_1, ..., \nabla_X X_i, ..., X_p).$$

The corresponding Dirac operator is the de Rham operator

$$\not{\partial} = \mathrm{d} + \mathrm{d}^*.$$

Then one can compute that $\not{\partial}^2 = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = \Delta$ is the Hodge Laplacian. Moreover, $(\Delta\eta, \eta) = \|\not{\partial}\eta\|^2$. This implies $\ker \not{\partial} = \ker \Delta$. By the Hodge theory on Riemannian manifolds, the Δ-harmonic p-forms has dimension equal to the Betti numbers $b_p(M)$. Hence

$$\mathrm{ind}(\not{\partial}) = \dim(\ker \not{\partial} \cap E^+) - \dim(\ker \not{\partial} \cap E^-) = \dim(\ker \Delta \cap E^+) - \dim(\ker \Delta \cap E^-)$$

$$= \dim\left(\bigoplus_{p \text{ even}} \mathcal{H}^p\right) - \dim\left(\bigoplus_{p \text{ odd}} \mathcal{H}^p\right) = \sum_{p \text{ even}} b_p - \sum_{p \text{ odd}} b_p = \chi(M).$$

Topologically, on a spin manifold, we have the bundle isomorphism $E \cong S \otimes S$. Indeed, in this case

$$E = \Lambda^\bullet T^*M \otimes \mathbb{C} \cong \mathbb{C}\mathrm{l}(M) \cong \mathrm{End}(S) \cong S^* \otimes S \cong S \otimes S$$

by (9.3), and the last isomorphism is given by the fiberwise inner product on S. Thus, the twisting bundle is $Q = S$. A fundamental identity in characteristic class theory states that $\hat{A}(TM)\,\mathrm{ch}(S) = e(TM)$. This concludes the proof. □

Another beautiful consequence of the index formula is the Hirzebruch signature theorem. Since M is a closed manifold of dimension $n = 2k$, the middle de Rham cohomology group $H^k(M)$ admits a bilinear form $B : H^k(M) \times H^k(M) \to \mathbb{C}$ given by

$$B(\alpha, \beta) = \int_M \alpha \wedge \beta.$$

Moreover $B(\alpha, \beta) = (-1)^{k^2} B(\beta, \alpha)$. So if we assume that $k = 2l$ so that $\dim M = 2k = 4l$, then B is symmetric. Poincaré duality implies that B is nondegenerate and hence the corresponding quadratic form has no null eigenvalue. We can then define the ***signature*** of M by

$$\mathrm{sign}(M) = b^+ - b^-$$

where $b^\pm$ is the number of positive (negative) eigenvalues of the corresponding quadratic forms counted with multiplicities. A consequence of the Hirzebruch signature theorem is that this quantity (which is a topological invariant) can be computed by integrating the ***Hirzebruch $\mathcal{L}$-class***, which is defined by the formal power series

$$\mathcal{L}(TM) = \prod_{i=1}^{k} \frac{r_i/2}{\tanh(r_i/2)}$$

where r_i's are the block entries of the curvature 2-forms F as in the previous section.

Corollary 9.30 (Hirzebruch Signature Theorem). *For a closed oriented $4l$-dimensional manifold, we have*

$$\mathrm{sign}(M) = \int_M \mathcal{L}(TM).$$

Proof Sketch. We still use the same Clifford bundle $E = \Lambda^\bullet T^*M \otimes \mathbb{C}$ and the same Clifford action $c : \mathbb{C}\mathrm{l}(M) \to \mathrm{End}(E)$. However, the grading of E is different in this case. We define an operation $\tau : E \to E$ given by

$$\tau = i^{p(p-1)+k} \star \qquad \text{on } \Lambda^p T^*M$$

where $\star$ is the Hodge star operator. Then one can check that $\tau^2 = 1$, and τ is self-adjoint, moreover $\tau c(v) = -c(v)\tau$. Hence $E = E^+ \oplus E^-$ decomposes as the (± 1)-eigenspace of τ. This defines the grading. The connection ∇^E is still the same as in the Gauss-Bonnet case, so is the Dirac operator

$$\not{\partial} = \mathrm{d} + \mathrm{d}^*.$$

But this time, because the change of the grading of E, we will get different results: $\mathrm{ind}(\not{\partial}) = \mathrm{sign}(M)$, and $\hat{A}(TM)\,\mathrm{ch}(S) = \mathcal{L}(TM)$. □

We conclude with the most celebrated theorem in algebraic geometry. Almost every algebraic geometer uses Hirzebruch-Riemann-Roch theorem in their daily life (The Hirzebruch-Riemann-Roch theorem implies the Riemann-Roch theorem on curves, so this is an undisputed statement!):

Corollary 9.31 (Hirzebruch-Riemann-Roch)**.** *Let X be a compact complex manifold and $\mathcal{E} \to X$ a holomorphic vector bundle on X. Let TX denote the holomorphic tangent bundle (which is isomorphic to the real tangent bundle as real vector bundles) on X and $\chi(X, \mathcal{E})$ denote the alternating sum of dimensions of sheaf cohomology groups of $\mathcal{E}$. Then we have*

$$\chi(X, \mathcal{E}) = \int_X \operatorname{td}(TX) \operatorname{ch}(\mathcal{E}).$$

Proof Sketch. This construction will just be the one given in Remark 9.16, with V replaced by TX.

Let our Clifford bundle be $E = \Lambda^{0,\bullet} T^* X \otimes \mathcal{E}$. Fix a hermitian metric on TX compatible with the complex structure, and a hermitian metric on $\mathcal{E}$. This will give a hermitian metric on E. Let ∇ be the Chern connection with respect to fiber metric on $\Lambda^{0,\bullet} T^* X$, and similarly let $\nabla^{\mathcal{E}}$ be the Chern connection on $\mathcal{E}$. Then we choose the connection ∇^E on E defined by

$$\nabla^E = \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^{\mathcal{E}}.$$

We have a natural $\mathbb{Z}_2$-grading on $\mathcal{E}$, where $e^{\pm}$ is the direct sum of $\Lambda^{0,p} T^* X \otimes \mathcal{E}$ where p is even (odd). Now for any $v \in TX \otimes \mathbb{C}$, the complex structure gives a decomposition

$$v = v^{1,0} + v^{0,1} \in T^{1,0} X \oplus T^{0,1} X.$$

Define the Clifford action $c : \mathrm{Cl}(M) \to \mathrm{End}(E)$ to be

$$c(v)\alpha = \sqrt{2}\Big((v^{0,1})^\flat \wedge \alpha - i_{v^{1,0}} \alpha \Big).$$

One can check that this is indeed a Clifford action. Then with these data, the Dirac operator is given by

$$\not{\partial} = \sqrt{2}(\bar{\partial}_{\mathcal{E}} + \bar{\partial}^*_{\mathcal{E}}).$$

Then one calculates that $\operatorname{ind}(\not{\partial}) = \chi(X, \mathcal{E})$.

For the right hand side, one needs to use a variant of the index theorem above called the Spin^c index theorem. The bundle $\Lambda^{0,\bullet} T^* X$ plays the role of the spinor bundle S associated to the canonical Spin^c structure of X. Thus, our bundle is $E = S \otimes \mathcal{E}$, and the twisting bundle is $Q = \mathcal{E}$. Using the Spin^c version of the index theorem, or simply the identity

$$\hat{A}(TX) \operatorname{ch}(S) = \hat{A}(TX) \mathrm{e}^{c_1(TX)/2} = \operatorname{td}(TX),$$

we obtain:

$$\operatorname{ind}(\not{\partial}) = \int_X \hat{A}(TX) \operatorname{ch}(S) \operatorname{ch}(\mathcal{E}) = \int_X \operatorname{td}(TX) \operatorname{ch}(\mathcal{E}).$$

This completes the proof. □

Chapter 10

Witten's Proof

We now present Witten's proof of the Atiyah-Singer index theorem on spin manifold using the technique of path integrals and supersymmetric quantum mechanics (SUSYQM). We will first develop the basic of SUSYQM on manifolds, then we set up the problem of computing the index of an elliptic operator on a spin manifold as a path integral computation. Finally, we will prove Atiyah-Singer index theorem by evaluating the said path integral.

It should be noted that the proof we present is at a formal level of rigor, since path integral is not well-defined mathematically. Moreover, we should mention that the proofs of the most general form of the Atiyah-Singer index theorem are much more complicated, involving deep ideas in algebraic K-theory or analysis.

Our treatment follows [19] closely, with more computational details carried out. For a more through introduction to SUSYQM, we refer the reader to [22], which also contains some materials on the proof of index theorem, though more details are omitted compared to here. Our approach is more computational, but for a more mathematical presentation of the subject, we recommend Witten's notes on the index of Dirac operators in [20], and Chapter 8 of [10]. Finally, we refer the interested readers to the original papers [23],[24].

10.1 Supersymmetric Quantum Mechanics

For simplicity, we first introduce SUSYQM in flat space, then we will upgrade our SUSYQM model to Riemannian manifolds. In the simplest case our space is $\mathbb{R}^d$, with coordinates x_k, and we have anticommuting Grassmannian ψ_k, for $k = 1, ..., d$. We propose to consider the Lagrangian

$$L = \frac{1}{2}\dot{x}_k\dot{x}_k + \frac{i}{2}\psi_k\dot{\psi}_k.$$

Note that we are summing over repeated indices, but we are not careful with upper and lower indices since in $\mathbb{R}^d$ the metric is so simple that there is no difference. Recall from classical mechanics that for a Lagrangian $L(\phi_k, \dot{\phi}_k)$ with generalized coordinates, the canonical momentum is defined by

$$\pi_k = \partial L/\partial\dot{\phi}_k,$$

and the Hamiltonian of the system is related to its Lagrangian via Legendre transformation by

$$H = \sum_k \pi_k\dot{\phi}_k - L(\phi_k, \dot{\phi}_k).$$

In our case the generalized coordinates are x_k, ψ_k, performing the procedure above, we find that the Hamiltonian is

$$H = \dot{x}_j p_j - \dot{\psi}_j \frac{i}{2}\psi_j - L = \frac{1}{2}p^2$$

where $p_j = \dot{x_j}$ is the canonical momentum of the even coordinates x_j. If we quantize this system, the momentum operator becomes proportional to spatial derivative, and we have that $H = -\Delta/2$, proportional to the Laplacian.

Variation of the Lagrangian L gives

$$\delta L = \dot{x}_j \frac{\mathrm{d}}{\mathrm{d}t}\delta x_j + \frac{i}{2}\delta\psi_j\dot{\psi}_j + \frac{i}{2}\psi_j \frac{\mathrm{d}}{\mathrm{d}t}\delta\psi_j.$$

Using the expression above, we investigate how L changes under supersymmetric transformation:

Theorem 10.1. *The action of the Lagrangian L defined above is invariant under the supersymmetric transformation*

$$\begin{aligned} \delta_\epsilon x_j &= i\epsilon\psi_j \\ \delta_\epsilon \psi_j &= -\epsilon\dot{x}_j. \end{aligned}$$

Here, ϵ is an infinitesimal ***Grassmann (odd)*** *real constant.*

Proof. A simple calculation yields

$$\begin{aligned} \delta_\epsilon L &= i\dot{x}_j\epsilon\dot{\psi}_j - \frac{i}{2}\epsilon\dot{x}_j\dot{\psi}_j - \frac{i}{2}\psi_j\epsilon\ddot{x}_j \\ &= i\dot{x}_j\epsilon\dot{\psi}_j - \frac{i}{2}\epsilon\dot{x}_j\dot{\psi}_j - \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_j\epsilon\dot{x}_j\right) + \frac{i}{2}\dot{\psi}_j\epsilon\dot{x}_j \\ &= -\frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_j\epsilon\dot{x}_j\right) \end{aligned}$$

which is a total derivative. Hence the action which is the integral of L is invariant under δ_ϵ. ☐

Let us define ***supercharge*** Q to be the associated conserved charge by Noether's theorem:

$$\epsilon Q \equiv i\epsilon p_j\psi_j = i\epsilon\psi_j p_j = i\epsilon\psi_j\dot{x}_j.$$

Let us now study how do these physical quantities change under δ_ϵ, and their relations.

Theorem 10.2. *Under the SUSY transformation defined in Theorem 10.1, we have:*

(1) $\delta_\epsilon L = \frac{\epsilon}{2}\frac{\mathrm{d}Q}{\mathrm{d}t}$*;*

(2) $\delta_\epsilon Q = -2i\epsilon L$*;*

(3) $[\delta_{\epsilon_2}, \delta_{\epsilon_1}] = \delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2} = -2i\epsilon_1\epsilon_2\frac{\partial}{\partial t}$*;*

(4) Let $H = -p^2/2$ *be the Hamiltonian defined before. Then we have* $Q^2 = -H$*, or equivalently* $H = (iQ)^2$*.*

Proof. (1) follows immediately from Theorem 10.1 and the definition of Q. The rest are just straightforward computation. For (2), we have

$$\begin{aligned}\delta_\epsilon Q &= i\left(\delta\psi_j\right)\dot{x}_j + i\psi_j\frac{\mathrm{d}}{\mathrm{d}t}\delta x_j = i\left(-\epsilon\dot{x}_j\right)\dot{x}_j + i\psi_j\left(i\epsilon\dot{\psi}_j\right)\\ &= -i\epsilon\dot{x}_j\dot{x}_j + \epsilon\psi_j\dot{\psi}_j = -2i\epsilon\left(\frac{1}{2}\dot{x}_j\dot{x}_j + \frac{i}{2}\psi_j\dot{\psi}_j\right)\\ &= -2i\epsilon L.\end{aligned}$$

For (3), note that

$$\begin{aligned}\delta_{\epsilon_2}\delta_{\epsilon_1}(x_j) &= \delta_{\epsilon_2}(x_j + i\epsilon_1\psi_j) = x_j + i(\epsilon_1+\epsilon_2)\psi_j - i\epsilon_1\epsilon_2\dot{x}_j\\ \delta_{\epsilon_2}\delta_{\epsilon_1}(\psi_j) &= \delta_{\epsilon_2}(\psi_j - \epsilon_1\dot{x}_j) = \psi_j - (\epsilon_1+\epsilon_2)\dot{x}_j - i\epsilon_1\epsilon_2\dot{\psi}_j\\ \delta_{\epsilon_1}\delta_{\epsilon_2}(x_j) &= x_j + i(\epsilon_1+\epsilon_2)\psi_j - i\epsilon_2\epsilon_1\dot{x}_j\\ \delta_{\epsilon_1}\delta_{\epsilon_2}(\psi_j) &= \psi_j - (\epsilon_1+\epsilon_2)\dot{x}_j - i\epsilon_2\epsilon_1\dot{\psi}_j\end{aligned}$$

and the claim follows from simple substraction. As for (4), we have

$$\begin{aligned}\{Q,Q\} = 2Q^2 &= 2\left(ip_j\psi_j\right)\left(ip_k\psi_k\right)\\ &= -p_jp_k\left(\psi_j\psi_k + \psi_k\psi_j\right) = -p_jp_k\delta_{jk}\\ &= -2H,\end{aligned}$$

concluding the proof. □

Let us now quantize the system. We take $d = 2n$ to be an even integer. The representation of ψ_j is achieved by defining

$$\psi_j = \frac{1}{\sqrt{2}}\gamma_j,$$

where γ_j are the gamma matrices and they satisfy the relation

$$\{\gamma_j, \gamma_k\} = 2\delta_{jk}.$$

Note that this is almost a Clifford algebra except $\gamma_k^2 = 2$ instead whereas technically for a Clifford algebra we need generators to square to -1. However, this algebra is isomorphic to the Clifford algebra generated by e_k because if $e_k^2 = -1$ we can assign a algebra homomorphism that sends γ_k to $-ie_k$. Physicists prefer the Clifford algebra represented by gamma matrices since these are more common in quantum field theory.

The supercharge operator now becomes

$$Q = i\psi_jp_j = \frac{1}{\sqrt{2}}\gamma_j\frac{\partial}{\partial x_j} = \frac{1}{\sqrt{2}}\not{\partial}$$

which is proportional to the Dirac operator (Compare with Definition 9.22).

Now let us upgrade our system to a Riemannian manifold. The calculation we have done before remains mostly the same, we just need to express things in a covariant way. We have a Riemannian manifold M with $\dim M = 2n$, and $g_{\mu\nu}$ its Riemannian metric tensor. Our coordinates now become x^μ, which commute, and ψ^μ, which anticommute.

The odd vector $\psi^\mu(t) \in T_{x(t)}M$ for each instant t (The rigorous characterization of the vector $\psi^\mu(t)$ is given by Proposition 10.3). Thus under a transformation $x^\mu \to x'^\mu(x^\nu)$ we have

$$\psi^\mu \to \psi'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\psi^\nu.$$

Thus

$$\delta_\epsilon x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\delta_\epsilon x^\nu = \frac{\partial x'^\mu}{\partial x^\nu} i\epsilon\psi^\nu = i\epsilon\psi'^\mu$$

and

$$\begin{aligned}
\delta\psi'^\mu &= \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\delta x^\lambda \psi^\nu + \frac{\partial x'^\mu}{\partial x^\nu}\delta\psi^\nu \\
&= \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda} i\epsilon\psi^\lambda\psi^\nu + \frac{\partial x'^\mu}{\partial x^\nu}\left(-\epsilon\dot{x}^\nu\right) = -\epsilon\dot{x}'^\mu.
\end{aligned}$$

Note that in the last equality, the first term is zero by anticommutativity of Grassmann numbers and symmetry of derivatives. Thus the SUSY transformation is covariant under change of coordinates.

The supercharge introduced before now generalizes to

$$Q = ig_{\mu\nu}(x)\dot{x}^\mu\psi^\nu.$$

Note that when we quantize this system, $Q \sim g_{\mu\nu}p^\mu\gamma^\nu = \not{\partial}$ is the Dirac operator on M.

Now the question is, how do we construct SUSY Lagrangian on M? Theorem 10.2 part (b) suggests that we compute $\delta_\epsilon Q$, and read off L from the expression. We have that

$$\begin{aligned}
\delta_\epsilon Q =& i\partial_\lambda g_{\mu\nu}\delta_\epsilon x^\lambda \dot{x}^\mu\psi^\nu + ig_{\mu\nu}\delta_\epsilon\dot{x}^\mu\psi^\nu + ig_{\mu\nu}\dot{x}^\mu\delta_\epsilon\psi^\nu \\
=& i\partial_\lambda g_{\mu\nu} i\epsilon\psi^\lambda\dot{x}^\mu\psi^\nu + ig_{\mu\nu}(i\epsilon\dot{\psi}^\mu)\psi^\nu + ig_{\mu\nu}\dot{x}^\mu\left(-\epsilon\dot{x}^\nu\right) \\
=& -2i\epsilon\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}\psi^\nu\dot{\psi}^\mu - \frac{i}{2}\dot{x}^\mu\frac{1}{2}\left(\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu}\right)\psi^\lambda\psi^\nu\right) \\
=& -2i\epsilon\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}\psi^\nu\dot{\psi}^\mu + \frac{i}{2}\dot{x}^\mu g_{\lambda\rho}\Gamma^\rho_{\mu\nu}\psi^\lambda\psi^\nu\right)
\end{aligned}$$

where

$$\Gamma^\nu_{\lambda\mu} = \frac{1}{2}g^{\nu\rho}\left(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu}\right)$$

is the Christoffel symbol associated with the Levi-Civita connection. In the third equality of the $\delta_\epsilon Q$ equation, we used

$$\dot{x}^\mu\frac{1}{2}(\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu})\psi^\lambda\psi^\nu = \dot{x}^\mu\partial_\lambda g_{\mu\nu}\psi^\lambda\psi^\nu.$$

To see this, first note that the last term is $\sim \partial_\mu g_{\lambda\nu}\psi^\lambda\psi^\nu = 0$ because $g_{\lambda\nu}$ is symmetric under the exchange of λ, ν while $\psi^\lambda\psi^\nu$ is antisymmetric under such an exchange. The second term is equal to the first term because

$$\dot{x}^\mu\partial_\lambda g_{\mu\nu}\psi^\lambda\psi^\nu = \dot{x}^\mu\partial_\nu g_{\mu\lambda}\psi^\nu\psi^\lambda = -\dot{x}^\mu\partial_\nu g_{\mu\lambda}\psi^\lambda\psi^\nu.$$

Thus we may read off the Lagrangian directly from the $\delta_\epsilon Q$ equation and conclude that our SUSYQM Lagrangian must be

$$\begin{aligned}
L &= \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}(x)\psi^\mu\left(\frac{\mathrm{d}\psi^\nu}{\mathrm{d}t} + \dot{x}^\lambda\Gamma^\nu{}_{\lambda\kappa}(x)\psi^\kappa\right) &(10.1)\\
&= \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{Dt} &(10.2)
\end{aligned}$$

where $D\psi^\nu/Dt$ is the covariant derivative of ψ^ν along the curve $x(t)$.

We now define some symbols that will be used in the following sections. The connection 1-form is

$$\Gamma^\mu_\nu = \mathrm{d}x^\lambda\Gamma^\mu_{\lambda\nu}$$

and the curvature 2-form is

$$\Omega^{\mu}_{\nu} = d\Gamma^{\mu}_{\nu} + \Gamma^{\mu}_{\sigma} \wedge \Gamma^{\sigma}_{\nu} = \frac{1}{2} R^{\mu}_{\nu\rho\sigma} dx^{\rho} \wedge dx^{\sigma},$$

where the curvature tensor component is given by

$$R^{\kappa}_{\lambda\mu\nu} = \mathrm{d}x^{\kappa}\left(\nabla_{\mu}\nabla_{\nu}\frac{\partial}{\partial x^{\lambda}} - \nabla_{\nu}\nabla_{\mu}\frac{\partial}{\partial x^{\lambda}}\right) = \partial_{\mu}\Gamma^{\kappa}_{\nu\lambda} - \partial_{\nu}\Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\eta}_{\nu\lambda}\Gamma^{\kappa}_{\mu\eta} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\kappa}_{\nu\eta}.$$

10.2 Connections to Supergeometry

We will now explain the connection between Dirac operators and supergeometry. Let $\mathbb{R}^{1|1}$ denote the super Euclidean space with one even variable t and one odd variable θ. Then $C^{\infty}(\mathbb{R}^{1|1}) = C^{\infty}(\mathbb{R}) \otimes \Lambda^{\bullet}\mathbb{R}$. We consider a quantum theory of maps

$$X : \mathbb{R}^{1|1} \to M$$

where M is a compact, oriented, $n = 2k$ dimensional spin manifold with a Riemannian metric. Consider the vector field (differential operator) on $\mathbb{R}^{1|1}$ given by

$$\mathcal{D} = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}.$$

A simple computation shows that

$$\mathcal{D}^2 = -\frac{\partial}{\partial t}.$$

This is not surprising, since $\mathcal{D}$ is essentially just the super conformal structure defined on a Super Riemann surface (except we are working with real manifolds now), and the computation follows identically. Now we consider the action

$$S = -\frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta g_{\mu\nu}(X)\frac{\mathrm{d}X^{\mu}}{\mathrm{d}t}\mathcal{D}X^{\nu} = -\frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta\langle\dot{X}, \mathcal{D}X\rangle.$$

We now try to simplify this expression. Since $X : \mathbb{R}^{1|1} \to M$ is a map between supermanifolds, we may Taylor expand the function $X = X(t, \theta)$ in powers of θ. Since $\theta^2 = 0$, we see that

$$X = x(t) + \theta\psi(t)$$

for some odd function $\psi(t)$. The precise characterization of ψ is the following:

Proposition 10.3. The function $\psi(t)$ is a section of the odd tangent bundle $\Pi(x^*TM)$ over $\mathbb{R}^1$, where x^*TM is the pullback of the tangent bundle $TM \to M$ over the map $x : \mathbb{R}^1 \to M$.

Proof. We already know from the expression of X that ψ must be an odd function, hence it remains to show that it transforms like a tangent vector. If we choose a different local coordinate $\tilde{x}^1, ..., \tilde{x}^n$ of M, then the new quantum map is given by $\tilde{X} = \tilde{x}(t) + \theta\tilde{\psi}(t)$. Now the ν-component of the new map after Taylor expansion at $\theta = 0$ is just

$$\tilde{X}^{\nu} = \tilde{x}^{\nu}(x + \theta\psi) = \tilde{x}^{\nu}(x) + \frac{\partial\tilde{x}^{\nu}}{\partial x^{\mu}}(\theta\psi^{\mu}) = \tilde{x}^{\nu} + \theta\tilde{\psi}^{\nu}.$$

Therefore, we see that

$$\tilde{\psi}^{\nu} = \psi^{\mu}\frac{\partial\tilde{x}^{\nu}}{\partial x^{\mu}}$$

which is the transformation rule of a vector in TM. Therefore, we conclude that ψ is a section the parity reversed tangent bundle $\Pi(x^*TM)$. $\square$

In order to obtain a local coordinate expression of the Lagrangian, we can do the dθ integral. Plugging everything in and collecting only the term linear in θ (since it is the only term that survives the Grassmann integral), we get

$$\begin{aligned}
S &= -\frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta g_{\mu\nu}(X)\frac{\mathrm{d}X^\mu}{\mathrm{d}t}\mathcal{D}X^\nu \\
&= -\frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta\bigg(g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\theta\psi^\rho\bigg)(\dot{x}^\mu + \theta\dot{\psi}^\mu)(\psi^\nu - \theta\dot{x}^\nu) \\
&= \frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta g_{\mu\nu}\theta(\dot{x}^\mu\dot{x}^\nu - \dot{\psi}^\mu\psi^\nu) - \theta\bigg(\frac{\partial g_{\mu\nu}}{\partial x^\rho}\psi^\rho\dot{x}^\mu\psi^\nu\bigg) \\
&= \frac{1}{2}\int_{\mathbb{R}^1} \mathrm{d}t\bigg(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - g_{\mu\nu}\dot{\psi}^\mu\psi^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\psi^\rho\dot{x}^\mu\psi^\nu\bigg) \\
&= \frac{1}{2}\int_{\mathbb{R}} \langle\dot{x}(t), \dot{x}(t)\rangle - \langle\nabla_t\psi(t), \psi(t)\rangle\mathrm{d}t,
\end{aligned}$$

where the last equality follows from the same computation as in the previous section when we carried out the $\delta_\epsilon Q$ computation. Comparing with (10.2) there is a missing factor of i in the second term, but this can be recovered by an appropriate rescaling of the odd coordinate θ and the spinor field ψ.

10.3 Index on Spin Manifolds

Now the spinor bundle $S = S(M)$ on M decomposes into the even and odd parts $S = S^\pm$. Recall that the index of the Dirac operator is given by

$$\mathrm{ind}(\partial\!\!\!/) = \dim\ker D - \dim\ker D^\dagger$$

where $D : \Gamma(S^+) \to \Gamma(S^-)$ and $D^\dagger : \Gamma(S^-) \to \Gamma(S^+)$ are the two pieces of the self-adjoint operator $\partial\!\!\!/$ (Theorem 9.23):

$$\partial\!\!\!/ = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}.$$

Proposition 10.4. $\mathrm{ind}(\partial\!\!\!/)$ is invariant under small perturbation of D.

Proof. First, note that $DD^\dagger, D^\dagger D$ are nonnegative operators, for $\langle\psi|DD^\dagger|\psi\rangle = \|D^\dagger|\psi\rangle\|^2$ which is nonnegative, and similarly for $D^\dagger D$. By the same formula we see that $\ker D^\dagger = \ker DD^\dagger$, and similarly $\ker D = \ker D^\dagger D$. Now let $\{|\phi_n\rangle\}$ be an orthonormal set of eigenvectors of $D^\dagger D$ with eigenvalue $\lambda_n > 0$. Then it is easy to see that $|\psi_n\rangle := D|\phi_n\rangle/\sqrt{\lambda_n}$ are eigenvectors of $DD^\dagger$ with the same eigenvalue λ_n. Also note that they are orthonormal, hence form a basis:

$$\langle\psi_n|\psi_m\rangle = \frac{1}{\sqrt{\lambda_n\lambda_m}}\langle\phi_n|D^\dagger D|\phi_m\rangle = \frac{\lambda_n}{\sqrt{\lambda_n\lambda_m}}\delta_{nm} = \delta_{nm}.$$

Note that $|\phi_n\rangle \in (\ker D)^\perp$ and $|\psi_n\rangle \in (\ker D^\dagger)^\perp$, and there is an isomorphism $(\ker D)^\perp \cong (\ker D^\dagger)^\perp$, which is a standard result in functional analysis. Hence we see that if under a small perturbation of D, $\dim\ker D$ decreases (or increases), then $\dim\ker D^\dagger$ must also decrease (or increase) by the same dimension to preserve the isomorphism $(\ker D)^\perp \cong (\ker D^\dagger)^\perp$. $\square$

Theorem 10.5. *Let $\not{\partial}$ be the dirac operator, then*

$$\operatorname{ind}(\not{\partial}) = \operatorname{Tr} \mathrm{e}^{-\beta D^\dagger D} - \operatorname{Tr} \mathrm{e}^{-\beta D D^\dagger},$$

where $\beta > 0$ is a real constant. In fact, $\operatorname{ind}(Q)$ *is independent of β.*

Proof. Let $|\phi_n\rangle, |\psi_n\rangle$ be defined as in the proof of the previous proposition. Then let $\{|\phi_{i,o}\rangle\}$ be orthonormal eigenvectors of $\ker D$ and similarly $\{|\psi_{i,o}\rangle\}$ be orthonormal eigenvectors of $\ker D^\dagger$. Note that both lists are finite since D is assumed to be Fredholm. Then simple computation yields

$$\begin{aligned}
&\operatorname{Tr} \mathrm{e}^{-\beta D^\dagger D} - \operatorname{Tr} \mathrm{e}^{-\beta D D^\dagger} \\
&= \sum_n \langle \phi_n | \mathrm{e}^{-\beta D^\dagger D} | \phi_n \rangle - \sum_n \langle \psi_n | \mathrm{e}^{-\beta D D^\dagger} | \psi_n \rangle + \sum_i \langle \phi_{i,o} | \phi_{i,o} \rangle - \sum_j \langle \psi_{j,o} | \psi_{j,o} \rangle \\
&= \sum_n \mathrm{e}^{-\beta \lambda_n} (\langle \phi_n | \phi_n \rangle - \langle \psi_n | \psi_n \rangle) + \sum_i 1 - \sum_j 1 \\
&= \dim \ker D - \dim \ker D^\dagger.
\end{aligned}$$

This shows the desired equality. It is immediately clear from the calculation above that $\operatorname{ind}(D)$ is independent of β. □

By the previous discussion, we could identify Q with the Dirac operator. However, it is more conventional to identify iQ with the Dirac operator $\not{\partial}$. Since we have a matrix representation

$$iQ = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}.$$

Then by the generalization of Theorem 10.2 part (4) on manifolds, the Hamiltonian should be

$$H = (iQ)^2 = \begin{pmatrix} D^\dagger D & 0 \\ 0 & D D^\dagger \end{pmatrix}.$$

Recall that the chirality operator gives ± 1 on $S^\pm$. However, in physicist's notation it is more common to denote the chirality operator by

$$(-1)^F.$$

Then Theorem 10.5 can be written in a more compact form, with the right hand side known as the ***Witten index***:

$$\operatorname{ind}(\not{\partial}) = \operatorname{Tr}(-1)^F \mathrm{e}^{-\beta H}. \tag{10.3}$$

In fact, if M is a spin manifold, and E its spin bundle, we already noted that $Q \sim \not{\partial}$ is the Dirac operator on M, and we note that $(-1)^F = \gamma_{2n+1} = i^n \gamma_1 \cdots \gamma_{2n}$. Since $\gamma_{2n+1}^2 = 1$, the eigenvalues of γ_{2n+1} are ± 1, which we call ***chirality***. Recall that $S = S^\pm$ is the eigenspace decomposition with respect to $(-1)^F$. We assign $F = 0$ to sections of $\Gamma(M, S^+)$ and $F = 1$ to sections of $\Gamma(M, S^-)$, which explains the notation $(-1)^F$. In this case, the index of the supercharge Q is defined to be

$$\operatorname{ind}(Q) := \operatorname{ind}(\not{\partial}) = \dim \ker D - \dim \ker D^\dagger.$$

10.4 The Proof

Let $H = (iQ)^2 = \frac{1}{2}g_{\mu\nu}p^\mu p^\nu$ be the Hamiltonian corresponding to the Dirac operator Q. By the generalized version of Theorem 10.2, in curved space the Lagrangian of this system should be the generalization to the flat space SUSYQM Lagrangian, which we introduced in (10.2). With the knowledge of (10.3) and (7.25), while noting the fact that (10.2) contains both bosonic and fermionic fields, we are motivated to conclude that the index can be computed by a path integral over all bosonic and fermionic field configurations:

$$\mathrm{ind}(Q) = \mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H} = \int_{\mathrm{PBC}} \mathcal{D}x\mathcal{D}\psi \mathrm{e}^{-\int_0^\beta \mathrm{d}tL}, \tag{10.4}$$

where L is the Euclidean Lagrangian after performing Wick rotation to (10.2). Note that since L in the formula above is in Euclidean time, to obtain L we should replace $t \to -it$ in (10.2) and replace $\mathrm{d}t \to -i\mathrm{d}t$ to get

$$L = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{Dt}.$$

As a sanity check, note that compared to (10.2), the sign of the momentum term is unchanged. We should explain why we integrate over PBC instead of APBC, since for grassmann-valued fields (8.12) seems to indicate we should integrate over APBC. This is because the factor $(-1)^F$ disappears if the APBC for the fermionic variable is changed into a periodic one. To see this, let us just instead consider the simple case of a fermionic oscillator. Recall that from (8.12) we have

$$\begin{aligned}\mathrm{Tr}(-1)^F\mathrm{e}^{-\beta H} &= \sum_n \langle n|(-1)^F\mathrm{e}^{-\beta H}|n\rangle \\ &= \int \mathrm{d}\theta^*\mathrm{d}\theta\langle -\theta|(-1)^F\mathrm{e}^{-\beta H}|\theta\rangle\mathrm{e}^{-\theta^*\theta}.\end{aligned}$$

Note that since $|\theta\rangle = |0\rangle + |1\rangle\,\theta$ and $F = c^\dagger c$ is the fermion number operator, we have that

$$(-1)^F|\theta\rangle = (-1)^0|0\rangle + (-1)^1|1\rangle\,\theta = |0\rangle - |1\rangle\,\theta = |-\theta\rangle\,,$$

hence the above integral becomes

$$\int \mathrm{d}\theta^*\mathrm{d}\theta\langle\theta|\mathrm{e}^{-\beta H}|\theta\rangle\mathrm{e}^{-\theta^*\theta}.$$

Therefore, we conclude that we must integrate over PBC. The rest of the section is dedicated to evaluate the path integral in (10.4).

Now the action in the exponential of (10.4) reads

$$S = \int_0^\beta \mathrm{d}t\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu\frac{D\psi^\nu}{Dt}\right).$$

Perform a change of variable $t = \beta s$, then $\dot{x}^\mu \to \dot{x}^\mu/\beta$ and $\mathrm{d}t \to \beta\mathrm{d}s$. Moreover, since the measure $\mathcal{D}\psi$ is invariant under scaling, we can perform a change of variable $\psi \to \psi/\sqrt{\beta}$. Then the action reads

$$S = \frac{1}{\beta}\int_0^1 ds\left(\frac{1}{2}g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} + \frac{1}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{Ds}\right).$$

In the $\beta \to 0$ limit, clearly the integal (10.4) localizes at constant paths x (ones with $\dot{x}^\mu = 0$). Moreover, since

$$\frac{D\psi^\nu}{Ds} = \frac{\mathrm{d}\psi^\nu}{\mathrm{d}s} + \Gamma^\nu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}\psi^\nu}{\mathrm{d}s},$$

and classical paths satisfies $\dot{x}^\alpha = 0$, so we conclude that $\dot{\psi}^\nu = 0$ as well. Hence (10.4) localizes at constant paths and constant spinor fields.

Having identified the region in which the integral (10.4) localizes in the small β limit, our next goal is to simplify the Lagrangian L in the small β limit as much as possible. To make our lives easier, we pick a normal coordinate centered at x_0, i.e., a coordinate system in which

$$\begin{aligned} g_{\mu\nu}(x_0) &= \delta_{\mu\nu}, \\ \partial_\lambda g_{\mu\nu}(x_0) &= 0. \end{aligned}$$

Moreover, we may expand the paths and fields in Fourier modes by the PBC condition:

$$x^\mu(t) = x_0^\mu + \xi^\mu(t) = x_0^\mu + \frac{1}{\sqrt{\beta}}\sum_{n\neq 0}\xi_n^\mu \mathrm{e}^{2\pi i n t/\beta}, \tag{10.5}$$

$$\psi^\mu(t) = \psi_0^\mu + \eta^\mu(t) = \psi_0^\mu + \frac{1}{\sqrt{\beta}}\sum_{n\neq 0}\eta_n^\mu \mathrm{e}^{2\pi i n t/\beta}. \tag{10.6}$$

In our normal coordinate, the Taylor expansion of the metric tensor is

$$g_{\mu\nu}(x_0 + x) = \delta_{\mu\nu} + \frac{1}{2!}g_{\mu\nu,\alpha\beta}x^\alpha x^\beta + \cdots .$$

Then the Lagrangian becomes

$$L = \frac{1}{2}\left(\delta_{\mu\nu} + \frac{1}{2!}g_{\mu\nu,\rho\sigma}x^\rho x^\sigma + \dots\right)\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu\left(\frac{\partial\psi^\nu}{\partial t} + \dot{x}^\lambda\Gamma^\nu_{\lambda k}(x)\psi^k\right)$$

Now expanding the Christoffel symbol above, and throwing away terms of quadratic order or higher, and using $\Gamma^\mu_{\nu\rho}(x_0) = 0$, we get that

$$L = \frac{1}{2}\dot{x}^\mu\dot{x}^\mu + \frac{1}{2}g_{\mu\nu}\psi^\mu\left(\frac{\partial\psi^\nu}{\partial t} + \dot{x}^\lambda\psi^k\Gamma^\nu_{\lambda k,\alpha}(x_0)\,x^\alpha\right).$$

But in the normal coordinate we have

$$\partial_\alpha\partial_\beta g_{\mu\nu}(x_0) = R_{\mu\alpha\nu\beta}(x_0),$$

and the second term in the L formula above is

$$\Gamma^\nu_{\lambda k,\alpha} = \frac{1}{2}g^{\delta\nu}\left(g_{\delta\lambda,k\alpha} + g_{\delta k,\lambda\alpha} - g_{\lambda k,\delta\alpha}\right).$$

Thus

$$g_{\delta\nu}\Gamma^\nu_{\lambda k,\alpha} = \frac{1}{2}\left(R_{\delta\alpha\lambda k} + R_{\delta\alpha k\lambda} - R_{\lambda\alpha k\delta}\right).$$

Note that the first two curvature tensor components cancel, because the curvature tensor is antisymmetric in the last two index. Thus

$$\Gamma^\nu_{\lambda k,\alpha} = -\frac{1}{2}R^\nu_{\lambda\alpha k}.$$

Hence putting everything together, after the expansion we have

$$L = \frac{1}{2}\dot{x}^\mu \dot{x}^\mu + \frac{1}{2} g_{\mu\nu}\psi^\mu \left(\frac{\partial \psi^\nu}{\partial t} + \dot{x}^\lambda \psi^k x^\alpha \left(-\frac{1}{2} R^\nu_{\lambda\alpha k}\left(x_0\right)\right)\right).$$

Now we plug in (10.5) and (10.6) into the above Lagrangian. However, we should note that L contains a curvature term $\sim R_{\lambda\alpha k\mu}\psi^\mu \dot{x}^\lambda \psi^k x^\alpha$. We can expand ψ^μ using (10.6), and if we include terms other than the zero mode ψ_0^μ, then this term would involve a power $1/\sqrt{\beta}$, which will be exponentially suppressed when we evaluate (10.4) in the small β limit. Therefore, we only need to look at the curvature term $\sim R_{\lambda\alpha k\mu}\psi_0^\mu \dot{\xi}^\lambda \psi_0^k \xi^\alpha$ in the expansion. Therefore, putting everything together, and relabeling the indices, we get that the effective action in (10.4) is

$$S = \int_0^\beta \mathrm{d}t \left(\frac{1}{2}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}t}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}t} + \frac{1}{2}\eta^\mu \frac{\mathrm{d}\eta^\mu}{\mathrm{d}t} - \frac{1}{2}\omega_{\mu\nu}(x_0)\xi^\mu \frac{\mathrm{d}\xi^\nu}{\mathrm{d}t}\right) \tag{10.7}$$

where we have put

$$\omega_{\mu\nu}(x_0) = \frac{1}{2} R_{\mu\nu\rho\sigma}(x_0)\psi_0^\rho \psi_0^\sigma.$$

To get rid of the β dependence in the integral, or rather to hide the β independence into the fields, we make the substitution

$$t \to \beta t' \qquad \eta \to \eta' \qquad \xi \to \sqrt{\beta}\xi' \qquad \psi_0 \to \psi_0'/\sqrt{\beta}.$$

The first substitution makes our range of integration goes from 0 to 1. The last substitution is fine since in (10.4), we are integrating over all ψ_0, so a scaling does not matter. The scaling $\xi \to \sqrt{\beta}\xi'$ is more tricky, but note that after the substitution the term $R_{\lambda\alpha k\mu}\psi_0^\mu \dot{\xi}^\lambda \psi_0^k \xi^\alpha$ still survives the small β limit, but any other terms in $R_{\lambda\alpha k\mu}\psi^\mu \dot{\xi}^\lambda \psi^k \xi^\alpha$ that contains other than the zero mode ψ_0^μ would involve a power $1/\sqrt{\beta}$, which will be exponentially suppressed in the small β limit. Hence our expansion is still valid. In conclusion, after the substitution and relabeling again by removing all the primes, we see that (10.7) can be written as

$$S = \int_0^1 \mathrm{d}t \left(\frac{1}{2}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}t}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}t} + \frac{1}{2}\eta^\mu \frac{\mathrm{d}\eta^\mu}{\mathrm{d}t} - \frac{1}{2}\omega_{\mu\nu}(x_0)\xi^\mu \frac{\mathrm{d}\xi^\nu}{\mathrm{d}t}\right). \tag{10.8}$$

From (10.8), one can directly read off the fluctuation operator for ξ. After performing integration by part, it is

$$-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t.$$

And the fluctuation operator for η is simply

$$\delta_{\mu\nu}\partial_t.$$

Now that we have identified that (10.4) localizes at constant paths in the small β limit, and we know what L looks like in such limit, we might be tempted to evaluate (10.4) at constant paths at x_0, ψ_0, and integrate over all such points. But we cannot just reduce the path integrals to ordinary integrals directly, since we have only calculated the small β limit, whereas the region of integration localizes to constant path only at $\beta = 0$. Hence, we should be reducing the path integral to a small neighborhood of constant paths instead of just constant paths. This means that we should do the path integral over the mode expansions ξ, η as well. Thus (10.4) becomes, after performing Gaussian integration using (8.8) and the path integral generalization of (7.2):

$$\begin{aligned}\operatorname{ind}(Q) &= \int \mathrm{d}x_0 \mathrm{d}\psi_0 \int \mathcal{D}\xi \mathcal{D}\eta \mathrm{e}^{-\int_0^1 \mathrm{d}t \frac{1}{2}[\xi^\mu(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t)\xi^\nu + \eta^\mu(\delta_{\mu\nu}\partial_t)\eta^\nu]} \\ &= \mathcal{N}\int \mathrm{d}x_0 \mathrm{d}\psi_0 \det(\delta_{\mu\nu}\partial_t)\det\left(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t\right)^{-1/2},\end{aligned}$$

where $\mathcal{N}$ is a formal constant, and $\det(\delta_{\mu\nu}\partial_t)$ can be evaluated using the ζ-function regulator. The computation for $\det(\delta_{\mu\nu}\partial_t)$ is done in Chapter 1 of [19]. But we will omit this since it is not of our concern: the result will only be a constant depending on the dimension d. Next we evaluate

$$\det\bigl(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t\bigr).$$

Note that $\omega_{\mu\nu}$ defined above is skew-symmetric with even matrix element. Thus there is an orthogonal transformation that turns $\omega_{\mu\nu}$ into the form

$$\omega = \begin{pmatrix} 0 & r_1 & \cdots & 0 & 0 \\ -r_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & r_n \\ 0 & 0 & \cdots & -r_n & 0 \end{pmatrix}$$

where $n = d/2$ is half the dimension of the spin manifold M. If we just look at one block of ω, which is of the form

$$A = \begin{pmatrix} -\partial_t^2 & -r \\ r & -\partial_t^2 \end{pmatrix}.$$

It is not hard to see that O has the same spectrum as

$$A' = \begin{pmatrix} -\partial_t^2 - ir\partial_t & 0 \\ 0 & -\partial_t^2 + ir\partial_t \end{pmatrix}.$$

Hence

$$\det A = \det A' = \det\bigl(-\partial_t^2 - ir\partial_t\bigr)\det\bigl(-\partial_t^2 + ir\partial_t\bigr).$$

We will only evaluate one of the determinants, since the other is similar. Define the differential operator

$$L = -\partial_t^2 - ir\partial_t.$$

A complete basis of eigenfunctions is given by

$$\phi_n(t) = \mathrm{e}^{2\pi int}$$

for all integers n. Then

$$L\phi_n = (-(2\pi in)^2 - ir(2\pi in))\phi_n = (4\pi^2n^2 - 2\pi rn)\phi_n.$$

So the eigenvalues are

$$\lambda_n = 4\pi^2n^2 - 2\pi nr = 2\pi n(2\pi n - r).$$

Hence, throwing everything that is independent of r into the formally infinite constant in front, we get

$$\begin{aligned} \det L &= \prod_{n\neq 0} 2\pi n(2\pi n - r) \\ &= (2\pi)^\infty \prod_{n\neq 0} n(2\pi n - r) \\ &= \mathcal{N} \prod_{n\geq 1} (2\pi n - r)(-2\pi n - r) \\ &= \mathcal{N}' \prod_{n\geq 1} \left(1 - \left(\frac{r}{2\pi n}\right)^2\right) \end{aligned}$$

and we will ignore $\mathcal{N}'$ as it can be thrown into the ζ-function regularization. Now using the identity

$$\prod_{n\geq 1}\left(1-\frac{x^2}{n^2}\right)=\frac{\sin(\pi x)}{\pi x},$$

we get

$$\det L=\frac{\sin r/2}{r/2}$$

up to a normalization constant. The other determinant will give exactly the same result by similar calculation. Hence

$$\det A=\left(\frac{\sin r/2}{r/2}\right)^2.$$

The quantity of interest $\det\left(-\delta_{\mu\nu}\partial_t^2-\omega_{\mu\nu}\partial_t\right)$ will be the product of determinants of the block operator above, with r running from $r_1,...,r_n$. Hence,

$$\det\left(-\delta_{\mu\nu}\partial_t^2-\omega_{\mu\nu}\partial_t\right)=\prod_{k=1}^{n}\left(\frac{\sin r_k/2}{r_k/2}\right)^2.$$

Using the identity $\sinh(ir)=i\sin(r)$, our calculation yields

$$\det\left(-\delta_{\mu\nu}\partial_t^2-\omega_{\mu\nu}\partial_t\right)^{-1/2}=\prod_{k=1}^{n}\left(\frac{r_k/2}{\sin r_k/2}\right)=j(i\Omega)^{-1/2},$$

using the identification of $\omega_{\mu\nu}$ on ΠTM with the curvature 2-form

$$\Omega_{\mu\nu}=\frac{1}{2}R_{\mu\nu\rho\sigma}\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma$$

on M, where $j(\Omega)$ is by definition

$$j(\Omega)=\det\left(\frac{\sinh\Omega/2}{\Omega/2}\right):=\prod_{k=1}^{n}\left(\frac{\sinh r_k/2}{r_k/2}\right)^2$$

and r_k are the entries of the block form of the curvature 2-form Ω, and the $\hat{A}$-class of M is by definition

$$\hat{A}(M)=j(\Omega)^{-1/2}.$$

Therefore, we see that (absorbing i^d from $j(i\Omega)^{-1/2}$ into C^{2d}) under the identification of $C^\infty(\Pi TM)\cong\mathcal{A}^\bullet(M)$, which we already used above to identify $\omega_{\mu\nu}$ with the curvature 2-form $\Omega_{\mu\nu}$ on M, that

$$\mathrm{ind}(Q)=C^{2d}\int_{\Pi TM}\mathrm{d}x_0\mathrm{d}\psi_0\,j(\Omega)^{-1/2}=C^{2d}\int_M\hat{A}(M).$$

In this derivation we have been careless with universal constants depending only on dimension (infinite but regularized in some manner). To evaluate the constant one can either follow through the regularization procedure or simply evaluate it on one example where the index is nontrivial.

Therefore, we have proven the celebrated Atiyah Singer index theorem for the index of a Dirac operator on a spin manifold:

Theorem 10.6 (Atiyah-Singer). *The index of the Dirac operator $\not{\partial}$ defined on a spin manifold M is given by*

$$\mathrm{ind}(\not{\partial})=\int_M\hat{A}(M).$$

Appendix A

Deformation Theory

We mentioned that for a curve X of genus g, the tangent space $T_X\mathcal{M}_g$ is given by $H^1(X, T_X)$. In this appendix we try to provide a heuristic reasoning why this is the case (Theorem A.3). The reference is Chapter 4 of [18] and Chapter 9 of [14].

Definition A.1. Let $\mathcal{X}, B$ be complex manifolds, and $\phi : \mathcal{X} \to B$ a holomorphic map. Let $X_t := \phi^{-1}(t)$ denote the fiber of ϕ above the point $t \in B$.

We say that $\phi : \mathcal{X} \to B$ is a ***family*** of complex manifolds if ϕ is a proper holomorphic submersion.

The following theorem is proved in p.220 of [14]:

Theorem A.2 (Ehresmann)**.** *Let $\phi : \mathcal{X} \to B$ be a proper submersion between two differentiable manifolds, where B is contractible with a base point $0 \in B$. Then there exists a diffeomorphism*

$$T : \mathcal{X} \xrightarrow{\cong} X_0 \times B$$

such that $\pi_2 \circ T = \phi$, where $\pi_2 : X_0 \times B \to B$ is just the projection to the second component.

Giving such a trivilization is thus equivalent to giving its first component

$$T_0 := \pi_1 \circ T : \mathcal{X} \to X_0,$$

as $\pi_2 \circ T = \phi$ is given. Note that when T_0 is restricted to the fiber X_t, it gives a diffeomorphism $X_t \cong X_0$. Therefore, under the identification of such a diffeomorphism, we can view that X_t still has the same topology and smooth structure, namely it is still X_0 as a smooth manifold, but the complex structure is different. Of course, the theorem above may not apply when B is not contractible. But since B is a manifold, at any base point $0 \in B$ we can always restrict to a sufficiently small locally Euclidean neighborhood of it, which will again be contractible.

Therefore, we conclude that infinitesimal deformation of X_0 is the same as infinitesimal deformation of complex structure on X_0. Hence the tangent space $T_X\mathcal{M}_g$ of a curve X should be identified with infinitesimal deformations of complex structures on X.

Next, let's do some explicit calculation in coordinates. How do we quantitatively describe deformation of complex structures? Instead of focusing on $X = X_0$, let us focus on X_t for general $t \in B$. Furthermore, let us first consider the simplified case where $\dim_{\mathbb{R}} B = 1$, hence t is a local coordinate on B. Since ϕ is proper, $X_t = \phi^{-1}(t)$ is a compact complex manifold. Hence we can cover X_t by finitely many open sets U_i for $i = 1, ..., m$, with chart $\varphi_i : U_i \to \varphi(U_i) \subset \mathbb{C}^n$ where $n = \dim_{\mathbb{C}} X_t$. By taking smaller charts if necessary, we may assume that U_i's are ***good covers***, meaning finite intersections $U_{i_1} \cap \cdots \cap U_{i_k}$ are contractible. On the overlap we have ***transition maps***

$$\begin{aligned} f_{ij} : \varphi_j(U_i \cap U_j) &\to \varphi_i(U_i \cap U_j) \\ (z_i^1, ..., z_i^n) &\mapsto (f_{ij}^1(z), ..., f_{ij}^n(z)) \end{aligned}$$

which are biholomorphisms. These maps satisfy the ***cocycle condition***

$$f_{ik} = f_{ij} \circ f_{jk} \tag{A.1}$$

on the overlap $U_i \cap U_j \cap U_k$. Recall that a ***complex structure*** of X_t is defined by the collection of biholomorphic transition maps f_{ij}'s. Hence, by a ***deformation of complex structure*** on X_t along B we simply mean that these transition maps are allowed to vary with respect to t. That is, now they have t dependence:

$$z_i^\alpha := f_{ij}^\alpha(z_j^1, ..., z_j^n, t) \qquad \alpha = 1, ..., n.$$

We use the above notation z_i^α to denote the αth coordinate in the chart U_i. In the calculation below, we will only contract greek letters like α, β, but not the letters i, j, k as they only suggests the charts we are in and the initial and end position of the transition maps.

With t dependence, the cocycle condition (A.1) becomes

$$f_{ik}^\alpha\left(z_k, t\right) = f_{ij}^\alpha(f_{jk}^1\left(z_k, t\right), \ldots, f_{jk}^n\left(z_k, t\right), t) \tag{A.2}$$

for every $\alpha = 1, ..., n$. Taking derivative of (A.2) with respect to t, and using the relation $z_j^\beta = f_{jk}^\beta(z_k, t)$ gives

$$\frac{\partial f_{ik}^\alpha\left(z_k, t\right)}{\partial t} = \frac{\partial f_{ij}^\alpha\left(z_j, t\right)}{\partial t} + \frac{\partial f_{ij}^\alpha\left(z_j, t\right)}{\partial z_j^\beta} \frac{\partial f_{jk}^\beta\left(z_k, t\right)}{\partial t}. \tag{A.3}$$

Now using $z_i^\alpha = f_{ij}^\alpha(z_j, t)$, (A.3) becomes

$$\frac{\partial f_{ik}^\alpha\left(z_k, t\right)}{\partial t} = \frac{\partial f_{ij}^\alpha\left(z_j, t\right)}{\partial t} + \frac{\partial z_i^\alpha}{\partial z_j^\beta} \frac{\partial f_{jk}^\beta\left(z_k, t\right)}{\partial t}. \tag{A.4}$$

Now using the expression (A.4) as the αth component function of a holomorphic vector field gives

$$\frac{\partial f_{ik}^\alpha(z_k, t)}{\partial t} \frac{\partial}{\partial z_i^\alpha} = \frac{\partial f_{ij}^\alpha(z_j, t)}{\partial t} \frac{\partial}{\partial z_i^\alpha} + \frac{\partial f_{jk}^\beta(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^\beta}, \tag{A.5}$$

where we used the chain rule

$$\frac{\partial}{\partial z_j^\beta} = \frac{\partial z_i^\alpha}{\partial z_j^\beta} \frac{\partial}{\partial z_i^\alpha}.$$

We introduce the holomorphic vector fields

$$\theta_{jk}(t) = \frac{\partial f_{jk}^\alpha(z_j, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha} \qquad z_k = f_{kj}(z_j, t). \tag{A.6}$$

Using this definition, (A.5) becomes the beautiful cocycle condition

$$\theta_{ik}(t) = \theta_{ij}(t) + \theta_{jk}(t). \tag{A.7}$$

From (A.7), we conclude that $\theta_{jk}(t)$ is a 1-cocycle with respect to the open cover $\mathfrak{U} = (U_i)_{i=1}^m$ on X_t, valued in the holomorphic tangent bundle T_{X_t}. Hence this determines an element of $H^1(\mathfrak{U}, T_{X_t})$. Since $\mathfrak{U}$ is a good cover, we actually have the isomorphism $H^1(\mathfrak{U}, T_{X_t}) \cong H^1(X_t, T_{X_t})$. Let $\theta(t)$ be the element in $H^1(X_t, T_{X_t})$ determined by $\theta_{jk}(t)$. One can show that this class is independent of the open cover we choose. We can now conclude:

Theorem A.3. *The first order deformation (Taylor expand up to first order with respect to t) of the complex structure f_{ij}^α on X_t*

$$f_{ij}^\alpha(t) = f_{ij}^\alpha|_{t=0} + t\theta_{ij}^\alpha + O(t^2)$$

determines a class in $H^1(X_t, T_{X_t})$.

In the above argument we assumed $\dim_{\mathbb{R}} B = 1$. But this assumption is in fact redundant. Suppose now B is an arbitrary complex manifold, with local coordinate $t_1, ..., t_m$. Then by a ***deformation of complex structure*** on X_t with respect to B we mean the chart transition function now depends on the coordinate of B:

$$z_j^\alpha = f_{jk}^\alpha(z_k^1, ..., z_k^n, t^1, ..., t^m) \qquad \alpha = 1, ..., n.$$

Now for a tangent vector

$$\frac{\partial}{\partial t} = c^\mu \frac{\partial}{\partial t^\mu}$$

of B, where $c^\mu \in \mathbb{C}$, we put

$$\theta_{jk}(t) = \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha} = c^\mu \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t^\mu} \frac{\partial}{\partial z_j^\alpha},$$

where again $z_k = f_{kj}(z_j, t)$. The previous argument all goes through and $\theta_{jk}(t)$ again defines a a 1-cocycle, hence determines an element $\theta(t) \in H^1(X_t, T_{X_t})$. This defines, for each $t \in B$, a $\mathbb{C}$-linear map

$$\rho_t : T_t B \to H^1(X_t, T_{X_t}) \qquad \rho_t : \frac{\partial}{\partial t} \mapsto \theta(t),$$

called the ***Kodaira-Spencer map***.

Remark A.4. There is a coordinate free way of defining Kodaira-Spencer map. Let $\phi : \mathcal{X} \to B$ be a family of complex manifolds. Then Lemma **??** gives an exact sequence

$$0 \to T_{X_t} \to T_{\mathcal{X}}|_{X_t} \xrightarrow{\phi_*} \phi^* T_B|_{X_t} \to 0.$$

Then the Kodaira-Spencer map is simply the induced map

$$\rho_t : H^0(X_t, \phi^* T_B|_{X_t}) \to H^1(X_t, T_{X_t})$$

in the long exact sequence of sheaf cohomology induced by the short exact sequence above. Note that $H^0(X_t, \phi^* T_B|_{X_t}) = \phi^* T_B(X_t) = T_t B$ since ϕ_* is assumed to be surjective. For a proof that these two definitions coincide, consult [14, p.225].

Appendix B

Supersymmetry

In this appendix we discuss physical aspects of supergeometry, which is supersymmetry. Perhaps the most convincing motivation for physicists to develop the theory of supersymmetry is the so called hierarchy problem.

A hierarchy problem occurs when the fundamental value of some physical parameter, such as a coupling constant or a mass, in some Lagrangian is vastly different from its effective value, which is the value that gets measured in an experiment. This happens because the effective value is related to the fundamental value by renormalization, which applies corrections to it. In Standard model, the Higgs mass is modified by corrective terms from every scale with which it interacts: terms that are proportional to those scales. For the Standard Model this scale can go all the way to

$$M_{pl} = \sqrt{\frac{Gh}{c^3}} \approx 10^{19}\text{GeV},$$

and so the QFT expectation for the Higgs mass is much higher than the experimental result (~ 125GeV). Currently these two values are brought together through a numerical cancellation of terms that results in the Higgs mass being reduced to its proper experimental values. This process is called ***fine tuning*** [33]. This has spurred a number of alternative theories which solve the problem of the disparity of expected and experimental Higgs mass values without requiring fine tuning in the terms of the calculation.

One of such alternative solution is called ***Supersymmetry*** (SUSY). In a word, it assumes there is some extra symmetry of spacetime (besides the Poincaré symmetry). In particular, a symmetry under the exchange of bosons and fermions. This is the object of study in this appendix.

Recall what we did with scalar/vector/spinor fields: We determined that the symmetry group of interest is the Lorentz group SO(1,3). We studied its Lie algebra so(1,3), in particular generators and commutation relations. We tried to find representations of Lorentz group $D[\Lambda], \Lambda \in SO(1,3)$. We constructed fields $\varphi(x^\mu)$ satisfying the transformation property

$$\varphi^a(x) \to D[\Lambda]^a_b \varphi^b(\Lambda^{-1}x).$$

Taking $D[\Lambda] = I$ (trivial representation), $D[\Lambda] = \Lambda$ (standard representation), and $D[\Lambda] = S[\Lambda]$ (spinor representation), we constructed scalar, vector, and spinor fields. Using these fields, we constructed Lagrangian that respects SO(1,3) symmetry.

What we will do now is similar to this procedure. First, we extend so(1,3) to the SUSY algebra, and we extend $\mathbb{R}^{1,3}$ to a supermanifold. We will construct super fields on this space, and construct a representation of SUSY algebra via differential operators on superspace. Finally, we will construct a SUSY Lagrangian, and solve the Hierarchy problem via miraculous cancellation (at least a similar version of it to the standard model).

B.1 SUSY Algebra

In general, a physical theory need to be invariant under translations and Lorentz transformations $x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu$. Mathematically, this means the symmetry group we are looking at this the ***Poincaré group*** defined by the semidirect product

$$\mathbb{R}^{1,3} \rtimes SO(1,3).$$

The infinitesimal generators of the Poincaré Lie algebra are the momentum operators P^μ (which corresponds to translations), and the 4×4 antisymmetric matrices $M^{\mu\nu}$, which are a total of 6 generators of so(1,3). A four dimensional representation is given by

$$(M^{\rho\sigma})^\mu{}_\nu = i(\eta^{\mu\nu}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\sigma{}_\nu)$$

where $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric on $\mathbb{R}^{1,3}$. One can verify the commutation relations of the Poincaré Lie algebra are given by

$$\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\sigma] &= i(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma}) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}).
\end{aligned}$$

The SUSY algebra includes all the generators and commutation relations of the Poincaré algebra ($P^\mu, M^{\rho\sigma}$ and their commutation relations) and a pair of left handed spinors Q_α and right handed spinors $\bar{Q}_{\dot\alpha}$ (where $\alpha = 1, 2$), called supercharge operators. SUSY algebra also includes the (anti)commutation relations involving the supercharges. In other words, the SUSY algebra is a $\mathbb{Z}_2$-graded Lie algebra (also known as a ***super Lie algebra***). All derivations of these commutation relations can be found in [34]. The (anti)commutation relations are given by the following:[1]

$$\begin{aligned}
&[M^{\mu\nu}, Q_\alpha] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \\
&[M^{\mu\nu}, \bar{Q}_{\dot\alpha}] = (\bar\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta} \\
&[Q_\alpha, P^\mu] = 0 = [\bar{Q}_{\dot\alpha}, P^\mu] \\
&\{Q_\alpha, Q_\beta\} = 0 \\
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu
\end{aligned}$$

where

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta$$

$$(\bar\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\beta} = \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)_{\dot\alpha}{}^{\dot\beta}.$$

[1] All internal symmetry must commute with the supercharges except possibly U(1)-symmetry: then $[R, Q_\alpha] = -Q_\alpha$ and $[R, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}$, where R generates U(1) symmetry.

B.2 Superspace, Superfields, and SUSY Lagrangian

Let $\mathfrak{g}$ be the SUSY Lie algebra. We can always exponentiate it to form the SUSY Lie group G, whose general element is of the form

$$g(\omega_{\mu\nu}, a_\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}) := \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right).$$

We define our superspace to be

$$X = G/\operatorname{SO}(1,3)$$

which is the set of equivalent classes under the equivalence relation $g \sim g \cdot h$ for all $h \in \operatorname{SO}(1,3)$. Since every element in SO(1,3) has the form $\exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$, we see that $X = G/\operatorname{SO}(1,3)$ is a supermanifold that can be parameterized by bosonic coordinates $x^\mu = a^\mu$, and fermionic coordinates $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$.

A ***superfield*** Y is a function $Y : X \to \mathbb{C}$ on superspace. We mostly write $Y(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$, or simply $Y(x, \theta, \bar{\theta})$.

Like how we found representations of so(1,3) on the space of fields in $\mathbb{R}^{1,3}$, we would also like to find a representation of SUSY algebra on space of superfields on X. That is, we want to find operators $\mathcal{M}^{\mu\nu}, \mathcal{P}^\mu, \mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}$ acting on superfields satisfying the SUSY Lie algebra commutation relation.

We already know how to define $so(1,3)$ acting on superfields:

$$\mathcal{M}^{\mu\nu}Y(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) := Y((M^{\mu\nu})^\rho{}_\sigma x^\sigma, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$$

That is, $\mathcal{M}^{\mu\nu}$ are still given by Lorentz transformations that we know well. What about $\mathcal{P}^\mu, \mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}$? There are a lot of tedious calculations involved, but one can check that

$$\begin{aligned}
\mathcal{P}_\mu &= -i\frac{\partial}{\partial x^\mu} \\
\mathcal{Q}_\alpha &= -i\frac{\partial}{\partial \theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \\
\bar{\mathcal{Q}}_{\dot{\alpha}} &= i\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}.
\end{aligned}$$

does the trick. To prove this, we need the following:

Lemma B.1. *Let* $(x, \theta, \bar{\theta}) := \exp\left(ix_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)$ *denote a point in superspace, and*

$$V(\epsilon, \bar{\epsilon}) := \exp\left(i\epsilon^\alpha Q_\alpha + i\bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)$$

Then the action of $V(\epsilon, \bar{\epsilon})$ *on* $(x, \theta, \bar{\theta})$ *is given by*

$$\begin{aligned}
x^\mu &\to x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}} - i\epsilon^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}} \\
\theta^\alpha &\to \theta^\alpha + \epsilon^\alpha \\
\bar{\theta}^{\dot{\alpha}} &\to \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}.
\end{aligned}$$

Proof. Recall that we have the commutator relation $[Q_\alpha, \bar{Q}_{\dot{\alpha}}] = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu$, we have that $\epsilon^\alpha(Q_\alpha\bar{Q}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}}Q_\alpha)\bar{\theta}^{\dot{\alpha}} = 2(\epsilon^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}})P_\mu$, so that $[\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}, \epsilon_\alpha Q^\alpha] = 2(\epsilon\sigma^\mu\bar{\theta})P_\mu$, where we have picked up a minus sign from the previous equation to the last equation because of anticommutitivity of $\bar{\theta}, \epsilon, Q, \bar{Q}$, turning $\{,\}$ into $[,]$.

Since $V(\epsilon, \bar{\epsilon})(x, \theta, \bar{\theta}) = \mathrm{e}^{i\epsilon Q + i\bar{\epsilon}\bar{Q}}\mathrm{e}^{ixP + i\theta Q + i\bar{\theta}\bar{Q}}$, evaluating this using the BCH formula: $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\cdots}$, we get that

$$V(\epsilon, \bar{\epsilon})(x, \theta, \bar{\theta}) = \mathrm{e}^{ixP + i(\theta+\epsilon)Q + i(\bar{\theta}+\bar{\epsilon})\bar{Q} + (\epsilon\sigma\bar{\theta})P - (\theta\sigma\bar{\epsilon})P}$$

which gives precisely the desired transformation by definition of points in superspace. □

Theorem B.2. *The operators $\mathcal{P}_\mu, \mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha}$ should indeed by given as above. Since they satisfy the SUSY relations, they define a representation of SUSY Lie algebra on the space of superfields.*

Proof. Let $U = \exp(ia^\mu P^\mu)$, then since P_μ generates translation, we have $UY(x,\theta,\bar\theta)U^\dagger = Y(x+a,\theta,\bar\theta)$. Expanding to 1st order: $U \simeq 1 + ia_\mu P^\mu$ and $Y(x+a) \simeq Y(x) + a^\mu \partial_\mu Y(x)$, and equating terms linear in a, we see:

$$[P_\mu, Y] = -i\partial_\mu Y := \mathcal{P}_\mu Y.$$

Similarly for $V(\epsilon,\bar\epsilon) = \exp\big(i\epsilon^\alpha Q_\alpha + i\bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\big)$: using the previous lemma, acting on superfields gives

$$VY(x,\theta,\bar\theta)V^\dagger = Y(x + i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta, \theta+\epsilon, \bar\theta+\bar\epsilon).$$

Now expand to leading order in ϵ gives:

$$[Q_\alpha, Y] = (-i\partial_\alpha - \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu)Y := \mathcal{Q}_\alpha Y$$
$$[\bar Q_{\dot\alpha}, Y] = (i\partial_{\dot\alpha} - \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu)Y := \bar{\mathcal{Q}}_{\dot\alpha} Y.$$

This concludes the proof. □

As mentioned earlier in the introduction, supersymmetry is a symmetry that exchange bosons and fermions. This can be illustrated most directly by applying these SUSY operators on a super field $Y(x,\theta,\bar\theta)$. The most general Taylor expansion of superfield in θ and $\bar\theta$ is given by

$$\begin{aligned} Y(x,\theta,\bar\theta) = \varphi(x) &+ \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x) + \theta^2 M(x) + \bar\theta^2 N(x) \\ &+ \bar\theta^\alpha\bar\theta^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}V_\mu(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \bar\theta^2\theta_\alpha\rho_\alpha(x) + \theta^2\bar\theta^2 D(x). \end{aligned} \tag{B.1}$$

Note that superfield Y contains all the things we usually care about: 4 complex scalar fields φ, M, N, D, two left-handed spinors ψ, ρ, two right-handed spinors $\bar\chi, \bar\lambda$, and a vector V_μ. To illustrate our point, let us take the simplest example where Y is just a bosonic field

$$Y = \varphi(x).$$

Then applying the supercharge on it gives

$$\mathcal{Q}_\alpha Y = -(\partial_\mu\varphi)(\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}).$$

Now $\mathcal{Q}_\alpha Y$ is an odd function on X, i.e., is a fermion. Hence we see the emergence of spinor components from scalars.

Since applying the super charge operator flips parity of the particle (quantum field), supersymmetric theory conjectures that for every type of elementary particles in the standard model, there exists a superpartner of it which has opposite parity (For example, the electron e which is a fermion has a superpartner called the selectron $\tilde e$ which is a boson). Ideally, we hope that these superpartners have the same mass as their counterparts, and we will see why when we do the calculation of hierarchy problem.

For now, let us return to (B.1). The infinitesimal change of superfield is defined by

$$\delta Y = i[\epsilon Q + \bar\epsilon\bar Q, Y] = i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})Y.$$

The natural question to ask is how to all the components in (B.1) transform under such an infinitesimal SUSY transform. This requires pages and pages of painful calculation, but the final answer is:

$$
\begin{aligned}
\delta\varphi &= \epsilon\psi + \bar{\epsilon}\bar{\chi} \\
\delta\psi &= 2\epsilon M + (\sigma^\mu\bar{\epsilon})\,(i\partial_\mu\varphi + V_\mu) \\
\delta\bar{\chi} &= 2\bar{\epsilon}N - (\epsilon\sigma^\mu)\,(i\partial_\mu\varphi - V_\mu) \\
\delta M &= \bar{\epsilon}\bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \\
\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi} \\
\delta V_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi}\right) \\
\delta\bar{\lambda} &= 2\bar{\epsilon}D + \frac{i}{2}\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon}\partial_\mu V_\nu + i\bar{\sigma}^\mu\epsilon\partial_\mu M \\
\delta\rho &= 2\epsilon D - \frac{i}{2}\sigma^\nu\bar{\sigma}^\mu\epsilon\partial_\mu V_\nu + i\sigma^\mu\bar{\epsilon}\partial_\mu N \\
\delta D &= \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon}).
\end{aligned}
$$

Using these transformations, we can now build a Lagrangian on $X = G/\,\mathrm{SO}(1,3)$ that is invariant under SUSY transformation. Here, the crucial fact is that linear combinations, products, spatial derivatives of superfields are again superfields. Hence using these operations, we can field a Lagrangian $\mathfrak{L}$ that respects SUSY. There are many such Lagrangians so we will not specify a particular one. We can define the action to be given by the superspace integral

$$
S = \int_X \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\mathfrak{L}.
$$

The transformation of field components above will ensure that under the SUSY action $\delta\mathfrak{L}$ will be a total derivative, hence the action S is invariant under SUSY.

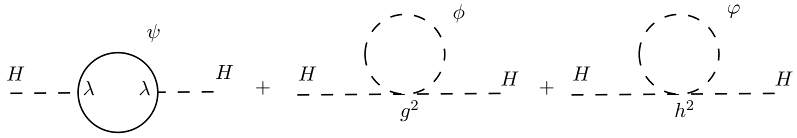
B.3 Miraculous Cancellation

Now we explain how SUSY is useful in solving the hierarchy problem. The ***Minimal Supersymmetric Standard Model*** (MSSM) is an extension to the Standard Model that realizes supersymmetry. MSSM is the minimal supersymmetrical model as it considers only the minimum number of new particle states and new interactions. The relevant part involving Higgs field reads

$$
\mathfrak{L} = -\lambda H\bar{\psi}\psi - g|H|^2|\phi|^2 - aH|\phi|^2 - h|H|^2|\varphi|^2 - bH|\varphi|^2 + \cdots,
$$

where ψ is a Dirac spinor, ϕ, φ are two complex scalar fields, λ, g, a, h, b are coupling constants.

The relevant interaction terms are given by the following three Feynman diagrams:



For example, ψ can be the top quark (Fermion), and ϕ, φ are superpartners (Boson) of t_L, t_R. The point is that SUSY puts constraints on the coefficients: one of such constraints is $\lambda^2 = g^2 = h^2$.

We can first compute the g-diagram (the same computation works for the h-diagram). Since the diagram is proportional to $-ig^2 \int d^4x \langle H_1 \bar{H}_2 H_x \bar{H}_x \phi_x \bar{\phi}_x \rangle$, the symmetry factor is $S_g = 1$. Thus

$$\begin{aligned} i\mathcal{M}_g &= (-ig^2) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{k^2 - m_\phi^2} = g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2 - m_\phi^2} \\ &\simeq g^2 \int \frac{k^3 \mathrm{d}k \mathrm{d}\Omega_{S^3}}{(2\pi)^4} \frac{1}{k^2} \simeq g^2 \frac{2\pi^2}{16\pi^4} \frac{\Lambda_{UV}^2}{2} = \frac{g^2}{16\pi^2} \Lambda_{UV}^2. \end{aligned}$$

The trickier to compute is the λ-diagram: The diagram is proportional to

$$\frac{(-i\lambda)^2}{2!} \int d^4x d^4y \langle H_1 \bar{H}_x \bar{\psi}_x \psi_x H_y \bar{\psi}_y \psi_y \bar{H}_2 \rangle,$$

hence $S_\lambda = 1/2$. Now

$$\begin{aligned} i\mathcal{M}_\lambda &= -\frac{(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{Tr} \left[\frac{i(\not{k} + m_\psi)}{k^2 - m_\psi^2} \frac{i(\not{p} + sk + m_\psi)}{(p+k)^2 - m_\psi^2} \right] \\ &= -\frac{\lambda^2}{2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\not{k}\not{p} + \not{k}\not{k} + m_\psi(\not{k} + \not{p}) + m_\psi^2 \right]}{(k^2 - m_\psi^2)((p+k)^2 - m_\psi^2)} \\ &= -2\lambda^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k \cdot p + k^2 + m_\psi^2}{(k^2 - m_\psi^2)((p+k)^2 - m_\psi^2)}. \end{aligned}$$

Keeping only quadratic divergences, we have

$$i\mathcal{M}_\lambda \simeq 2\lambda^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^2}{k^4} = 2\lambda^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^3 \mathrm{d}k \mathrm{d}\Omega_{S^3}}{k^2} \simeq -2\lambda^2 \frac{2\pi^2}{16\pi^4} \frac{\Lambda_{UV}^2}{2} = -\frac{\lambda^2}{8\pi^2} \Lambda_{UV}^2.$$

Hence the over all correction is

$$i\mathcal{M}_g + i\mathcal{M}_h + i\mathcal{M}_\lambda = \left(\frac{g^2}{16\pi^2} + \frac{h^2}{16\pi^2} - \frac{\lambda^2}{8\pi^2} \right) \Lambda_{UV}^2.$$

Now the important point. One can show that if $\mathfrak{L}$ respects SUSY, then the constants need to saisfy

$$g^2 = h^2 = \lambda^2.$$

Hence the quadratic divergence nicely cancels. This is an elegant solution to the hierarchy problem, because the cancellation of quadratic divergence is enforced by a spacetime symmetry, hence is natural compared to fine tuning.

Unfortunately, overall SUSY requires these constant to be proportional to the masses of the fields interacting with the Higgs field. This means that the masses of particles in the standard model and their superpartners must have identical masses for SUSY to cancel off the quadratic divergence exactly. But if this were the case, then the energy level required to find superpartners of these particles should be the same as that required to find the particles in the standard model. However, physicists haven't found any evidence that these superpartners exist in the particle accelerators. This means SUSY is weakly violated at best. The higher the difference of masses between particles and their superpartners, the less effective SUSY is to cancel off the quadratic divergences in the MSSM model.

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