THE ATIYAH-SINGER INDEX THEOREM FOR DIRAC OPERATORS: A SUPERGEOMETRY AND FUNCTIONAL INTEGRAL PERSPECTIVE

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Preface

The goal of this thesis is to present a heuristic yet brilliant argument by Edward Witten that recovers the celebrated Atiyah-Singer index formula for Dirac operator on spin manifold, using the techniques of path integral localization. The idea is that the index of the Dirac operator can be identified with the partition function of a supersymmetric quantum mechanical system on the spin manifold. Computing the partition function using a localized functional integral over the superspace of bosonic and fermionic paths to the manifold yields the integral of the $\hat{A}$-class after renormalization. Although this is not a rigorous proof, as the mathematical foundation of path integrals in quantum field theory is still missing, the author thinks this topic is worth studying, as similar techniques also allowed Witten to recover many other beautiful results from geometry and topology, including his Fields medal winning work on Jones polynomials, and even inspire new mathematics such as Gromov-Witten theory and mirror symmetry.

Chapter 1 is a short introduction to supermanifolds. These are appropriate backgrounds for studying supersymmetric quantum mechanics and supersymmetric field theories. Supermanifold is a deep and intriguing topic that is worth writing an thesis on, but we will only collect the most relevant fact for our purposes.

Chapter 2 is really a lesson from quantum mechanics and quantum field theory. We introduce the important notion of functional integrals over all paths into a space, and its fermionic generalization. Path integral takes many forms, and they are central objects in quantum field theory. However, in this thesis, we are only going to study Gaussian path integrals, since this is the only type that we will need in Witten's proof of the index theorem. Fortunately, these are also the only type of path integrals that one can directly evaluate.

In Chapter 3, we study the mathematical aspects of the index theorem. Although the spirit of this thesis is to primarily rely on physical intuitions and less on rigorous proofs, it would be a pity if we miss the beautiful mathematical sides of the story. I'm sure the readers will appreciate the mathematical richness of the Atiyah-Singer index theory after reading this chapter, as topology, geometry, analysis, and algebra all merge together elegantly to form the backbone of the index theory. We aim to present all the relevant mathematical background so that we can understand the statement of the index theorem. A fully rigorous mathematical proof, however, is beyond the scope of this thesis. Hence, we will just present the statement of the theorem, and see several nice applications of it.

Finally, the real story starts in Chapter 4. We start off by introducing supersymmetric quantum mechanics (SUSYQM) in flat and curved space. This will give us the correct Lagrangian for the SUSYQM system on the spin manifold that Witten was motivated to consider. In the next section we will reveal the connection between this SUSYQM system with supergeometry, where our preparations in Chapter 1 will turn out to be extremely useful. After that, we introduce the set up of the problem. The last section identifies the index of the Dirac operator with the Witten index, which can be calculated by the path integral technique, as we will see. The remaining part of this thesis will be dedicated to the computation of the path integral, from which the index theorem follows directly.

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Contents

Chapter 1

Supermanifolds

1.1 Basic Definitions

One can define a C^∞-manifold M in two ways: First is the usual way of defining to be a topological space that is locally diffeomorphic to $\mathbb{R}^n$, and with a set of chart transition maps satisfying C^∞-compatibility condition. However, one can also note that the smooth manifold M comes with a sheaf of smooth functions $\mathcal{O}_M$ on M, which algebraic geometers like to refer to as the structure sheaf. Then one can define a C^∞-manifold to be a locally ringed space that is locally isomorphic to the model space $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. The first approach is more intuitive, however the second approach is easier to generalize to supermanifold, which we will define below.

Definition 1.1 (supercommutativity)**.** A $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ is said to be **supercommutative** if $\alpha\beta = (-1)^{|\alpha|\cdot|\beta|}\beta\alpha$.

Definition 1.2 (split supermanifold)**.** Now let M be a manifold and V a vector bundle over M. Then we define the **split supermanifold** $S := S(M, V)$ to be the locally ringed space $(M, \mathcal{O}_S)$, where $\mathcal{O}_S := \mathcal{O}_M \otimes \bigwedge^\bullet V^*$ is the sheaf of $\mathcal{O}_M$-valued sections of the exterior algebra $\bigwedge^\bullet V^*$. In other words,

$$S(M, V) = (M, \mathcal{O}_M \otimes \bigwedge^\bullet V^*).$$

If we interpret V as a locally free sheaf of $\mathcal{O}_M$-modules, then $\mathcal{O}_S$ is simply $\bigwedge^\bullet V^*$. Then $\mathcal{O}_S$ is $\mathbb{Z}_2$-graded (as there is a natural notion of eveness and oddness in the exterior algebra), and supercommutative, and its stalks are local rings.

Example 1.3 (examples of split supermanifolds)**.** The **affine superspace** $\mathbb{A}^{m|n}$ is defined by

$$(\mathbb{A}^m, \mathcal{O}_{\mathbb{A}^{m|n}}) = S(\mathbb{A}^m, \mathcal{O}_{\mathbb{A}^m}^{\oplus n}),$$

where for M a manifold $V = \mathcal{O}_M^{\oplus n}$ is the trivial rank n bundle on it.

Here the affine space $\mathbb{A}^n = \mathbb{A}^n_K$ depends on the choice of the base field K. In our case, we almost always consider either $K = \mathbb{R}$ or $K = \mathbb{C}$.

More specifically, let us consider the case where $K = \mathbb{R}$. Then $S(\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^m}^{\oplus n})$ is just the smooth manifold $\mathbb{R}^m$, with a sheaf of supercommutative ring

$$\mathcal{O} : U \mapsto C^\infty(U)[\theta^1, ..., \theta^n],$$

that sends U to the Grassmann algebra with generators $\theta^1, ..., \theta^n$ over $C^\infty(U) = \mathcal{O}_{\mathbb{R}^m}(U)$. This is the structure sheaf. An element of the structure sheaf looks like $f = \sum_I f_I \theta^I$ where $I = (i_1, ..., i_k) \subset \{1, ..., n\}$ is an index set and $\theta^I = \theta^{i_1}\theta^{i_2} \cdots \theta^{i_k}$. We will call this model space $\mathbb{R}^{m|n}$. Thus $\mathbb{R}^{m|n}$ has even coordinates $x^1, ..., x^m$ and odd coordinates $\theta^1, ..., \theta^n$. For brevity, we will usually write an element f in the structure sheaf as $f(x^\mu, \theta^\nu)$, and we will denote the structure sheaf $\mathcal{O}$ by $\mathcal{O}_{\mathbb{R}^{m|n}}$ if we want to be specific.

Similarly, one can define $\mathbb{C}^{m|n}$.

Definition 1.4. A **supermanifold** of dimension $m|n$ is a locally ringed space $(S, \mathcal{O}_S)$ that is locally isomorphic to some model space $S(M, V)$, where $\dim M = m$ and $\operatorname{rank} V = n$. We will usually just refer to this supermanifold as S with the understanding that it also comes with the structure sheaf $\mathcal{O}_S$. The manifold M is called the **reduced space** of S and we usually write $|S| = M$.

In particular, a real supermanifold of dimension $m|n$ is a locally ringed space locally isomorphic to $(\mathbb{R}^{m|n}, \mathcal{O}_{\mathbb{R}^{m|n}})$, and a complex supermanifold of dimension $m|n$ is a locally ringed space locally isomorphic to $(\mathbb{C}^{m|n}, \mathcal{O}_{\mathbb{C}^{m|n}})$.

Remark 1.5. In the C^0, C^1, C^∞, or analytic settings, we may as well restrict our local models to affine superspaces $\mathbb{A}^{m|n}$. But in the algebraic setting we muts allow all $S(M, V)$ with affine M, as in the bosonic algebraic case.

Remark 1.6 (morphisms of supermanifolds)**.** In the definition above, by an isomorphism we mean an isomorphism of locally ringed spaces. Recall that a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a map $f : X \to Y$ between topological spaces together with a morphism of sheaves $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$, where $f_*\mathcal{O}_X$ is a sheaf on Y defined to be $\mathcal{O}_X(f^{-1}(U))$ for any open set $U \subset Y$. That is, every morphism f of locally ringed space comes with the data of a ring homomorphism $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, called **pullback** on U. With this notion of morphism, supermanifolds form a category.

It is also important to note that the isomorphisms above are isomorphisms of $\mathbb{Z}_2$-graded algebras, but they need not preserve the $\mathbb{Z}$-grading of $\bigwedge^\bullet V^*$. For example, let z be a coordinate on M and θ^i are the fiber coordinates of V. Then z cannot go to $z + \theta_1$ under such isomorphisms, since this changes parity, but it can go to $z + \theta^1\theta^2$.

Theorem 1.7. *Let $f : S \to S'$ be a morphism between supermanifolds. Choose a local coordinate $(x^1, ..., x^m, \theta^1, ..., \theta^n)$ on $U \subset S$ and $(y^1, ..., y^s, \chi^1, ..., \chi^\ell)$ a local coordinates on an open subset containing $f(U) \subset S'$. Then f has the local expression*

$$f(x^\mu, \theta^\nu) = (f^*y^1, ..., f^*y^s, f^*\chi^1, ..., f^*\chi^\ell)$$

where $f^* : \mathcal{O}_{S'} \to f_*\mathcal{O}_S$ is the morphism on locally ringed spaces. Conversely, the information above on all local charts completely determines $f : |S| \to |S'|$ and $f_* : \mathcal{O}_{S'} \to f_*\mathcal{O}_S$.

Proof. To obtain the local expression, note that y^α, χ^β are the (even or odd) coordinate functions on S'. Since the local expression of f is obtained by postcomposing f with the coordinate functions, the theorem follows. Note that in the case that the odd dimensions are zero, we reduce to the case of a map between smooth manifolds. □

Remark 1.8. In the familiar case where $f : M \to M'$ is simply a smooth map between manifolds, we have similar results, where if we take local coordinates $x^1, ..., x^m$ and $y^1, ..., y^n$ containing $U \subset M$ and $f(U) \subset M'$ then we have

$$f(x^1, ..., x^m) = (f^*y^1, ..., f^*y^n).$$

Hence we can also define manifolds as locally ringed space isomorphic to the Euclidean model. However, note that in the case of supermanifolds, there are way more morphisms than ordinary manifolds, because a morphism can mix even and odd part as long as it preserves grading. For example, the map $\varphi : \mathbb{R}^{1|2} \to \mathbb{R}^{1|2}$ defined by

$$\varphi(x, \theta^1, \theta^2) = (x + \theta^1\theta^2, \theta^1, \theta^2)$$

is an isomorphism. But note that $\varphi^* x = x + \theta^1\theta^2$, so it need not be true that in the above expression $f^* y^j$ only involves even coordinates, it can also depends on odd coordinates.

Remark 1.9. Given a supermanifold $(S, \mathcal{O}_S)$, its structure sheaf $\mathcal{O}_S$ contains an ideal J consisting of all nilpotents. This is the ideal generated by all odd functions. Note that the underlying topological space of S is already given, which we call $M = |S|$. To recover the structure sheaf $\mathcal{O}_M$, we note that $\mathcal{O}_M = \mathcal{O}_S/J$. The dual bundle V^* can also be recovered because $V^* = J/J^2$. Hence from the supermanifold $S = S(M, V)$ we can recover the data of S and V. In other words, a supermanifold S determines its reduced space.

Note that the definition of a supermanifold $(S, \mathcal{O}_S)$ gives $\mathcal{O}_S$ the structure of a sheaf of (real or complex) algebras, and also with a surjective homomorphism to $\mathcal{O}_M$ by setting $\theta^\nu = 0$ for all odd coordinates. In other words, $\mathcal{O}_M = \mathcal{O}_S/J$ induces an exact sequence of sheaves

$$0 \to J \to \mathcal{O}_S \to \mathcal{O}_M \to 0.$$

However, this does not endow $\mathcal{O}_S$ the structure of $\mathcal{O}_M$-modules, because although there are open sets U on which $\mathcal{O}_S$ is isomorphic to $\mathcal{O}_M(U)[\theta^1, ..., \theta^n]$, and multiplication by elements in $\mathcal{O}_M(U)$ is commutative there, but it need not commute for larger open sets. In other words, multiplication by $\mathcal{O}_M$ need not to commute with gluing isomorphisms.

Definition 1.10. A **sheaf** $\mathcal{F}$ on a supermanifold S is simply a sheaf on the reduced space M, and sheaf cohomology $H^*(S, \mathcal{F})$ of $\mathcal{F}$ on S is simply the sheaf cohomology $H^*(M, \mathcal{F})$ on the reduced space.

Remark 1.11. Despite the fact that sheaves and cohomology are defined to be the same as the reduced space, it is usually more illuminating to denote the cohomology as $H^*(S, \mathcal{F})$ instead of $H^*(M, \mathcal{F})$. For example, it is much more natural to think of a sheaf of $\mathcal{O}_S$-modules as a sheaf on S rather than a sheaf on M.

Example 1.12 (tangent bundle)**.** The **tangent bundle** T_S of supermanifold S is an example of a sheaf of $\mathcal{O}_S$-modules. It can be defined in terms of derivations, i.e., as sheaves

$$T_S := \mathrm{Der}(\mathcal{O}_S).$$

It is a $\mathbb{Z}_2$-graded vector bundle, or a locally free sheaf of $\mathcal{O}_S$-modules.

In the simplest example where $S = \mathbb{A}^{m|n}$, TS is the free $\mathcal{O}_S$-module generated by even tangent vectors $\frac{\partial}{\partial x^\mu}$ for $\mu = 1, ..., m$ and odd tangent vectors $\frac{\partial}{\partial \theta^\nu}$ for $\nu = 1, ..., n$.

Remark 1.13. In general, for a split supermanifold $S = S(M, V)$, the tangent bundle T_S need not have distinguished complementary even and odd subbundles: The even and odd parts are not sheaves of $\mathcal{O}_S$ modules. For example, $\frac{\partial}{\partial \theta^1}$ is an odd vector field, but multiplication by an element of $\mathcal{O}_S$ might change parity: Consider $\theta^1 \frac{\partial}{\partial \theta^1}$. This is now an even vector field. However, we note that since elements in $\mathcal{O}_M$ are even, if we restrict ourselves to multiplying only elements in $\mathcal{O}_M$ then parity is preserved. More specifically, what we are saying is that the restriction $T_S|_M$ of T_S to the reduced space M does split (By restriction to M we mean pullback, under the natural inclusion $i : M \to S$, of a sheaf of $\mathcal{O}_S$-modules to a sheaf of $\mathcal{O}_M$-modules. In this case, this is accomplished by setting all odd functions to 0). Explicitly, the splitting is given by

$$T_{S,+} := T_M,$$
$$T_{S,-} := V.$$

For a general supermanifold S, we only have local isomorphisms between S and $S(M, V)$ for some model space M and vector bundle V, but there is no canonical identification $T_{S,-} = V$. However, there is still a way to define $T_{S,-}$. We consider the natural embedding $M \to S$, where we view M as the subspace that is locally given by $\theta^\alpha = 0$ for all α. Then then we have a normal bundle sequence

$$0 \to T_M \to T_S|_M \to N_{M/S} \to 0.$$

Now $N_{M/S}$ is a purely odd bundle. Therefore, we may define $T_{S,+} = T_M$ and $T_{S,-} = N_{M/S}$.

1.2 Examples of Supermanifolds

Example 1.14. The simplest example is the affine superspace $\mathbb{A}^{m|n}$, which we already saw. In the algebraic case, the global function ring is

$$\mathcal{O}_{\mathbb{A}^{m|n}}(\mathbb{A}^{m|n}) = K[x_1, ..., x_m, \theta_1, ..., \theta_n]$$

is just the polynomial ring in m commuting variables and n anticommuting variables. In the C^0, C^1, C^∞ or analytic case we allow the even part of the function to be C^0, C^1, C^∞ or analytic, but the anticommuting part is unchange, still the polynomial ring in θ_j's.

Example 1.15. The complex **projective superspace** $\mathbb{P}^{m|n}$ is defined by the quotient

$$\mathbb{P}^{m|n} = \{(x^0, x^1, ..., x^m, \theta^1, ..., \theta^n) \in \mathbb{A}^{m+1|n} : x^\mu \neq 0 \text{ for some } \mu\}/\sim_{\mathbb{C}^*}$$

where $(x^\mu, \theta^\nu) \sim_{\mathbb{C}^*} (x^{\mu'}, \theta^{\nu'})$ iff $x^\mu = \lambda x^{\mu'}$ and $\theta^\nu = \lambda\theta^{\nu'}$ for some $\lambda \in \mathbb{C}^*$. Note that the group $\mathbb{C}^*$ is purely even, i.e., λx^μ is even and $\lambda\theta^\nu$ is odd for all $\lambda \in \mathbb{C}^*$.

The complex projective superspace are covered by local charts of open sets

$$U_\alpha = \{(x^0, ..., x^m, \theta^1, ..., \theta^n) \in \mathbb{P}^{m|n} : x^\alpha \neq 0\}$$

for $\alpha = 0, ..., m$. Each U_α can be identified with affine superspace $\mathbb{A}^{m|n}$ via the ratios $x^\beta/x^\alpha, \beta \neq \alpha$, and θ^ν/x^α, for $\nu = 1, ..., n$.

Note that for $m = 0$ there is a unique open set U_α, so $\mathbb{P}^{0|n} \cong \mathbb{A}^{0|n}$.

Example 1.16 (submanifolds)**.** Given a supermanifold, one can construct super submanifolds by imposing equations. For example in $\mathbb{P}^{m|n}$ we can impose the equation $f(z^0, ..., z^m, \theta^1, ..., \theta^n) = 0$ where f is a homogeneous polynomial in the homogeneous coordinate of $\mathbb{P}^{m|n}$ that is either even or odd. If f is even and sufficiently generic, this will give a complex supermanifold of dimension $m-1|n$. For suitable odd f, this gives a supermanifold of dimension $m|n-1$. A **divisor** is a super submanifold of *codimension* $1|0$, i.e., when the defining polynomial is even.

Example 1.17 (Parity Reversed Tangent Bundle ΠTM)**.** Let M be a manifold of dimension p and $E \to M$ a vector bundle of rank q. Then there is an associated supermanifold ΠE of dimension

$p|q$ with $|\Pi E| = M$, defined by revering the parity of the fiber coordinates. Namely, for every local chart $U \subset M$, we note that the space $C^\infty(U, E)$ of smooth sections of E over U is a free $C^\infty(U)$-module generated by q sections $\sigma_1, ..., \sigma_q$. We now identify these sections with grassmann variables $\theta^1, ..., \theta^q$, and define the structure sheaf

$$\mathcal{O}_{\Pi E}(U) = C^\infty(U)[\theta^1, ..., \theta^n].$$

In the particular case where $E = TM$, we have the **parity reversed tangent bundle** ΠTM. Let $C^\infty(\Pi TM)$ be the smooth global sections of the structure sheaf $\mathcal{O}_{\Pi TM}$, namely the superalgebra of smooth functions on ΠTM. Then there is an isomorphism

$$\rho : \mathcal{A}^\bullet(M) \xrightarrow{\cong} C^\infty(\Pi TM) \tag{1.1}$$

between smooth functions on ΠTM and differential forms on M, where we identify

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} a_{i_1 \ldots i_p}(x) \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \in \mathcal{A}^p(M)$$

with

$$\rho(\omega) = \sum_{1 \le i_1 < \cdots < i_p \le n} a_{i_1 \ldots i_p}(x) \theta^{i_1} \cdots \theta^{i_p} \in C^\infty(\Pi TM).$$

Using the identification (1.1), we can integrate smooth functions on ΠTM by defining

$$\int_{\Pi TM} \rho(\omega) \mathrm{d}x^p \cdots \mathrm{d}x^p \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^p := \int_M \omega$$

where $\mathrm{d}x^p \cdots \mathrm{d}x^p \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^p$ is the canonical volume form on ΠTM, which we will denote as $\mathrm{d}x\mathrm{d}\theta$ for short.

Chapter 2

Path Integral Techniques

Thirty-one years ago Richard Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes', he said. 'It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function.' I said to him, 'You're crazy'. But he wasn't. – F. J. Dyson

2.1 Path Integrals in Quantum Mechanics

Perhaps the best way to introduce the path integral formalism is by starting with the double slit experiment. A particle emitted from a source at time $t = 0$ passes through one or the other of two holes drilled in a board and is detected at time $t = T$ by a detector located on the screen, as shown in Fig.2.1. The amplitude for detection is given by a fundamental postulate of quantum mechanics, the superposition principle, as the sum of the amplitude for the particle to propagate from the source through one hole and then onward to the detector and the amplitude for the particle to propagate from the source through the other hole and then onward to the detector. There is an obvious generalization to this situation: If we drill more than two holes, say $A_1, ..., A_n$ of the holes, and we want to measure the amplitude $\mathcal{A}_{S,O}$ for the particle to propagate from the source S to the detector O, it should be given by

$$\mathcal{A}_{S,O} = \mathcal{A}_{S,A_1,O} + \mathcal{A}_{S,A_2,O} + \cdots + \mathcal{A}_{S,A_n,O},$$

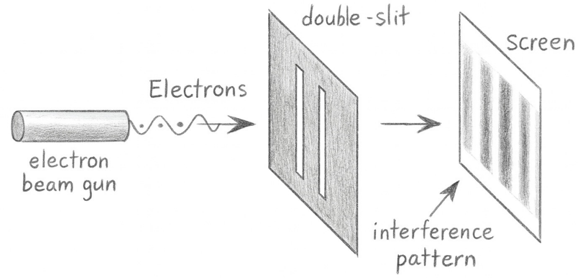

Figure 2.1: The double slit experiment

where $\mathcal{A}_{S,A_j,O}$ is the amplitude of the particle starts from S, passes A_j, then ending up at O. If we drill so many holes on the board that it feels like the screen is not really there, then we should get the contribution $\mathcal{A}_{S,P,O}$ from every point P on the board. Similarly, if we have a sequence of boards, say $B_1, ..., B_m$ of them, then the amplitude $\mathcal{A}_{S,O}$ should recieve contributions from every

$$\mathcal{A}_{S,P_1,P_2,...,P_m,O}$$

where P_j is an arbitrary point on B_j. Taking $n, m \to \infty$, the amplitude $\mathcal{A}_{S,O}$ should recive contribution from amplitude of every possible path from S to O. This clever thought experiment was due to Richard Feynman, who later formulated his deep insights to the path integral formalism.

To understand the path integral formalism, we have to be more precise about how do we interpret $\mathcal{A}_{S,O}$. The symbol is not totally precise because we didn't consider the time in which detection is made. Suppose we want to express the amplitude of a particle started at an initial position q_I at time t_I, and ended up at a final position q_F at time t_F. Let $|q_I\rangle$ denote the position eigenstate at q_I, and similarly define $|q_F\rangle$. Suppose the particle is governed by the Hamiltonian H. Then if we denote $T = t_F - t_I$, after we waited T units of time, the particle started off as $|q_I\rangle$ would have been in the state $\mathrm{e}^{-iHT}|q_I\rangle$ by Schrödinger equation, and the amplitude for it to end up at q_F at time t_F is given by the inner product

$$\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle,$$

which is the quantity we are interested in evaluating. Now we use Feynman's idea described above to slice up the path into small intervals. Namely, we divide T into N segments by defining $\delta t = T/N$.

Thus

$$\langle q_F | \, \mathrm{e}^{-iHT} \, |q_I\rangle = \langle q_F | \, \mathrm{e}^{-iH\delta t} \mathrm{e}^{-iH\delta t} \cdots \mathrm{e}^{-iH\delta t} \, |q_I\rangle \, .$$

Since the position eigenstates are complete, we insert a bunch of identity operators of the form $\int \mathrm{d}q_j |q_j\rangle\langle q_j| = 1$. So now we have that

$$\begin{aligned} &\langle q_F|\mathrm{e}^{-iHT}|q_i\rangle \\ &= \left(\prod_{j=1}^{N-1} \int \mathrm{d}q_j\right) \langle q_F| \, \mathrm{e}^{-iH\delta t} \, |q_{N-1}\rangle \, \langle q_{N-1}| \, \mathrm{e}^{-iH\delta t} \, |q_{N-2}\rangle \cdots \langle q_1| \, \mathrm{e}^{-iH\delta t} \, |q_I\rangle \, . \end{aligned}$$

Let us focus on one factor $\langle q_{j+1}|\mathrm{e}^{-iH\delta t}|q_j\rangle$. Suppose first that our Hamiltonian is just given by the Hamiltonian of a free particle $H = \hat{p}^2/2m.$, since the general case $H = \hat{p}^2/2m + V(\hat{x})$ with potential is similar. Inserting the identity using momentum eigenstates $\int \frac{\mathrm{d}p}{2\pi} |p\rangle\langle p| = 1$ and using $\langle q|p\rangle = \mathrm{e}^{ipq}$ gives

$$\begin{aligned} \langle q_{j+1}| \, \mathrm{e}^{-i\delta t(\hat{p}^2/2m)} \, |q_j\rangle &= \int \frac{\mathrm{d}p}{2\pi} \, \langle q_{j+1}| \, \mathrm{e}^{-i\delta t(\hat{p}^2/2m)} |p\rangle \, \langle p|q_j\rangle \\ &= \int \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{-i\delta t(p^2/2m)} \, \langle q_{j+1}|p\rangle \, \langle p|q_j\rangle \\ &= \int \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{-i\delta t(p^2/2m)} \mathrm{e}^{ip(q_{j+1}-q_j)}. \end{aligned}$$

The integral over p is an Gaussian integral, which can be evaluated. Doing this integral yields

$$\begin{aligned} \langle q_{j+1}| \, \mathrm{e}^{-i\delta t(\hat{p}^2/2m)} \, |q_j\rangle &= \left(\frac{-im}{2\pi\delta t}\right)^{1/2} \mathrm{e}^{[im(q_{j+1}-q_j)^2]/2\delta t} \\ &= \left(\frac{-im}{2\pi\delta t}\right)^{1/2} \mathrm{e}^{i\delta t(m/2)[(q_{j+1}-q_j)/\delta t]^2}. \end{aligned}$$

Putting all these pieces back we get

$$\langle q_F| \, \mathrm{e}^{-iHT} \, |q_I\rangle = \left(\frac{-im}{2\pi\delta t}\right)^{N/2} \left(\prod_{k=1}^{N-1} \int \mathrm{d}q_k\right) \mathrm{e}^{i\delta t(m/2) \sum_{j=0}^{N-1} [(q_{j+1}-q_j)/\delta t]^2},$$

where $q_0 = q_I, q_N = q_F$. Now we take the limit where $N \to \infty$, or equivalently where $\delta t \to 0$. Then the exponent in exponential above becomes $\int_0^T \mathrm{d}t \frac{1}{2} m\dot{q}^2$. Thus the formula above becomes

$$\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle = \int \mathcal{D}q \mathrm{e}^{i\int_0^T \mathrm{d}t \frac{1}{2} m\dot{q}^2}$$

where we define

$$\int \mathcal{D}q = \lim_{N\to\infty}\left(\frac{-im}{2\pi\delta t}\right)^{N/2}\prod_{k=1}^{N-1}\int \mathrm{d}q_k = \lim_{N\to\infty}\left(\frac{-imN}{2\pi T}\right)^{N/2}\prod_{k=1}^{N-1}\int \mathrm{d}q_k.$$

The integral with respect to dq_k above should be interpreted as integrating over all points on a screen in which the particle passes at time $t = k\delta t$. So what we are doing is assigning a weight of $\mathrm{e}^{i\int_0^T \mathrm{d}t \frac{1}{2} m\dot{q}^2}$ to each path $q(t)$ that the particle can possibly take and integrate over all such paths. However, everything we have done here is at a formal level, and immediately we see two concerning things from the formula above:

The first thing is that the formula above is just nonsense. There is a constant in front that diverges at $N \to \infty$. However, what physicists usually do is to calculate time ordered correlation function (as we will see later), which is the ratio of two path integrals. Hence, this infinite constant will be canceled because both the numerator and the denominator will contribute the same diverging constant.

The second thing is that $\mathcal{D}q$ can not be interpreted as a measure on the space of paths. This is a more mathematical statement, but it is known that in many cases that such measure cannot exist. Therefore, we should interpret the path integral as intergrating over all possible paths only at a heuristic level, while in reality it always means we do the procedure above and consider the limit of infinite product of integrals in small time interval. In fact, the only path integral that can be evaluated exactly are those that are Gaussian, and for non-Gaussian integral we will need to use perturbation theory (as we will see later).

As a second step, repeat the derivation above with the Hamiltonian now replaced with its most general form: $\hat{H} = \hat{p}^2/2m + V(\hat{q})$. I claim that the result we will get is

$$\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle = \int \mathcal{D}q \mathrm{e}^{i\int_0^T \mathrm{d}t\left(\frac{1}{2} m\dot{q}^2 - V(q)\right)}. \tag{2.1}$$

We recognize that what is inside the exponential is just i times the action $S = \int_0^T \mathrm{d}t L(q, \dot{q})$. This is the path integral formalism: The amplitude $\langle q_F|\mathrm{e}^{-iHT}|q_I\rangle = \langle q_F, t_F|q_I, t_I\rangle$ recieves contribution from all possible path $q(t)$ from q_I to q_F, each weighed by a factor of $\mathrm{e}^{iS(q,\dot{q})}$.

We will prove (2.1) shortly, but let us discuss first why the path integral formalism is more favourable compared to the Schrödinger picture that we have developed. This formalism is highly advantages because classical solutions emerges naturally in the limit $\hbar \to 0$. Note that in the derivation above (in fact in pretty much every other places of this writing), we assumed $\hbar = 1$. If we restore the unit in (2.1), the actually formula should read

$$\langle q_F|\mathrm{e}^{-iHT/\hbar}|q_I\rangle = \int \mathcal{D}q\mathrm{e}^{iS(q,\dot{q})/\hbar}.$$

When $\hbar \to 0$, the integrand $\mathrm{e}^{iS(q,\dot{q})/\hbar}$ will oscillate rapidly. For any q, if q and its nearby path does not minimize the action $S(q,\dot{q})$, then they will destructively interfere and the contribution from them cancels out with each other. Therefore, the only significant contribution to the amplitude come from those paths that minimizes the action, which are the classical paths by the principle of least action.

To prove (2.1), note that each term in the insertion will be of the form

$$\begin{aligned}
\langle q_{j+1}|\mathrm{e}^{-i\delta t(\hat{p}^2/2m)}\mathrm{e}^{-i\delta t V(\hat{q})}|q_j\rangle &= \int \frac{\mathrm{d}p}{2\pi}\langle q_{j+1}|\mathrm{e}^{-i\delta t(\hat{p}^2/2m)}|p\rangle\langle p|\mathrm{e}^{-i\delta t V(\hat{q})}|q_j\rangle \\
&= \int \frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-i\delta t(p^2/2m)}\langle q_{j+1}|p\rangle\langle p|q_j\rangle\mathrm{e}^{-i\delta t V(q_j)} \\
&= \mathrm{e}^{-i\delta t V(q_j)}\left(\frac{-im}{2\pi\delta t}\right)^{1/2}\mathrm{e}^{i\delta t(m/2)[(q_{j+1}-q_j)/\delta t]^2} \\
&= \left(\frac{-im}{2\pi\delta t}\right)^{1/2}\mathrm{e}^{i\delta t\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 - V(q_j)\right)}.
\end{aligned}$$

So it follows that when we multiply all the insertion, and take limit as $N \to \infty$, we would get (2.1).

Example 2.1 (Gaussian integral in one variable)**.** As a warm up, in the finite dimensional case, we consider the integral

$$\int \mathrm{d}p\ \mathrm{e}^{-\alpha p^2 + Jp}.$$

Recalling that $\int \mathrm{d}p\ \mathrm{e}^{-\alpha p^2} = \sqrt{\pi/\alpha}$, we just need to complete the square, and the above expression becomes

$$\int \mathrm{d}p\, \mathrm{e}^{-\alpha\left(p-\frac{J}{2\alpha}\right)^2+\frac{J^2}{4a}} = \sqrt{\frac{\pi}{\alpha}}\mathrm{e}^{J^2/4\alpha}.$$

A somewhat equivalent method is called the method of stationary phase. Taking the derivative of the exponent to find the critical point:

$$0 = \frac{\partial}{\partial p}(-\alpha p^2 + Jp) = 2\alpha p + J$$

gives the critical point

$$p_* = \frac{J}{2\alpha}.$$

Then up to the constant $\sqrt{\pi/\alpha}$, we see that

$$\int \mathrm{d}p\, \mathrm{e}^{-\alpha p^2+Jp} \sim \mathrm{e}^{-\alpha p_*^2+Jp_*} = \mathrm{e}^{J^2/4\alpha}.$$

Example 2.2 (Gaussian integral in finite dimension)**.** Suppose now J be a n-dimensional vector, and A an $n \times n$ matrix. We now consider the n-dimensional integral

$$\int \mathrm{d}^n p \mathrm{e}^{-\frac{1}{2}p^\dagger Ap+J^\dagger p}.$$

After diagonalizing A the integral becomes just a product of integrals over the p_i, and the result is the product of one-dimensional Gaussian integrals, with a being replaced by an eigenvalue of A. That is,

$$\int \mathrm{d}^n p \mathrm{e}^{-\frac{1}{2}p^\dagger Ap+J^\dagger p} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}\mathrm{e}^{\frac{1}{2}J^\dagger A^{-1}J}. \tag{2.2}$$

Any path integrals that we can directly evaluate will be the infinite dimensional generalization of (2.2). That is, they will be of the form

$$\int \mathcal{D}\xi \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}t(\xi A\xi)}$$

where ξ runs over all paths that maps to a given space. In fact, we will not even need the exact value of this path integral (because the formally infinite constant can always be renormalized using techniques in quantum field theory). All we need to remember is that according to (2.2), the answer will be of the form

$$\int \mathcal{D}\xi \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}t(\xi A \xi)} \propto (\det A)^{-1/2}. \tag{2.3}$$

We will use this result in the proof of the Atiyah-Singer index theorem.

2.2 Path Integrals and Partition Function

The Green's functions of quantum theory are related to time-evolution operators, which (roughly speaking) look like a complex exponential e^{-iEt}, with E an energy and t the time. The probabilities (and density matrices) of statistical physics are based on the Boltzmann factor, given (roughly) by the real exponential $\mathrm{e}^{-E/T}$, where T is the temperature and k_B is the Boltzmann constant. The fundamental elements of the two theories are therefore both exponentials, albeit one is real and the other is complex. This similarity can be exploited by making the identification $it = T^{-1}$. The claim is that this will map a quantum field theory on to a statistical field theory. The advantage of this mapping is that it allows us to transfer results from one subject directly to the other, and vice versa. Within the framework of the mapping, inverse temperature $\beta = 1/T$ in statistical physics behaves like imaginary time in a quantum field theory. It turns out that the concept of imaginary time is a remarkably powerful one with uses across quantum and statistical field theory. We examine the imaginary time formalism now. A **Wick rotation** is the substitution

$$\tau = it,$$

i.e., allowing imaginary time. With this substitution, we replace the Minkowski metric $(+,-,-,-)$ to the Euclidean metric $(+,+,+,+)$ up to a minus sign. We will see the benefit of this trick very soon.

Arguably the most important object in statistical physics is the **partition function** of a system. Suppose the system is governed by a Hamiltonian H, the partition function is given by

$$Z(\beta) = \operatorname{Tr} \mathrm{e}^{-\beta H}.$$

If we have an complete orthonormal basis of our Hilbert space, say $|\lambda\rangle$ for λ in some index set, then the partition function is expressed by

$$Z(\beta) = \sum_{\lambda} \langle \lambda | \mathrm{e}^{-\beta H} | \lambda \rangle.$$

In particular if these are chosen to be eigenstates of the Hamiltonian, $H \left| E_n \right\rangle = E_n \left| E_n \right\rangle$, then

$$Z(\beta) = \sum_{n} \langle E_n | \mathrm{e}^{-\beta H} | E_n \rangle = \sum_{n} \mathrm{e}^{-\beta E_n}.$$

Our next claim is that the partition function is really a path integral in disguise. Note that the expression $\langle \lambda | \mathrm{e}^{-\beta H} | \lambda \rangle$ can be interpreted as an amplitude $\langle q_F | \mathrm{e}^{-iHt} | q_I \rangle$ if we choose $|\lambda\rangle$ to be the position eigenstate at $q_I = q_F = \lambda$ (of course then instead of suming over λ in the $Z(\beta)$ expression we should integrate over all position λ), and make the replacement $\beta = it$. Then we have

$$Z(\beta) = \int \mathrm{d}\lambda \langle \lambda | \mathrm{e}^{-iHt} | \lambda \rangle = \int \mathrm{d}\lambda \int \mathcal{D}q \mathrm{e}^{i \int_0^t \mathrm{d}t L(q(t))}.$$

But we already made the replacement $t = -i\beta$. So our limit of integration goes from $t = 0$ to $t = -i\beta$. To make the expression nicer, we apply Wick rotation $t = -i\tau$. After performing change of variable, we end up with

$$Z(\beta) = \int \mathrm{d}\lambda \int \mathcal{D}q \mathrm{e}^{-\int_0^\beta \mathrm{d}\tau L_E(q(\tau))}, \tag{2.4}$$

where we denote the Lagrangian $L(q(t))$ under the change of variable by $L_E(q(\tau))$, the **Euclidean Lagrangian**. For example, if the Lagrangian we started with is the standard Lagrangian

$$L(q(t)) = \frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}t} \right)^2 - V(q),$$

then under the Wick rotation $\mathrm{d}q/\mathrm{d}t = i\mathrm{d}q/\mathrm{d}\tau$, thus

$$\begin{aligned} i \int_0^{-i\beta} \mathrm{d}t L(q(t)) &= i \int_0^\beta -i\mathrm{d}\tau \left[-\frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} \right)^2 - V(q) \right] \\ &= -\int_0^\beta \mathrm{d}\tau \left[\frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} \right)^2 + V(q) \right]. \end{aligned}$$

Hence we see that

$$L_E(q(\tau)) = \frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q).$$

Note that the sign of the kinetic term of L_E is not changed compared to L, but the sign of the potential term is reversed. But there is still more to (2.4). Note that in the formula above it we had to integrate $\langle\lambda|\mathrm{e}^{-iHt}|\lambda\rangle$ where t goes from 0 to $-i\beta$, so τ goes from 0 to β. Since $q_I = q_F = \lambda$, we have $q(t=0) = q(t=-i\beta) = \lambda$. Thus $q(\tau=0) = q(\tau=\beta) = \lambda$. But that's not all. If we imagine τ extending from $-\infty$ to ∞ then the boundary conditions are periodic: after a period β, the state returns to where it started. The integral over λ means that we integrate over all path subject to these periodic boundary conditions (PBC). Thus (2.4) becomes

$$Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta H} = \int_{\mathrm{PBC}} \mathcal{D}q\mathrm{e}^{-\int_0^\beta \mathrm{d}\tau L_E(q(\tau))} \tag{2.5}$$

where PBC stands for the periodic boundary condition with period β. This result will be very useful, because as we will see later, we can write the index of a Dirac operator on spin manifold as a partition function, which can then be computed using the right hand side of (2.5) using path integral techniques, which will lead to the famous Atiyah-Singer index theorem.

2.3 Fermionic Path Integrals

We already know how to integrate all paths that maps onto a manifold from the previous sections. Now we are also interested in integrating paths that maps onto a supermanifold. Unavoidably we will encounter odd maps: That is, we will need to integrate odd sections of the structure sheaf $\mathcal{O}_X$ on a supermanifold X. These are anticommuting fields on supermanifolds, or more commonly refered to by physicists as fermionic fields. For fermionic fields we have $\psi(x_1)\psi(x_2) = -\psi(x_2)\psi(x_1)$.

As in the case of ordinary path integrals, it helps to first look at finite dimensional toy model. Thus we need to define integration over anticommuting variables on a complex manifold will coordinates $\theta_1, ..., \theta_m$. Recall that $\theta_i\theta_j = -\theta_j\theta_i$. A function $f(x,\theta)$ may depends on the position x^μ and Grassmann variables θ^α for $\alpha = 1, ..., m$. We have the nice property when we Taylor expand

$$f(x,\theta) = f_0(x) + \theta_\alpha f^\alpha(x) + \theta_\alpha\theta_\beta f^{\alpha\beta}(x) + \cdots + \theta_1\theta_2\cdots\theta_m f^{1\cdots m}(x).$$

The Taylor series terminates because $\theta_i^2 = 0$. We now introduce calculus of Grassmann numbers. The derivative operator satisfies the usual property:

$$\frac{\partial}{\partial\theta}(\theta) = 1 \qquad \frac{\partial}{\partial\theta}(1) = 0.$$

Integration is more interesting, it is exactly the same as derivatives:

$$\int \mathrm{d}\theta(\theta) = 1 \qquad \int \mathrm{d}\theta(1) = 0.$$

If we have multivariable integration, we require

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n (\theta_n \cdots \theta_1) = 1.$$

We require the measure $\mathrm{d}\theta_i$ and θ_j's to anticommute, so derivative and integration are order sensitive. We can also have conjugation variables $\bar{\theta}_j$'s, just like holomorphic and antiholomorphic coordinates $z_j, \bar{z}_j$. Let us now evaluate an important Grassmann integral:

$$\mathcal{I} = \int \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_1 \cdots \mathrm{d}\bar{\theta}_n \mathrm{d}\theta_n \exp\bigl(-\bar{\theta}_i A_{ij} \theta_j\bigr).$$

Recall that in the bosonic case, we have the Gaussian integral

$$\int \mathrm{d}\bar{z}_1 \mathrm{d}\bar{z}_1 \cdots \mathrm{d}\bar{z}_n \mathrm{d}z_n \exp(-\bar{z}_i A_{ij} z_j) \propto \frac{1}{\det A} \exp\bigl(-J^T A^{-1} J\bigr). \tag{2.6}$$

In fermionic case, we will show that actually $\mathcal{I} \propto \det A$. Our strategy of evaluating $\mathcal{I}$ is to do the Taylor expansion, since it always terminates. Let us expand $\exp\bigl(-\bar{\theta}_i A_{ij}\theta_j\bigr)$. We have

$$\exp\bigl(-\bar{\theta}_i A_{ij} \theta_j\bigr) = 1 - \bar{\theta}_i A_{ij} \theta_j + \cdots + \frac{1}{n!}(-\bar{\theta}_i A_{ij} \theta_j)^n + \mathcal{O}(\theta^{n+1}).$$

Note that the Grassmann integral only picks out the order n term in the Taylor expansion, hence

$$\mathcal{I} = \int \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_1 \cdots \mathrm{d}\bar{\theta}_n \mathrm{d}\theta_n \frac{1}{n!}(-\bar{\theta}_i A_{ij} \theta_j)^n = \sum_{\sigma \in S_n} \frac{1}{n!} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)} \propto \det A.$$

Thus, we conclude that

$$\int \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_1 \cdots \mathrm{d}\bar{\theta}_n \mathrm{d}\theta_n \exp\bigl(-\bar{\theta}_i A_{ij} \theta_j\bigr) \propto \det A. \tag{2.7}$$

Now we do the analogues source term in the bosonic case, by introducing additional Grassmannians $\bar{\eta}_i, \eta_i$'s. We consider the integral

$$\mathcal{I}' = \int \mathrm{d}\bar{\theta}\mathrm{d}\theta \exp\bigl(-\bar{\theta}_i A_{ij}\theta_j + \bar{\eta}_i \theta_i + \bar{\theta}_i \eta_i\bigr),$$

where $\mathrm{d}\bar{\theta}\mathrm{d}\theta$ is the shorthand for $\prod_i \mathrm{d}\bar{\theta}_i \mathrm{d}\bar{\theta}_i$, as before. Using matrix notation, we have

$$\mathcal{I}' = \int \mathrm{d}\bar{\theta}\mathrm{d}\theta \exp\bigl(-(\bar{\theta} - \bar{\eta}A^{-1})A(\theta - A^{-1}\eta) + \bar{\eta}A^{-1}\eta\bigr) = \exp\bigl(\bar{\eta}A^{-1}\eta\bigr) \det A. \tag{2.8}$$

Compare (2.8) with (2.6). Now, (2.7) and (2.8) have direct generalizations to grassmannian path integrals. That is, if we want to integrate over all grassmann-valued paths η, then

$$\int \mathcal{D}\eta \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}t \eta A \eta} \propto \det A. \tag{2.9}$$

This identity will also be used in the proof of the Atiyah-Singer index theorem.

2.4 Fermionic Partition Functions

The fermionic harmonic oscillator is the direct analogue of the bosonic harmonic oscillator. The fermionic oscillator has Hamiltonian

$$H = \frac{1}{2}(c^\dagger c - c c^\dagger)\omega.$$

Recall that the bosonic oscillator hamiltonian is of the same form, with $c^\dagger, c$ replaced by the creation and annihilation operator $a^\dagger, a$, satisfying commutation relations $[a, a^\dagger] = 1$ and $[a, a] = [a^\dagger, a^\dagger] = 0$. Naturally, we would like to require the fermionic analogue of such relation:

$$\{c, c^\dagger\} = 1 \qquad \{c, c\} = \{c^\dagger, c^\dagger\} = 0.$$

Similar to the bosonic case, we can define the **number operator**

$$N = c^\dagger c,$$

which will put the Hamiltonian into simpler forms

$$H = \left(N - \frac{1}{2}\right)\omega.$$

Since $N^2 = c^\dagger c c^\dagger c = N$, the eigenvalue of N must be either 0 or 1. Let $|n\rangle$ denote the eigenvector of H with eigenvalue n, where $n = 0, 1$ as argued earlier. Then it is easy to verify that

$$c^\dagger |0\rangle = |1\rangle \qquad c\,|0\rangle = 0 \qquad c^\dagger |1\rangle = 0 \qquad c\,|1\rangle = |0\rangle\,.$$

Hence it follows that $c^\dagger c\,|n\rangle = n\,|n\rangle$, for $n = 0, 1$. Now consider hilbert space

$$\mathbf{H} = \operatorname{span}\{|0\rangle\,, |1\rangle\}.$$

Now we consider the states

$$|\theta\rangle = |0\rangle + |1\rangle\,\theta \qquad \langle\theta| = \langle 0| + \theta^* \langle 1|\,,$$

where θ and θ^* are Grassmann numbers. These states are called **coherent states** and are eigenstates of $c, c^\dagger$ respectively, since

$$c\,|\theta\rangle = |0\rangle\,\theta = |\theta\rangle\,\theta \qquad \langle\theta|\,c^\dagger = \theta^* \langle 0| = \theta^* \langle\theta|\,.$$

Now we have a very useful completeness relation:

Lemma 2.3. *Let $|\theta\rangle$ and $\langle\theta|$ be defined as above. Then we have*

$$\int \mathrm{d}\theta^* \mathrm{d}\theta\,|\theta\rangle\langle\theta|\,\mathrm{e}^{-\theta^*\theta} = 1. \tag{2.10}$$

Proof. This is just a straightforward computation:

$$
\begin{aligned}
\int \mathrm{d}\theta^* \mathrm{d}\theta \, |\theta\rangle\langle\theta| \, \mathrm{e}^{-\theta^*\theta} &= \int \mathrm{d}\theta^* \ \mathrm{d}\theta(|0\rangle + |1\rangle\theta) \, (\langle 0| + \theta^*\langle 1|) \, (1-\theta^*\theta) \\
&= \int \mathrm{d}\theta^* \ \mathrm{d}\theta \, (|0\rangle\langle 0| + |1\rangle\theta\langle 0| + |0\rangle\theta^*\langle 1| + |1\rangle\theta\theta^*\langle 1|) \, (1-\theta^*\theta) \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \\
&= 1.
\end{aligned}
$$

This completes the proof. □

Now we use the completeness relation to calculate the partition function of fermionic oscillator.

Lemma 2.4. *Let H be the Hamiltonian of a fermionic oscillator. Then the partition function of the system is given by*

$$\mathrm{Tr}\,\mathrm{e}^{-\beta H} = \int \mathrm{d}\theta^*\mathrm{d}\theta\langle -\theta|\mathrm{e}^{-\beta H}|\theta\rangle\mathrm{e}^{-\theta^*\theta}. \tag{2.11}$$

Proof. We insert the completeness relation (2.10) into the definition of a partition function to get

$$
\begin{aligned}
Z(\beta) = \sum_{n=0,1}\langle n|\mathrm{e}^{-\beta H}|n\rangle &= \sum_n \int \mathrm{d}\theta^*\mathrm{d}\theta\mathrm{e}^{-\theta^*\theta}\langle n|\theta\rangle\langle\theta|\mathrm{e}^{-\beta H}|n\rangle \\
&= \sum_n \int \mathrm{d}\theta^*\mathrm{d}\theta(1-\theta^*\theta)(\langle n|0\rangle + \langle n|1\rangle\theta)(\langle 0|\mathrm{e}^{-\beta H}|n\rangle + \theta^*\langle 1|\mathrm{e}^{-\beta H}|n\rangle) \\
&= \sum_n \int \mathrm{d}\theta^*\mathrm{d}\theta(1-\theta^*\theta)[\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle - \theta^*\theta\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle \\
&\qquad + \theta\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle + \theta^*\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle].
\end{aligned}
$$

The last term of the last line does not contribute to the integral, since $\langle n|0\rangle \neq 0$ only when $n=0$, in which case $\langle 1|\mathrm{e}^{-\beta H}|0\rangle = 0$. Hence we may change θ^* to $-\theta^*$ in the last term. Therefore,

$$
\begin{aligned}
Z(\beta) &= \sum_n \int \mathrm{d}\theta^*\mathrm{d}\theta(1-\theta^*\theta)[\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle - \theta^*\theta\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle \\
&\qquad + \theta\langle 0|\mathrm{e}^{-\beta H}|n\rangle\langle n|1\rangle - \theta^*\langle 1|\mathrm{e}^{-\beta H}|n\rangle\langle n|0\rangle] \\
&= \int \mathrm{d}\theta^*\mathrm{d}\theta\mathrm{e}^{-\theta^*\theta}\langle -\theta|\mathrm{e}^{-\beta H}|\theta\rangle.
\end{aligned}
$$

This completes the proof. □

Chapter 3

Mathematical Aspects of the Atiyah-Singer Index Theorem

In this chapter we put together the mathematical backgrounds required to understand the statement of the Atiyah-Singer index theorem for Dirac operators on a spin manifold. The mathematical aspect involves a beautiful combination of analysis, algebraic topology, and Riemannian geometry, which demonstrates the richness of this theorem. Our treatment follows [10] quite closely.

3.1 Clifford Algebras and Spin Groups

The goal of this section is to construct, for every $n \geq 3$, the unique double cover of the rotation group $\mathrm{Spin}_n \to \mathrm{SO}_n$. When $n = 3$, using the standard quaternion argument we have a double cover $\mathrm{SU}(2) \to \mathrm{SO}_3$. This argument does not work for higher dimensions. To generalize this construction, we need to introduce the notion of Clifford algebra.

Definition 3.1. Let V be an (real) inner product space. The **Clifford algebra** $\mathrm{Cl}(V)$ of V is the quotient of the tensor algebra

$$\mathrm{Cl}(V) = \otimes^{\bullet} V / \sim$$

where the relation is $vw + wv = -2\langle v, w\rangle$ for all $v, w \in V$. Of course $\otimes^0 V \cong \mathbb{R}$.

Equivalently, we may choose an orthonormal basis $e_1, ..., e_n$ of V. Then $\mathrm{Cl}(V)$ is the algebra generated by $e_1, ..., e_n$ with the relation

$$e_i^2 = -1 \qquad e_i e_j = -e_j e_i$$

for $i \neq j$. Therefore, we see that as a vector space, we have $\mathrm{Cl}(V) \cong \Lambda^\bullet V$, where the isomorphism is given by $e_{i_1} \cdots e_{i_p} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_p}$. This identification gives $\mathrm{Cl}(V)$ a natural $\mathbb{Z}_2$-grading

$$\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V)$$

which is given by the parity of the number of vectors. Moreover, using the isomorphism with the exterior algebra, the number of vectors gives a grading

$$\mathrm{Cl}(V) = \bigoplus_{k=0}^{n} \mathrm{Cl}^{(k)}(V).$$

Now we study the case where $V = \mathbb{R}^n$ with the standard inner product and the standard orthonormal basis. Then we use shorthand Cl_n to denote $\mathrm{Cl}(\mathbb{R}^n)$. We first study its center $\mathscr{Z}(\mathrm{Cl}_n)$.

Proposition 3.2. If n is even, then $\mathscr{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)}$. If n is odd, then $\mathscr{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)} \oplus \mathrm{Cl}_n^{(n)}$. In any case, $\mathscr{Z}(\mathrm{Cl}_n) \cap \mathrm{Cl}_n^0 = \mathrm{Cl}_n^{(0)}$.

Proof. Clearly $\mathscr{Z}(\mathrm{Cl}_n) = \{a \in \mathrm{Cl}_n : av = va, \forall v \in \mathbb{R}^n\}$. Now let $e_I = e_{i_1} \cdots e_{i_p}$. For $j \neq I$ we have $e_I e_j = (-1)^p e_j e_I$, and if $j \in I$ then $e_I e_j = (-1)^{p-1} e_j e_I$.

Clearly $\mathrm{Cl}_n^{(0)} \subset \mathscr{Z}(\mathrm{Cl}_n)$ for any n. For $1 \leq p \leq n-1$, since if $e_I \in \mathscr{Z}(\mathrm{Cl}_n)$ we must have $e_I e_j = e_j e_I$ for all j, we must have that p is both even and odd, which is impossible. Now if n is even, and $e_I \in \mathrm{Cl}_n^{(n)}$, above calculation shows that $e_I e_j = -e_j e_I$ for all j, hence e_I cannot be in the center. If n is odd, then it is easy to see from above that $\mathrm{Cl}_n^{(n)} \subset \mathscr{Z}(\mathrm{Cl}_n)$. This concludes the proof. $\square$

Now we consider the group $\mathrm{Cl}_n^* \subset \mathrm{Cl}_n$, which is the group of multiplicative invertible elements in Cl_n. If $v \in \mathbb{R}^n$ is a nonzero vector then we have $v^{-1} = -v/\|v\|^2$. Therefore $\mathbb{R}^n \setminus \{0\} \subset \mathrm{Cl}_n^*$. In particular $S^{n-1} \subset \mathrm{Cl}_n^*$.

Definition 3.3. The **pin group** Pin_n is the subgroup of Cl_n^* generated by the unit sphere S^{n-1}. The **spin group** is the even part of the pin group: $\mathrm{Spin}_n := \mathrm{Pin}_n \cap \mathrm{Cl}_n^0$.

We now define an automorphism of Cl_n called **transpose**, namely $(e_{i_1} \cdots e_{i_p})^t = e_{i_p} \cdots e_{i_1}$.

Proposition 3.4. We have that

$$\mathrm{Spin}_n = \{a \in \mathrm{Pin}_n : aa^t = 1\} = \{v_1 \cdots v_p : v_i \in S^{n-1},\ p \text{ even}\}.$$

In particular, $\mathrm{Spin}_n \subset \mathrm{Cl}_n^0$.

Proof. By definition, $a \in \mathrm{Pin}_n$ iff $a = v_1 \cdots v_p$ with $v_i \in S^{n-1}$. Thus $aa^t = (v_1 \cdots v_p)(v_p \cdots v_1) = (-1)^p$, and $aa^t = 1$ iff p is even. □

Now we consider the group homomorphism

$$\rho : \mathrm{Pin}_n \to \mathrm{GL}_n(\mathbb{R}) \qquad \rho(a)v = ava^t.$$

In general, a is a product of vectors, and ava^t is an element in Cl_n. To show that it is an honest vector, we note that for a unit vector $u \in \mathbb{R}^n$, the Clifford relation gives

$$uvu^t = uvu = -v + 2\langle u, v\rangle u$$

which is a linear combination of vectors, hence is a vector. Now $a = u_1 \cdots u_p$ is a product of unit vectors by definition, hence ava^t is a successive conjugate of v by unit vectors, hence is a vector. Therefore $\rho : \mathrm{Pin}_n \to \mathrm{GL}_n(\mathbb{R})$ is indeed well-defined.

Proposition 3.5. $\rho(\mathrm{Pin}_n) = \mathrm{O}_n$.

Proof. We first look at the image of S^{n-1} under ρ. If $a \in S^{n-1}$ and $v \in \mathbb{R}^n$. By choosing a suitable orthonormal basis we may decompose $v = \lambda a + a'$ for some $\lambda \in \mathbb{R}$ and such that $\langle a, a'\rangle = 0$. Then

$$\rho(a)v = ava^t = a(\lambda a + a')a = \lambda a^3 + aa'a = -\lambda a + a'$$

since $aa' = -a'a - 2\langle a, a'\rangle = -a'a$. Therefore, $\rho(a)$ is the reflection with respect to the hyperplane perpendicular to a and the result follows from the classical fact that any element in O_n is a product of reflections. □

Corollary 3.6. $\rho(\mathrm{Spin}_n) = \mathrm{SO}_n$.

Proof. This follows immediately from the previous results, since any element of SO_n is a product of an even number of reflections. $\square$

Theorem 3.7. *If $n \geq 3$, then the map $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is the universal double covering map. In particular,* Spin_n *is compact.*

Proof. By looking at the long exact sequence of homotopy group induced by the fibration

$$\mathrm{SO}_n \to \mathrm{SO}_{n+1} \to S^n$$

one can compute that $\pi_1(\mathrm{SO}_n) \cong \pi_1(\mathrm{SO}_{n+1})$ for $n \geq 3$. Moreover, from the $n = 3$ long exact sequence, one computes $\pi_1(\mathrm{SO}_3) = \mathbb{Z}_2$. Hence we see that $\pi_1(\mathrm{SO}_n) = \mathbb{Z}_2$ for $n \geq 3$. Therefore, SO_n has a unique double cover. We claim that $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is that cover.

We have $a \in \ker\rho$ iff $va = av$ for any $v \in \mathbb{R}^n$ since $a^t = a^{-1}$ by Proposition 3.4. But this means by Proposition 3.2 that

$$a \in \mathrm{Spin}_n \cap \mathscr{Z}(\mathrm{Cl}_n) \subset \mathrm{Cl}_n^0 \cap \mathscr{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)} = \mathbb{R}$$

and hence $\ker\rho = \{\pm 1\}$. This already shows that $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is a double cover (local invertibility is obvious because we can conjugate by a^t). Since $\pi_1(\mathrm{SO}_n) = \mathbb{Z}_2$ for $n \geq 3$, and since the universal cover of SO_n must have degree $|\pi_1(\mathrm{SO}_n)| = 2$, we conclude that $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ is the universal cover.

Finally, since Spin_n is a finite cover of a compact space SO_n, it is compact. $\square$

3.2 Spinor Representation

Let us return to the general theory of Clifford algebra. Now our goal is to study representations of the Clifford algebra. As always, complex representations are much nicer to work with. Thus, let us complexify:

Definition 3.8. The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ is obtained after tensoring with $\mathbb{C}$ the real

Clifford algebra Cl_n:

$$\mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}.$$

We have the natural inclusion $\mathrm{Cl}_n \subset \mathbb{C}\mathrm{l}_n$, and the gradings ($\mathbb{Z}$ or $\mathbb{Z}_2$) have direct counterparts in the complex case. In particular, with obvious notation we have inclusion $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n^0 \subset \mathbb{C}\mathrm{l}_n$. Now we study complex representations of $\mathbb{C}\mathrm{l}_n$:

Definition 3.9. A complex **representation** of a complex algebra $\mathcal{A}$ is an algebra homomorphism $\zeta : \mathcal{A} \to \mathrm{Hom}(V)$ where V is a finite dimensional vector space. We say that V is a (left) $\mathcal{A}$-module.

A representation ζ is **irreducible** if there exists no nontrivial proper subspace $V_0 \subset V$ such that $\zeta(a)(V_0) \subset V_0$ for any $a \in \mathcal{A}$. Otherwise, ζ is said to be **reducible**.

A representation ζ is called **completely reducible** if there exists a direct sum decomposition $V = \bigoplus_{k=1}^{r} V_k$ of V into irreducible submodules.

There is a priori no reason why a representation of an arbitrary algebra should be completely reducible. But of course the situation for Clifford algebra is different.

Proposition 3.10. Any representation of $\mathbb{C}\mathrm{l}_n$ is completely reducible.

Proof. We consider the **Clifford group** Cliff_n formed by all elements of the type $\pm e_{i_1} \cdots e_{i_p}$ for $0 \leq p \leq n$. This is a finite multiplicative group of order 2^{n+1}. And clearly there exists a bijective correspondence between algebra representations $\zeta : \mathbb{C}\mathrm{l}_n \to \mathrm{Hom}(V)$ with group representations $\xi : \mathrm{Cliff}_n \to \mathrm{Aut}(V)$ with $\xi(-1) = -\mathrm{id}_V$. This means we just have to check any representation of Cliff_n is completely reducible. Indeed, let ξ be such a representation and let $\langle \cdot, \cdot \rangle$ be any hermitian product on V. Define

$$(v, w) = \frac{1}{2^{n+1}} \sum_{g \in \mathrm{Cliff}_n} \langle \xi(g)v, \xi(g)w \rangle$$

for all $v, w \in V$. It follows that ξ acts by isometries with respect to $(\cdot, \cdot)$. In particular, if V_o is a submodule then $V_o^{\perp} = \{v \in V : (v, w) = 0 \ \forall w \in V_o\}$ is also a submodule. The result follows from splitting off one irreducible module at a time (such irreducible module exists by picking a representation with the least dimension). $\square$

Remark 3.11. The averaging trick above shows that any representation of a finite group G is completely reducible. In the case where G is a compact Lie group, we integrate with respect to the

Haar measure μ_G to get a continuous averaging expression

$$(v,w) = \int_G \langle gv, gw\rangle \mathrm{d}\mu_G(g)$$

for $v, w \in V$. This shows that if G is a compact Lie group, then any representation of G is completely reducible. In particular, every representation of Spin_n is completely reducible.

Now that we have established every representation of $\mathbb{C}\mathrm{l}_n$ is completely reducible, let's classify all irreducible representations of $\mathbb{C}\mathrm{l}_n$. The crucial ingredient to do this is the **chirality operator**

$$\Gamma_n = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n.$$

Simple computation shows

Lemma 3.12. *The chirality operator Γ_n satisfies $\Gamma_n^2 = 1$, and $\Gamma_n v = (-1)^{n+1} v \Gamma_n$ for any $v \in \mathbb{R}^n$. In particular, $\Gamma_n \in \mathscr{Z}(\mathbb{C}\mathrm{l}_n)$ if n is odd and $\Gamma_n \in \mathscr{Z}(\mathbb{C}\mathrm{l}_n^0)$ if n is even.*

The main theorem of this section is:

Theorem 3.13. *If $n = 2k$ there exists a unique irreducible module of dimension 2^k over $\mathbb{C}\mathrm{l}_n$. If $n = 2k+1$, there exists exactly two inequivalent irreducible modules of dimension 2^k over $\mathbb{C}\mathrm{l}_n$.*

Proof. Since $\mathbb{C}\mathrm{l}_0 = \mathrm{Cl}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R} \otimes \mathbb{C} = \mathbb{C}$, the base case $n = 0$ follows immediately. Now assume the theorem holds for $n = 0, 1, ..., 2k-2$, and let V be a $\mathbb{C}\mathrm{l}_{2k-1}$-module. By Lemma 3.12, $\Gamma_{2k-1} = i^{k-1} e_1 ... e_{2k-1} \in \mathscr{Z}(\mathbb{C}\mathrm{l}_{2k-1})$ and $\Gamma_{2k-1}^2 = 1$. Thus we have a decomposition

$$V = V^+ \oplus V^-$$

into $\mathbb{C}\mathrm{l}_{2k-1}$-modules with $V^{\pm} = \{x \in V : \Gamma_{2k-1} x = \pm x\}$. Since $\mathbb{C}\mathrm{l}_{2k-2} \subset \mathbb{C}\mathrm{l}_{2k-1}$, both $V^{\pm}$ are $\mathbb{C}\mathrm{l}_{2k-2}$-modules and the inductive hypothesis gives the decomposition

$$V^{\pm} = \bigoplus_j W_j^{\pm}$$

into irreducible $\mathbb{Cl}_{2k-2}$-modules of dimension 2^{k-1}. But $x \in W_j^{\pm}$ implies

$$e_{2k-1}x = \pm e_{2k-1}\Gamma_{2k-1}x = \mp i^{k-1}e_1...e_{2k-2}x \in W_j^{\pm}.$$

Hence each $W_j^{\pm}$ is also a $\mathbb{C}_{2k-1}$-module. Moreover, as $\mathbb{Cl}_{2k-1}$-modules $W_j^+ \neq W_j^-$ because left multiplication by e_{2k-1} gives distinct results, and W_j^+'s (respectively W_j^-'s) are mutually equivalent since multiplication by e_{2k-1} gives the same result. This proves the theorem in this case.

Now assume the theorem holds for $n = 0, 1, ..., 2k-1$ and let V be a $\mathbb{Cl}_{2k}$-module. As before, $\mathbb{Cl}_{2k-1} \subset \mathbb{Cl}_{2k}$, V is a $\mathbb{Cl}_{2k-1}$-module and the induction hypothesis splits V into a sum of $\mathbb{Cl}_{2k-1}$-modules $V = V^+ \oplus V^-$, where $V^{\pm}$ is a direct sum of irreducible $\mathbb{Cl}_{2k-1}$-modules $W_j^{\pm}$ as above. Now since $\Gamma_{2k-1}e_{2k} = -e_{2k}\Gamma_{2k-1}$, multiplication μ by e_{2k} satisfies $\mu(V^{\pm}) \subset V^{\mp}$. Writing $W_j = W_j^+ \oplus \mu(W_j^+)$ we get a decomposition

$$V = \bigoplus_j W_j$$

into mutually equivalent $\mathbb{Cl}_{2k}$-modules of dimension 2^k. □

Next, we are ready of introduce the class of representations of Spin_n we have been searching for:

Definition 3.14. The **spinor representation** $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ is obtained by restricting any irreducible representation of $\mathbb{Cl}_n$ under the inclusion $\mathrm{Spin}_n \subset \mathbb{Cl}_n$.

Since the only case we care is the even dimensional case, we have the following:

Corollary 3.15. *If $n = 2k$ is even, then the spin representation $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ splits as a sum*

$$\Psi = \Psi^+ \oplus \Psi^-$$

of two irreducible modules of dimension 2^{k-1}, where $\Psi^{\pm} = \{x \in \Psi : \Gamma_{2k}x = \pm x\}$ and we have the exchange relation

$$e_i(\Psi^{\pm}) \subset \Psi^{\mp} \qquad i = 1, ..., n.$$

Proof. The decomposition follows from Theorem 3.13 and the exchange relation follows from Lemma 3.12. □

Remark 3.16. Let V be a real vector space of dimension n with an inner product and an almost complex structure, i.e., a endomorphism $J : V \to V$ with $J^2 = -\operatorname{id}_V$. Then we can give an explicit description of the Ψ above. We have a decomposition

$$V \otimes \mathbb{C} = V^+ \oplus V^-$$

where $V^\pm$ is the $(\pm i)$-eigenspace of J. Then we can define

$$\Psi := \Lambda^\bullet (V^-)^*.$$

(Compare this with supermanifold case: The structure sheaf of a supermanifold is locally given by $\mathcal{O}_M \otimes \Lambda^\bullet V$ for a vector bundle $V \to M$, and odd sections of this sheaf is the fermionic functions, which are spinors) The action of $\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n(V)$ on Ψ is as follows: Any vector $a \in V$ has a unique decomposition $a = a^+ + a^-$ where $a^\pm \in V^\pm$. Now for any $\psi \in \Psi$, we define

$$a \cdot \psi = \sqrt{2}\Big((a^-)^\flat \wedge \psi - i_{a^+}\psi\Big)$$

where $\flat : V^- \to (V^-)^*$ is given by the inner product $(v^-)^\flat = \langle -, v^- \rangle$ and i_{a^+} is contraction by the vector a^+. Through tedious computation one can check that $(a^\pm)^2 = 0$ and $a^+a^- + a^-a^+ = -2\langle a^+, a^- \rangle$. Thus this indeed defines a representation of $\mathbb{C}\mathrm{l}_n$ on Ψ. One can think of wedging by a^+ as the creation operator (since it increases degree) and contracting by a^- as the annihilation operator (since it decreases degree). Now since $\dim(\Lambda^\bullet V^{-*}) = 2^k$ we conclude that it is the Ψ given in Theorem 3.13. Moreover, note that

$$\dim(\mathbb{C}\mathrm{l}_n) = 2^n = 2^{2k} = (2^k)^2 = \dim \operatorname{End}(\Lambda^\bullet V^{-*}) = \dim(\Psi).$$

Moreover, the map $\mathbb{C}\mathrm{l}_n \to \operatorname{End}(\Lambda^\bullet V^{-*})$ given by this representation is surjective: One can reach any states $v_1 \wedge \cdots \wedge v_p$ from 1 by successive creations, and one can go back to 1 from any state by successive annihilation, it follows that this map is in fact an isomorphism. Therefore, we conclude that

$$\mathbb{C}\mathrm{l}_n = \operatorname{End}(\Psi). \tag{3.1}$$

In fact this last equation holds for the most general case, even when V is not an almost complex vector space.

3.3 Spin Bundles and Spin Manifolds

Throughout this section, let $n = 2k$ be an even number.

In the previous section, we constructed irreducible representation Ψ of $\mathbb{C}\mathrm{l}_n$, and we showed that $\mathbb{C}\mathrm{l}_n = \mathrm{End}(\Psi)$ when the vector space has an almost complex structure, but we remarked that this holds in general.

Now we want to make the pointwise vector space picture into family of vector spaces over a Riemannian manifold M of dimension n. Namely, we want to consider a bundle $\mathbb{C}\mathrm{l}(M)$ such that at every point $x \in M$ its fiber is $\mathbb{C}\mathrm{l}(T_xM)$. That is,

$$\mathbb{C}\mathrm{l}(M) = \coprod_{x \in M} \mathbb{C}\mathrm{l}(T_xM)$$

as a set. To make sure this construction works globally, we need the following construction:

Definition 3.17. Let $\pi : P \to M$ be a principal G-bundle and $\mu : G \to \mathrm{Aut}(V)$ a representation of G. Note that G acts on the product $P \times V$ on the right via

$$(p, v) \cdot g := (pg, \mu(g^{-1})v)$$

for $(p, v) \in P \times V$. The **associated bundle** $E := P \times_\mu V$ is the quotient $P \times V / \sim$ where $(p, v) \sim (p, v) \cdot g$ for all $g \in G$. Then it can be checked that with the obvious projection E is a bundle over M with fiber V.

We can consider the principal SO_n-bundle P_M^{SO} (the frame bundle), whose fiber over any $x \in M$ is the set of orthonormal frames of T_xM. Then we can define the **Clifford algebra bundle** to be the associated bundle

$$\mathbb{C}\mathrm{l}(M) := P_M^{\mathrm{SO}} \times_\mu \mathbb{C}\mathrm{l}_n$$

where $\mu : \mathrm{SO}_n \to \mathrm{Aut}(\mathbb{C}\mathrm{l}_n)$ is defined by

$$\mu(g)(v_1...v_p) = (gv_1)...(gv_p).$$

Proposition 3.18. At any point $x \in M$, the fiber $\mathbb{C}\mathrm{l}(M)_x$ is canonically isomorphic to $\mathbb{C}\mathrm{l}(T_xM)$.

Proof. A point $p \in P_M^{\mathrm{SO}}$ is an orthonormal basis $(e_1, ..., e_n)$ of T_xM. This induces an isometry $i_p : \mathbb{R}^n \to T_xM$ that sends the standard basis of $\mathbb{R}^n$ to the basis $(e_1, ..., e_n)$ of T_xM. Therefore, this induces an isomorphism of Clifford algebras $i_p : \mathbb{C}\mathrm{l}_n \to \mathbb{C}\mathrm{l}(T_xM)$. Therefore we can define an map $\Phi_x : \mathbb{C}\mathrm{l}(M)_x \to \mathbb{C}\mathrm{l}(T_xM)$ by $\Phi_x([p, v]) = i_p(v)$. Note that there is an obvious inverse induced by $i_p^{-1} : T_xM \to \mathbb{R}^n$. To check that this is well-defined, let $[p', v'] = [p, v]$ in $\mathbb{C}\mathrm{l}(M)_x$. This means that $p' = pg$ and $v' = \mu(g^{-1})v$. Thus

$$i_{p'}(v') = i_{pg}(\mu(g^{-1})v) = i_p(\mu(g)\mu(g^{-1})v) = i_p(v).$$

Since Φ_x is an isomorphism, the proof is complete. □

We can ask whether there exists a bundle S over M such that for each $x \in M$, the fiber S_x is an irreducible module over $\mathbb{C}\mathrm{l}(M)_x$ and $\mathbb{C}\mathrm{l}(M) = \mathrm{End}(S)$. However, there exists topological obstructions to the existence of S, which we shall discuss now. We first need a description of space that parameterize the set of principal G-bundles over M:

Theorem 3.19. *The sheaf cohomology group $H^1(M, G)$ parameterizes the set of principal G-bundles over M.*

Proof. Let $M = \bigcup_\alpha U_\alpha$ be a good cover. Let P be a principal G-bundle. By shrinking the good cover we might as well assume that P trivializes over each U_α with transition functions $\xi_{\alpha\beta}$. The transition functions satisfy the cocycle condition

$$\xi_{\alpha\beta}\xi_{\beta\gamma} = \xi_{\alpha\gamma}.$$

Moreover, P' is another principal G-bundle that is isomorphic to P (by passing to common trivializing cover we might as well assume P' also trivializes over U_α) iff there exists a family of maps

$\xi_\alpha : U_\alpha \to G$ such that

$$\xi'_{\alpha\beta} = \xi_\beta^{-1}\xi_{\alpha\beta}\xi_\alpha,$$

so they must differ by a coboundary. $\square$

Now let $\mathcal{E}$ be an oriented vector bundle of rank r over the Riemannian manifold M. Thus we can consider the principal SO_r-bundle $P_{\mathcal{E}}^{\mathrm{SO}}$ which is its frame bundle. By Theorem 3.7, we have a short exact sequence of groups

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_r \xrightarrow{\rho} \mathrm{SO}_r \to 0.$$

This induces a long exact sequence in sheaf cohomology

$$\cdots \to H^0(M, \mathrm{SO}_r) \xrightarrow{\partial_*} H^1(M, \mathbb{Z}_2) \to H^1(M, \mathrm{Spin}_r) \xrightarrow{\rho_*} H^1(M, \mathrm{SO}_r) \xrightarrow{\partial_*} H^2(M, \mathbb{Z}_2) \to \cdots \tag{3.2}$$

The key observation is that the second Wtiefel-Whitney class $w_2(\mathcal{E})$ of $\mathcal{E}$ is involved here:

Proposition 3.20. $w_2(\mathcal{E}) = \partial_*(P_{\mathcal{E}}^{\mathrm{SO}})$.

Proof. This can be proven by looking at the axiomatic characterization of Stiefel-Whitney classes. $\square$

From the exactness of (3.2), we see that $w_2(\mathcal{E}) = \partial_*(P_{\mathcal{E}}^{\mathrm{SO}}) = 0$ iff there exists a principal Spin_r-bundle $P_{\mathcal{E}}^{\mathrm{Spin}}$ over M such that

$$\rho_*(P_{\mathcal{E}}^{\mathrm{Spin}}) = P_{\mathcal{E}}^{\mathrm{SO}}.$$

This means that $P_{\mathcal{E}}^{\mathrm{SO}}$ is obtained from $P_{\mathcal{E}}^{\mathrm{Spin}}$ as the quotient by the $\mathbb{Z}_2$-action on the fibers. Therefore, we obtain

Proposition 3.21. Let $\mathcal{E}$ be a real oriented real vector bundle of rank r over M. The following statements are equivalent: (1) $w_2(\mathcal{E}) = 0$; (2) There exists a Spin_r-bundle $P_{\mathcal{E}}^{\mathrm{Spin}}$ over M and a nontrivial two-sheeted covering map $\gamma : P_{\mathcal{E}}^{\mathrm{Spin}} \to P_{\mathcal{E}}^{\mathrm{SO}}$ such that the diagram

$$\begin{tikzcd}
P_{\mathcal{E}}^{\mathrm{Spin}} \arrow[rr, "\gamma"] \arrow[dr] & & P_{\mathcal{E}}^{\mathrm{SO}} \arrow[dl] \\
& M &
\end{tikzcd}$$

commutes and $\gamma(pg) = \gamma(p)\rho(g)$, for $p \in P_{\mathcal{E}}^{\mathrm{Spin}}$ and $g \in \mathrm{Spin}_r$.

If either of the conditions above happens we say that $\mathcal{E}$ is a **spin bundle**. An oriented Riemannian manifold of dimension $n \geq 3$ is said to be **spin** if its tangent bundle TM is spin in the above sense.

3.4 The Dirac Operator on Spin Manifolds

Now we turn our attention to geometry of spin manifolds. Let us now assume that M is a spin manifold of dimension $n = 2k$ with a Riemannian metric. We fix a spin structure, i.e, we fix a lift $\gamma : P_M^{\mathrm{Spin}} \to P_M^{\mathrm{SO}}$ of the frame bundle. If $A \in \mathcal{A}^1(P_M^{\mathrm{SO}}, \mathfrak{so}_n)$ is the connection 1-form on the frame bundle induced from the Levi-Civita connection ∇^{LC} on M, then $A^s := \gamma^* A \in \mathcal{A}^1(P_M^{\mathrm{Spin}}, \mathfrak{so}_n)$ is a connection 1-form on the spin bundle, and it defines a connection $\nabla^s := \mathrm{d} + A^s$ called the **spin connection**. Recall that we have a spin representation $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$. Hence we can use the associated bundle construction to form the **spinor bundle** on M:

$$S(M) = P_M^{\mathrm{Spin}} \times_\tau \Psi.$$

This is a complex vector bundle of rank 2^k over M. Elements in $\Gamma(S(M))$ are referred to as **spinors**. For each $x \in M$, the fiber $S(M)_x$ is an irreducible module over $\mathbb{C}\mathrm{l}(M)_x$ and in fact we have the global generalization of (3.1):

$$\mathbb{C}\mathrm{l}(M) = \mathrm{End}(S(M)). \tag{3.3}$$

Moreover, one can check that Ψ is unitary. In the case where Remark 3.16 applies, one can this direct. Therefore the Clifford action $c : \mathbb{C}\mathrm{l}_n \to \Psi$ given in Remark 3.16 satisfies $c(a)^\dagger = -c(a)$. Since $c^2(a) = c(a^2) = -1$, a skew-Hermitian operator that squares to minus one must be unitary.

Since Ψ is unitary, $S(M)$ can be equipped with a natural metric $\langle \cdot, \cdot \rangle$ that is compatible with the connection ∇^s. We point out two properties of these structures:

1. Clifford multiplication by unit tangent vectors is an isometry: For any $\varphi, \varphi' \in \Gamma(S(M))$ and any $u \in \Gamma(TM)$ with $\|u\| = 1$, we have

$$\langle u\varphi, u\varphi' \rangle = \langle \varphi, \varphi' \rangle. \tag{3.4}$$

2. Let ∇^c be the natural connections on $\mathbb{C}\mathrm{l}(M)$, which is the connection $\mathrm{End}(S(M))$ induced from the connection on $S(M)$, is compatible with the connection ∇^s on $S(M)$: For any $a \in \Gamma(\mathbb{C}\mathrm{l}(M))$ and $\varphi \in \Gamma(S(M))$ we have

$$\nabla^s(a\varphi) = \nabla^c a \cdot \varphi + a\nabla^s \varphi. \tag{3.5}$$

Now we can finally define the Dirac operator on a spin manifold.

Definition 3.22. In terms of a local orthonormal frame $e_1, ..., e_n$ of TM, the **Dirac operator** $\partial\!\!\!/ : \Gamma(S(M)) \to \Gamma(S(M))$ is defined by

$$\partial\!\!\!/ := \sum_{i=1}^{n} e_i \cdot \nabla^s_{e_i}.$$

We see that $\partial\!\!\!/$ is a first order linear differential operator. Moreover, indeed we can view $\partial\!\!\!/$ as an endomorphism on $\Gamma(S(M))$ because via the musical isomorphism $\sharp : T^*M \to TM$ given by the metric on M and the Clifford $\Gamma(TM \otimes S(M)) \to \Gamma(S(M))$, the sequence of arrows

$$\Gamma(S(M)) \xrightarrow{\nabla^s} \Gamma(T^*M \otimes S(M)) \xrightarrow{\sharp} \Gamma(TM \otimes S(M)) \xrightarrow{\cdot} \Gamma(S(M))$$

starts and ends at the correct spaces. With respect to the inner product on $\Gamma(S(M))$ defined by

$$(\varphi, \varphi') = \int_M \langle \varphi, \varphi' \rangle \mathrm{d}\Omega$$

where Ω is the Riemannian volume form on M, one can show that $\partial\!\!\!/$ is self-adjoint:

Theorem 3.23. *With respect to the inner product on $\Gamma(S(M))$ defined above, $\partial\!\!\!/$ is self-adjoint: For* *any* $\varphi, \varphi' \in \Gamma(S(M))$ *one has*

$$(\partial\!\!\!/\varphi, \varphi') = (\varphi, \partial\!\!\!/\varphi').$$

Proof. If u is a unit vector, since $u^2 = -1$, (3.4) implies that $\langle u\varphi, \varphi' \rangle = -\langle \varphi, u\varphi' \rangle$. Since the connection 1-form $A \in \mathcal{A}^1(P_M^{\mathrm{SO}}, \mathfrak{so}_n)$ is induced from the Levi-Civita connection on M, the connection ∇^c on $\mathbb{C}\mathrm{l}(M)$ coincides with the Levi-Civita connection on vectors. That is, at any point $x \in M$ we can choose local orthonormal frames $e_1, ..., e_n$ of TM such that $\nabla^c_{e_i} e_i|_x = 0$. Then at this point, with respect to this frame, using (3.4) and (3.5) we have

$$\begin{aligned}
\langle \partial\!\!\!/\varphi, \varphi' \rangle - \langle \varphi, \partial\!\!\!/\varphi' \rangle &= \sum_i \langle e_i \nabla^s_{e_i} \varphi, \varphi' \rangle - \langle \varphi, e_i \nabla^s_{e_i} \varphi' \rangle \\
&= \sum_i \langle e_i \nabla^s_{e_i} \varphi, \varphi' \rangle + \langle e_i \varphi, \nabla^s_{e_i} \varphi' \rangle \\
&= \sum_i \langle \nabla^s_{e_i} (e_i \varphi), \varphi' \rangle + \langle e_i \varphi, \nabla^s_{e_i} \varphi' \rangle \\
&= \sum_i e_i \langle e_i \varphi, \varphi' \rangle,
\end{aligned}$$

where the last equality is because ∇^s is compatible with the fiber metric on $S(M)$. Now we can define a vector field $X \in \Gamma(TM)$ by $\langle X, Y \rangle = \langle Y\varphi, \varphi' \rangle$, where the left hand side is the Riemannian inner product and on the right hand side the inner product is the fiberwise inner product on $S(M)$, where $Y\varphi$ denotes the Clifford multiplication. Therefore, we have that

$$\langle \partial\!\!\!/\varphi, \varphi' \rangle - \langle \varphi, \partial\!\!\!/\varphi' \rangle = \sum_i e_i \langle e_i \varphi, \varphi' \rangle = \sum_i e_i \langle X, e_i \rangle = \sum_i \langle \nabla_{e_i} X, e_i \rangle = \operatorname{div} X.$$

The rest is an easy application of Stokes' theorem. Since $\mathcal{L}_X \Omega = (\operatorname{div} X)\Omega$, and Cartan's magic formula gives $\mathcal{L}_X \Omega = \mathrm{d}(i_X \Omega)$ since $\mathrm{d}\Omega = 0$, one has

$$\int_M (\operatorname{div} X)\Omega = \int_M \mathrm{d}(i_X \Omega) = \int_{\partial M} i_X \Omega = 0$$

since $\partial M = \emptyset$. Therefore, we see that $(\partial\!\!\!/\varphi, \varphi') - (\varphi, \partial\!\!\!/\varphi') = 0$, as desired. $\square$

By Corollary 3.15, we have a decomposition

$$S(M) = S^+(M) \oplus S^-(M)$$

where $S^{\pm}(M) := P_M^{\mathrm{Spin}} \times_{\tau^{\pm}} \Psi^{\pm}$. This decomposition is preserved by ∇^s and is orthogonal with respect to the fiber metric of $S(M)$. Now from the parity exchange relation in Corollary 3.15, we have $\partial\!\!\!/(\Gamma(S^{\pm}(M))) \subset \Gamma(S^{\mp}(M))$. Therefore, in matrix form, there exists some differential operator $D : \Gamma(S^+(M)) \to \Gamma(S^-(M))$ such that

$$\partial\!\!\!/ = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$$

since $\partial\!\!\!/$ is self adjoint by Theorem 3.23. Using some technical analysis argument, one can show that $\partial\!\!\!/$ is Fredhom, meaning both $\dim\ker D$ and $\dim\ker D^\dagger$ are finite. Hence we may define the **index** of the Dirac operator as

$$\mathrm{ind}(\partial\!\!\!/) = \dim\ker D - \dim\ker D^\dagger.$$

The Atiyah-Singer index theorem is a striking result, which says the purely analytical quantity $\mathrm{ind}(\partial\!\!\!/)$ is already determined by the topology of M. Specifically, the curvature 2-form

$$F_{\mu\nu} = R_{\mu\nu\alpha\beta} \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta.$$

of TM. The splitting principle for real vector bundle (See [10], Theorem 7.2.2) shows that if we want to define characteristic classes using the curvature 2-form F, we may assume that it splits into block diagonal form

$$F = \begin{pmatrix} 0 & r_1 & \cdots & 0 & 0 \\ -r_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & r_n \\ 0 & 0 & \cdots & -r_n & 0 \end{pmatrix}$$

Then the **$\hat{A}$-class** of M is defined by the Taylor expansion of

$$\hat{A}(M) := \prod_k \frac{r_k/2}{\sinh(r_k/2)}.$$

Extracting the top-dimensional piece of $\hat{A}(M)$ and integrate, the Atiyah-Singer index theorem in this case is

Theorem 3.24 (=Theorem 4.6)**.** *On a spin Riemannian manifold M of even dimension, we have*

$$\mathrm{ind}(\not{\partial}) = \int_M \hat{A}(M).$$

In the next chapter we will present a proof of this result on a heuristic level of rigor using path integral techniques.

3.5 General Atiyah-Singer Index Formula and Applications

In fact, there is a more general version of the index theorem without the need of a spin structure. The characteristics of this theory is very similar to the spin case discussed previously.

Throughout this section, let M be an oriented closed Riemannian manifold of dimension $n = 2k$.

Definition 3.25. A vector bundle $E \to M$ is called a **Clifford bundle** if the following holds:

1. E is a (complex) vector bundle whose typical fiber is a Clifford module. This means we have a Clifford action $c : \mathbb{C}\mathrm{l}(M) \to \mathrm{End}(E)$.

2. E has a fiber metric such that Clifford multiplication by unit tangent vectors is a unitary isomorphism. For any $\varphi, \varphi' \in \Gamma(E)$ and any $u \in \Gamma(TM)$ with $\|u\| = 1$, we have

$$\langle c(u)\varphi, c(u)\varphi' \rangle = \langle \varphi, \varphi' \rangle. \tag{3.6}$$

3. There is a connection ∇^E on E that is compatible with the fiber metric and with the Clifford connection ∇^c on $\mathbb{C}\mathrm{l}(M)$ (induced by the Levi-Civita connection). For any $a \in \Gamma(\mathbb{C}\mathrm{l}(M))$ and

$\varphi \in \Gamma(E)$, we have the Leibniz rule:

$$\nabla^E(a \cdot \varphi) = (\nabla^c a) \cdot \varphi + a \cdot \nabla^E \varphi. \tag{3.7}$$

For a Clifford bundle E, we define the Dirac operator $\not{\partial} : \Gamma(E) \to \Gamma(E)$ in a local orthonormal frame $\{e_i\}$ of TM:

$$\not{\partial} = \sum_{i=1}^{n} e_i \cdot \nabla^E_{e_i}.$$

Definition 3.26. We say a Clifford bundle E is $\mathbb{Z}_2$-graded if there is a decomposition $E = E^+ \oplus E^-$ preserved by the connection, such that Clifford multiplication by a tangent vector v maps $E^\pm \to E^\mp$ (odd parity). The Dirac operator then takes the form:

$$\not{\partial} = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

The **index** of the Dirac operator is then defined as:

$$\operatorname{ind}(\not{\partial}) = \dim\ker D^+ - \dim\ker D^-.$$

Proposition 3.27. Let $S = S(M)$ be the spinor bundle as before. If E is a Clifford bundle over M, then E is always isomorphic to a twisted spinor bundle, in the sense that there exists a vector bundle Q and an isomorphism

$$E \cong S \otimes Q.$$

Explicitly, $Q = \operatorname{Hom}_{\mathbb{C}\mathrm{l}(M)}(S, E)$.

Proof. We claim that the map

$$\Upsilon : S \otimes \operatorname{Hom}_{\mathbb{C}\mathrm{l}(M)}(S, E) \to E \qquad \Upsilon(s \otimes \phi) = \phi(s)$$

is an isomorphism. Since this is a bundle map, we only need to check that it is an isomorphism on each fiber. At each $x \in M$, the map is $\Upsilon_x : S_x \otimes \operatorname{Hom}_{\mathbb{C}\mathrm{l}(M)}(S_x, E_x) \to E_x$. Since S_x is irreducible, we must have $E_x \cong S_x^{\oplus p}$ for some p. Under this identification we have $\Upsilon_x : S_x \otimes \operatorname{Hom}(S_x, S_x^{\oplus p}) \to S_x^{\oplus p}$.

Now since S_x is irreducible, Schur's lemma says that every map in $\mathrm{Hom}(S_x, S_x)$ must be a constant multiple of identity, namely $\mathrm{Hom}(S_x, S_x) \cong \mathbb{C}$. Now clearly $\mathrm{Hom}(S_x, S_x^{\oplus p}) = \mathrm{Hom}(S_x, S_x)^{\oplus p} \cong \mathbb{C}^p$. Under this identification the we have

$$\Upsilon : S_x \otimes \mathbb{C}^p \to S_x^{\oplus p} \qquad \Upsilon(s \otimes (\lambda_1, ..., \lambda_p)) = (\lambda_1 s, ..., \lambda_p s).$$

Clearly this is a surjective map, hence it is an isomorphism since $\dim(S_x \otimes \mathbb{C}^p) = \dim(S_x^{\oplus p})$. $\square$

Theorem 3.28 (Atiyah-Singer)**.** *Let M be a compact, oriented, even-dimensional Spin manifold and let $\not{\partial}$ be the Dirac operator associated to the twisted Clifford bundle $E = S \otimes Q$. The analytical index is equal to the topological index:*

$$\mathrm{ind}(\not{\partial}) = \int_M \hat{A}(TM)\,\mathrm{ch}(Q).$$

This theorem is considered one of the greatest achievements of twentieth-century mathematics. We illustrate its power by sketching the proofs of several classical consequences.

Corollary 3.29 (Poincaré-Hopf-Gauss-Bonnet)**.** *The integral of the Euler class of TM is the Euler characteristic of M:*

$$\chi(M) = \int_M e(TM).$$

Proof Sketch. Let $E = \Lambda^\bullet T^*M \otimes \mathbb{C}$. We define the Clifford action $c(v)$ on a form α by:

$$c(v)\alpha = v^\flat \wedge \alpha - i_v \alpha.$$

Here, $\flat : TM \to T^*M$ is the isomorphism $v^\flat = g(v, -)$ given by the metric. Then one can verify that $c(v)c(w) + c(w)c(v) = -2g(v, w)$. Hence this indeed defines a Clifford action. Now the bundle E is naturally $\mathbb{Z}_2$-graded by even or odd degree forms. Moreover, the metric on M gives E a fiberwise inner product by $\langle \mathrm{d}x^i, \mathrm{d}x^j \rangle = g^{ij}$ and can be extended to all p-forms by taking determinants, and integrating this over M gives an inner product on E, so we can make sense of the adjoint d^* of the exterior differentiation operator. The connection ∇^E is the Levi-Civita connection extended to differential forms: Namely,

$$(\nabla_X^E \alpha)(X_1, ..., X_p) = X\alpha(X_1, ..., X_p) - \sum_{i=1}^{p} \alpha(X_1, ..., \nabla_X X_i, ..., X_p).$$

The corresponding Dirac operator is the de Rham operator

$$\not{\partial} = \mathrm{d} + \mathrm{d}^*.$$

Then one can compute that $\not{\partial}^2 = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = \Delta$ is the Hodge Laplacian. Moreover, $(\Delta\eta, \eta) = \|\not{\partial}\eta\|^2$. This implies $\ker \not{\partial} = \ker \Delta$. By the Hodge theory on Riemannian manifolds, the Δ-harmonic p-forms has dimension equal to the Betti numbers $b_p(M)$. Hence

$$\begin{aligned} \mathrm{ind}(\not{\partial}) &= \dim(\ker \not{\partial} \cap E^+) - \dim(\ker \not{\partial} \cap E^-) = \dim(\ker \Delta \cap E^+) - \dim(\ker \Delta \cap E^-) \\ &= \dim\left(\bigoplus_{p \text{ even}} \mathcal{H}^p\right) - \dim\left(\bigoplus_{p \text{ odd}} \mathcal{H}^p\right) = \sum_{p \text{ even}} b_p - \sum_{p \text{ odd}} b_p = \chi(M). \end{aligned}$$

Topologically, on a spin manifold, we have the bundle isomorphism $E \cong S \otimes S$. Indeed, in this case

$$E = \Lambda^\bullet T^*M \otimes \mathbb{C} \cong \mathbb{C}\mathrm{l}(M) \cong \mathrm{End}(S) \cong S^* \otimes S \cong S \otimes S$$

by (3.3), and the last isomorphism is given by the fiberwise inner product on S. Thus, the twisting bundle is $Q = S$. A fundamental identity in characteristic class theory states that $\hat{A}(TM)\,\mathrm{ch}(S) = e(TM)$. This concludes the proof. $\square$

Another beautiful consequence of the index formula is the Hirzebruch signature theorem. Since M is a closed manifold of dimension $n = 2k$, the middle de Rham cohomology group $H^k(M)$ admits a bilinear form $B : H^k(M) \times H^k(M) \to \mathbb{C}$ given by

$$B(\alpha, \beta) = \int_M \alpha \wedge \beta.$$

Moreover $B(\alpha, \beta) = (-1)^{k^2} B(\beta, \alpha)$. So if we assume that $k = 2l$ so that $\dim M = 2k = 4l$, then B is symmetric. Poincaré duality implies that B is nondegenerate and hence the corresponding quadratic form has no null eigenvalue. We can then define the **signature** of M by

$$\operatorname{sign}(M) = b^+ - b^-$$

where $b^\pm$ is the number of positive (negative) eigenvalues of the corresponding quadratic forms counted with multiplicities. A consequence of the Hirzebruch signature theorem is that this quantity (which is a topological invariant) can be computed by integrating the **Hirzebruch $\mathcal{L}$-class**, which is defined by the formal power series

$$\mathcal{L}(TM) = \prod_{i=1}^{k} \frac{r_i/2}{\tanh(r_i/2)}$$

where r_i's are the block entries of the curvature 2-forms F as in the previous section.

Corollary 3.30 (Hirzebruch Signature Theorem)**.** *For a closed oriented $4l$-dimensional manifold, we have*

$$\operatorname{sign}(M) = \int_M \mathcal{L}(TM).$$

Proof Sketch. We still use the same Clifford bundle $E = \Lambda^\bullet T^*M \otimes \mathbb{C}$ and the same Clifford action $c : \mathbb{C}\mathrm{l}(M) \to \operatorname{End}(E)$. However, the grading of E is different in this case. We define an operation $\tau : E \to E$ given by

$$\tau = i^{p(p-1)+k} \star \qquad \text{on } \Lambda^p T^*M$$

where $\star$ is the Hodge star operator. Then one can check that $\tau^2 = 1$, and τ is self-adjoint, moreover $\tau c(v) = -c(v)\tau$. Hence $E = E^+ \oplus E^-$ decomposes as the (± 1)-eigenspace of τ. This defines the grading. The connection ∇^E is still the same as in the Gauss-Bonnet case, so is the Dirac operator

$$\not{\partial} = \mathrm{d} + \mathrm{d}^*.$$

But this time, because the change of the grading of E, we will get different results: $\operatorname{ind}(\not{\partial}) = \operatorname{sign}(M)$, and $\hat{A}(TM)\operatorname{ch}(S) = \mathcal{L}(TM)$. □

We conclude with the most celebrated theorem in algebraic geometry. Almost every algebraic geometer uses Hirzebruch-Riemann-Roch theorem in their daily life (The Hirzebruch-Riemann-Roch theorem implies the Riemann-Roch theorem on curves, so this is an undisputed statement!):

Corollary 3.31 (Hirzebruch-Riemann-Roch)**.** *Let X be a compact complex manifold and $\mathcal{E} \to X$ a holomorphic vector bundle on X. Let TX denote the holomorphic tangent bundle (which is isomorphic to the real tangent bundle as real vector bundles) on X and $\chi(X, \mathcal{E})$ denote the alternating sum of dimensions of sheaf cohomology groups of $\mathcal{E}$. Then we have*

$$\chi(X, \mathcal{E}) = \int_X \mathrm{td}(TX)\,\mathrm{ch}(\mathcal{E}).$$

Proof Sketch. This construction will just be the one given in Remark 3.16, with V replaced by TX.

Let our Clifford bundle be $E = \Lambda^{0,\bullet}T^*X \otimes \mathcal{E}$. Fix a hermitian metric on TX compatible with the complex structure, and a hermitian metric on $\mathcal{E}$. This will give a hermitian metric on E. Let ∇ be the Chern connection with respect to fiber metric on $\Lambda^{0,\bullet}T^*X$, and similarly let $\nabla^{\mathcal{E}}$ be the Chern connection on $\mathcal{E}$. Then we choose the connection ∇^E on E defined by

$$\nabla^E = \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^{\mathcal{E}}.$$

We have a natural $\mathbb{Z}_2$-grading on $\mathcal{E}$, where $e^{\pm}$ is the direct sum of $\Lambda^{0,p}T^*X \otimes \mathcal{E}$ where p is even (odd). Now for any $v \in TX \otimes \mathbb{C}$, the complex structure gives a decomposition

$$v = v^{1,0} + v^{0,1} \in T^{1,0}X \oplus T^{0,1}X.$$

Define the Clifford action $c : \mathbb{C}\mathrm{l}(M) \to \mathrm{End}(E)$ to be

$$c(v)\alpha = \sqrt{2}\Big((v^{0,1})^\flat \wedge \alpha - i_{v^{1,0}}\alpha\Big).$$

One can check that this is indeed a Clifford action. Then with these data, the Dirac operator is given by

$$\not{\partial} = \sqrt{2}(\bar{\partial}_{\mathcal{E}} + \bar{\partial}^*_{\mathcal{E}}).$$

Then one calculates that $\mathrm{ind}(\not{\partial}) = \chi(X, \mathcal{E})$.

For the right hand side, one needs to use a variant of the index theorem above called the Spin^c

index theorem. The bundle $\Lambda^{0,\bullet}T^*X$ plays the role of the spinor bundle S associated to the canonical Spin^c structure of X. Thus, our bundle is $E = S \otimes \mathcal{E}$, and the twisting bundle is $Q = \mathcal{E}$. Using the Spin^c version of the index theorem, or simply the identity

$$\hat{A}(TX)\,\mathrm{ch}(S) = \hat{A}(TX)\mathrm{e}^{c_1(TX)/2} = \mathrm{td}(TX),$$

we obtain:

$$\mathrm{ind}(\not{\partial}) = \int_X \hat{A}(TX)\,\mathrm{ch}(S)\,\mathrm{ch}(\mathcal{E}) = \int_X \mathrm{td}(TX)\,\mathrm{ch}(\mathcal{E}).$$

This completes the proof. $\square$

Chapter 4

Witten's Proof

We now present Witten's proof of the Atiyah-Singer index theorem on spin manifold using the technique of path integrals and supersymmetric quantum mechanics (SUSYQM). We will first develop the basic of SUSYQM on manifolds, then we set up the problem of computing the index of an elliptic operator on a spin manifold as a path integral computation. Finally, we will prove Atiyah-Singer index theorem by evaluating the said path integral.

It should be noted that the proof we present is at a formal level of rigor, since path integral is not well-defined mathematically. Moreover, we should mention that the proofs of the most general form of the Atiyah-Singer index theorem are much more complicated, involving deep ideas in algebraic K-theory or analysis.

Our treatment follows [4] closely, with more computational details carried out. For a more through introduction to SUSYQM, we refer the reader to [7], which also contains some materials on the proof of index theorem, though more details are omitted compared to here. Our approach is more computational, but for a more mathematical presentation of the subject, we recommend Witten's notes on the index of Dirac operators in [5], and Chapter 8 of [2]. Finally, we refer the interested readers to the original papers [8],[9].

4.1 Supersymmetric Quantum Mechanics

For simplicity, we first introduce SUSYQM in flat space, then we will upgrade our SUSYQM model to Riemannian manifolds. In the simplest case our space is $\mathbb{R}^d$, with coordinates x_k, and we have anticommuting Grassmannian ψ_k, for $k = 1, ..., d$. We propose to consider the Lagrangian

$$L = \frac{1}{2}\dot{x}_k\dot{x}_k + \frac{i}{2}\psi_k\dot{\psi}_k.$$

Note that we are summing over repeated indices, but we are not careful with upper and lower indices since in $\mathbb{R}^d$ the metric is so simple that there is no difference. Recall from classical mechanics that for a Lagrangian $L(\phi_k, \dot{\phi}_k)$ with generalized coordinates, the canonical momentum is defined by

$$\pi_k = \partial L/\partial\dot{\phi}_k,$$

and the Hamiltonian of the system is related to its Lagrangian via Legendre transformation by

$$H = \sum_k \pi_k\dot{\phi}_k - L(\phi_k, \dot{\phi}_k).$$

In our case the generalized coordinates are x_k, ψ_k, performing the procedure above, we find that the Hamiltonian is

$$H = \dot{x}_j p_j - \dot{\psi}_j\frac{i}{2}\psi_j - L = \frac{1}{2}p^2$$

where $p_j = \dot{x_j}$ is the canonical momentum of the even coordinates x_j. If we quantize this system, the momentum operator becomes proportional to spatial derivative, and we have that $H = -\Delta/2$, proportional to the Laplacian.

Variation of the Lagrangian L gives

$$\delta L = \dot{x}_j\frac{\mathrm{d}}{\mathrm{d}t}\delta x_j + \frac{i}{2}\delta\psi_j\dot{\psi}_j + \frac{i}{2}\psi_j\frac{\mathrm{d}}{\mathrm{d}t}\delta\psi_j.$$

Using the expression above, we investigate how L changes under supersymmetric transformation:

Theorem 4.1. *The action of the Lagrangian L defined above is invariant under the supersymmetric transformation*

$$\delta_\epsilon x_j = i\epsilon\psi_j$$

$$\delta_\epsilon\psi_j = -\epsilon\dot{x}_j.$$

Here, ϵ is an infinitesimal ***Grassmann (odd)*** *real constant.*

Proof. A simple calculation yields

$$\begin{aligned}\delta_\epsilon L &= i\dot{x}_j\epsilon\dot{\psi}_j - \frac{i}{2}\epsilon\dot{x}_j\dot{\psi}_j - \frac{i}{2}\psi_j\epsilon\ddot{x}_j \\ &= i\dot{x}_j\epsilon\dot{\psi}_j - \frac{i}{2}\epsilon\dot{x}_j\dot{\psi}_j - \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_j\epsilon\dot{x}_j\right) + \frac{i}{2}\dot{\psi}_j\epsilon\dot{x}_j \\ &= -\frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_j\epsilon\dot{x}_j\right)\end{aligned}$$

which is a total derivative. Hence the action which is the integral of L is invariant under δ_ϵ. □

Let us define **supercharge** Q to be the associated conserved charge by Noether's theorem:

$$\epsilon Q \equiv i\epsilon p_j\psi_j = i\epsilon\psi_j p_j = i\epsilon\psi_j\dot{x}_j.$$

Let us now study how do these physical quantities change under δ_ϵ, and their relations.

Theorem 4.2. *Under the SUSY transformation defined in Theorem 4.1, we have:*

(1) $\delta_\epsilon L = \frac{\epsilon}{2}\frac{\mathrm{d}Q}{\mathrm{d}t}$;

(2) $\delta_\epsilon Q = -2i\epsilon L$;

(3) $[\delta_{\epsilon_2}, \delta_{\epsilon_1}] = \delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2} = -2i\epsilon_1\epsilon_2\frac{\partial}{\partial t}$;

(4) Let $H = -p^2/2$ *be the Hamiltonian defined before. Then we have* $Q^2 = -H$*, or equivalently* $H = (iQ)^2$.

Proof. (1) follows immediately from Theorem 4.1 and the definition of Q. The rest are just straightforward computation. For (2), we have

$$\begin{aligned}\delta_\epsilon Q &= i\left(\delta\psi_j\right)\dot{x}_j + i\psi_j\frac{\mathrm{d}}{\mathrm{d}t}\delta x_j = i\left(-\epsilon\dot{x}_j\right)\dot{x}_j + i\psi_j\left(i\epsilon\dot{\psi}_j\right) \\ &= -i\epsilon\dot{x}_j\dot{x}_j + \epsilon\psi_j\dot{\psi}_j = -2i\epsilon\left(\frac{1}{2}\dot{x}_j\dot{x}_j + \frac{i}{2}\psi_j\dot{\psi}_j\right) \\ &= -2i\epsilon L.\end{aligned}$$

For (3), note that

$$\delta_{\epsilon_2}\delta_{\epsilon_1}(x_j) = \delta_{\epsilon_2}(x_j + i\epsilon_1\psi_j) = x_j + i(\epsilon_1+\epsilon_2)\psi_j - i\epsilon_1\epsilon_2\dot{x}_j$$
$$\delta_{\epsilon_2}\delta_{\epsilon_1}(\psi_j) = \delta_{\epsilon_2}(\psi_j - \epsilon_1\dot{x}_j) = \psi_j - (\epsilon_1+\epsilon_2)\dot{x}_j - i\epsilon_1\epsilon_2\dot{\psi}_j$$
$$\delta_{\epsilon_1}\delta_{\epsilon_2}(x_j) = x_j + i(\epsilon_1+\epsilon_2)\psi_j - i\epsilon_2\epsilon_1\dot{x}_j$$
$$\delta_{\epsilon_1}\delta_{\epsilon_2}(\psi_j) = \psi_j - (\epsilon_1+\epsilon_2)\dot{x}_j - i\epsilon_2\epsilon_1\dot{\psi}_j$$

and the claim follows from simple substraction. As for (4), we have

$$\begin{aligned}\{Q,Q\} &= 2Q^2 = 2\left(ip_j\psi_j\right)\left(ip_k\psi_k\right)\\ &= -p_jp_k\left(\psi_j\psi_k+\psi_k\psi_j\right) = -p_jp_k\delta_{jk}\\ &= -2H,\end{aligned}$$

concluding the proof. □

Let us now quantize the system. We take $d = 2n$ to be an even integer. The representation of ψ_j is achieved by defining

$$\psi_j = \frac{1}{\sqrt{2}}\gamma_j,$$

where γ_j are the gamma matrices and they satisfy the relation

$$\{\gamma_j, \gamma_k\} = 2\delta_{jk}.$$

Note that this is almost a Clifford algebra except $\gamma_k^2 = 2$ instead whereas technically for a Clifford algebra we need generators to square to -1. However, this algebra is isomorphic to the Clifford algebra generated by e_k because if $e_k^2 = -1$ we can assign a algebra homomorphism that sends γ_k to $-ie_k$. Physicists prefer the Clifford algebra represented by gamma matrices since these are more common in quantum field theory.

The supercharge operator now becomes

$$Q = i\psi_jp_j = \frac{1}{\sqrt{2}}\gamma_j\frac{\partial}{\partial x_j} = \frac{1}{\sqrt{2}}\not{\partial}$$

which is proportional to the Dirac operator (Compare with Definition 3.22).

Now let us upgrade our system to a Riemannian manifold. The calculation we have done before remains mostly the same, we just need to express things in a covariant way. We have a Riemannian manifold M with $\dim M = 2n$, and $g_{\mu\nu}$ its Riemannian metric tensor. Our coordinates now become x^μ, which commute, and ψ^μ, which anticommute.

The odd vector $\psi^\mu(t) \in T_{x(t)}M$ for each instant t (The rigorous characterization of the vector $\psi^\mu(t)$ is given by Proposition 4.3). Thus under a transformation $x^\mu \to x'^\mu(x^\nu)$ we have

$$\psi^\mu \to \psi'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\psi^\nu.$$

Thus

$$\delta_\epsilon x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\delta_\epsilon x^\nu = \frac{\partial x'^\mu}{\partial x^\nu} i\epsilon\psi^\nu = i\epsilon\psi'^\mu$$

and

$$\begin{aligned}\delta\psi'^\mu &= \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\delta x^\lambda \psi^\nu + \frac{\partial x'^\mu}{\partial x^\nu}\delta\psi^\nu \\ &= \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda} i\epsilon\psi^\lambda\psi^\nu + \frac{\partial x'^\mu}{\partial x^\nu}(-\epsilon\dot{x}^\nu) = -\epsilon\dot{x}'^\mu.\end{aligned}$$

Note that in the last equality, the first term is zero by anticommutativity of Grassmann numbers and symmetry of derivatives. Thus the SUSY transformation is covariant under change of coordinates.

The supercharge introduced before now generalizes to

$$Q = ig_{\mu\nu}(x)\dot{x}^\mu\psi^\nu.$$

Note that when we quantize this system, $Q \sim g_{\mu\nu}p^\mu\gamma^\nu = \not{\partial}$ is the Dirac operator on M.

Now the question is, how do we construct SUSY Lagrangian on M? Theorem 4.2 part (b) suggests that we compute $\delta_\epsilon Q$, and read off L from the expression. We have that

$$\begin{aligned}\delta_\epsilon Q =& i\partial_\lambda g_{\mu\nu}\delta_\epsilon x^\lambda \dot{x}^\mu\psi^\nu + ig_{\mu\nu}\delta_\epsilon\dot{x}^\mu\psi^\nu + ig_{\mu\nu}\dot{x}^\mu\delta_\epsilon\psi^\nu \\ =& i\partial_\lambda g_{\mu\nu} i\epsilon\psi^\lambda\dot{x}^\mu\psi^\nu + ig_{\mu\nu}(i\epsilon\dot{\psi}^\mu)\psi^\nu + ig_{\mu\nu}\dot{x}^\mu(-\epsilon\dot{x}^\nu) \\ =& -2i\epsilon\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}\psi^\nu\dot{\psi}^\mu - \frac{i}{2}\dot{x}^\mu\frac{1}{2}\left(\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu}\right)\psi^\lambda\psi^\nu\right) \\ =& -2i\epsilon\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}\psi^\nu\dot{\psi}^\mu + \frac{i}{2}\dot{x}^\mu g_{\lambda\rho}\Gamma^\rho_{\mu\nu}\psi^\lambda\psi^\nu\right)\end{aligned}$$

where

$$\Gamma^{\nu}_{\lambda\mu} = \frac{1}{2} g^{\nu\rho} \left(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu} \right)$$

is the Christoffel symbol associated with the Levi-Civita connection. In the third equality of the $\delta_\epsilon Q$ equation, we used

$$\dot{x}^\mu \frac{1}{2} (\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu}) \psi^\lambda \psi^\nu = \dot{x}^\mu \partial_\lambda g_{\mu\nu} \psi^\lambda \psi^\nu.$$

To see this, first note that the last term is $\sim \partial_\mu g_{\lambda\nu} \psi^\lambda \psi^\nu = 0$ because $g_{\lambda\nu}$ is symmetric under the exchange of λ, ν while $\psi^\lambda \psi^\nu$ is antisymmetric under such an exchange. The second term is equal to the first term because

$$\dot{x}^\mu \partial_\lambda g_{\mu\nu} \psi^\lambda \psi^\nu = \dot{x}^\mu \partial_\nu g_{\mu\lambda} \psi^\nu \psi^\lambda = -\dot{x}^\mu \partial_\nu g_{\mu\lambda} \psi^\lambda \psi^\nu.$$

Thus we may read off the Lagrangian directly from the $\delta_\epsilon Q$ equation and conclude that our SUSYQM Lagrangian must be

$$L = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \left(\frac{\mathrm{d}\psi^\nu}{\mathrm{d}t} + \dot{x}^\lambda \Gamma^\nu{}_{\lambda\kappa}(x) \psi^\kappa \right) \tag{4.1}$$

$$= \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \frac{D\psi^\nu}{Dt} \tag{4.2}$$

where $D\psi^\nu / Dt$ is the covariant derivative of ψ^ν along the curve $x(t)$.

We now define some symbols that will be used in the following sections. The connection 1-form is

$$\Gamma^\mu_\nu = \mathrm{d}x^\lambda \Gamma^\mu_{\lambda\nu}$$

and the curvature 2-form is

$$\Omega^\mu_\nu = d\Gamma^\mu_\nu + \Gamma^\mu_\sigma \wedge \Gamma^\sigma_\nu = \frac{1}{2} R^\mu_{\nu\rho\sigma} dx^\rho \wedge dx^\sigma,$$

where the curvature tensor component is given by

$$R^\kappa_{\lambda\mu\nu} = \mathrm{d}x^\kappa \left(\nabla_\mu \nabla_\nu \frac{\partial}{\partial x^\lambda} - \nabla_\nu \nabla_\mu \frac{\partial}{\partial x^\lambda} \right) = \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\eta_{\nu\lambda} \Gamma^\kappa_{\mu\eta} - \Gamma^\eta_{\mu\lambda} \Gamma^\kappa_{\nu\eta}.$$

4.2 Connections to Supergeometry

We will now explain the connection between Dirac operators and supergeometry. Let $\mathbb{R}^{1|1}$ denote the super Euclidean space with one even variable t and one odd variable θ. Then $C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R}) \otimes \Lambda^\bullet \mathbb{R}$. We consider a quantum theory of maps

$$X : \mathbb{R}^{1|1} \to M$$

where M is a compact, oriented, $n = 2k$ dimensional spin manifold with a Riemannian metric. Consider the vector field (differential operator) on $\mathbb{R}^{1|1}$ given by

$$\mathcal{D} = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}.$$

A simple computation shows that

$$\mathcal{D}^2 = -\frac{\partial}{\partial t}.$$

This is not surprising, since $\mathcal{D}$ is essentially just the super conformal structure defined on a Super Riemann surface (except we are working with real manifolds now), and the computation follows identically. Now we consider the action

$$S = -\frac{1}{2} \int_{\mathbb{R}^{1|1}} \mathrm{d}t \mathrm{d}\theta g_{\mu\nu}(X) \frac{\mathrm{d}X^\mu}{\mathrm{d}t} \mathcal{D} X^\nu = -\frac{1}{2} \int_{\mathbb{R}^{1|1}} \mathrm{d}t \mathrm{d}\theta \langle \dot{X}, \mathcal{D} X \rangle.$$

We now try to simplify this expression. Since $X : \mathbb{R}^{1|1} \to M$ is a map between supermanifolds, we may Taylor expand the function $X = X(t, \theta)$ in powers of θ. Since $\theta^2 = 0$, we see that

$$X = x(t) + \theta \psi(t)$$

for some odd function $\psi(t)$. The precise characterization of ψ is the following (Recall Example 1.17):

Proposition 4.3. The function $\psi(t)$ is a section of the odd tangent bundle $\Pi(x^*TM)$ over $\mathbb{R}^1$, where x^*TM is the pullback of the tangent bundle $TM \to M$ over the map $x : \mathbb{R}^1 \to M$.

Proof. We already know from the expression of X that ψ must be an odd function, hence it remains to show that it transforms like a tangent vector. If we choose a different local coordinate $\tilde{x}^1, ..., \tilde{x}^n$ of M, then the new quantum map is given by $\tilde{X} = \tilde{x}(t) + \theta\tilde{\psi}(t)$. Now the ν-component of the new map after Taylor expansion at $\theta = 0$ is just

$$\tilde{X}^\nu = \tilde{x}^\nu(x + \theta\psi) = \tilde{x}^\nu(x) + \frac{\partial \tilde{x}^\nu}{\partial x^\mu}(\theta\psi^\mu) = \tilde{x}^\nu + \theta\tilde{\psi}^\nu.$$

Therefore, we see that

$$\tilde{\psi}^\nu = \psi^\mu \frac{\partial \tilde{x}^\nu}{\partial x^\mu}$$

which is the transformation rule of a vector in TM. Therefore, we conclude that ψ is a section the parity reversed tangent bundle $\Pi(x^*TM)$. □

In order to obtain a local coordinate expression of the Lagrangian, we can do the dθ integral. Plugging everything in and collecting only the term linear in θ (since it is the only term that survives the Grassmann integral), we get

$$\begin{aligned} S &= -\frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta g_{\mu\nu}(X)\frac{\mathrm{d}X^\mu}{\mathrm{d}t}\mathcal{D}X^\nu \\ &= -\frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta \left(g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\theta\psi^\rho \right)(\dot{x}^\mu + \theta\dot{\psi}^\mu)(\psi^\nu - \theta\dot{x}^\nu) \\ &= \frac{1}{2}\int_{\mathbb{R}^{1|1}} \mathrm{d}t\mathrm{d}\theta g_{\mu\nu}\theta(\dot{x}^\mu\dot{x}^\nu - \dot{\psi}^\mu\psi^\nu) - \theta\left(\frac{\partial g_{\mu\nu}}{\partial x^\rho}\psi^\rho\dot{x}^\mu\psi^\nu \right) \\ &= \frac{1}{2}\int_{\mathbb{R}^1} \mathrm{d}t \left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - g_{\mu\nu}\dot{\psi}^\mu\psi^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\psi^\rho\dot{x}^\mu\psi^\nu \right) \\ &= \frac{1}{2}\int_{\mathbb{R}} \langle\dot{x}(t), \dot{x}(t)\rangle - \langle\nabla_t\psi(t), \psi(t)\rangle \mathrm{d}t, \end{aligned}$$

where the last equality follows from the same computation as in the previous section when we carried out the $\delta_\epsilon Q$ computation. Comparing with (4.2) there is a missing factor of i in the second term, but this can be recovered by an appropriate rescaling of the odd coordinate θ and the spinor field ψ.

4.3 Index on Spin Manifolds

Now the spinor bundle $S = S(M)$ on M decomposes into the even and odd parts $S = S^{\pm}$. Recall that the index of the Dirac operator is given by

$$\mathrm{ind}(\not{\partial}) = \dim\ker D - \dim\ker D^{\dagger}$$

where $D : \Gamma(S^+) \to \Gamma(S^-)$ and $D^{\dagger} : \Gamma(S^-) \to \Gamma(S^+)$ are the two pieces of the self-adjoint operator $\not{\partial}$ (Theorem 3.23):

$$\not{\partial} = \begin{pmatrix} 0 & D^{\dagger} \\ D & 0 \end{pmatrix}.$$

Proposition 4.4. $\mathrm{ind}(\not{\partial})$ is invariant under small perturbation of D.

Proof. First, note that $DD^{\dagger}, D^{\dagger}D$ are nonnegative operators, for $\langle\psi|DD^{\dagger}|\psi\rangle = \|D^{\dagger}|\psi\rangle\|^2$ which is nonnegative, and similarly for $D^{\dagger}D$. By the same formula we see that $\ker D^{\dagger} = \ker DD^{\dagger}$, and similarly $\ker D = \ker D^{\dagger}D$. Now let $\{|\phi_n\rangle\}$ be an orthonormal set of eigenvectors of $D^{\dagger}D$ with eigenvalue $\lambda_n > 0$. Then it is easy to see that $|\psi_n\rangle := D|\phi_n\rangle/\sqrt{\lambda_n}$ are eigenvectors of $DD^{\dagger}$ with the same eigenvalue λ_n. Also note that they are orthonormal, hence form a basis:

$$\langle\psi_n|\psi_m\rangle = \frac{1}{\sqrt{\lambda_n\lambda_m}}\langle\phi_n|D^{\dagger}D|\phi_m\rangle = \frac{\lambda_n}{\sqrt{\lambda_n\lambda_m}}\delta_{nm} = \delta_{nm}.$$

Note that $|\phi_n\rangle \in (\ker D)^{\perp}$ and $|\psi_n\rangle \in (\ker D^{\dagger})^{\perp}$, and there is an isomorphism $(\ker D)^{\perp} \cong (\ker D^{\dagger})^{\perp}$, which is a standard result in functional analysis. Hence we see that if under a small perturbation of D, $\dim\ker D$ decreases (or increases), then $\dim\ker D^{\dagger}$ must also decrease (or increase) by the same dimension to preserve the isomorphism $(\ker D)^{\perp} \cong (\ker D^{\dagger})^{\perp}$. □

Theorem 4.5. *Let $\not{\partial}$ be the dirac operator, then*

$$\mathrm{ind}(\not{\partial}) = \mathrm{Tr}\,\mathrm{e}^{-\beta D^{\dagger}D} - \mathrm{Tr}\,\mathrm{e}^{-\beta DD^{\dagger}},$$

where $\beta > 0$ is a real constant. In fact, $\mathrm{ind}(Q)$ *is independent of β.*

Proof. Let $|\phi_n\rangle, |\psi_n\rangle$ be defined as in the proof of the previous proposition. Then let $\{|\phi_{i,o}\rangle\}$ be orthonormal eigenvectors of $\ker D$ and similarly $\{|\psi_{i,o}\rangle\}$ be orthonormal eigenvectors of $\ker D^\dagger$. Note that both lists are finite since D is assumed to be Fredholm. Then simple computation yields

$$\begin{aligned}
&\operatorname{Tr} \mathrm{e}^{-\beta D^\dagger D} - \operatorname{Tr} \mathrm{e}^{-\beta D D^\dagger} \\
&= \sum_n \langle \phi_n | \mathrm{e}^{-\beta D^\dagger D} | \phi_n \rangle - \sum_n \langle \psi_n | \mathrm{e}^{-\beta D D^\dagger} | \psi_n \rangle + \sum_i \langle \phi_{i,o} | \phi_{i,o} \rangle - \sum_j \langle \psi_{j,o} | \psi_{j,o} \rangle \\
&= \sum_n \mathrm{e}^{-\beta \lambda_n} (\langle \phi_n | \phi_n \rangle - \langle \psi_n | \psi_n \rangle) + \sum_i 1 - \sum_j 1 \\
&= \dim \ker D - \dim \ker D^\dagger.
\end{aligned}$$

This shows the desired equality. It is immediately clear from the calculation above that $\operatorname{ind}(D)$ is independent of β. □

By the previous discussion, we could identify Q with the Dirac operator. However, it is more conventional to identify iQ with the Dirac operator $\not{\partial}$. Since we have a matrix representation

$$iQ = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}.$$

Then by the generalization of Theorem 4.2 part (4) on manifolds, the Hamiltonian should be

$$H = (iQ)^2 = \begin{pmatrix} D^\dagger D & 0 \\ 0 & D D^\dagger \end{pmatrix}.$$

Recall that the chirality operator gives ± 1 on $S^\pm$. However, in physicist's notation it is more common to denote the chirality operator by

$$(-1)^F.$$

Then Theorem 4.5 can be written in a more compact form, with the right hand side known as the **Witten index**:

$$\operatorname{ind}(\not{\partial}) = \operatorname{Tr}(-1)^F \mathrm{e}^{-\beta H}. \tag{4.3}$$

In fact, if M is a spin manifold, and E its spin bundle, we already noted that $Q \sim \not{\partial}$ is the Dirac operator on M, and we note that $(-1)^F = \gamma_{2n+1} = i^n \gamma_1 \cdots \gamma_{2n}$. Since $\gamma_{2n+1}^2 = 1$, the eigenvalues of γ_{2n+1} are ± 1, which we call **chirality**. Recall that $S = S^{\pm}$ is the eigenspace decomposition with respect to $(-1)^F$. We assign $F = 0$ to sections of $\Gamma(M, S^+)$ and $F = 1$ to sections of $\Gamma(M, S^-)$, which explains the notation $(-1)^F$. In this case, the index of the supercharge Q is defined to be

$$\mathrm{ind}(Q) := \mathrm{ind}(\not{\partial}) = \dim\ker D - \dim\ker D^{\dagger}.$$

4.4 The Proof

Let $H = (iQ)^2 = \frac{1}{2} g_{\mu\nu} p^{\mu} p^{\nu}$ be the Hamiltonian corresponding to the Dirac operator Q. By the generalized version of Theorem 4.2, in curved space the Lagrangian of this system should be the generalization to the flat space SUSYQM Lagrangian, which we introduced in (4.2). With the knowledge of (4.3) and (2.5), while noting the fact that (4.2) contains both bosonic and fermionic fields, we are motivated to conclude that the index can be computed by a path integral over all bosonic and fermionic field configurations:

$$\mathrm{ind}(Q) = \mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H} = \int_{\mathrm{PBC}} \mathcal{D}x \mathcal{D}\psi \mathrm{e}^{-\int_0^{\beta} \mathrm{d}t L}, \tag{4.4}$$

where L is the Euclidean Lagrangian after performing Wick rotation to (4.2). Note that since L in the formula above is in Euclidean time, to obtain L we should replace $t \to -it$ in (4.2) and replace $\mathrm{d}t \to -i\mathrm{d}t$ to get

$$L = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^{\mu} \dot{x}^{\nu} + \frac{1}{2} g_{\mu\nu}(x) \psi^{\mu} \frac{D\psi^{\nu}}{Dt}.$$

As a sanity check, note that compared to (4.2), the sign of the momentum term is unchanged. We should explain why we integrate over PBC instead of APBC, since for grassmann-valued fields (2.11) seems to indicate we should integrate over APBC. This is because the factor $(-1)^F$ disappears if the APBC for the fermionic variable is changed into a periodic one. To see this, let us just instead consider the simple case of a fermionic oscillator. Recall that from (2.11) we have

$$\begin{aligned}\mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H} &= \sum_n \langle n | (-1)^F \mathrm{e}^{-\beta H} | n \rangle \\ &= \int \mathrm{d}\theta^* \mathrm{d}\theta \langle -\theta | (-1)^F \mathrm{e}^{-\beta H} | \theta \rangle \mathrm{e}^{-\theta^* \theta}.\end{aligned}$$

Note that since $|\theta\rangle = |0\rangle + |1\rangle\,\theta$ and $F = c^\dagger c$ is the fermion number operator, we have that

$$(-1)^F\,|\theta\rangle = (-1)^0\,|0\rangle + (-1)^1\,|1\rangle\,\theta = |0\rangle - |1\rangle\,\theta = |-\theta\rangle\,,$$

hence the above integral becomes

$$\int \mathrm{d}\theta^*\mathrm{d}\theta\langle\theta|\mathrm{e}^{-\beta H}|\theta\rangle\mathrm{e}^{-\theta^*\theta}.$$

Therefore, we conclude that we must integrate over PBC. The rest of the section is dedicated to evaluate the path integral in (4.4).

Now the action in the exponential of (4.4) reads

$$S = \int_0^\beta \mathrm{d}t\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu\frac{D\psi^\nu}{Dt}\right).$$

Perform a change of variable $t = \beta s$, then $\dot{x}^\mu \to \dot{x}^\mu/\beta$ and $\mathrm{d}t \to \beta\mathrm{d}s$. Moreover, since the measure $\mathcal{D}\psi$ is invariant under scaling, we can perform a change of variable $\psi \to \psi/\sqrt{\beta}$. Then the action reads

$$S = \frac{1}{\beta}\int_0^1 ds\left(\frac{1}{2}g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} + \frac{1}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{Ds}\right).$$

In the $\beta \to 0$ limit, clearly the integal (4.4) localizes at constant paths x (ones with $\dot{x}^\mu = 0$). Moreover, since

$$\frac{D\psi^\nu}{Ds} = \frac{\mathrm{d}\psi^\nu}{\mathrm{d}s} + \Gamma^\nu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}\psi^\nu}{\mathrm{d}s},$$

and classical paths satisfies $\dot{x}^\alpha = 0$, so we conclude that $\dot{\psi}^\nu = 0$ as well. Hence (4.4) localizes at constant paths and constant spinor fields.

Having identified the region in which the integral (4.4) localizes in the small β limit, our next goal is to simplify the Lagrangian L in the small β limit as much as possible. To make our lives easier, we pick a normal coordinate centered at x_0, i.e., a coordinate system in which

$$g_{\mu\nu}(x_0) = \delta_{\mu\nu},$$

$$\partial_\lambda g_{\mu\nu}(x_0) = 0.$$

Moreover, we may expand the paths and fields in Fourier modes by the PBC condition:

$$x^{\mu}(t) = x_0^{\mu} + \xi^{\mu}(t) = x_0^{\mu} + \frac{1}{\sqrt{\beta}} \sum_{n \neq 0} \xi_n^{\mu} \mathrm{e}^{2\pi int/\beta}, \tag{4.5}$$

$$\psi^{\mu}(t) = \psi_0^{\mu} + \eta^{\mu}(t) = \psi_0^{\mu} + \frac{1}{\sqrt{\beta}} \sum_{n \neq 0} \eta_n^{\mu} \mathrm{e}^{2\pi int/\beta}. \tag{4.6}$$

In our normal coordinate, the Taylor expansion of the metric tensor is

$$g_{\mu\nu}\left(x_0 + x\right) = \delta_{\mu\nu} + \frac{1}{2!} g_{\mu\nu,\alpha\beta} x^{\alpha} x^{\beta} + \cdots .$$

Then the Lagrangian becomes

$$L = \frac{1}{2}\left(\delta_{\mu\nu} + \frac{1}{2!} g_{\mu\nu,\rho\sigma} x^{\rho} x^{\sigma} + \dots\right) \dot{x}^{\mu} \dot{x}^{\nu} + \frac{1}{2} g_{\mu\nu} \psi^{\mu} \left(\frac{\partial \psi^{\nu}}{\partial t} + \dot{x}^{\lambda} \Gamma^{\nu}_{\lambda k}(x) \psi^{k}\right)$$

Now expanding the Christoffel symbol above, and throwing away terms of quadratic order or higher, and using $\Gamma^{\mu}_{\nu\rho}(x_0) = 0$, we get that

$$L = \frac{1}{2} \dot{x}^{\mu} \dot{x}^{\mu} + \frac{1}{2} g_{\mu\nu} \psi^{\mu} \left(\frac{\partial \psi^{\nu}}{\partial t} + \dot{x}^{\lambda} \psi^{k} \Gamma^{\nu}_{\lambda k,\alpha}\left(x_0\right) x^{\alpha}\right).$$

But in the normal coordinate we have

$$\partial_{\alpha} \partial_{\beta} g_{\mu\nu}\left(x_0\right) = R_{\mu\alpha\nu\beta}\left(x_0\right),$$

and the second term in the L formula above is

$$\Gamma^{\nu}_{\lambda k,\alpha} = \frac{1}{2} g^{\delta\nu}\left(g_{\delta\lambda,k\alpha} + g_{\delta k,\lambda\alpha} - g_{\lambda k,\delta\alpha}\right).$$

Thus

$$g_{\delta\nu} \Gamma^{\nu}_{\lambda k,\alpha} = \frac{1}{2}\left(R_{\delta\alpha\lambda k} + R_{\delta\alpha k\lambda} - R_{\lambda\alpha k\delta}\right).$$

Note that the first two curvature tensor components cancel, because the curvature tensor is antisymmetric in the last two index. Thus

$$\Gamma^{\nu}_{\lambda k,\alpha} = -\frac{1}{2} R^{\nu}_{\lambda\alpha k}.$$

Hence putting everything together, after the expansion we have

$$L = \frac{1}{2}\dot{x}^{\mu}\dot{x}^{\mu} + \frac{1}{2} g_{\mu\nu}\psi^{\mu}\left(\frac{\partial\psi^{\nu}}{\partial t} + \dot{x}^{\lambda}\psi^{k}x^{\alpha}\left(-\frac{1}{2}R^{\nu}_{\lambda\alpha k}(x_0)\right)\right).$$

Now we plug in (4.5) and (4.6) into the above Lagrangian. However, we should note that L contains a curvature term $\sim R_{\lambda\alpha k\mu}\psi^{\mu}\dot{x}^{\lambda}\psi^{k}x^{\alpha}$. We can expand ψ^{μ} using (4.6), and if we include terms other than the zero mode ψ_0^{μ}, then this term would involve a power $1/\sqrt{\beta}$, which will be exponentially suppressed when we evaluate (4.4) in the small β limit. Therefore, we only need to look at the curvature term $\sim R_{\lambda\alpha k\mu}\psi_0^{\mu}\dot{\xi}^{\lambda}\psi_0^{k}\xi^{\alpha}$ in the expansion. Therefore, putting everything together, and relabeling the indices, we get that the effective action in (4.4) is

$$S = \int_0^{\beta} \mathrm{d}t \left(\frac{1}{2}\frac{\mathrm{d}\xi^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}\xi^{\mu}}{\mathrm{d}t} + \frac{1}{2}\eta^{\mu}\frac{\mathrm{d}\eta^{\mu}}{\mathrm{d}t} - \frac{1}{2}\omega_{\mu\nu}(x_0)\xi^{\mu}\frac{\mathrm{d}\xi^{\nu}}{\mathrm{d}t}\right) \tag{4.7}$$

where we have put

$$\omega_{\mu\nu}(x_0) = \frac{1}{2}R_{\mu\nu\rho\sigma}(x_0)\psi_0^{\rho}\psi_0^{\sigma}.$$

To get rid of the β dependence in the integral, or rather to hide the β independence into the fields, we make the substitution

$$t \to \beta t' \qquad \eta \to \eta' \qquad \xi \to \sqrt{\beta}\xi' \qquad \psi_0 \to \psi_0'/\sqrt{\beta}.$$

The first substitution makes our range of integration goes from 0 to 1. The last substitution is fine since in (4.4), we are integrating over all ψ_0, so a scaling does not matter. The scaling $\xi \to \sqrt{\beta}\xi'$ is more tricky, but note that after the substitution the term $R_{\lambda\alpha k\mu}\psi_0^{\mu}\dot{\xi}^{\lambda}\psi_0^{k}\xi^{\alpha}$ still survives the small β limit, but any other terms in $R_{\lambda\alpha k\mu}\psi^{\mu}\dot{\xi}^{\lambda}\psi^{k}\xi^{\alpha}$ that contains other than the zero mode ψ_0^{μ} would involve a power $1/\sqrt{\beta}$, which will be exponentially suppressed in the small β limit. Hence our expansion is still valid. In conclusion, after the substitution and relabeling again by removing all the primes, we see that (4.7) can be written as

$$S = \int_0^1 \mathrm{d}t \left(\frac{1}{2} \frac{\mathrm{d}\xi^\mu}{\mathrm{d}t} \frac{\mathrm{d}\xi^\mu}{\mathrm{d}t} + \frac{1}{2} \eta^\mu \frac{\mathrm{d}\eta^\mu}{\mathrm{d}t} - \frac{1}{2} \omega_{\mu\nu}(x_0) \xi^\mu \frac{\mathrm{d}\xi^\nu}{\mathrm{d}t} \right). \tag{4.8}$$

From (4.8), one can directly read off the fluctuation operator for ξ. After performing integration by part, it is

$$-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t.$$

And the fluctuation operator for η is simply

$$\delta_{\mu\nu}\partial_t.$$

Now that we have identified that (4.4) localizes at constant paths in the small β limit, and we know what L looks like in such limit, we might be tempted to evaluate (4.4) at constant paths at x_0, ψ_0, and integrate over all such points. But we cannot just reduce the path integrals to ordinary integrals directly, since we have only calculated the small β limit, whereas the region of integration localizes to constant path only at $\beta = 0$. Hence, we should be reducing the path integral to a small neighborhood of constant paths instead of just constant paths. This means that we should do the path integral over the mode expansions ξ, η as well. Thus (4.4) becomes, after performing Gaussian integration using (2.3) and (2.9):

$$\begin{aligned} \mathrm{ind}(Q) &= \int \mathrm{d}x_0 \mathrm{d}\psi_0 \int \mathcal{D}\xi \mathcal{D}\eta \mathrm{e}^{-\int_0^1 \mathrm{d}t \frac{1}{2}[\xi^\mu(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t)\xi^\nu + \eta^\mu(\delta_{\mu\nu}\partial_t)\eta^\nu]} \\ &= \mathcal{N} \int \mathrm{d}x_0 \mathrm{d}\psi_0 \det(\delta_{\mu\nu}\partial_t) \det\bigl(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t\bigr)^{-1/2}, \end{aligned}$$

where $\mathcal{N}$ is a formal constant, and $\det(\delta_{\mu\nu}\partial_t)$ can be evaluated using the ζ-function regulator. The computation for $\det(\delta_{\mu\nu}\partial_t)$ is done in Chapter 1 of [4]. But we will omit this since it is not of our concern: the result will only be a constant depending on the dimension d. Next we evaluate

$$\det\bigl(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t\bigr).$$

Note that $\omega_{\mu\nu}$ defined above is skew-symmetric with even matrix element. Thus using the splitting principle of real vector bundles, one can assume that there is an orthogonal transformation that turns $\omega_{\mu\nu}$ into the form

$$\omega = \begin{pmatrix} 0 & r_1 & \cdots & 0 & 0 \\ -r_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & r_n \\ 0 & 0 & \cdots & -r_n & 0 \end{pmatrix}$$

where $n = d/2$ is half the dimension of the spin manifold M. If we just look at one block of ω, which is of the form

$$A = \begin{pmatrix} -\partial_t^2 & -r \\ r & -\partial_t^2 \end{pmatrix}.$$

It is not hard to see that O has the same spectrum as

$$A' = \begin{pmatrix} -\partial_t^2 - ir\partial_t & 0 \\ 0 & -\partial_t^2 + ir\partial_t \end{pmatrix}.$$

Hence

$$\det A = \det A' = \det\bigl(-\partial_t^2 - ir\partial_t\bigr) \det\bigl(-\partial_t^2 + ir\partial_t\bigr).$$

We will only evaluate one of the determinants, since the other is similar. Define the differential operator

$$L = -\partial_t^2 - ir\partial_t.$$

A complete basis of eigenfunctions is given by

$$\phi_n(t) = \mathrm{e}^{2\pi i n t}$$

for all integers n. Then

$$L\phi_n = (-(2\pi i n)^2 - ir(2\pi i n))\phi_n = (4\pi^2 n^2 - 2\pi r n)\phi_n.$$

So the eigenvalues are

$$\lambda_n = 4\pi^2 n^2 - 2\pi n r = 2\pi n(2\pi n - r).$$

Hence, throwing everything that is independent of r into the formally infinite constant in front, we get

$$\begin{aligned}
\det L &= \prod_{n\neq 0} 2\pi n(2\pi n - r) \\
&= (2\pi)^\infty \prod_{n\neq 0} n(2\pi n - r) \\
&= \mathcal{N} \prod_{n\geq 1} (2\pi n - r)(-2\pi n - r) \\
&= \mathcal{N}' \prod_{n\geq 1} \left(1 - \left(\frac{r}{2\pi n}\right)^2\right)
\end{aligned}$$

and we will ignore $\mathcal{N}'$ as it can be thrown into the ζ-function regularization. Now using the identity

$$\prod_{n\geq 1}\left(1 - \frac{x^2}{n^2}\right) = \frac{\sin(\pi x)}{\pi x},$$

we get

$$\det L = \frac{\sin r/2}{r/2}$$

up to a normalization constant. The other determinant will give exactly the same result by similar calculation. Hence

$$\det A = \left(\frac{\sin r/2}{r/2}\right)^2.$$

The quantity of interest $\det\bigl(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t\bigr)$ will be the product of determinants of the block operator above, with r running from $r_1, ..., r_n$. Hence,

$$\det\bigl(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t\bigr) = \prod_{k=1}^{n}\left(\frac{\sin r_k/2}{r_k/2}\right)^2.$$

Using the identity $\sinh(ir) = i\sin(r)$, our calculation yields

$$\det\bigl(-\delta_{\mu\nu}\partial_t^2 - \omega_{\mu\nu}\partial_t\bigr)^{-1/2} = \prod_{k=1}^{n}\left(\frac{r_k/2}{\sin r_k/2}\right) = j(i\Omega)^{-1/2},$$

using the identification of $\omega_{\mu\nu}$ on ΠTM with the curvature 2-form

$$\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma} \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma$$

on M, where $j(\Omega)$ is by definition

$$j(\Omega) = \det\left(\frac{\sinh \Omega/2}{\Omega/2}\right) := \prod_{k=1}^{n} \left(\frac{\sinh r_k/2}{r_k/2}\right)^2$$

and r_k are the entries of the block form of the curvature 2-form Ω, and the $\hat{A}$-class of M is by definition

$$\hat{A}(M) = j(\Omega)^{-1/2}.$$

Therefore, we see that (absorbing i^d from $j(i\Omega)^{-1/2}$ into C^{2d}) under the identification of $C^\infty(\Pi TM) \cong \mathcal{A}^\bullet(M)$, which we already used above to identify $\omega_{\mu\nu}$ with the curvature 2-form $\Omega_{\mu\nu}$ on M, that

$$\mathrm{ind}(Q) = C^{2d} \int_{\Pi TM} \mathrm{d}x_0 \mathrm{d}\psi_0 \; j(\Omega)^{-1/2} = C^{2d} \int_M \hat{A}(M).$$

In this derivation we have been careless with universal constants depending only on dimension (infinite but regularized in some manner). To evaluate the constant one can either follow through the regularization procedure or simply evaluate it on one example where the index is nontrivial.

Therefore, we have proven the celebrated Atiyah Singer index theorem for the index of a Dirac operator on a spin manifold:

Theorem 4.6 (Atiyah-Singer)**.** *The index of the Dirac operator $\not{\partial}$ defined on a spin manifold M is given by*

$$\mathrm{ind}(\not{\partial}) = \int_M \hat{A}(M).$$

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