

Homological Algebra: The study of complexes of R modules where R is a ring ($1 \in R, 1_R \neq 0_R$), where complex defined as follows:

Notation. $R\text{-Mod}$: category of all left R -modules.

$R\text{-mod}$: " finitely generated left R -modules.

$$(\text{Im } d_n \subseteq \text{Ker } d_{n-1})$$

Defⁿ. • Let $A_n \in R\text{-Mod}$ for $n \in \mathbb{Z}$, $d_n \in \text{Hom}_R(A_n, A_{n-1})$ s.t. $d_{n-1} \circ d_n = 0$

$\forall n \in \mathbb{Z}$. Then the seq

$$\cdots \rightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \rightarrow \cdots$$

is called a **complex** of R -modules. This seq is abbrev. as A .

- If only finitely many nonzero terms:

$$0 \rightarrow A_m \rightarrow A_{m-1} \rightarrow \cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow 0.$$

- A complex A is called an **Exact Sequence** if $\text{Im } d_n = \text{Ker } d_{n-1}, \forall n$.

A short Exact seq:

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

Given a complex A , we call

$$H_n(A) := \text{Ker } d_{n-1} / \text{Im } d_n$$

the n^{th} **homology module** of A . A measure of how far A deviates from an Exact seq.

Remarks & Examples:

$$\bullet \quad 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

SES. If $A \xrightarrow{f} B$ known. Then f is a monomorphism (injective module hom), and g is a epimorphism (surjective "). Then

$$C \cong B / \text{Ker } g \cong B / \text{Im } f.$$

Not possible to know what C is if f is unknown. Then infinitely

many C is possible to make a SES.

E.g. $R = K$ a field. Then $R\text{-Mod} = \text{vector spaces}$.

$$A = B = K^{(\mathbb{N})} = \bigoplus_{i \in \mathbb{N}} K b_i.$$

a) $0 \rightarrow A \xrightarrow{1} B \rightarrow 0 \rightarrow 0$ SES.

b) Take $f: A \rightarrow B$ $f(b_i) = b_{2i}$, $i \in \mathbb{N}$

$$g(b_i) = \begin{cases} 0 & i \text{ even} \\ b_{\tau(i)} & i \text{ odd} \end{cases}$$

where τ bijection $\{\text{odds in } \mathbb{N}\} \rightarrow \mathbb{N}$.

Then $\ker g = \text{Im } f$. Thus

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

where $A = B = C = K^{(\mathbb{N})}$ is a SES.

Ex. 1) $R = \mathbb{Z}$.

$$0 \rightarrow 3\mathbb{Z} \xrightarrow{\text{incl}} \mathbb{Z} \xrightarrow{\text{quo}} \mathbb{Z}/3\mathbb{Z} \rightarrow 0$$

is a SES.

$$0 \rightarrow 6\mathbb{Z} \xrightarrow{A_1} \mathbb{Z} \xrightarrow{A_0} \mathbb{Z}/3\mathbb{Z} \rightarrow 0$$

is a complex, but not exact.

$$H_0 = 3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

2a) $R = K[X, Y]$, K a field.

ideal of R generated by X

$$0 \rightarrow (X) \xrightarrow[f]{\text{inc}} R \xrightarrow{g} K[Y] \quad A$$

where

$$g\left(\sum_{i,j} a_{ij} X^i Y^j\right) = \sum_j a_{0j} Y^j.$$

A is a complex. f mono. g epi. This is a SES.

$$2b) \quad 0 \rightarrow (X) \hookrightarrow R \xrightarrow{g} K = R/(X, Y) \rightarrow 0.$$

$$\sum_{i,j} a_{ij} x^i y^j \mapsto a_{00}.$$

$\ker g = \text{poly with } 0 \text{ constant term} = (X, Y).$

$\text{Im } f = (X).$

So this is a complex that is not exact.

$$H_0(A) = (X, Y)/(X) \cong (Y) \xrightarrow{\text{as } R\text{-module}} K[X, Y]$$

$K[X, Y]$ is an integral domain,
so every principal ideal is iso to $K[X, Y]$

$$K[X, Y] \rightarrow (Y)$$

$$a \mapsto aY.$$

Surjective. Injective
since $K[X, Y]$ integral domain,
so $aY = bY \Leftrightarrow (a-b)Y = 0 \Leftrightarrow a=b.$

module homomorphism!

Q1. Suppose

$$A \quad \dots \rightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \rightarrow \dots$$

is exact in $R\text{-Mod}$ and

$$F: R\text{-Mod} \rightarrow S\text{-Mod}$$

a functor. Then is

$$F(A) \quad \dots \rightarrow F(A_{n+1}) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \rightarrow \dots$$

an exact seq?

- No in general.

- $F(A)$ is a complex in case F is an additive functor.

Q2. Given A, C in $R\text{-Mod}$. Characterize all $B \in R\text{-Mod}$ st. $\exists B$ s.t.:

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Specifically: $R = \mathbb{Z}$, $A = B = \mathbb{Z}/p\mathbb{Z}$ p prime

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$$

$$x \mapsto (x, 0)$$

$$(x, y) \mapsto y$$

is exact.

Also:

$$\begin{aligned} 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0 \\ x+p\mathbb{Z} \mapsto px+p^2\mathbb{Z} \\ y+p^2\mathbb{Z} \mapsto y+p\mathbb{Z} \end{aligned}$$

is a SES.

In this case, $A=C=\mathbb{Z}/p$, then $B=\mathbb{Z}/p \oplus \mathbb{Z}/p$ or $B=\mathbb{Z}/p^2$ will be only possibilities. By Lagrange Thm, $|B|=p^2$. And only abelian gps of order p^2 is of this form.

Example from topology:

- 1) Standard n -simplex $\Delta_n \subseteq \mathbb{R}^n$ is the convex hull of $v_0, v_1, \dots, v_n$ where $v_0=0$, $v_i = (0, \dots, 1, \dots, 0)$.
 $\leftarrow i^{\text{th}} \text{ slot}$
 $\leftarrow \text{equiv class.}$
- 2) An oriented simplex is $(\Delta_x, [\pi])$ where π is a permutation of $\{0, \dots, n\}$, where $\pi \sim \pi' \Leftrightarrow \text{sign}(\pi) = \text{sign}(\pi')$. Write $(\Delta_x, [\pi]) = [v_{\pi(0)}, \dots, v_{\pi(n)}]$.
Identify $\Delta_n = [0, 1, \dots, n]$. $-[w_0, \dots, w_n] = [w_1, w_0, w_2, \dots, w_n]$
- 3) Let X be a topological space. A simplex in X is a continuous map $\sigma : \Delta_n \rightarrow X$
 $\Delta_n \stackrel{\text{def}}{=} \mathbb{R}^n$, Euclidean topology.
- 4) The group of n -chains of X :
 $S_n(X) = \text{FAG}$ with basis all n -simplices in X .
- 5) The singular chain complex of X :
 $\dots \rightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \rightarrow S_0(X) \rightarrow 0.$ ☺
Consider the map $\tau_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$
 $\tau_i(a_1, \dots, a_{n-1}) = (a_1, \dots, a_{i-1}, 0, a_i, \dots, a_{n-1})$
Then τ_i is linear (hence continuous) and
 $\tau_i(\Delta_{n-1}) = \Delta_n.$

Define

$$\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma \tau_i.$$

Fact: $\odot$ is a complex, since $\partial_{n-1} \partial_n = 0 \ \forall n$ ($\partial^2 = 0$).

$$Z_n(X) = \ker(\partial_n) \subseteq S_n(X)$$

$$B_n(X) = \operatorname{Im}(\partial_{n+1}) \subseteq S_n(X).$$

$$H_n(X) := Z_n(X) / B_n(X) \quad \text{singular homology groups.}$$

Chapter 1. Categories and Functors

VOCABULARY AND EXAMPLES

1. Definitions. A *category* $\mathcal{C}$ consists of

- A class $\text{obj } \mathcal{C}$ whose elements are called *objects*;
- A set $\text{Hom}_{\mathcal{C}}(A, B)$ for each $A, B \in \text{obj } \mathcal{C}$, whose elements are called *morphisms from A to B*;
- A map $\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$ for each $A, B, C \in \text{obj } \mathcal{C}$, written $(f, g) \mapsto gf$ and called *composition*;

satisfying

- (*Identity Axiom*) For each $A \in \text{obj } \mathcal{C}$, there exists a morphism 1_A (or id_A) in $\text{Hom}_{\mathcal{C}}(A, A)$ such that $f1_A = f$ for all $f \in \text{Hom}_{\mathcal{C}}(A, X)$ and $1_Ag = g$ for all $g \in \text{Hom}_{\mathcal{C}}(X, A)$, for any $X \in \text{obj } \mathcal{C}$;
- (*Associativity Axiom*) $h(gf) = (hg)f$ for all $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $g \in \text{Hom}_{\mathcal{C}}(B, C)$, $h \in \text{Hom}_{\mathcal{C}}(C, D)$, for any $A, B, C, D \in \text{obj } \mathcal{C}$.

2. Notes and further terminology. Let $\mathcal{C}$ be a category.

(1) Morphisms $f \in \text{Hom}_{\mathcal{C}}(A, B)$ are often written $f : A \rightarrow B$ or $A \xrightarrow{f} B$, even when they are not maps.

(2) It is tacitly assumed that the sets $\text{Hom}_{\mathcal{C}}(A, B)$ are pairwise disjoint. Thus, any morphism f in $\mathcal{C}$ belongs to a unique Hom-set $\text{Hom}_{\mathcal{C}}(A, B)$, in which case A and B are called the *domain* and *codomain* of f , respectively.

If this disjointness condition does not hold, it can be achieved by redefining $\text{Hom}_{\mathcal{C}}(A, B)$ to be the set of triples (A, f, B) for original morphisms $f : A \rightarrow B$.

(3) Identities are unique in the sense that for each $A \in \text{obj } \mathcal{C}$, there is exactly one morphism in $\text{Hom}_{\mathcal{C}}(A, A)$ satisfying the requirements of the Identity Axiom.

(4) A morphism $f \in \text{Hom}_{\mathcal{C}}(A, B)$ is called an *isomorphism* if it has an inverse, i.e., if there exists a morphism $g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$. If an inverse for f exists, it is unique, and it is denoted f^{-1} . Objects A and B in $\mathcal{C}$ are *isomorphic* (in $\mathcal{C}$), written $A \cong B$ if there exists an isomorphism from A to B .

3. Examples. (1) **Set.** Objects: all sets; morphisms: all maps between sets; composition: the usual composition of functions. Here the isomorphisms are the bijections.

(2) **Top.** Objects: all topological spaces; morphisms: all continuous maps; composition: usual. The description of a category of this type is usually abbreviated to “the category of topological spaces and continuous maps”. The isomorphisms in this category are the homeomorphisms.

(3) **Grp.** Groups and group homomorphisms. Here isomorphisms are the usual group isomorphisms.

The examples described so far are *concrete categories* – in the sense that the objects are sets paired with some kind of structure, the morphisms are functions which respect the structure in a specified way, and composition of morphisms is the usual composition of functions.

(4) Let P be any *poset*, i.e., a partially ordered set with respect to a partial order relation $\leq$. There is a category $\mathcal{C}$ with

- $\text{obj } \mathcal{C} = P$;
- $\text{Hom}_{\mathcal{C}}(x, y) = \begin{cases} \{(x, y)\} & (\text{if } x \leq y) \\ \emptyset & (\text{if } x \not\leq y) \end{cases}$ for all $x, y \in P$;
- $(y, z)(x, y) = (x, z)$ for any $x, y, z \in P$ with $x \leq y \leq z$.

The only isomorphisms in this category are the identity morphisms.

(5) The *dual* or *opposite* of a category $\mathcal{C}$ is the category $\mathcal{C}^{\text{op}}$ with

- $\text{obj } \mathcal{C}^{\text{op}} = \text{obj } \mathcal{C}$;
- $\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$ for all $A, B \in \text{obj } \mathcal{C}^{\text{op}}$;
- composition $*$ equal to the reverse of the composition in $\mathcal{C}$, that is, $g * f = fg$ for all $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(B, C)$.

16) R -ring. R -Mod, R -mod with ring homo as morphisms.

$\mathbb{Z}$ -Mod usually denoted by $\underline{Ab}$.

17) R -Comp. $\text{Obj}(R\text{-Comp}) = \text{class of all complexes of left } R\text{-modules.}$

Let $A, A' \in R\text{-Comp}$.

$$\begin{array}{ccccccc} A: & \cdots & \rightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \rightarrow & \cdots \\ & & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ A': & \cdots & \rightarrow & A'_{n+1} & \xrightarrow{d'_{n+1}} & A'_n & \xrightarrow{d'_n} & A'_{n-1} & \rightarrow & \cdots \end{array}$$

$$\text{Hom}_{R\text{-Comp}}(A, A') = \{ (f_n)_{n \in \mathbb{Z}} : f_n \in \text{Hom}_R(A_n, A'_n) \text{ s.t. } f_{n-1}d_n = d'_n f_n \forall n \}$$

18) Rings. Obj: Rings with identity.

Morphisms: $R \xrightarrow{f} S$ ring hom s.t. $f(1_R) = 1_S$.

Defⁿ. A category $\mathcal{C}$ is called **preadditive** if $\forall A, B \in \text{Obj}(\mathcal{C})$, the set $\text{Hom}_{\mathcal{C}}(A, B)$ is an (additive) abelian group s.t.

$$h(f+g) = hf + hg \quad (f+g)k = fk + gk$$

$\forall f, g, h, k \in \text{Hom}_{\mathcal{C}}(A, B)$ with suitable domains and codomains (so addition and composition is well-defined).

e.g. $R\text{-Mod}$ and $\underline{Ab}$ are additive.

$R\text{-Comp}$ with componentwise addition and composition:

$$(f_n) + (g_n) = (f_n + g_n) \quad (f_n)(g_n) = (f_n g_n).$$

Defⁿ. Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of

- an assignment $F: \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$, $A \mapsto F(A) \forall A \in \text{Obj}(\mathcal{C})$.

- For each pair $A, B \in \text{Obj}(\mathcal{C})$, a map

$$F: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$$

- st. (a) $F(\text{id}_A) = \text{id}_{F(A)}$
 (b) $F(fg) = F(f)F(g)$

e.g. 1) $\mathcal{C}$ category. $\text{id}_{\mathcal{C}}$ the identity functor.

2) Functor $\pi_n : \text{Top} \rightarrow \text{Grp}$, $X \mapsto \pi_n(X)$, and use induced maps.

3) Functor $\mathbb{S} : \text{Top} \rightarrow \mathbb{Z}\text{-comp}$

$$X \mapsto \mathbb{S}(X) : (\cdots \rightarrow S_{n+1}(X) \xrightarrow{\partial} S_n(X) \xrightarrow{\partial} S_{n-1}(X) \rightarrow \cdots)$$

let $\phi : X \rightarrow Y$ be continuous map.

$$[\mathbb{S}(\phi)]_n : S_n(X) \rightarrow S_n(Y)$$

$$\sigma \mapsto \phi \sigma$$

and extend this to a homomorphism linearly.

lec 4.

Defⁿ. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $T: \mathcal{C} \rightarrow \mathcal{D}$ is a function s.t.

- (i) If $A \in \text{Obj}(\mathcal{C})$, then $T(A) \in \text{Obj}(\mathcal{D})$.
- (ii) If $f: A \rightarrow A'$ in $\mathcal{C}$, then $T(f): T(A) \rightarrow T(A')$ in $\mathcal{D}$.
- (iii) If $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{C}$, then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathcal{D}$ and $T(gf) = T(g)T(f)$.
- (iv) $T(1_A) = 1_{T(A)}$.

A **contravariant functor** $T: \mathcal{C} \rightarrow \mathcal{D}$ amounts to a covariant functor $T: \mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$:

- (i) $C \in \text{Obj}(\mathcal{C}) \Rightarrow T(C) \in \text{Obj}(\mathcal{D}^{\text{op}}) = \text{Obj}(\mathcal{D})$.
- (ii) If $f \in \text{Hom}_{\mathcal{C}}(A, B)$, then $T(f) \in \text{Hom}_{\mathcal{D}^{\text{op}}}(T(A), T(B)) = \text{Hom}_{\mathcal{D}}(T(B), T(A))$
- (iii) If $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $g \in \text{Hom}_{\mathcal{C}}(B, C)$, then
$$T(gf) = \underbrace{T(g) * T(f)}_{\text{in } \mathcal{D}^{\text{op}}} = \underbrace{T(f) T(g)}_{\text{in } \mathcal{D}}.$$
- (iv) $T(1_A) = 1_{T(A)}$.

e.g. Homology functor $H_n: R\text{-Comp} \rightarrow \mathbb{Z}\text{-Mod}$.

Defⁿ. Let $\mathcal{C}, \mathcal{D}$ be preadditive categories. Then a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called **additive** if $\forall A, B \in \text{Obj}(\mathcal{C})$, the map

$$F: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$$

is a homomorphism of abelian groups.

Note. $H_n: R\text{-Comp} \rightarrow \mathbb{Z}\text{-Mod}$ is an additive functor.

E.g. • Forgetful functors :

$$\text{ex. } F: R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod} \quad \text{or} \quad F: R\text{-Mod} \rightarrow \text{Sets}$$

$$M \mapsto M \qquad \qquad M \mapsto M$$

If $\mathcal{C}, \mathcal{D}$ are preadditive and $F: \mathcal{C} \rightarrow \mathcal{D}$ is forgetful, then F is additive.

- Let $F: R\text{-Mod} \rightarrow S\text{-Mod}$ be an additive functor. Then F induces an additive functor $\tilde{F}: R\text{-Comp} \rightarrow S\text{-Comp}$:

$$\begin{array}{ccccccc} \text{IA:} & \cdots & \rightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \rightarrow \cdots \\ \downarrow & & & & & & & & \\ \tilde{F}(\text{IA}) & \cdots & \rightarrow & F(A_{n+1}) & \xrightarrow{F(d_{n+1})} & F(A_n) & \xrightarrow{F(d_n)} & F(A_{n-1}) & \rightarrow \cdots \end{array}$$

- Let $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{E}$ be functors. Then $G \circ F: \mathcal{C} \rightarrow \mathcal{E}$ is a functor.

e.g. (of interest): $F: R\text{-Mod} \rightarrow S\text{-Mod}$. Then $\tilde{F}: R\text{-Comp} \rightarrow S\text{-Comp}$.

then $H_n \circ \tilde{F}: R\text{-Comp} \rightarrow \text{Ab}$

$$A \mapsto H_n(\tilde{F}(\text{IA})).$$

Rmk. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ sends isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$.

§ 2: Hom & $\otimes$ functors.

(I) Hom-functors: Let $\mathcal{C}$ be a category. There is a **bifunctor**

$$\text{Hom}_{\mathcal{C}}(-, -): \mathcal{C} \times \mathcal{C} \rightarrow \text{Sets}$$

$$(A, B) \mapsto \text{Hom}_{\mathcal{C}}(A, B)$$

On maps. Let $f: A \rightarrow A'$, $g: B \rightarrow B'$ in $\mathcal{C}$. Then

$$\text{Hom}(f, g): \text{Hom}_{\mathcal{C}}(A', B) \rightarrow \text{Hom}_{\mathcal{C}}(A, B')$$

$$\varphi \mapsto g \circ \varphi \circ f.$$

Lec 5. Hom Functor.

Let $\mathcal{C}$ be category. $\text{Hom}(-, -): \mathcal{C} \times \mathcal{C} \rightarrow \text{Sets}$
 $(C, D) \mapsto \text{Hom}_{\mathcal{C}}(C, D).$

Let $f: C \rightarrow C', g: D \rightarrow D'$, then
 $\text{Hom}(f, g): \text{Hom}(C', D) \rightarrow \text{Hom}(C, D')$
 $\varphi \mapsto g \circ \varphi \circ f.$

Splits into 2 parts:

$F = \text{Hom}_{\mathcal{C}}(C, -)$ a covariant functor:

$$F(C') = \text{Hom}_{\mathcal{C}}(C, C').$$

$$f: A \rightarrow B, \quad F(f): \text{Hom}(C, A) \rightarrow \text{Hom}(C, B), \quad \varphi \mapsto f \circ \varphi$$

Similarly, $\text{Hom}_{\mathcal{C}}(-, D)$ is a contravariant functor.

Of major interest: $\mathcal{C} = R\text{-Mod}$. Then $\nearrow R\text{-modules are Ab groups under addition.}$
 $\text{Hom}_R(M, -): R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ covariant
 $\text{Hom}_R(-, M): R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ contravariant.

Additional structure on $\text{Hom}_R(M, N)$:

Prop. Suppose S is a ring and M is a left R right S bimodule, i.e.,
 $(rm)s = r(ms).$ $\downarrow$
 ${}_R M_S$ is the notation.

And ${}_R N_T$ is a left R right T bimodule, then $\text{Hom}_R(M, N)$
is a left S right T bimodule.

pf. Let $f \in \text{Hom}_R(M, N)$. Define

$$(sf)(m) = f(ms). \quad (ft)(m) = f(m) \cdot t.$$

□

Mnemonic: ${}_S \text{Hom}_R({}_R M_S, {}_R N_T)_T$.

- o If R is commutative, then $\text{Hom}_R(M, -): R\text{-Mod} \rightarrow R\text{-Mod} = \text{Mod-}R$
 $\text{Hom}_R(-, N)$ is the same. $\downarrow$
by $mr = rm$
- o If ${}_R M_S$ bimodule, then $\text{Hom}_R(M, -): R\text{-Mod} \rightarrow S\text{-Mod}$.
- o If ${}_R N_T$ bimodule, then $\text{Hom}_R(-, N): R\text{-Mod} \rightarrow \text{Mod-}T$.

Prop. Let $M, N, M_i, N_i \in R\text{-Mod}$. Then

- (1) $\text{Hom}_R(R, M) \cong M$ in $R\text{-Mod}$.
- (2) $\text{Hom}_R(M, \prod_{i \in I} N_i) \cong \prod_{i \in I} \text{Hom}_R(M, N_i)$.
- (3) $\text{Hom}_R(\bigoplus_{i \in I} M_i, N) \cong \prod_{i \in I} \text{Hom}_R(M_i, N)$.

These are all isomorphisms in Ab.

Pf. (1) Consider

$$\begin{aligned} \psi: \text{Hom}_R(R, M) &\rightarrow M \\ f &\mapsto f(1). \end{aligned}$$

This is a homomorphism: $\psi(f+g) = (f+g)(1) = f(1) + g(1) = \psi(f) + \psi(g)$.

This map is injective: If $f(1) = g(1)$, then since f, g are R -homomorphisms,
 $f(r) = g(r) \quad \forall r \in R$.

This map is surjective. Suppose $m \in M$. Then define $h: R \rightarrow M$ by

$h(1) = m$ and $h(r) = r f(1) = rm \quad \forall r \in R$, and extend this to a R -hom.

(2). See [Rotman] Thm 2.30 - (i) proof.

(3). [Rotman] Thm 2.31 - (i).

□

Tensor Functor.

Defⁿ. Let $M \in \text{Mod-}R$, $N \in R\text{-Mod}$. Then T is called a **tensor product** of M and N if there exists a map

$$\tau: M \times N \rightarrow T$$

which is $\mathbb{Z}$ -bilinear and **R -balanced** (i.e., $\tau(mr, n) = \tau(m, rn)$)

s.t. the following universal property holds:

Whenever A is an Ab-group, $\sigma: M \times N \rightarrow A$ is $\mathbb{Z}$ -bilinear and R -balanced, $\exists!$ $\mathbb{Z}$ -linear map $\sigma': T \rightarrow A$ s.t.

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau} & T \\ \sigma \searrow & \tau & \nearrow \exists! \sigma' \\ & A & \end{array}$$

Thm. $M \in \text{Mod-}R$, $N \in R\text{-Mod}$. Then a tensor product $M \otimes_R N$ exists and is unique up to isomorphism.

pf. Let F be the free abelian gp on basis $M \times N$:

$$F = \bigoplus_{(m,n) \in M \times N} \mathbb{Z}(m,n).$$

Define $M \otimes_R N = F/U$ where U is the submodule generated by

$$\left\langle \begin{array}{l} (m_1+m_2, n) - (m_1, n) - (m_2, n) \\ (m, n_1+n_2) - (m, n_1) - (m, n_2) \\ (mr, n) - (m, rn) \end{array} \middle| \begin{array}{l} \forall m_i, m \in M \\ \forall n_i, n \in N \\ \forall r \in R \end{array} \right\rangle.$$

Define $\tau: M \times N \rightarrow M \otimes_R N$ by $\tau(m,n) = m \otimes n =: (m,n) + U$.

Then τ is $\mathbb{Z}$ -bilinear and R -balanced.

Claim: $M \otimes_R N$ satisfies the universal property.

Let $\sigma: M \times N \rightarrow A$ be $\mathbb{Z}$ -linear, R -balanced. Let $\tilde{\sigma}: F \rightarrow A$ be the map $(m,n) \mapsto \sigma(m,n)$ extend linearly. By construction

$\tilde{\sigma}(U) = 0$. Thus $\exists$ unique map $\sigma': F/U (= M \otimes_R N) \rightarrow A$ s.t.

$$\begin{aligned} \sigma'(m \otimes n) &= \sigma'(\tau(m,n)) \\ &= \tilde{\sigma}(m,n) = \sigma(m,n). \end{aligned}$$

Uniqueness: Exercise

(See [Rotman] pf of Prop 2.44).

□

lec 6.

$$\mathbb{Z}\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$$

$$A \mapsto \mathbb{Z} \quad f \mapsto \text{id}_{\mathbb{Z}}.$$

This is a functor, but not additive.

Tensor Functor:

Note: $\{m \otimes n : m \in M, n \in N\}$ form a generating set of $M \otimes_R N$, but in general not a basis.

We have a bifunctor

$$\begin{aligned} - \otimes - : \text{Mod-}R \times R\text{-Mod} &\longrightarrow \mathbb{Z}\text{-Mod} \\ (M, N) &\longmapsto M \otimes_R N \end{aligned}$$

Let $(f, g) \in \text{Hom}(M, M') \times \text{Hom}(N, N')$. Define

$$\begin{aligned} f \otimes g : M \otimes_R N &\longrightarrow M' \otimes_R N' \\ m \otimes n &\longmapsto f(m) \otimes g(n) \end{aligned}$$

To show well-definedness, use UP. Check $\phi: M \times N \rightarrow M \otimes_R N$
 $(m, n) \mapsto f(m) \otimes g(n)$

is $\mathbb{Z}$ -bilinear and R -balanced.

The $- \otimes -$ functor is covariant in both argument.

Additional Structure:

Suppose ${}_S M_R$ and ${}_R N_T$ bimodule. Then $M \otimes_R N$ is S - T bimodule.

$$s(m \otimes n)t = (sm) \otimes (nt) \quad (\text{check well-defined}).$$

Basic Properties:

- $R \otimes_R M \cong M$ (isomorphism of R -modules & Abelian gps).
 $1 \otimes m \mapsto m$
- $M \otimes_R (\bigoplus_i N_i) \cong \bigoplus_i (M \otimes_R N_i)$.

Warning: $m \otimes n$ only make sense when $M \ni m, N \ni n$ are fixed.

e.g. Take $2 \otimes \bar{1} \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Now $2 \otimes \bar{1} = 1 \otimes \bar{2} = 1 \otimes 0 = 0 \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$.

But $2 \otimes \bar{1} \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is nonzero. To see this:

$$f: \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \quad x \mapsto \frac{1}{2}x.$$

$$\text{Then } f \otimes \text{id}_{\mathbb{Z}_2}: \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$$

$$x \otimes y \mapsto f(x) \otimes y.$$

is again an iso (functors take isos to isos).

$$f: 2 \otimes \bar{1} \mapsto 1 \otimes \bar{1} \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \text{ is nonzero.}$$

$$\downarrow \varphi$$

$$\bar{1} \in \mathbb{Z}_2$$

$$\text{via iso } \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \xrightarrow{\varphi} \mathbb{Z}_2 \quad \varphi(x, \bar{y}) = \overline{xy}.$$

§3. Natural Transformations, Equivalences, and Dualities.

Defⁿ. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. An **morphism** of functors, or a **natural transformation**, from F to G , is a family $(\phi(C))$ of morphisms, $C \in \text{Obj}(\mathcal{C})$, $\phi(C): F(C) \rightarrow G(C)$ s.t. for all $f \in \text{Hom}_{\mathcal{C}}(C, C')$, the square commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ \phi(C) \downarrow & \curvearrowright & \downarrow \phi(C') \\ G(C) & \longrightarrow & G(C') \end{array}$$

Call $(\phi(C))_{C \in \text{Obj}(\mathcal{C})}$ an **isomorphism** of functors (& **Natural Equivalence**) if all of the $\phi(C)$ are isomorphisms in $\mathcal{D}$.

Defⁿ. Two categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent**, if $\exists$ two functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$, s.t. $G \circ F \cong \text{id}_{\mathcal{C}}$, $F \circ G \cong \text{id}_{\mathcal{D}}$, where $\cong$ denote "...is naturally equivalent to..."

Rmk. R, S ring are called **Morita equivalent** if $R\text{-Mod}$ and $S\text{-Mod}$ are equivalent as categories.

Defⁿ. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be functors. We say (F, G) form an **adjoint pair** if the bifunctor $\mathcal{C} \times \mathcal{D} \rightarrow \text{Sets}$

$$\text{Hom}_{\mathcal{D}}(F(-), -) \cong \text{Hom}_{\mathcal{C}}(-, G(-))$$

are isomorphic.

↑
Viewed as
a new Cat
where
 $\text{Obj}(\mathcal{C} \times \mathcal{D})$ are
 (C, D) .

e.g. 1) R a ring. Then $R \otimes_R - \cong \text{id}_{R\text{-Mod}}$.
 $\text{Hom}_R(R, -) \cong \text{id}_{\text{Mod-}R}$.

2) $R \sim_{\text{Morita Equ}} M_n(R)$. $n \in \mathbb{N}$.

pf. Let $F_R = R^n$ (free R -mod with a basis of n elements). Then $S = \text{End}_R(F) \cong M_n(R)$, which can be proved just like the case $F = \text{field}$. Then ${}_S F_R$ a bimodule. Let

$$F^* = {}_R [\text{Hom}_R(F, R)]_S$$

• Claim: The functor $\text{Hom}_R(F, -) : \text{Mod-}R \rightarrow \text{Mod-}S$ & $\text{Hom}_S(F^*, -) : \text{Mod-}S \rightarrow \text{Mod-}R$ are mutually inverse functors.

• WTS: for $M \in \text{Mod-}R$,

$$\begin{array}{ccc} \Phi(M) : \text{id}_{\text{Mod-}R} & \xrightarrow{\cong} & \text{Hom}_S(F^*, \text{Hom}_R(F, -)) \\ M \ni m & \longmapsto & (F^* = \text{Hom}(F, R) \ni f \mapsto \underbrace{(mf(x))}_{\text{Hom}_R(F, -)}) \end{array}$$

check: $\Phi(M)$ is a R -module isomorphism.

3) Let $R \cong K$ be a field. Then there is a duality

$$\begin{array}{ccc} K\text{-mod} & \longrightarrow & K\text{-mod} : \quad (\text{finite dim vector space}) \\ \text{vector space iso} \rightarrow \Phi(V) : V & \longrightarrow & V^{**} = \text{Hom}_K(\text{Hom}_K(V, K), K) \\ x & \longmapsto & (\text{Hom}_K(V, K) \ni f \mapsto f(x) \in K). \end{array}$$

Then $(\Phi(V))_{V \in K\text{-mod}}$ is a duality $K\text{-mod} \rightarrow K\text{-mod}$.
(a contravariant equivalence)

Namely, this shows

$$\text{id}_{K\text{-mod}} \cong (-)^{**}$$

Extending Φ to a functor $K\text{-Mod} \rightarrow K\text{-Mod}$ where $\Phi(V) = V^{**}$ is not surjective if $\dim V = \infty$.

4) Adjoint pair: Let ${}_S B_R$ be a bimodule. Then the functor

$$B \otimes_R - : R\text{-Mod} \rightarrow S\text{-Mod}$$

is left adjoint to

$$\text{Hom}_S(B, -) : S\text{-Mod} \rightarrow R\text{-Mod}$$

Chapter II, Section 4. Additive and abelian categories

Definitions. In the following, let $\mathcal{C}$ be a pre-additive category, $A, B \in \text{obj } \mathcal{C}$, and $f \in \text{Hom}_{\mathcal{C}}(A, B)$.

(in Ab or $R\text{-mod}$, $\text{mono} = \text{inj}$, $\text{epi} = \text{surj}$).

(1) • f is called a *categorical monomorphism* in case $fg = fh$ implies $g = h$ for all morphisms $g, h \in \text{Hom}_{\mathcal{C}}(D, A)$ and $D \in \text{obj } \mathcal{C}$.

• f is called a *categorical epimorphism* in case $gf = hf$ implies $g = h$ for all morphisms $g, h \in \text{Hom}_{\mathcal{C}}(B, D)$ and $D \in \text{obj } \mathcal{C}$.

(2) • A *kernel* of f is a pair (K, k) with $K \in \text{obj } \mathcal{C}$ and $k \in \text{Hom}_{\mathcal{C}}(K, A)$ such that $fk = 0$ and the following universal property is satisfied: Whenever $h \in \text{Hom}_{\mathcal{C}}(K', A)$ with $K' \in \text{obj } \mathcal{C}$ and $fh = 0$, there exists a unique morphism $h' \in \text{Hom}_{\mathcal{C}}(K', K)$ such that $h = kh'$.

We write $K = \text{Ker}(f)$ in the above case and, by an abuse of language, we refer also to the object K , which is unique up to isomorphism, as “the” kernel of f .

• A *cokernel* of f is defined dually. In parallel with the definition of a kernel, the term cokernel is also applied to the object involved in the pertinent pair, and this object (unique up to isomorphism) is denoted by $\text{Coker}(f)$.

(3) The pre-additive category $\mathcal{C}$ is called *additive* if the following conditions (i) and (ii) hold:

(i) $\mathcal{C}$ has a *zero object*, i.e., there exists $O \in \text{obj } \mathcal{C}$ such that, for every $A \in \text{obj } \mathcal{C}$, both of the sets $\text{Hom}_{\mathcal{C}}(O, A)$ and $\text{Hom}_{\mathcal{C}}(A, O)$ are singletons.

(ii) $\mathcal{C}$ has *finite direct sums* and *finite direct products*, where direct sums and direct products are defined in terms of the familiar universal properties and the qualifier “finite” refers to the *corresponding index sets*.

(4) The pre-additive category $\mathcal{C}$ is said to be *abelian* if $\mathcal{C}$ is additive and the following additional conditions (iii) and (iv) hold:

(iii) Every morphism in $\mathcal{C}$ has a kernel and a cokernel.

? (iv) Every categorical monomorphism in $\mathcal{C}$ is a kernel, and every categorical epimorphism in $\mathcal{C}$ is a cokernel.

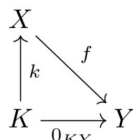
Kernels and Cokernels:

Let $\mathbf{C}$ be a [category](#). In order to define a kernel in the general category-theoretical sense, $\mathbf{C}$ needs to have [zero morphisms](#). In that case, if $f: X \rightarrow Y$ is an arbitrary [morphism](#) in $\mathbf{C}$, then a kernel of f is an [equaliser](#) of f and the zero morphism from X to Y . In symbols:

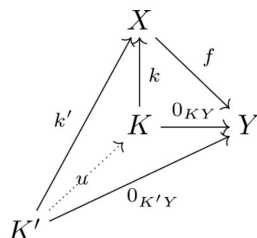
$$\ker(f) = \text{eq}(f, 0_{XY})$$

To be more explicit, the following [universal property](#) can be used. A kernel of f is an object K together with a morphism $k: K \rightarrow X$ such that:

- $f \circ k$ is the zero morphism from K to Y ;

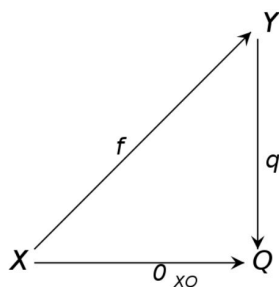


- Given any morphism $k': K' \rightarrow X$ such that $f \circ k'$ is the zero morphism, there is a unique morphism $u: K' \rightarrow K$ such that $k \circ u = k'$.

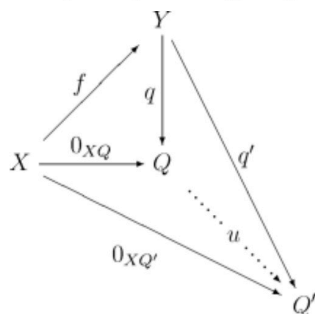


Note that in many [concrete](#) contexts, one would refer to the object K as the "kernel", rather than the morphism k . In those situations, K would be a [subset](#) of X , and that would be sufficient to reconstruct k as an [inclusion map](#); in the nonconcrete case, in contrast, we need the morphism k to describe *how* K is to be interpreted as a [subobject](#) of X . In any case, one can show that k is always a [monomorphism](#) (in the categorical sense). One may prefer to think of the kernel as the pair (K, k) rather than as simply K or k alone.

Explicitly, this means the following. The cokernel of $f: X \rightarrow Y$ is an object Q together with a morphism $q: Y \rightarrow Q$ such that the diagram



[commutes](#). Moreover, the morphism q must be [universal](#) for this diagram, i.e. any other such $q': Y \rightarrow Q'$ can be obtained by composing q with a unique morphism $u: Q \rightarrow Q'$:



As with all universal constructions the cokernel, if it exists, is unique [up to](#) a unique [isomorphism](#), or more precisely: if $q: Y \rightarrow Q$ and $q': Y \rightarrow Q'$ are two cokernels of $f: X \rightarrow Y$, then there exists a unique isomorphism $u: Q \rightarrow Q'$ with $q' = u \circ q$.

e.g. (1) In Rings, $\exists$ categorical epi but fails to be surjective. Let
 $f: \mathbb{Z} \hookrightarrow \mathbb{Q}$ embedding.

Let $g, h \in \text{Hom}_{\text{Ring}}(\mathbb{Q}, R)$ s.t. $gf = hf$. i.e. $g|_{\mathbb{Z}} = h|_{\mathbb{Z}}$. It follows that $g = h$:

$$[b \in \mathbb{Z} \setminus \{0\}, 1_R = g(1_{\mathbb{Q}}) = g(b \cdot \frac{1}{b}) = g(b)g(\frac{1}{b}) = g(\frac{1}{b})g(b),$$

Hence $g(b)$ is a unit in R and $g(\frac{1}{b}) = g(b)^{-1}$.]

In $R\text{-Mod}$, mono = injective, epi = surjective.

(2). Abelian Categories: $R\text{-Mod}$, Ab , $R\text{-Comp}$.

(3). K a field, $R = K^{\mathbb{N}} = \prod_{i \in \mathbb{N}} K_i$ w/ $K_i = K \ \forall i$.

$I = K^{(\mathbb{N})} = \{ (k_i)_{i \in \mathbb{N}} \in K^{\mathbb{N}} : k_i = 0 \text{ for all but finitely many } i \}$.

$\pi: R \rightarrow R/I$ ^{s.g.} does not have a kernel in $R\text{-mod}$ ^{f.g.}

$\ker \pi = I$ not f.g.

Chap 2. On the Road to Derived Functors.

§1. Exactness Properties of Functors.

Note: We'll develop the theory for the abelian category $R\text{-Mod}$. But it can be easily adapted to arbitrary Abelian category.

Defⁿ. Let R, S be rings, F an additive functor $R\text{-Mod} \rightarrow S\text{-Mod}$.

① F is **exact** if $\forall$ SES $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, the sequence

$$0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \rightarrow 0$$

is exact in $S\text{-Mod}$ (F covariant), or

$$0 \rightarrow F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A) \rightarrow 0$$

is exact in $S\text{-Mod}$ (F contravariant).

② F is called **left exact** if $\forall$ exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

in $R\text{-Mod}$, the sequence

$$0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

in $S\text{-Mod}$ is exact, i.e., takes monomorphisms to monomorphisms.

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Rmk. 1) $F: R\text{-Mod} \rightarrow S\text{-Mod}$ induces a functor $F: R\text{-Comp} \rightarrow S\text{-Comp}$.

2) $F \cong G: R\text{-Mod} \rightarrow S\text{-Mod}$, then F and G have the same exactness properties.

3) $F: R\text{-Mod} \rightarrow S\text{-Mod}$ is an equivalence, then F is exact.

Thm 3. (Our favourite functors).

① Let $M \in R\text{-Mod}$. Then $\text{Hom}_R(M, -)$ and $\text{Hom}_R(-, M): R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ are left exact.

② Let $M \in R\text{-Mod}$, then $M \otimes_R -: R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is right exact.

pf. ① Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$ be exact in $R\text{-Mod}$. Then

$$0 \rightarrow \text{Hom}_R(M, A) \xrightarrow{\text{Hom}(M, f)} \text{Hom}_R(M, B) \xrightarrow{\text{Hom}(M, g)} \text{Hom}_R(M, C)$$

is a complex: $\text{Hom}(M, f): \phi \mapsto f \circ \phi$ and $\text{Hom}(M, g): \psi \mapsto g \circ \psi$.

• $\text{Hom}(M, f)$ a mono: Indeed, from $f \circ \phi = 0$, we obtain $\phi = 0$ since f is a mono.

• We know $\text{Im}(\text{Hom}(M, f)) \subseteq \text{Ker}(\text{Hom}(M, g))$ since $g \circ f \circ \phi = (g \circ f) \circ \phi = 0$.

To show " $\supseteq$ ": Let $\psi \in \text{Ker}(\text{Hom}(M, g))$, i.e. $g \circ \psi = 0$. So $\text{Im } \psi \subseteq \text{Ker } g$.

Consider

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \xrightarrow{\tilde{f}} & \overset{\text{Ker } g}{\text{Im } f} & \hookrightarrow & B \rightarrow C \rightarrow 0 \\ & & x \mapsto f(x) & & \uparrow \psi & & \uparrow \psi \\ & & & & \psi & & M \end{array}$$

Set $\phi = \tilde{f}^{-1} \circ \psi$ and check

$$\text{Hom}(M, f)(\phi) = f \circ \tilde{f}^{-1} \circ \psi = \psi.$$

The case for $\text{Hom}_R(-, M)$ follows similarly.

III

② Let $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow D$ be exact in $R\text{-Mod}$. Then the sequence

$$M \otimes_R A \xrightarrow{M \otimes f} M \otimes_R B \xrightarrow{M \otimes g} M \otimes_R C \rightarrow D$$

is a complex. Recall: $M \otimes g: m \otimes b \rightarrow m \otimes g(b)$. Since g is epi, so

- $M \otimes g$ is an epimorphism.
- Since $(M \otimes g)(M \otimes f)(m \otimes b) = m \otimes (f \circ g)(b) = m \otimes 0 = 0$. So

$\text{Im}(M \otimes f) \subseteq \text{Ker}(M \otimes g)$. WTS " $\supseteq$ ": Factor $M \otimes g$:

$:= E$

$$\begin{array}{ccccc} & & M \otimes g & & \\ & \searrow & & \nearrow & \\ M \otimes B & \xrightarrow{\pi} & M \otimes B / E & \xrightarrow{G} & M \otimes C \\ & & m \otimes b + E & \longmapsto & m \otimes g(b). \end{array}$$

$\text{Ker } G = \text{Ker}(M \otimes g) / E$. Thus it suffices to show G is mono.

Plan: Construct a left inverse $H: M \otimes C \rightarrow M \otimes B / E$ s.t.

$H \circ G = \text{id}_{M \otimes B / E}$. Define

$$H': M \times C \longrightarrow M \otimes_R B / E$$

$$(m, c) \longmapsto m \otimes b + E, \quad b \in B \text{ s.t. } g(b) = c.$$

check well-definedness of H' : Suppose $g(b) = g(b')$ where $b, b' \in B$.

Then $b - b' \in \text{Ker } g = \text{Im } f$. Hence

$$m \otimes b - m \otimes b' = m \otimes (b - b') = m \otimes f(a) \in \text{Im}(M \otimes f) = E.$$

Check that H' is $\mathbb{Z}$ -bilinear and R -balanced.

UP of tensor product gives H :

$$\begin{array}{ccc} M \times C & \xrightarrow{H'} & M \otimes_R B / E \\ & \searrow \quad \nearrow \exists H & \\ & M \otimes_R C & \end{array}$$

with $H(M \otimes C) = H'(m, c) = m \otimes b + E$ where $g(b) = c$.

Then

$$(H \circ G)(m \otimes b + E) = H(m \otimes g(b)) = m \otimes b + E.$$

Hence G is a mono. Other case follows similarly. $\square$

Recall: Let $A \in \mathbb{Z}\text{-Mod}$ be an abelian group. $a \in A$ is a torsion element if $\exists n \in \mathbb{Z} \setminus \{0\}$ with $na = 0$.

$$T(A) = \{a \in A : a \text{ torsion}\}.$$

We say A is torsion iff $A = T(A)$.

We say A is torsionfree iff $T(A) = 0$.

An element $a \in A$ is divisible in A iff $a \in nA \ \forall n \in \mathbb{Z} \setminus \{0\}$. i.e., $\exists b \in A$ s.t. $a = nb$.

A is divisible if all of its elements are divisible.

Comment: If $f \in \text{Hom}_{\mathbb{Z}}(A, B)$ and A divisible, then $f(A)$ also divisible subgp of B .

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- ... Then
- $\text{Hom}_{\mathbb{Z}}(\text{torsion}, \text{torsion-free}) = 0$
 - $\text{Hom}_{\mathbb{Z}}(\text{divisible}, \text{reduced}) = 0$.

e.g. (Thm 3 is the best we can hope):

(I) $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, -)$ not right exact, $n \geq 2$, applied to $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_n$.

$$\begin{array}{ccc} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) & \xrightarrow{\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \pi)} & \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_n) \\ \parallel & & \parallel \\ 0 & & \mathbb{Z}_n \neq 0 \end{array}$$

So $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \pi)$ not an epimorphism.

$\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, -)$ not right exact. Consider $g: \mathbb{Z}^{(\mathbb{N})} \twoheadrightarrow \mathbb{Q}$.

$$\begin{array}{ccc} 0 \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}^{(\mathbb{N})}) & \xrightarrow{\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, g)} & \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \cong 0 \\ \uparrow & & \uparrow \\ \text{divisible} & & \text{reduced} \end{array}$$

↙ not epi.

(II) let $R = \mathbb{Z}$. let $F = \mathbb{Z}_n \otimes_{\mathbb{Z}} -$, $n \geq 2$. Apply F to $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

$$\begin{array}{ccc} \mathbb{Z}_n & & 0 \\ \parallel & & \parallel \\ \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

divisible, so $\mathbb{Q} = n\mathbb{Q}$.
 $\dots \cong \mathbb{Z}_n \otimes n\mathbb{Q}$
 $\cong n\mathbb{Z}_n \otimes \mathbb{Q} \cong 0$.

So the map $F(\hookrightarrow)$ not a mono. (Rmk. torsion $\otimes_{\mathbb{Z}}$ divisible = 0).

Short-term Goal:

- ① Find Exact functors.
- ② characterize exact seq A in $R\text{-Mod}$ s.t. $F(A)$ is exact in $S\text{-Mod}$ for any additive functor $F: R\text{-Mod} \rightarrow S\text{-Mod}$.

Next Thm: If $F(A)$ exact where A is a SES, then $F(A')$ exact for any LES A' .

Thm 4. If $F: R\text{-Mod} \rightarrow S\text{-Mod}$ is an **exact functor** (i.e. F takes SES to SES), then $F(A)$ is exact in $S\text{-Mod}$ where

$$A: \dots \rightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \rightarrow \dots$$

is exact in $S\text{-Mod}$.

pf. Factor each $f_n: A_n \rightarrow A_{n-1}$ as follows:

$$\begin{array}{c} A_n \xrightarrow{\tilde{f}_n} \text{Im}(f_n) \xrightarrow{L_n} A_{n-1} \\ x \mapsto f_n(x) \longrightarrow f_n(x). \end{array}$$

Now split A as follows: $0 \rightarrow \text{Im}(f_{n+1}) \xrightarrow{L_{n+1}} A_n \xrightarrow{\tilde{f}_n} \text{Im}(f_n) \rightarrow 0$. These are all SES. Since F exact, applying F gives a SES.

$$0 \rightarrow F(\text{Im}(f_{n+1})) \xrightarrow{F(L_{n+1})} F(A_n) \xrightarrow{F(\tilde{f}_n)} F(\text{Im}(f_n)) \rightarrow 0.$$

Thus $\text{Im } F(L_{n+1}) = \text{Ker } F(\tilde{f}_n)$. Now:

$$\dots \rightarrow F(A_{n+1}) \xrightarrow{F(f_{n+1})} F(A_n) \xrightarrow{F(f_n)} F(A_{n-1}) \rightarrow \dots$$

Claim: (a) $\text{Ker } F(f_n) = \text{Ker } F(\tilde{f}_n)$

$$(b) \text{Im } F(f_{n+1}) = \text{Im } F(L_{n+1})$$

pf. (a) $f_n = L_n \circ \tilde{f}_n$. So $F(f_n) = F(L_n) \circ F(\tilde{f}_n)$. Since F exact, $F(\text{mono}) = \text{mono}$, $F(\text{epi}) = \text{epi}$. So $F(L_n)$ a mono. Thus $\text{Ker } F(f_n) = \text{Ker } F(\tilde{f}_n)$.

(b) Similarly $F(f_{n+1}) = F(L_{n+1}) \circ F(\tilde{f}_{n+1})$ and $F(\tilde{f}_{n+1})$ is an epi. So $\text{Im } F(f_{n+1}) = \text{Im } F(L_{n+1})$. ///

Therefore, $F(A)$ is exact. □

First Installment re ①:

Prop 5. Let I be a set. Then

$$\begin{aligned} \prod_{i \in I} (R\text{-Mod})^I &\longrightarrow R\text{-Mod} \\ (M_i, f_i: M_i \rightarrow N_i)_{i \in I} &\longmapsto (\prod_{i \in I} M_i, \prod_{i \in I} f_i). \end{aligned}$$

and

$$\begin{aligned}\bigoplus_{i \in I} : (R\text{-Mod})^I &\longrightarrow R\text{-Mod} \\ (M_i, f_i : M_i \rightarrow N_i)_{i \in I} &\longmapsto (\bigoplus_{i \in I} M_i, \bigoplus_{i \in I} f_i).\end{aligned}$$

are both exact.

pf. Obvious (???)

□

First Installment re ② :

Defⁿ. 1) $A, B \in R\text{-Mod}$, $f \in \text{Hom}_Z(A, B)$. Then f is called **split** if $\text{Ker } f$ is a direct summand of A , and $\text{Im } f$ is a direct summand of B : $\exists A', B' \in R\text{-Mod}$ with $A = \text{Ker } f \oplus A'$, and $B = \text{Im}(f) \oplus B'$. We abbreviate: $\text{Ker } f \subseteq {}^\oplus A$, $\text{Im } f \subseteq {}^\oplus B$.

Lemma 6. *Let $A, B, C \in R\text{-Mod}$ and f, g, h R -module homomorphisms such that the following triangle commutes.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h \downarrow & \swarrow g & \\ C & & \end{array}$$

- (i) *If h is a monomorphism, then $\text{Im}(f) \cap \text{Ker}(g) = 0$.*
- (ii) *If h is an epimorphism, then $B = \text{Im}(f) + \text{Ker}(g)$.*
- (iii) *If h is an isomorphism, then $B = \text{Im}(f) \oplus \text{Ker}(g)$.*

Proof. (i) Suppose h is a mono, and let $b \in \text{Im}(f) \cap \text{Ker}(g)$. Then $b = f(a)$ for some $a \in A$ and $h(a) = g(f(a)) = g(b) = 0$. Since h is injective, this shows $a = 0$, and hence $b = 0$.

(ii) Suppose h is an epi. Given any $b \in B$, choose $a \in A$ with $h(a) = g(b)$. We deduce $b - f(a) \in \text{Ker}(g)$. $\square$

Observation 7. Characterization of short exact sequences which are split. *For a short exact sequence $\mathbb{A} : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ in $R\text{-Mod}$, the following conditions are equivalent:*

- (1) *The sequence $\mathbb{A}$ is split.*
- (2) *There exists $f' \in \text{Hom}_R(B, A)$ such that $f'f = \text{id}_A$.*
- (3) *There exists $g' \in \text{Hom}_R(C, B)$ such that $gg' = \text{id}_C$.*

Proof. “(1) $\implies$ (2)” By hypothesis $B = \text{Im}(f) \oplus B'$ for some R -submodule B' of B . Let $\tilde{f} : A \rightarrow \text{Im}(f)$ be the isomorphism given by $\tilde{f}(a) = f(a)$, and define $f' \in \text{Hom}_R(B, A)$ via $f'(b) = (\tilde{f})^{-1}(x)$, if $b = x + y$ with $x \in \text{Im}(f)$ and $y \in B'$; this is a well-defined homomorphism, due to the uniqueness of x and y in the present context.

“(2) $\implies$ (1)” is immediate from Lemma 6, and the remaining equivalence is shown analogously. $\square$

The next Observation supplements this criterion. It addresses exactness in tandem with splitness.

Observation 7*. *Let A, B, C be left R -modules, $f \in \text{Hom}_R(A, B)$ and $g \in \text{Hom}_R(B, C)$. Then the following conditions are equivalent:*

- (1) *The sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split exact.*
- (2) *There exist $f' \in \text{Hom}_R(B, A)$ and $g' \in \text{Hom}_R(C, B)$ such that $ff' + g'g = \text{id}_B$ and both triangles in the diagram below commute:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \parallel & & \swarrow f' & & \nwarrow g' \\ A & & & & C \end{array}$$

Proof. The proof of “(1) $\implies$ (2)” relies on the same type of argument as was used to justify Observation 7.

“(2) $\implies$ (1)” We only need to verify exactness of the sequence; splitness then follows from the previous observation. Since $f'f = \text{id}_A$, we find f to be a mono; $gg' = \text{id}_C$ yields that g is an epi and g' a mono.

To see that $\text{Im}(f) \subseteq \text{Ker}(g)$, note that $f = \text{id}_B f = ff'f + g'gf$. The first summand on the righthand side of this equation equals f since $f'f = \text{id}_A$, which shows that $g'gf = 0$. Since g' is a mono, this implies $fg = 0$ as required.

To see that $\text{Ker}(g) \subseteq \text{Im}(f)$, let $b \in \text{Ker}(g)$. Then $b = \text{id}_B(b) = ff'(b) + g'g(b) = ff'(b)$ clearly belongs to $\text{Im}(f)$. $\square$

Lec 11

Convention: $M, N \in R\text{-Mod}$, N is a direct summand of M (written $N \subseteq^\oplus M$) in case there exists $U \in R\text{-Mod}$ with $M = N \oplus U$.

Def ① $f \in \text{Hom}_K(A, B)$ is split if $\ker(f) \subseteq^\oplus A$ and $\text{Im}(f) \subseteq^\oplus B$.

② Let $A: \dots \rightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \rightarrow \dots \in R\text{-Comp}$, then A is split if all f_n 's are split.

We'll see, every additive functor $R\text{-Mod} \rightarrow S\text{-Mod}$ preserves exactness ~~for~~ of split exact sequences.

Theorem 8 Let $A: \dots \rightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \rightarrow \dots \in R\text{-Comp}$. Then TFAE

① For all additive functors, $F: R\text{-Mod} \rightarrow S\text{-Mod}$ (S any ring), the sequence $F(A)$ is split exact.

② A is split exact.

proof ① $\Rightarrow$ ② Apply $F = \text{id}_{R\text{-Mod}}$.

② $\Rightarrow$ ① (prove only in case $A: 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split exact. Move in the general case as in the proof of theorem 4).

By Observation 7*, $\exists f' \in \text{Hom}_R(B, A)$, $g' \in \text{Hom}_R(C, B)$ with $\text{id}_B = ff' + g'g$. Let F be an additive functor as in ①, then $\text{id}_{F(B)} = F(f)F(f') + F(g')F(g)$.

Using observation 7* again, the sequence $0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \rightarrow 0$ is split exact.

Examples: $\exists$ exact sequences $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0^{(*)}$ s.t. $B \cong A \oplus C$ but the sequence fails to be split.

(Cannot occur if all the modules are finitely generated)

Take $R = \mathbb{Z}$, let $n \geq 2$, $A = n\mathbb{Z}$, $B = \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{(N)}$, $C = (\mathbb{Z}/n\mathbb{Z})^{(N)}$

Then $A \oplus C \cong B$ because $A \cong \mathbb{Z}$.

Find $f \in \text{Hom}_{\mathbb{Z}}(A, B)$, $g \in \text{Hom}_{\mathbb{Z}}(B, C)$ s.t. $(*)$ is exact but not split exact.

Chapter II

Section 2 Projective modules

Aim: Characterize those $M \in R\text{-Mod}$ for which $\text{Hom}_R(M, -): R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact.

Def ① R^I is free if $R^I \cong R R^{(I)}$ (Know: R^I free iff I has an R basis)
 ② $P \in R\text{-Mod}$ is called projective if P is isomorphic to the direct summand of a free module.

First examples:

1) Vector space
 2) Let K be a field and $R = K \times K$. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$
 Then $Re_1 = K \times 0$, $Re_2 = 0 \times K$, have $R = Re_1 \oplus Re_2$, so the Re_i 's are projective, but not free. ($\dim_K R = 2$, $\dim_K Re_i = 1$)

Remark: 1) Let $(P_i)_{i \in I}$ be a family of left R -modules

Then $\bigoplus_{i \in I} P_i$ is projective iff P_i projective $\forall i \in I$.

2) but $\prod P_i$ need not be projective for ~~every~~ infinite I .

Example: $\mathbb{Z}^{\mathbb{N}}$ is not free and hence not ~~free~~ projective (proof to come).

3) Serre's conjecture

If R is a polynomial $R = K[x_1, \dots, x_n]$, K - a field

Is every projective R -modules free.

Answer is yes, independent proofs by Suslin and Quillen

Lec 12.

Thm 9. For $M \in R\text{-Mod}$, TFAE:

- ① ${}_R M$ is projective.
- ② $\text{Hom}_R(M, -) : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact.
- ③ Whenever $f \in \text{Hom}_R(B, C)$ is an epi and $g \in \text{Hom}_R(M, C)$, there exists $\phi \in \text{Hom}_R(M, B)$ with $f \circ \phi = g$.

$$\begin{array}{ccccc} B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & \searrow \text{dotted } f & \uparrow g & & \\ & & M & & \end{array}$$

$\exists \phi$ (indicated by a curved arrow from M to B)

- ④ Every epi onto M is split.

pf: ① $\Rightarrow$ ③ $\Rightarrow$ ② $\Rightarrow$ ④ $\Rightarrow$ ①.

Abbreviate: $\text{Hom}_R(M, -) =: [M, -]$.

① $\Rightarrow$ ③: Have $M \subseteq {}^\oplus_R F$, F free module with basis $(x_i)_{i \in I}$, $F = M \oplus N$.

Let f, g be as in ③.

$$\begin{array}{ccccc} B & \xrightarrow{f} & C \\ & \searrow \text{dotted } \psi & \uparrow g \\ & & M \\ & \searrow \text{dotted } \psi & \uparrow \pi \\ & & F \end{array}$$

(A curved arrow labeled ι points from F to M)

where $\pi: F \rightarrow M$ be the proj-
 $\iota: M \hookrightarrow F$ embedding.

Define $\psi \in \text{Hom}_R(F, B)$ via $\psi(x_i) = b_i$ if $f(b_i) = g\pi(x_i)$. This is well-defined. Then $f \circ \psi = g \circ \pi$. So $f \circ \psi \circ \iota = g \circ \pi \circ \iota = g \circ \text{id}_M = g$.

Hence $\varphi = \psi \circ \iota$ is as required.

Since ψ is defined on basis, extend by linearity. Note b_i exist since f is an epi.

③ $\Rightarrow$ ②: Since the functor $[M, -] = \text{Hom}_R(M, -)$ is left exact, it suffices to show $[M, -]$ takes epis to epis. Let $B \xrightarrow{f} C \rightarrow 0$ be an epi. To see that,

$$[M, f]: [M, B] \rightarrow [M, C]$$

Let $g \in [M, C]$. Then we have a diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ & \searrow \exists \phi & \uparrow g \\ & & M \end{array}$$

so $\exists \phi \in [M, B]$ s.t. $f \circ \phi = g$. But note $[M, f](\phi) = f \circ \phi$.

Thus $[M, -]$ takes epis to epis. ///

② $\Rightarrow$ ④: Let $f: N \twoheadrightarrow M$ be epi. Then by ②, $[M, f]: [M, N] \rightarrow [M, M]$ is epi. So $\exists \phi \in [M, N]$ with $f \circ \phi = \text{id}_M \in [M, M]$. So:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \parallel & \swarrow f & \\ M & & \end{array}$$

By lemma 6 of the lecture notes, $N = \text{Im } \phi \oplus \text{Ker } f$. So $\text{Ker } f \subseteq {}^\circ N$.

Thus f is split, since $\text{Im } f = M \subseteq {}^\circ M$, since f is epi. ///

④ $\Rightarrow$ ①: If $(m_i)_{i \in I}$ generates M , then $F := R^{(I)} \xrightarrow{f} M$
 $(r_i) \mapsto \sum_{i \in I} r_i m_i$

is an epimorphism. By hypothesis, f splits. So $\text{Ker } f \oplus N = F$.

Thus $N \cong F / \text{Ker } f \cong M$ (1st iso thm). This proves ①. ///

□

e.g. (1) $R = \text{ring}$. All $M \in R\text{-Mod}$ are projective iff R is semisimple.

Lec 13.

In general, ${}_R P$ projective $\not\Rightarrow P$ is a direct sum of f.g. modules.

Thm [Kaplansky]. $P \in R\text{-Mod}$ projective $\Rightarrow P = \bigoplus (\text{countably generated modules})$

Cor. If R is local, then all projective modules are free.

§3. Projective Resolutions and Projective Dimension.

Idea: $M \in R\text{-Mod}$. Approximate $f_0: P_0 \twoheadrightarrow M \rightarrow 0$, P_0 projective.

Error: $\text{Ker } f_0$. Next approximation: $\tilde{f}_1: P_1 \twoheadrightarrow \text{Ker } f_0 \rightarrow 0$, P_1 projective.

Then consider

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 \xrightarrow{f_0} M \\ & & \searrow \tilde{f}_2 & & \nearrow \tilde{f}_1 & & \nearrow \\ & & \text{Ker } f_1 & & \text{Ker } f_0 & & \end{array}$$

Defⁿ. $M \in R\text{-Mod}$. A **projective resolution** of M is an exact sequence

$$\cdots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$$

where all P_i are projective.

Call such a resolution **finite** of length n if $P_n \neq 0$, $P_m = 0 \ \forall m > n$.

Defⁿ. The **projective dimension** $\text{pdim } M$ of M is infinite if no finite projective resl. Otherwise

$$\text{pdim } M = \min \{ n : \exists \text{ a proj resl of } M \text{ of length } n \}.$$

Defⁿ. $\text{lgl dim}(R) = \sup \{ \text{pdim } M : M \in R\text{-Mod} \}$

↑
left global dimension.

e.g. • $\text{gl dim } \mathbb{Z} = 1$ • $M \in \text{Ab} = \mathbb{Z}\text{-Mod}$:

$$\cdots 0 \rightarrow \underset{\substack{P_1'', \text{ free}}}{\text{Ker}(f_1)} \xrightarrow{f_1} \underset{\substack{P_0''}}{\text{FAB}(M)} \twoheadrightarrow M$$

• $\text{gl dim } R = 0 \Leftrightarrow R \text{ is semisimple.}$

• Hilbert [Syzygy Thm]: $\text{gl dim } K[X_1, \dots, X_n] = n$, $K = \text{field}$.

• $R = \mathbb{Z}/p^n\mathbb{Z}$, p prime, $n \geq 2$. Let $\bar{x} = x + p^n\mathbb{Z} \in R$.

$$\begin{array}{ccccc} \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{f_0} & \mathbb{Z}/p^{n-1}\mathbb{Z} \rightarrow 0 \\ & \searrow \bar{i} \mapsto \bar{p} & \nearrow \underbrace{R = p_0} & & \underbrace{M} \\ & \bar{p}\mathbb{Z}/p^n\mathbb{Z} & & & \end{array}$$

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Lec 14.

$M \in R\text{-Mod}$.

Recall. A projective resolution of M is any exact sequence

$$\cdots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$$

with P_i proj for all $i \geq 0$.

Expl. $R = \mathbb{Z}/p^n\mathbb{Z}$, p prime, $n \geq 2$.

Write $\bar{x} = x + p^n\mathbb{Z}$, and consider

$M = R/\bar{p}^{n-1}$. A proj resolution of M :

$$\begin{array}{ccccccc} R & \xrightarrow{f_2} & R & \xrightarrow{f_1} & R & \xrightarrow{f_0} & M = R/\bar{p}^{n-1} \rightarrow 0 \\ \downarrow \pi & \searrow & \downarrow \pi & \searrow & \downarrow \pi & \searrow & \\ R/\bar{p}^{n-1} & & R/\bar{p}^{n-1} & & R/\bar{p}^{n-1} & & \end{array}$$

$\pi \circ f_2 = f_1 \circ \pi$ and $\pi \circ f_1 = f_0 \circ \pi$

Defⁿ. Let $M \in R\text{-Mod}$, $\cdots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$ a projective resolution.

Then $\text{Im } f_n = \text{Ker } f_{n-1}$ is called a n^{th} **syzygy**. [Hilbert]

↑
not unique up to isomorphism.

So First syzygies of a module is unique up to projective summands:

Lemma 15 [Scharnel]. Let $f: P \twoheadrightarrow M$ and $g: Q \twoheadrightarrow M$ be epis in $R\text{-Mod}$, P, Q projective. Then $\text{Ker } f \oplus Q \cong \text{Ker } g \oplus P$. In particular, $\text{Ker } f$ projective iff $\text{Ker } g$ projective.

pf. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 0 & \rightarrow & \text{Ker } f & \hookrightarrow & P & \xrightarrow{f} & M \rightarrow 0 \\
 & & \uparrow \scriptstyle u \mapsto (u,0) & & \uparrow \scriptstyle g' & & \uparrow \scriptstyle g \\
 0 & \rightarrow & \text{Ker } f & \hookrightarrow & L & \xrightarrow{f'} & Q \rightarrow 0 \\
 & & \uparrow \scriptstyle (0,v) & & \uparrow \scriptstyle (x,y) & & \uparrow \scriptstyle y \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Ker } g & = & \text{Ker } g & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

where $L = \{(x, y) \in P \oplus Q : f(x) = g(y)\}$.

Claim: The 2nd row is exact. $x \in \text{Ker } f$.

- $(x, y) \in \text{Ker } f' \Leftrightarrow y = 0$ and $f(x) = g(y) = 0$.

- f' is an epi: let $y \in Q$. Since f is epi, $\exists x \in P$ with $f(x) = g(y)$.
Hence $(x, y) \in L$ and $y = f'(x, y)$.

Since P is projective, the epi $g': L \twoheadrightarrow P$ splits. i.e. $L \cong P \oplus \ker g$.
..... Q $f': L \rightarrow Q$ $L \cong Q \oplus \ker f$.

□

Lemma 16. All n^{th} syzygies are unique up to projective summands:
Suppose U_n, V_n are n^{th} syzygies of $M \in R\text{-Mod}$. Then $\exists$ projective modules X_n, Y_n , s.t. $U_n \oplus X_n \cong V_n \oplus Y_n$.

Pf. Schanuel + induction. See Gaochospace.

□

Thm 17. ① For $M \in R\text{-mod}$, TFAE:

- (a) $\text{pdim } M \leq n < \infty$.
- (b) $\exists$ a proj. resl. $0 \rightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_0} M \rightarrow 0$ s.t. $\text{Im } f_n = \text{Ker } f_{n-1}$ is projective. $\underbrace{\hspace{10em}}_{\text{a } n^{\text{th}} \text{ syzygy.}}$
- (c) In ALL proj. resl. of M , the n^{th} syzygy is projective.

② $M \in R\text{-Mod}$, TFAE:

- (a) $\text{pdim } M = \infty$.
- (b) $\exists$ a proj. resl. of M with a non-projective n^{th} syzygy.
- (c) In ALL proj. resl. of M the n^{th} syzygy is non-projective.

Pf. ① (a) $\Rightarrow$ (b): $\text{pdim } M \leq n \Rightarrow \exists$ proj. resl.

$$0 \rightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_0} M \rightarrow 0$$

Then $\text{Im } f_n \cong P_n$ is projective by exactness.

(b) $\Rightarrow$ (c): Suppose (b) holds. Let

$$\dots \rightarrow Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} M \rightarrow 0$$

be another proj. resl. of M . By lemma 16, $\text{Im } g_n = \text{Ker } g_{n-1}$ differs only by proj. summands from $\text{Im}(f_n) = \text{Ker}(f_{n-1})$. Since $\text{Ker}(f_{n-1})$ is projective, so is $\text{Ker}(g_{n-1})$.

(c) $\Rightarrow$ (a): Assume (c). Let

$$\rightarrow P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \dots \rightarrow P_0 \xrightarrow{f_0} M \rightarrow 0$$

be a proj. resl. Then $\text{Im}(f_n) = \text{Ker}(f_{n-1})$ is proj. by hypothesis.

Consider the seq.

$$0 \rightarrow \text{Ker } f_{n-1} \hookrightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \dots \rightarrow M \rightarrow 0$$

This is a proj. resl. of length $\leq n$.

② Immediate from ①.

□

Lec 15.

$$\text{gl dim } \mathbb{Z}_{p^n} = \begin{cases} 0 & \text{if } n=1 \\ \infty & \text{if } n \geq 2. \end{cases}$$

By Chinese remainder theorem:

$$\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_s^{r_s}}.$$

Thus

$$\text{gl dim } \mathbb{Z}_m = \begin{cases} 0 & \text{if } r_i = 1 \ \forall i \\ \infty & \text{if } r_i \geq 2 \text{ for some } i. \end{cases}$$

§4. Injective Modules and Injective Resolutions.

A) Generalities.

Defⁿ. $M \in R\text{-Mod}$ is **injective** iff $\text{Hom}_R(-, M) : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact.

Thm 18. For $M \in R\text{-Mod}$, TFAE:

- ① R is injective.
- ② Whenever $f: X \hookrightarrow Y$ monomorphism, $g: X \rightarrow M$ between $R\text{-Mods}$,
 $\exists \psi: Y \rightarrow M$, s.t. $g = \psi \circ f$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & \nearrow \exists \psi & \\ M & & \end{array}$$

- ③ Every monomorphism with domain M is split.

pf. Analogous to proof of Thm 9.

□

Rmks & E.g.:

(a) $M_i \in R\text{-Mod}$ for $i \in I$. Then $\prod_{i \in I} M_i$ injective $\Leftrightarrow M_i$ injective $\forall i$.

(b) All left R -modules injective iff R is semisimple iff all right R -module is injective.

lem 19 (Baer's Criterion). $M \in R\text{-Mod}$. Then TFAE:

(1) M is injective.

(2) For every diagram where I is a left ideal of R ,

$$\begin{array}{ccc} I & \xhookrightarrow{\text{embedding}} & R \\ \varphi \downarrow & \nearrow \exists \Phi & \\ M & & \end{array}$$

$\exists \Phi \in \text{Hom}_R(R, M)$ with $\Phi|_I = \varphi$.

pf. (1) $\Rightarrow$ (2): Obvious from def of injectivity.

(2) $\Rightarrow$ (1): Assume (2). To show injectivity of M , consider a diagram

$$\begin{array}{ccc} X & \xhookrightarrow{f} & Y \\ g \downarrow & \nearrow ? & \\ M & & \end{array}$$

Note: Can assume $X \subseteq Y$ and f is embedding: $X \xrightarrow{\cong} f(X) \subseteq Y$.

Thus if we can show the submodule case, inverting the iso $\cong$ gives the general case.

$$\begin{array}{ccc} X & \xrightarrow[\cong]{\varphi=f} f(X) & \rightarrow Y \\ g \downarrow & \nearrow g \circ \varphi^{-1} & \nearrow \cong \\ M & & \end{array} \quad \begin{array}{c} \Phi \\ \nearrow \end{array}$$

Consider

$$\mathcal{P} = \{(\gamma', g') : X \subseteq \gamma' \subseteq Y \text{ and } g' \in \text{Hom}_R(\gamma', M) \text{ with } g'|_X = g\}.$$

$\mathcal{P}$ is partially ordered:

$$(\gamma', g') \leq (\gamma'', g'') \stackrel{\text{def}}{\iff} \gamma' \subseteq \gamma'' \text{ and } g''|_{\gamma'} = g'.$$

$\mathcal{P} \neq \emptyset$: $(X, g) \in \mathcal{P}$. Every chain $\{(\gamma_\alpha, g_\alpha)\}_{\alpha \in \mathcal{M}}$ has an upper bound

$$(\gamma^*, g^*) \text{ where } \gamma^* = \bigcup_{\alpha} \gamma_{\alpha}, \quad g^*: \gamma^* \rightarrow M$$

$$g^*|_{\gamma_{\alpha}} = g_{\alpha} \quad \left(\begin{array}{l} \text{well defined by} \\ \text{def above} \end{array} \right).$$

By Zorn's lemma, $\mathcal{S}$ contains a maximal element (Y_0, g_0) . WTS $Y = Y_0$. Then $\psi = g_0$ extends g to Y . Thus we will be done.

Suppose $Y_0 \subsetneq Y$. Pick $\hat{y} \in Y \setminus Y_0$. Set $Y_1 = Y_0 + R\hat{y}$. Define
$$I = \{r \in R : r\hat{y} \in Y_0\} \subseteq R.$$

$$\uparrow$$

$$\text{left ideal}$$

Define $\varphi: I \rightarrow M$ by $\varphi(r) = g_0(r\hat{y})$. Then $\varphi \in \text{Hom}_R(I, M)$.

By hypothesis, $\exists \phi \in \text{Hom}_R(R, M)$ with $\phi|_I = \varphi$.

Define: $g_1 \in \text{Hom}_R(Y_1, M)$ via $Y_1 = Y_0 + R\hat{y} \xrightarrow{\phi} M$

$$\begin{matrix} y + r\hat{y} \\ \uparrow \\ Y_0 \end{matrix} \mapsto g_0(y) + \phi(r).$$

Show only well-definedness. Then $g_1 \in \text{Hom}_R(Y_1, M)$, done.

Suppose $y_1 + r_1\hat{y} = y_2 + r_2\hat{y}$, $y_i \in Y_0$, $r_i \in R$. Then $r_1\hat{y} - r_2\hat{y} = y_2 - y_1 \in Y_0$.

So $(r_1 - r_2)\hat{y} \in Y_0$. By def of I , $r_1 - r_2 \in I$.

$$\begin{aligned} \text{Moreover: } 0 &= g_0(0) = g_0((y_1 + r_1\hat{y}) - (y_2 + r_2\hat{y})) \\ &= g_0(\underbrace{(y_1 - y_2)}_{Y_0} + \underbrace{(r_1 - r_2)\hat{y}}_{Y_0}) \\ &= g_0(y_1 - y_2) + g_0((r_1 - r_2)\hat{y}). \\ &= g_0(y_1) - g_0(y_2) + \varphi(r_1 - r_2) \\ &= g_0(y_1) - g_0(y_2) + \phi(r_1 - r_2) \\ &= g_0(y_1) + \phi(r_1) - (g_0(y_2) + \phi(r_2)) \\ &= g_1(y_1 + r_1\hat{y}) - g_1(y_2 + r_2\hat{y}) = 0. \end{aligned}$$

□

Lec 16.

B) Injective Modules over a PID.

Thm 20. Let R be a PID, $M \in R\text{-Mod}$. Then TFAE:

- ① M is injective
- ② M is divisible ($aM = M \ \forall a \in R \setminus \{0\}$).

pf. ① $\Rightarrow$ ②: Assume ①. To show M divisible, let $a \in R \setminus \{0\}$. Fix $x \in M$ and consider the diagram

$$\begin{array}{ccc}
 r_a & I = Ra & \hookrightarrow R \\
 \downarrow & g \downarrow & \searrow \psi \\
 r_x & M &
 \end{array}$$

(well-defined since R is a domain)

So $g(a) = g(a \cdot 1) = \psi(a \cdot 1) = a \psi(1) \in aM$.

② $\Rightarrow$ ①: Apply Baer: Let ${}_R I \hookrightarrow {}_R R$. Let $\varphi \in \text{Hom}_R(I, M)$. Since R is a PID, $I = Ra$ (WLOG $a \neq 0$). By hypothesis, $\exists x \in M$ with $\varphi(a) = ax$. Define $\phi \in \text{Hom}_R(R, M)$ via $\phi(r) = rx$. Check $\phi|_I = \varphi$. $\square$

Supplement: Divisible Abelian Group.

examples: $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$ Prüfer groups for p prime.

$$\mathbb{Z}(p^\infty) = (\bigoplus \mathbb{Z}x_i) / U(p)$$

where $U(p)$ is the subgp of $\bigoplus \mathbb{Z}x_i$ generated by $px_1, \dots, px_i, \dots, px_{i+1} - x_i \ \forall i \in \mathbb{N}$.

So we get embeddings:

$$\begin{array}{lcl}
 \mathbb{Z}\bar{x}_i \cong \mathbb{Z}/p^i\mathbb{Z} & \hookrightarrow & \mathbb{Z}/p^{i+1}\mathbb{Z} \cong \mathbb{Z}/p^{i+1}\mathbb{Z} \\
 \mathbb{Z} + p^i\mathbb{Z} & \hookrightarrow & p\mathbb{Z} + p^{i+1}\mathbb{Z}
 \end{array}$$

i.e. $p\bar{x}_i = 0$ and $p\overline{x_{i+1}} = \bar{x}_i$

Alternate Description:

$$\mathbb{Z}(p^\infty) = T_p(\mathbb{Q}/\mathbb{Z}) = \{x + \mathbb{Z} : p^n x \in \mathbb{Z} \text{ for some } n \in \mathbb{N}\}.$$

Thm 21. $A \in \mathbb{Z}\text{-Mod}$ is divisible iff $A \cong \mathbb{Q}^{(I)} \oplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(I_p)}$ for some suitable sets I and I_p .

pf. Exercise.

□

c) Injective Resolutions.

Thm 22 [Eckmann]. Every left R -module is a submodule of an injective module.

lem 23. If $M \in \mathbb{Z}\text{-Mod}$, then $\exists$ divisible $D \in \mathbb{Z}\text{-Mod}$ with $M \subseteq D$.

pf. $M \cong \mathbb{Z}^{(I)}/K$ for some subgroup $K \subseteq \mathbb{Z}^{(I)}$. Now

$M \cong \mathbb{Z}^{(I)}/K \subseteq \mathbb{Q}^{(I)}/K$ divisible.

□

Lec 17.

Note : The bimodule ${}_Z R_R$ and note $R \otimes_R - \cong \text{id}_{R\text{-Mod}}$ is exact.

lem 24. If $D \in Z\text{-Mod}$ is divisible abelian gp, then the left R -module ${}_R \text{Hom}_Z(R_R, D)$ is injective.

pf. Set ${}_R E = \text{Hom}_Z(R, D) \in R\text{-Mod}$. To show injectivity of ${}_R E$, let

$$0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$$

be exact in $R\text{-Mod}$. Hence

$$0 \rightarrow R \otimes_R U \rightarrow R \otimes_R V \rightarrow R \otimes_R W \rightarrow 0$$

is exact. Since ${}_Z D$ injective, the upper seq is exact.

$$\begin{array}{ccccccc} 0 \rightarrow \text{Hom}_Z(R \otimes_R W, D) & \xrightarrow{g'} & \text{Hom}_Z(R \otimes_R V, D) & \xrightarrow{f'} & \text{Hom}_Z(R \otimes_R U, D) & \rightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 \rightarrow \text{Hom}_R(W, E) & \longrightarrow & \text{Hom}_R(V, E) & \longrightarrow & \text{Hom}_R(U, E) & \longrightarrow & 0 \end{array}$$

The $\cong$ are given by $\text{Hom}-\otimes$ adjointness. Thus exactness of upper row implies exactness of lower row. This shows injectivity of E . $\square$

Proof of Thm 22 :

We have a left R -module iso ${}_R M \cong {}_R \text{Hom}_R(R_R, M) \subset \text{Hom}_Z(R, M)$.

By lem 23, ${}_Z M \subset {}_Z D$ where D is divisible, and $\text{Hom}_Z(R, -)$

is left exact. So

$${}_R M \cong {}_R \text{Hom}_R(R_R, M) \subset {}_R \text{Hom}_Z(R_R, M) \subset {}_R \text{Hom}_Z(R_R, D) \xrightarrow{\text{injective by lem 24. Done}}$$

(All iso & embedding above respect left R -module structure). $\square$

Consequence: Each $M \in R\text{-Mod}$ has an injective resolution.

$$\begin{array}{ccccccc} 0 \rightarrow M & \xrightarrow{f_0} & E_0 & \xrightarrow{f_1} & E_1 & \xrightarrow{f_2} & E_2 \rightarrow \dots \\ & & \uparrow \text{injective} & & \downarrow \text{Coker } f_0 & & \uparrow \text{inj} \\ & & & & \downarrow \text{Coker } f_1 & & \end{array}$$

Defⁿ. Let $M \in R\text{-Mod}$.

- ① An **injective resolution** of M is any exact sequence

$$0 \rightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \quad (*)$$

where E_i 's are injective. Define $\text{length} (*)$ in analogy with proj case.

- ② The n^{th} **cosyzzyggy** of M in $(*)$ is

$$\text{Im}(f_n) \cong \text{Coker}(f_{n-1})$$

[unique up to direct summands].

- ③ $\text{idim } M = \min \{ \text{len}(IF) : IF \text{ injective resolution of } M \}$.

Thm 25. For $M \in R\text{-Mod}$, $n \in \mathbb{N} \cup \{0\}$, TFAE:

- ① (a) $\text{idim } M \leq n$

(b) $\exists$ injective resolution of M with injective n^{th} cosyzzyggy.

(c) In all injective resolutions of M , the n^{th} cosyzzyggy is injective.

- ② $\text{idim } M = \infty$ iff all (one) inj. resl. have no injective cosyzzyggy.

Pf. Analogous of Thm 17. Just turn everything around. □

Chapter II, Section 5. Flat Modules

Definition. A right R -module M is *flat* in case the functor $M \otimes_R - : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact. Flatness of a left R -module is defined symmetrically.

Remarks and Examples. (1) We know that any tensor functor $M \otimes_R -$ is right exact. Hence M_R is flat precisely, when $M \otimes_R -$ takes monomorphisms to monomorphisms.

(2) As we saw before, the abelian group $M = \mathbb{Z}/2\mathbb{Z}$ fails to be flat in $\mathbb{Z}\text{-Mod}$. Indeed, $M \otimes_{\mathbb{Z}} -$ takes the monomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$, $z \mapsto 2z$, to the zero map $M \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \cong M \otimes_{\mathbb{Z}} \mathbb{Z}$.

(3) Since tensor products commute with arbitrary direct sums, a direct sum $\bigoplus_{i \in I} M_i$ is flat if and only if each M_i is flat.

(4) The regular R -module R_R is flat, because $R \otimes_R - \cong \text{id}_{R\text{-Mod}}$. In light of the preceding remark, we find that every free R -module is flat, and hence so is every projective R -module. In particular, if R is semisimple, then all R -modules are flat.

Caution: Here the converse does not hold. In fact, all right R -modules are flat (equivalently, all left R -modules are flat) if and only if R is von Neumann regular, meaning that, for any $r \in R$, there exists $s \in R$ with $rsr = r$. For instance: The endomorphism rings of infinite dimensional vector spaces are von Neumann regular, but not semisimple.

(5) Using the adjointness of tensor and hom functors, one obtains: If ${}_S M_R$ is a bimodule such that M_R is flat and $N \in S\text{-Mod}$ is injective, then ${}_R \text{Hom}_S(M, N)$ is injective in $R\text{-Mod}$. The proof is a carbon copy of that for injectivity of ${}_R \text{Hom}_{\mathbb{Z}}(R, D)$ in case D is injective in $\mathbb{Z}\text{-Mod}$.

FLAT FACTS. (PROOFS WILL FOLLOW.)

Theorem 26. For any M_R , TFAE:

- (a) M is flat.
- (b) For any left ideal I of R , the homomorphism $\text{id}_M \otimes (I \xrightarrow{\text{incl}} R) : M \otimes_R I \rightarrow M \otimes_R R$ is monic.
- (c) For any finitely generated left ideal I of R , the homomorphism $M \otimes (I \xrightarrow{\text{incl}} R) : M \otimes_R I \rightarrow M \otimes_R R$ is monic.

Our argument for Theorem 26 will be based on a converse to Remark 4 above in the special case where $S = \mathbb{Z}$ and $N = \mathbb{Q}/\mathbb{Z}$ (Lemma 28 below).

Lemma 27. Let A, B be abelian groups and $f \in \text{Hom}_{\mathbb{Z}}(A, B)$. Then f is a monomorphism if and only if $\text{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z}) : \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is an epimorphism.

We will find that the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is in fact a testing device for flatness. Indeed, considering the left R -module $M^+ := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ for $M \in \text{Mod-}R$ (the “character module of M ”), we obtain

Lemma 28. For any $M \in \text{Mod-}R$,

$$M_R \text{ is flat} \iff {}_R M^+ \text{ is injective.}$$

Combined with Baer’s Criterion, Lemma 28 will yield a smooth proof of Theorem 26. We next specialize to a PID. The following fact is readily derived from Theorem 26 (Homework 2).

Corollary 29. *Let R be a PID and M an R -module. Then M is flat precisely when M is torsionfree (the latter meaning that $rx \neq 0$ for $r \in R \setminus \{0\}$ and $x \in M \setminus \{0\}$.)*

PROOFS OF THE ABOVE FACTS

Proof of Lemma 27. First suppose that f is a mono. Then surjectivity of $\text{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z})$ follows from divisibility of $\mathbb{Q}/\mathbb{Z}$, alias injectivity of $\mathbb{Q}/\mathbb{Z}$.

For the converse, we prove an **auxiliary fact**: If $a \in A \setminus \{0\}$, there exists a map $h \in \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ such that $h(a) \neq 0$. Once this fact is established, our claim will follow. Indeed, surjectivity of the map $\text{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z})$ will then guarantee the existence of $g \in \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ with $h = \text{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z})(g) = g \circ f$, and consequently $h(a) \neq 0$ will imply $f(a) \neq 0$.

Proof of the auxiliary fact: Let $a \in A \setminus \{0\}$. We know that every nonzero finitely generated module contains a maximal submodule. Applying this to $\mathbb{Z}a$, we obtain a maximal subgroup $C \subseteq \mathbb{Z}a$. This means that $\mathbb{Z}a/C$ is a simple abelian group; in other words $\mathbb{Z}a/C \cong \mathbb{Z}/\mathbb{Z}p$ for some prime p . Now $\mathbb{Z}/\mathbb{Z}p$ embeds into $\mathbb{Z}(p^\infty)$ which is the p -torsion subgroup of $\mathbb{Q}/\mathbb{Z}$, and hence $\mathbb{Z}a/C$ embeds into $\mathbb{Q}/\mathbb{Z}$. Given that $\mathbb{Q}/\mathbb{Z}$ is injective, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}a & \hookrightarrow & A \\ \text{can} \downarrow & & \nearrow \exists h \\ \mathbb{Z}a/C & & \\ \downarrow & & \\ \mathbb{Q}/\mathbb{Z} & & \end{array}$$

and from the fact that $a \notin C$ we conclude $h(a) \neq 0$. $\square$

Proof of Lemma 28. First suppose that M_R is flat. Then injectivity of $M^+ \in R\text{-Mod}$ follows from the well-known tensor-hom adjointness as discussed before.

Now suppose that M^+ is an injective left R -module. To check flatness of M_R , let $f \in \text{Hom}_R(X, Y)$ be a monomorphism. By hypothesis

$$\text{Hom}_R(f, M^+) : \text{Hom}_R(Y, \text{Hom}_{\mathbb{Z}}(Y, \mathbb{Q}/\mathbb{Z})) \rightarrow \text{Hom}_R(X, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$$

is an epimorphism, and the naturality of the adjointness isomorphisms

$$\text{Hom}_R(-, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(M \otimes_R -, \mathbb{Q}/\mathbb{Z})$$

therefore shows that $\text{Hom}_{\mathbb{Z}}(M \otimes_R f, \mathbb{Q}/\mathbb{Z}) : \text{Hom}_{\mathbb{Z}}(M \otimes_R Y, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(M \otimes_R X, \mathbb{Q}/\mathbb{Z})$ is onto as well. By Lemma 27, we conclude that $M \otimes_R f$ is a mono. $\square$

Proof of Theorem 26. The implications “(a) $\implies$ (b) $\implies$ (c)” are obvious.

We first show “(b) $\implies$ (a)”: Suppose that (b) is satisfied. To establish flatness, we will prove that ${}_R M^+$ is injective and apply Lemma 28.

Our proof for injectivity of ${}_R M^+$ uses Baer’s Criterion: Let $\iota : {}_R I \hookrightarrow {}_R R$ be the embedding of a left ideal I of R . By hypothesis, the map $M \otimes \iota : M \otimes_R I \rightarrow M \otimes_R R$ is in turn injective. Consequently, injectivity of $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{Z}\text{-Mod}$ shows that the upper row of the following commutative diagram is exact.

$$\begin{array}{ccccc}
 \text{Hom}_{\mathbb{Z}}(M \otimes_R R, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{Hom}(M \otimes \iota, \mathbb{Q}/\mathbb{Z})} & \text{Hom}_{\mathbb{Z}}(M \otimes_R I, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\
 \downarrow \cong & & \downarrow \cong & & \\
 \text{Hom}_R(R, M^+) & \xrightarrow{h \mapsto h \circ \iota} & \text{Hom}_R(I, M^+) & \longrightarrow & 0
 \end{array}$$

Hence the lower row is exact as well. This means that every map in $\text{Hom}_R(I, M^+)$ is a restriction of a map in $\text{Hom}_R(R, M^+)$, whence Baer indeed guarantees injectivity of M^+ , and the argument is complete.

“(c) $\implies$ (b)”: Assume (c) and suppose there exists an embedding $\iota : I \hookrightarrow R$, where I is a left ideal of R , such that $M \otimes f : M \otimes_R I \rightarrow M \otimes_R R$ fails to be a monomorphism; say $z = \sum_{1 \leq i \leq n} m_i \otimes x_i$ is a nonzero element of $M \otimes_R I$ which belongs to the kernel of $M \otimes f$. Consider the finitely generated left ideal $I_0 = \sum_{1 \leq i \leq n} R x_i$. The element $z_0 = \sum_{1 \leq i \leq n} m_i \otimes x_i$ is a fortiori nonzero in $M \otimes I_0$ and the canonical map $M \otimes_R I_0 \rightarrow M \otimes_R I$ sends z_0 to z ; in other words z_0 is a nonzero element in the kernel of $M \otimes$ (embedding of I_0 into R). Thus condition (b) fails as well in this situation. $\square$

Lec 18.

Chap III Derived Functors

§1. The Long Exact Homology Sequence.

Recall: $H_n: R\text{-Comp} \rightarrow \mathbb{Z}\text{-Mod}$

$$A = (\dots \rightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \rightarrow \dots) \mapsto H_n(A) = \text{Ker } d_n / \text{Im } d_{n+1}.$$

$f \in \text{Hom}_{R\text{-Comp}}(A, A')$, say $(f_n)_{n \in \mathbb{Z}}$ s.t. $f d = d f$. Then

$$H_n(f): H_n(A) \rightarrow H_n(A'), \quad a_n + \text{Im}(d_{n+1}) \mapsto f_n(a_n) + \text{Im}(d'_{n+1}).$$

We know $R\text{-Comp}$ is an Ab category. So has SES:

$$0 \rightarrow A \xrightarrow{f} A' \xrightarrow{f'} A'' \rightarrow 0.$$

H_n is not exact (left nor right) in general. But

$$H_n(A) \xrightarrow{H_n(f)} H_n(A') \xrightarrow{H_n(f')} H_n(A'')$$

is exact.

Problem: Find Connecting homomorphism $\partial_n: H_n(A') \rightarrow H_{n-1}(A)$, s.t.

$$\dots \rightarrow H_{n+1}(A'') \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(f)} H_n(A') \xrightarrow{H_n(f')} H_n(A'') \xrightarrow{\partial_n} H_{n-1}(A) \rightarrow \dots$$

is exact.

30. Snake lemma. Suppose the following is a commutative diagram in $R\text{-Mod}$ with exact rows

$$\begin{array}{ccccccc} A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' & \rightarrow & 0 \\ \downarrow h & & \downarrow h' & & \downarrow h'' & & \\ 0 & \rightarrow & B & \xrightarrow{g} & B' & \xrightarrow{g'} & B'' \end{array}$$

(a) $\exists!$ homomorphisms p, p', σ, σ' to make (*) commute with exact rows.

(b) $\exists \partial \in \text{Hom}_R(\text{Ker } h'' \rightarrow \text{Coker } h)$ s.t. the sequence

$$\text{Ker } h \xrightarrow{p} \text{Ker } h' \xrightarrow{p'} \text{Ker } h'' \xrightarrow{\partial} \text{Coker } h \xrightarrow{\sigma} \text{Coker } h' \xrightarrow{\sigma'} \text{Coker } h''$$

is exact.

$$\begin{array}{ccccc}
 \text{Ker } h & \xrightarrow{f} & \text{Ker } h' & \xrightarrow{f'} & \text{Ker } h'' \\
 \downarrow i & & \downarrow i' & & \downarrow i'' \\
 A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' \\
 \downarrow & & \downarrow & & \downarrow \\
 B & \xrightarrow{g} & B' & \xrightarrow{g'} & B'' \\
 \downarrow j & & \downarrow j' & & \downarrow j'' \\
 \text{Coker } h & \xrightarrow{\sigma} & \text{Coker } h' & \xrightarrow{\sigma'} & \text{Coker } h''
 \end{array}$$

(*)

where i, i', i'' are inclusions, j, j', j'' are canonical epis.

(c) If f is a mono, so is p . If g' is an epi, so is σ' .

Pf (Detail see Ganchospace Notes):

(a) Existence of P : By UP of $\text{Ker } h'$, we obtain P making the diagram

$$\begin{array}{ccc}
 \text{Ker } h & \xrightarrow{P} & \text{Ker } h' \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{f'} & A'
 \end{array}$$

commute. Similarly P' exists.

(*)

Claim Upper row of DA is exact. i.e. $\exists a \in A$ with $a' = f(a)$

Clearly, $g'g = 0$ i.e. $\text{Ker}(g') \supseteq \text{Im}(g)$

Since $\text{Im}(g) \supseteq \text{Ker}(g')$, let $a' \in \text{Ker}(g')$, i.e. $a' \in A'$ with $f'(a') = 0$. Hence $a' \in \text{Ker}(f') = \text{Im}(f)$

i.e. $\exists a \in A$ with $a' = f(a)$

To see $a \in \text{Ker}(h)$, we only need to show $gh(a) = 0$, since g is a mono.

Note $gh(a) = h'f'(a) = 0$



↑
MY GOAT

Lec 19.

Theorem 31 [Long Exact Homology sequence]

Suppose $0 \rightarrow A \xrightarrow{f} A' \xrightarrow{f'} A'' \rightarrow 0$ (*) is an exact sequence in $R\text{-Comp}$. Then, for each $n \in \mathbb{Z}$, $\partial_n \in \text{Hom}_R(H_n(A''), H_{n-1}(A))$ s.t. the following sequence is exact.

$$\cdots \rightarrow H_{n+1}(A'') \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(f)} H_n(A') \xrightarrow{H_n(f')} H_n(A'') \xrightarrow{\partial_n} H_{n-1}(A) \rightarrow \cdots$$

Naturality: For any $\mathcal{A}$ as above, the ~~following~~ family $(\partial_n)_{n \in \mathbb{Z}} = \mathcal{A}(\partial_n)_{n \in \mathbb{Z}}$ satisfies the following condition.

whenever

$$\begin{array}{ccccccc} \mathcal{A}: & 0 & \rightarrow & A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' & \rightarrow 0 \\ & & & \downarrow h & & \downarrow h' & & \downarrow h'' & \\ & & & & & & & & \text{is commutative.} \end{array}$$

$$\mathcal{B}: 0 \rightarrow B \xrightarrow{g} B' \xrightarrow{g'} B'' \rightarrow 0$$

The following diagram commutes.

$$\begin{array}{ccc} H_n(A'') & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \downarrow H_n(h'') & & \downarrow H_{n-1}(h) \\ H_n(B'') & \xrightarrow{\partial_n} & H_{n-1}(B) \end{array}$$

Proof (*) translates into an infinite commutative grid with exact rows.

$$\begin{array}{ccccccc} 0 & \rightarrow & A_{n+1} & \rightarrow & A_{n+1}' & \rightarrow & A_{n+1}'' \rightarrow 0 \\ & & \downarrow d_{n+1} & & \downarrow d_{n+1}' & & \downarrow d_{n+1}'' \\ 0 & \rightarrow & A_n & \rightarrow & A_n' & \rightarrow & A_n'' \rightarrow 0 \\ & & \downarrow d_n & & \downarrow d_n' & & \downarrow d_n'' \\ 0 & \rightarrow & A_{n-1} & \rightarrow & A_{n-1}' & \rightarrow & A_{n-1}'' \rightarrow 0 \\ & & \vdots & & \vdots & & \vdots \end{array} \Rightarrow \begin{array}{ccc} \frac{A_n}{\text{Im}(d_{n+1})} & \xrightarrow{\bar{f}_n} & \frac{A_n'}{\text{Im}(d_{n+1}')} \\ \downarrow \bar{d}_n & & \downarrow \bar{d}_n' \\ 0 & \rightarrow & \ker(d_{n-1}) \rightarrow \ker(d_{n-1}) \rightarrow \ker(d_{n-1}) \rightarrow 0 \end{array}$$

Apply snake lemma to this diagram.

Step 1: Define $\bar{d}_n: \frac{A_n}{\text{Im}(d_{n+1})} \rightarrow \ker(d_{n-1})$ by $\bar{d}_n \mapsto d_n(a_n)$

$\ker(\bar{d}_n) = \frac{\ker(d_n)}{\text{Im}(d_{n+1})} = H_n(A)$ well-defined since $\text{Im}(d_{n+1}) \subseteq \ker(d_n)$
 $\text{coker}(\bar{d}_n) = \ker(d_{n-1}) / \text{Im}(d_n)$
 $\Rightarrow$ diagram commutes.

Step 2: The snake lemma yields $\partial_n \in \text{Hom}_R(H_n(A''), H_{n-1}(A))$ s.t. (∂_n) commutes.

and the sequence $\cdots \rightarrow H_n(A') \rightarrow H_n(A'') \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(A') \rightarrow \cdots$ is exact.

So we obtain exact sequences

$$0 \rightarrow H_n(A) \xrightarrow{\text{in}} \frac{A_n}{\text{Im}(d_{n+1})} \xrightarrow{\bar{d}_n} \ker(d_{n-1}) \xrightarrow{\text{in}} H_{n-1}(A) \rightarrow 0 \Rightarrow \text{exact columns.}$$

Section 2: Homotopy of complexes

Def Let $u: A \rightarrow A'$ be a morphism in $R\text{-Comp}$.

say $u = (u_n)_{n \in \mathbb{Z}}$.

Call u null-homotopic if there exists $s_n \in \text{Hom}_R(A_n, A'_{n+1})$ s.t.

$$u_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$$

Visualize:

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} \\ u_{n+1} \downarrow & s_n \swarrow & u_n \downarrow & s_{n-1} \swarrow & u_{n-1} \downarrow \\ A'_{n+1} & \xrightarrow{d'_{n+1}} & A'_n & \xrightarrow{d'_n} & A'_{n-1} \end{array}$$

Proposition 32: ~~Suppose~~ Suppose $u = (u_n)_{n \in \mathbb{Z}} \in \text{Hom}_{R\text{-comp}}(A, A')$ is nullhomotopic, then $H_n(u) = 0 \quad \forall n \in \mathbb{Z}$.

proof. Let $s_n \in \text{Hom}(A_n, A'_{n+1})$ be as in the definition, i.e.,

$$u_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$$

$$H_n(u): \frac{\ker(d_n)}{\text{Im}(d_{n+1})} \rightarrow \frac{\ker(d'_n)}{\text{Im}(d'_{n+1})}$$

$$\overline{a_n} \mapsto \overline{u_n(a_n)}$$

~~Suppose~~ Suppose $a_n \in \ker(d_n)$, then $u_n(a_n) = d'_{n+1} \circ s_n(a_n) + s_{n-1} \circ d_n(a_n) = 0$.

$$\Rightarrow u_n(a_n) = d'_{n+1}(s_n(a_n)) \in \text{Im}(d'_{n+1})$$

$$\Rightarrow \overline{u_n(a_n)} = 0$$

□

Lemma 30. [Snake Lemma].

Suppose that the following is a commutative diagram with exact rows in R -Mod.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' & \rightarrow & 0 \\ \downarrow h & & \downarrow h' & & \downarrow h'' & & \\ 0 \rightarrow B & \xrightarrow{g} & B' & \xrightarrow{g'} & B'' & & \end{array}$$

(a) There exist unique R -module homom's $\sigma, \sigma', \sigma'', \sigma'''$ such that the following commutative diagram is commutative with exact rows and columns:

$$\begin{array}{ccccccc} \text{Ker}(h) & \xrightarrow{\sigma} & \text{Ker}(h') & \xrightarrow{\sigma'} & \text{Ker}(h'') & \xrightarrow{\sigma''} & 0 \\ \downarrow i & & \downarrow i' & & \downarrow i'' & & \\ A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' & \rightarrow & 0 \\ \downarrow h & & \downarrow h' & & \downarrow h'' & & \\ 0 \rightarrow B & \xrightarrow{g} & B' & \xrightarrow{g'} & B'' & & \\ \downarrow j & & \downarrow j' & & \downarrow j'' & & \\ \text{Coker}(h) & \xrightarrow{\sigma} & \text{Coker}(h') & \xrightarrow{\sigma'} & \text{Coker}(h'') & \xrightarrow{\sigma''} & 0 \end{array}$$

Here i, i', i'' are the inclusions, and j, j', j'' the canonical epis.

(b) There exists $\gamma \in \text{Hom}_R(\text{Ker}(h''), \text{Coker}(h))$

such that the sequence

$$\text{Ker}(h) \xrightarrow{g} \text{Ker}(h') \xrightarrow{g'} \text{Ker}(h'') \xrightarrow{\gamma} \text{Coker}(h) \xrightarrow{\sigma} \text{Coker}(h') \xrightarrow{\sigma'} \text{Coker}(h'')$$

is exact.

(c) • If f is a mono, then so is g .

• If g' is an epi, then so is σ' .

Proof. (a) • The unique existence of g such that $\text{Ker}(h) \xrightarrow{g} \text{Ker}(h')$ commutes

$$\begin{array}{ccc} i \downarrow & & i' \downarrow \\ A & \xrightarrow{f} & A' \end{array}$$

comes from the (U.P.) of $\text{Ker}(h')$: Indeed,

$$h'f i = g h' i = 0. \text{ Analogous for } g'.$$

Similarly, the unique existence of σ comes from the (U.P.) of $\text{Coker}(h)$: $j' g h = j' h' f = 0$.

• Exactness of the upper row: $g'g = 0$, since

$$i'' g'g = f'f i = 0, \text{ so } \text{Im}(g) \subseteq \text{Ker}(g').$$

To see $\text{Ker}(g') \subseteq \text{Im}(g)$, let $a' \in \text{Ker}(g')$,
 i.e. $a' \in \text{Ker}(h')$ with $g'(a') = 0$. Then $a' = f(a)$
 $a' \in \text{Ker}(f') = \text{Im}(f)$, say $a' = f(a)$
 for some $a \in A$. Show only $a \in \text{Ker}(h)$. Since
 g is a mono, it suffices to show $gh(a) = 0$.
 Compute $gh(a) = h'f(a) = h'(a') = 0$.
 Exactness of lowest row similar.

(b) Step 1: Definition of ∂ .

Let $a'' \in \text{Ker}(h'')$. Since f' is onto, there
 exists $a' \in A'$ with $a'' = f'(a')$. We infer
 $g'h'(a') = h''f'(a') = h''(a'') = 0$, i.e.,
 $h'(a') \in \text{Ker}(g') = \text{Im}(g)$. This yields $b \in B$
 with $g(b) = h'(a')$.

Define: $\partial(a'') = b + \text{Im}(h)$.

Well-definedness of ∂ . Suppose $\tilde{a}' \in A'$
 with $f'(\tilde{a}') = a''$ and $\tilde{b} \in B$ with
 $g(\tilde{b}) = h'(\tilde{a}')$. In order to see that $b - \tilde{b} \in \text{Im}(h)$,
 note that $\tilde{a}' - a' \in \text{Ker}(f') = \text{Im}(f)$, and choose
 $a \in A$ with $f(a) = \tilde{a}' - a'$. We deduce

$$\text{that } gh(a) = h'f(a) = h'(\tilde{a}' - a') = h'(\tilde{a}') - h'(a') \\ = g(\tilde{b}) - g(b) = g(\tilde{b} - b).$$

In light of injectivity of g , this yields

$$b - \tilde{b} \in h(a) \in \text{Im}(h) \text{ as required.}$$

That ∂ is an R -module isomorphism now follows easily.

Step 2. Exactness of

$$\text{Ker}(h') \xrightarrow{g'} \text{Ker}(h'') \xrightarrow{\partial} \text{Coker}(h).$$

$\text{Im}(g') \subseteq \text{Ker}(\partial)$: let $a'' \in \text{Im}(g')$, i.e., $a'' = f'(a')$ with $a' \in \text{Ker}(h')$. Then $0 = h'(a') = g(0_B)$, whence $\partial(a'') = 0_B + \text{Im}(h) = 0 \in \text{Coker}(h)$.

$\text{Ker}(\partial) \subseteq \text{Im}(g')$: let $a'' \in \text{Ker}(\partial)$. Pick $a' \in A'$ with $f'(a') = a''$ and $b \in B$ with $g(b) = h'(a')$. From $0 = \partial(a'') = b + \text{Im}(h) \in \text{Coker}(h)$, we obtain $b \in \text{Im}(h)$, i.e., there is $a \in A$ with $h(a) = b$.

Then $h'f(a) = gh(a) = g(b) = h'(a')$, which shows $a' - f(a) \in \text{Ker}(h') = \text{Dom}(g')$, and we obtain $g'(a' - f(a)) = f'(a' - f(a)) = f'(a') - f'(f(a)) = f'(a) = a''$. This shows $a'' \in \text{Im}(g')$.

Step 3. Exactness of

$$\text{Ker}(h'') \xrightarrow{\partial} \text{Coker}(h) \xrightarrow{\sigma} \text{Coker}(h').$$

$\text{Im}(\partial) \subseteq \text{Ker}(\sigma)$: let $z \in \text{Im}(\partial)$, i.e., there exists $a'' \in \text{Ker}(h'')$, $a' \in A'$ with $f'(a') = a''$ and $b \in B$ with $g(b) = h'(a')$ such that $z = b + \text{Im}(h)$.
Then $\sigma(z) = \sigma j(b) = j'g(b) = j'h'(a') = 0$ as required.

$\text{Ker}(\sigma) \subseteq \text{Im}(\partial)$: let $b + \text{Im}(h) \in \text{Ker}(\sigma)$, i.e. $g(b) \in \text{Im}(h')$, say $g(b) = h'(a')$ for some $a' \in A$. We next deduce that $f'(a') \in \text{Ker}(h'')$:
Indeed, $h''f'(a') = g'h'(a') = g'g(b) = 0$.
Thus $f'(a') \in \text{Dom}(\partial)$ and $\partial f'(a') = b + \text{Im}(h)$ by the definition of ∂ . In particular, $b + \text{Im}(h) \in \text{Im}(\partial)$. $\square$

The proof of part (c) is trivial. $\square$

[This tried my patience!]

§3. Deleted Resolutions.

Defⁿ. ① Let $M \in R\text{-Mod}$, and

$$\rightarrow P_n \xrightarrow{d_n} \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

a projective resolution. Then the **deleted projective resolution** is

$$\rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0.$$

This is only a complex. Exactness fails on the right.

② If

$$0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots$$

is an inj. resl., then

$$0 \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots$$

is a **deleted injective resolution**.

Lemma 33. Let $M, M' \in R\text{-Mod}$, $f \in \text{Hom}_R(M, M')$. Moreover, let IP, IP' be deleted proj. resl. of M, M' . Then $\exists \bar{f} = (f_n)_{n \geq 0} \in \text{Hom}_{R\text{-comp}}(IP, IP')$ st. TFDC:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\ \cdots & \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{d'_0} & M' \longrightarrow 0 \end{array}$$

Moreover, given another map $g \in \text{Hom}_{R\text{-comp}}(IP, IP')$ with this property, then $\bar{f} \simeq \bar{g}$.

(Compare with Hatcher Lem 3.1).

pf. Existence of $\bar{f}$: By induction on $n \geq 0$.

• $n=0$: Consider the diagram on the right.

Since $d'_0: P'_0 \rightarrow M'$ epimorphism, and P_0 projective, f_0 exists.

$$\begin{array}{ccc} & & P_0 \\ \exists f_0 \swarrow & & \downarrow d_0 \\ & M & \\ & \downarrow f & \\ P'_0 & \xrightarrow{d'_0} & M' \end{array}$$

Let $n \geq 1$, and $f_0, \dots, f_{n-1}$ are given as required. Consider the diagram

$$\begin{array}{ccc} & P_n & \\ & \downarrow d_n & \\ & P_{n-1} & \\ & \downarrow f_{n-1} & \\ P'_n & \xrightarrow{d'_n} & P'_{n-1} \end{array}$$

But d'_n not epi. So want to replace P'_{n-1} by $\text{Im } d'_n$. So WTS
 $\text{Im}(f_{n-1} \circ d_n) \subseteq \text{Im}(d'_n) = \text{Ker}(d'_{n-1})$.

Compute

$$d_{n-1}' f_{n-1} d_n = d_{n-1}' d'_n f_n = 0.$$

Lemma 33 and terminology. [Comparison Lemma]

Let $M, M' \in R\text{-Mod}$, and let

$$(I) \quad \dots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

and

$$(II) \quad \dots \rightarrow P'_2 \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} M' \rightarrow 0$$

projective resolutions of M, M' , respectively.

Moreover, let P and P' be the corresponding deleted resolutions.

Given $f \in \text{Hom}_R(M, M')$, there exists

$\bar{f} = (f_n)_{n \geq 0} \in \text{Hom}_{R\text{-comp}}(P, P')$ making the

diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 \xrightarrow{d_0} M \rightarrow 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ \dots & \rightarrow & P'_2 & \xrightarrow{d'_2} & P'_1 & \xrightarrow{d'_1} & P'_0 \xrightarrow{d'_0} M' \rightarrow 0 \end{array}$$

commutative. Any such map $\bar{f}$ is called a morphism of complexes over f .

If $\bar{f}, \bar{g} \in \text{Hom}_{R\text{-comp}}(P, P')$ are both morphisms over f , then $\bar{f} \simeq \bar{g}$.

Proof. Existence of $\bar{f} = (f_n)_{n \geq 0}$. We construct the maps $f_n \in \text{Hom}_R(P_n, P'_n)$ by induction on $n \geq 0$.

$n=0$: Consider the diagram

$$\begin{array}{ccc} & & P_0 \\ & \swarrow \exists f_0 & \downarrow d_0 \\ & & M \\ & \swarrow & \downarrow f \\ P'_0 & \xrightarrow{d'_0} & M' \rightarrow 0 \end{array}$$

The map f_0 making the triangle commutative exists due to projectivity of P_0 and surjectivity of d'_0 .

• Suppose $f_0, \dots, f_n$ with the claimed properties are given. In particular

$$\begin{array}{ccc} P_n & \xrightarrow{d_n} & P_{n-1} \\ f_n \downarrow & & \downarrow f_{n-1} \\ P'_n & \xrightarrow{d'_n} & P'_{n-1} \end{array}$$

commutes. Then $\text{Im}(f_n d_{n+1}) \subseteq \text{Ker}(d'_n)$: indeed,

$d'_n f_n d_{n+1} = f_{n-1} d_n d_{n+1} = 0$. Thus $\text{Im}(f_n d_{n+1}) \subseteq \text{Im}(d'_{n+1})$,

and the following diagram

$$\begin{array}{ccc} \exists f_{n+1}, & & P_{n+1} \\ & \swarrow & \downarrow d_{n+1} \\ & & P_n \\ & \swarrow & \downarrow \\ P'_{n+1} & \xrightarrow{d'_{n+1}} & \text{Im}(d'_{n+1}) \end{array}$$

yields f_{n+1} as desired.

Uniqueness of $\bar{f}$ up to homotopy.

Suppose also $\bar{g} \in \text{Hom}_{R\text{-comp}}(P, P')$ lies over f .

Construction of suitable $s_i \in \text{Hom}_R(P_i, P'_{i+1})$

so that $f_i - g_i = d'_{i+1}s_i + s_{i-1}d_i$ for $i \geq -1$. $(\#)$

$n=0$: Consider

$$\begin{array}{ccccc} P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 \\ & \swarrow \exists s_0, f_0 - g_0 & \downarrow & \nearrow s_{-1} := 0 & \\ P'_1 & \xrightarrow{d'_1} & P'_0 & \longrightarrow & 0 \end{array}$$

Since P_0 is proj. and d'_1 surjective, s_0 exists so that the left-hand triangle commutes. This means $f_0 - g_0 = d'_1 s_0$ as required.

$n \geq 1, n-1 \rightsquigarrow n$: Suppose that, for $1 \leq i \leq n-1$ we have $s_i \in \text{Hom}_R(P_i, P'_{i+1})$ such that $(\#)$ holds.

In particular, $f_{n-1} - g_{n-1} = d'_n s_{n-1} + s_{n-2} d_{n-1}$

We check that $\text{Im}(f_n - g_n - s_{n-1} d_n) \subseteq \text{Ker}(d'_n)$:

On the board

Since $\text{Ker}(d_n') = \text{Im}(d_{n+1}')$, we obtain a commutative diagram

$$\begin{array}{ccc}
 & P_n & \\
 S_n \swarrow & \downarrow f_n g_n - S_n d_n & \\
 P_{n+1}' & \xrightarrow{d_{n+1}'} & \text{Im}(d_{n+1}')
 \end{array}$$

where S_n exists, because P_n is projective and the horizontal map is a surjection. $\square$

Comment: Our proof did not use the full hypotheses. We only needed that (I) is a complex with P_i projective for all i , and that (II) is an exact sequence.

Corollary 34: Suppose $M \in R\text{-Mod}$. Any two deleted projective resolutions of M are homotopy equivalent.

Proof. let

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

and

$$\dots \rightarrow P'_2 \rightarrow P'_1 \rightarrow P'_0 \rightarrow M \rightarrow 0$$

be projective resolutions, and P, P' the corresponding deleted resolutions. By Lemma 33, we obtain $\bar{f} \in \text{Hom}_{R\text{-comp}}(P, P')$ and $\bar{g} \in \text{Hom}_{R\text{-comp}}(P', P)$ such that $\bar{f}$ and $\bar{g}$ both lie over id_M . Since both $\bar{g} \circ \bar{f}$ and id_P lie over id_M , we obtain $\bar{g} \circ \bar{f} \cong \text{id}_P$ from the uniqueness statement of Lemma 33. Symmetrically, $\bar{f} \circ \bar{g} \cong \text{id}_{P'}$. $\square$

Lemma 35 [Horseshoe Lemma]

Let $0 \rightarrow M \xrightarrow{f} M' \xrightarrow{f'} M'' \rightarrow 0$ be an exact sequence in $R\text{-Mod}$, and suppose that

$$\dots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0 \text{ and } \dots \rightarrow P_1'' \xrightarrow{d_1''} P_0'' \xrightarrow{d_0''} M'' \rightarrow 0$$

are projective resolutions. Then there exists a projective resolution

$$\dots \rightarrow P_1' \xrightarrow{d_1'} P_0' \xrightarrow{d_0'} M' \rightarrow 0,$$

together with morphisms $f_n \in \text{Hom}_R(P_n, P_n')$ and $f_n' \in \text{Hom}_R(P_n', P_n'')$ for $n \geq 0$ such that the following diagram has exact rows and commutes:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \rightarrow & P_1 & \xrightarrow{f_1} & P_1' & \xrightarrow{f_1'} & P_1'' \rightarrow 0 \\
 & d_1 \downarrow & & d_1' \downarrow & & d_1'' \downarrow & \\
 0 & \rightarrow & P_0 & \xrightarrow{f_0} & P_0' & \xrightarrow{f_0'} & P_0'' \rightarrow 0 \\
 & d_0 \downarrow & & d_0' \downarrow & & d_0'' \downarrow & \\
 0 & \rightarrow & M & \xrightarrow{f} & M' & \xrightarrow{f'} & M'' \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Proof, Step 1: We first show that any diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \downarrow & & & & \downarrow \\
 & & K & & & & K'' \\
 & & \downarrow u & & & & \downarrow u'' \\
 & & P & & & & P'' \\
 & & \downarrow v & & & & \downarrow v'' \\
 0 \rightarrow & L & \xrightarrow{\alpha} & L' & \xrightarrow{\alpha'} & L'' & \rightarrow 0 \\
 & \downarrow & & & & \downarrow & \\
 & 0 & & & & 0 &
 \end{array}$$

in $R\text{-Mod}$ with exact rows and columns, and P, P'' projective can be completed to a commutative 3×3 diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \downarrow & & & & \downarrow \\
 0 \rightarrow & K & \xrightarrow{\gamma} & K' & \xrightarrow{\gamma'} & K'' & \rightarrow 0 \\
 & \downarrow u & & \downarrow u' & & \downarrow u'' & \\
 0 \rightarrow & P & \xrightarrow{\beta} & P' & \xrightarrow{\beta'} & P'' & \rightarrow 0 \\
 & \downarrow v & & \downarrow v' & & \downarrow v'' & \\
 0 \rightarrow & L & \xrightarrow{\alpha} & L' & \xrightarrow{\alpha'} & L'' & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

with exact rows and columns and P' projective.

Set $P' = P \oplus P''$. Since α' is an epi,
there exists $W \in \text{Hom}_R(P'', L')$ such that
the triangle

$$\begin{array}{ccc} & P'' & \\ W \swarrow & \downarrow v'' & \\ L' & \xrightarrow[\alpha']{} & L'' \end{array} \quad \text{commutes,}$$

and since v'' is an epi, we obtain

$$L' = \text{Im}(W) + \text{Ker}(\alpha') = \text{Im}(W) + \text{Im}(\alpha).$$

[Lemma 6]. Let $\beta: P \rightarrow P'$ be the inclusion map
and $\beta': P' \rightarrow P''$ the canonical projection.
Moreover, define $v': P' = P \oplus P'' \rightarrow L'$
$$x + x'' \mapsto \alpha v(x) + W(x'')$$

Then v' makes the diagram below commutative:

$$\begin{array}{ccccccc} 0 \rightarrow & P & \xrightarrow{\beta} & P' & \xrightarrow{\beta'} & P'' & \rightarrow 0 \\ & \downarrow v & & \downarrow v' & & \downarrow v'' & \\ 0 \rightarrow & L & \xrightarrow{\alpha} & L' & \xrightarrow{\alpha'} & L' & \rightarrow 0 \end{array}$$

To see that also v' is an epi, observe that
 $\alpha v(P) = \text{Im}(\alpha)$ and use $L' = \text{Im}(\alpha) + \text{Im}(W)$
(see above).

Now we apply the Snake Lemma to the diagram

$$\begin{array}{ccccccc}
 & \text{Ker}(V) & & \text{Ker}(V') & & \text{Ker}(V'') & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & P & \xrightarrow{\beta} & P' & \xrightarrow{\beta'} & P'' & \rightarrow 0 \\
 & \downarrow \gamma & & \downarrow \gamma' & & \downarrow \gamma'' & \\
 0 \rightarrow & L & \xrightarrow{\alpha} & L' & \xrightarrow{\alpha'} & L'' & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

to obtain homomorphisms γ and γ' which make the sequence

$$0 \rightarrow \text{Ker}(V) \xrightarrow{\gamma} \text{Ker}(V') \xrightarrow{\gamma'} \text{Ker}(V'') \rightarrow 0$$

exact. This completes Step 1.

Step 2. Construction of the P_n', d_n', f_n, f_n' as in the statement of the Lemma by induction on $n \geq 0$.

$n=0$. We obtain P_0', d_0', f_0, f_0' by applying Step 1 to the diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \downarrow & & & & \downarrow \\
 & & \text{Ker}(d_0) & \xrightarrow{\quad} & & & \text{Ker}(d_0'') \\
 & & \downarrow & & & & \downarrow \\
 & & P_0 & & & & P_0'' \\
 & & \downarrow d_0 & & & & \downarrow d_0'' \\
 0 \rightarrow & M & \xrightarrow{f} & M' & \xrightarrow{f'} & M'' & \rightarrow 0 \\
 & \downarrow & & & & \downarrow & \\
 & 0 & & & & 0 &
 \end{array}$$

In particular, this yields an exact upper row $0 \rightarrow \text{Ker}(d_0) \rightarrow \text{Ker}(d_0') \rightarrow \text{Ker}(d_0'') \rightarrow 0$.

$n \rightsquigarrow n+1$. Suppose we have all the quantities we are trying to construct for $i \leq n$ such that, moreover, the resulting sequences

$0 \rightarrow \text{Ker}(d_i) \xrightarrow{(UP)} \text{Ker}(d_i') \xrightarrow{(UP)} \text{Ker}(d_i'') \rightarrow 0$ are exact for $i \leq n$.

Now we apply step 1 to the diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \text{Im}(d_{n+2}) & & \text{Im}(d''_{n+2}) \\
 \parallel & & \parallel \\
 \text{Ker}(d_{n+1}) & & \text{Ker}(d''_{n+1}) \\
 \downarrow \text{incl.} & & \downarrow \text{incl.} \\
 P_{n+1} & & P''_{n+1} \\
 \downarrow d_{n+1} & & \downarrow d''_{n+1} \\
 \text{Im}(d_{n+1}) & & \text{Im}(d''_{n+1})
 \end{array}$$

$$0 \rightarrow \text{Ker}(d_n) \rightarrow \text{Ker}(d'_n) \rightarrow \text{Ker}(d''_n) \rightarrow 0,$$

where exactness of the bottom row is guaranteed by the induction hypothesis.

In particular, this yields another term of a projective resolution $\dots \rightarrow P'_{n+1} \rightarrow P'_n \rightarrow \dots \rightarrow M' \rightarrow 0$,

and the resulting map $d''_{n+1} \in \text{Hom}_R(P'_{n+1}, \text{Ker}(d'_n))$

provides the next rung of the ladder of short exact sequences over

$$0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0. \quad \square$$

Lec 21

Chapter III, Section 4 Left derived functors.

Agenda for this section

- Show that $L_n T$ is a well-defined functor $R\text{-Mod} \rightarrow S\text{-Mod}$.
Check that, if we use a different choice of ~~deleted projective~~ ~~projective~~ resolution for $M \in R\text{-Mod}$, the resulting functor $\hat{L}_n T$ is naturally equivalent to $L_n T$.
- Explore the exact homology sequence corresponding to an exact sequence $0 \rightarrow M \xrightarrow{f} M' \xrightarrow{f'} M'' \rightarrow 0$.
- Relate $L_n T$ to T .
- Help toward computing the higher $L_n T$'s. ~~(n \geq 2)~~

Have: Covariant functor $T: R\text{-Mod} \rightarrow S\text{-Mod}$

Goal: Construct functor $L_n T$ for $n \geq 0$ such that, for $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ in $R\text{-Mod}$, we obtain an exact sequence.

$$\dots \rightarrow L_n T(M) \rightarrow L_n T(M') \rightarrow L_n T(M'') \xrightarrow{\partial} L_{n-1} T(M) \rightarrow L_{n-1} T(M') \rightarrow L_{n-1} T(M'') \rightarrow 0$$

Moreover we want: If T is right exact, then $L_0 T \cong T$.

Construction:

Pick and fix a projective resolution for M ^(each) $\dots \rightarrow P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \dots \rightarrow P_0 \xrightarrow{d_0} M \rightarrow 0$
and let $\hat{P}_M: \dots \rightarrow P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \dots \rightarrow P_0 \rightarrow 0$ be the deleted complex

Define $L_n T$ on objects $L_n T(M) := H_n(T(\hat{P}_M))$

$L_n T$ on maps $f \in \text{Hom}_R(M, M')$

By lemma 33, we choose (and fix) $\bar{f} \in \text{Hom}_{R\text{-comp}}(\hat{P}_M, \hat{P}_{M'})$ lying over f

Define $L_n T(f) := H_n(T(\bar{f}))$

Theorem 36 $L_n T$ is a well-defined covariant functor $R\text{-Mod} \rightarrow S\text{-Mod}$

Moreover, if $\hat{P}_M$ is a different deleted projective resolution of $M \in R\text{-Mod}$, then the functor $\hat{L}_n T$ defined on the basis of the $\hat{P}_M$ satisfies $\hat{L}_n T \cong L_n T$.

proof Show that for $f \in \text{Hom}_R(M, M')$, $g \in \text{Hom}_R(M', M'')$, we have ~~III~~

$$L_n T(gf) = L_n T(g) \circ L_n T(f)$$

$\bar{g} \in \text{Hom}_{R\text{-comp}}(\hat{P}_{M'}, \hat{P}_{M''})$ and $\bar{f} \in \text{Hom}_{R\text{-comp}}(\hat{P}_M, \hat{P}_{M'})$ lie over g and f , respectively. Then both $\bar{g} \circ \bar{f}$ and $\overline{g \circ f}$ both lie over $g \circ f$

By lemma 33, $\bar{g} \circ \bar{f} = \overline{g \circ f}$.

Then $T(\bar{g} \circ \bar{f}) \simeq T(\bar{g} \circ f)$. By Proposition 32, this implies
 $L_n T(g) \circ L_n T(f) = \text{H}_n T(\bar{g}) \circ \text{H}_n T(\bar{f}) = \text{H}_n T(\bar{g} \circ \bar{f}) = \text{H}_n T(\bar{g} \circ f) = L_n T(g \circ f)$

Analogously, $L_n T(\text{id}_M) = \text{id}_{L_n T(M)}$

~~Relative~~ Relative independence of the $L_n T$ of the choice of P_M

By Proposition 32, P_M and $\hat{P}_M$ are homotopy equivalent

Let $i \in \text{Hom}_{R\text{-comp}}(P_M, \hat{P}_M)$ and $j \in \text{Hom}_{R\text{-comp}}(\hat{P}_M, P_M)$

with $ji \simeq \text{id}_{P_M}$, $ij \simeq \text{id}_{\hat{P}_M}$

Then $T(ji) \simeq \text{id}_{T(P_M)}$, $T(ij) \simeq \text{id}_{T(\hat{P}_M)}$

Hence

$$\text{H}_n T(j) \circ \text{H}_n T(i) = \text{H}_n (T(ji)) = \text{H}_n (\text{id}_{T(P_M)}) = \text{id}_{\text{H}_n T(P_M)}$$

Same for ij

So $\text{H}_n T(i) : \text{H}_n T(P_M) \rightarrow \text{H}_n T(\hat{P}_M)$ is an isomorphism in $R\text{-Mod}$.

$$\varphi(M) \quad L_n T(M) \quad \hat{L}_n T(M)$$

This shows $L_n T \simeq \hat{L}_n T$

Show $(\varphi(M))_{M \in R\text{-Mod}}$ is a natural transformation $L_n T \rightarrow \hat{L}_n T$.

[Use lemma 33 and proposition 32]

Theorem 37. All notations kept.

Suppose $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ (*) is a short exact sequence in $R\text{-Mod}$.

Then there exists an exact sequence

$$\cdots \rightarrow L_{n+1} T(M'') \xrightarrow{\partial_{n+1}} L_n T(M'') \xrightarrow{\partial_n} L_{n-1} T(M'') \xrightarrow{\partial_{n-1}} \cdots$$

$$L_1 T(M'') \xrightarrow{\partial_1} L_0 T(M'') \xrightarrow{\partial_0} 0$$

Moreover:

(a) If T is right exact, then $L_0 T = T$.

(b) If M is projective, then $L_n T(M) = 0 \quad \forall n \geq 1$.

Finally, the ∂_n for $n \geq 1$ are natural in the sense specified in theorem 31.

proof. By the horseshoe lemma, we can choose a short exact sequence of complexes

$$0 \rightarrow P_M \xrightarrow{\bar{f}} P_{M'} \xrightarrow{\bar{f}'} P_{M''} \rightarrow 0 \quad \text{lying over } (*)$$

This yields exact sequences $0 \rightarrow P_n \xrightarrow{f_n} P_n' \xrightarrow{f_n'} P_n'' \rightarrow 0$ in $R\text{-Mod}$.

Projectivity of P_n'' makes this sequence split exact, ~~so $0 \rightarrow P_n \rightarrow P_n' \rightarrow P_n'' \rightarrow 0$ is split exact~~

$$\text{so } 0 \rightarrow T(P_n) \xrightarrow{T(f_n)} T(P_n') \xrightarrow{T(f_n')} T(P_n'') \rightarrow 0 \text{ is still exact.}$$

Lec 22.

$T: R\text{-Mod} \rightarrow S\text{-Mod}$ covariant additive functor

For $M \in R\text{-Mod}$ $L_n T(M) = H_n T(P_n)$ where P_n is a deleted projective resolution of M . $L_n T(f)$ for $f: M \rightarrow M'$ is defined via the comparison lemma.

Theorem 37 For any exact sequence $0 \rightarrow M \xrightarrow{f} M' \xrightarrow{g} M'' \rightarrow 0$ in $R\text{-Mod}$, there is an exact sequence

$$\cdots \xrightarrow{d_2} L_1 T(M) \rightarrow L_1 T(M') \rightarrow L_1 T(M'') \xrightarrow{d_1} L_0 T(M) \rightarrow L_0 T(M') \rightarrow L_0 T(M'') \rightarrow 0$$

Moreover, (a) if M is projective, then $L_n T(M) = 0 \quad \forall n \geq 1$.

(b) If T is right exact, then $L_0 T \cong T$.

proof Let P, P', P'' be deleted projective resolutions of M, M', M'' such that we get a commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \rightarrow & P & \xrightarrow{\bar{f}} & P' & \xrightarrow{\bar{f}'} & P'' \rightarrow 0 \\ & & d_0 \downarrow & & d_0' \downarrow & & d_0'' \downarrow \\ 0 & \rightarrow & M & \xrightarrow{f} & M' & \xrightarrow{f'} & M'' \rightarrow 0 \end{array}$$

with $\bar{f} = (f_n)_{n \geq 0}$
 $\bar{f}' = (f'_n)_{n \geq 0}$

Since each $0 \rightarrow P_n \xrightarrow{f_n} P'_n \xrightarrow{f'_n} P''_n \rightarrow 0$ is ~~not~~ exact in $R\text{-Mod}$, the sequence splits and $0 \rightarrow TP_n \xrightarrow{Tf_n} TP'_n \xrightarrow{Tf'_n} TP''_n \rightarrow 0$ is exact, i.e., ~~$0 \rightarrow TP \xrightarrow{T\bar{f}} TP' \xrightarrow{T\bar{f}'} TP'' \rightarrow 0$~~ is exact in $R\text{-Comp}$.

By theorem 31, we get an exact sequence.

$$\cdots \rightarrow H_1(TP'') \rightarrow H_0(TP) \rightarrow H_0(TP') \rightarrow H_0(TP'') \rightarrow 0$$

$$L_1 T(M) \quad L_0 T(M) \quad L_0 T(M') \quad L_0 T(M'')$$

(a) M projective. A projective resolution of M is $\cdots \rightarrow 0 \rightarrow 0 \rightarrow M \rightarrow 0$
the deleted projective resolution is $0 \rightarrow M \rightarrow 0$.

Hence $H_n T(P) = 0 \quad \forall n \geq 1$.
 $L_n T(M)$.

(b) Suppose T is right exact, ~~M, P as before~~. $\cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$.
projective resolution. By hypothesis, $TP_1 \xrightarrow{Td_1} TP_0 \xrightarrow{Td_0} TM \rightarrow 0$ is still exact.
Hence $TM \cong TP_0 / \text{Im}(Td_1) = H_0(TP)$ where P is the corresponding deleted resolution.
 $L_0 T(M)$.

Dimension shifting

Theorem 38 [Help for computing the $L_n T$'s for $n \geq 2$ from $L_1 T$]

Let M, T etc. be as before. Then

$$L_n T(M) \cong L_{n-1} T(K_1) \cong L_{n-2} T(K_2) \cong \cdots \cong L_1 T(K_{n-1})$$

where each K_i is ~~the~~ an ^{i th} syzygy of M

True for $n \geq 2$.

proof. Let $\dots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$

A deleted projective resolution K_1

of K_1 is $\dots \rightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \rightarrow 0$

$$L_1 T(K_1) \cong H_1 T\tilde{P} = \frac{\ker(d_2)}{\text{Im}(d_3)} = H_2 T\tilde{P}$$

where $\tilde{P}$ is $\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0$

We'll show $L_2 T(M) \cong L_1 T(K_1)$

Model situation is

Section 4. Other derived functors (Sketch).

a) Right derived functors of covariant functors. $T: R\text{-Mod} \rightarrow S\text{-Mod}$.

For $M \in R\text{-Mod}$, let

$$T = \text{Hom}(X_s, -)$$

$$0 \rightarrow M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \rightarrow \dots$$

Consider the deleted injective resolution $0 \rightarrow E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \rightarrow \dots \cong E_M$

$$H_n T(E_M) = R^n T(M)$$

Obtain an exact sequence

$$0 \rightarrow R^0 T M \rightarrow R^1 T M \rightarrow R^2 T M \xrightarrow{\partial_2} R^3 T M \rightarrow \dots$$

$R^0 T \cong T$ if T is left exact.

Moreover $R^n T(M) = 0$ if M is injective.

b) Right derived functors of contravariant additive T .

For $M \in R\text{-Mod}$, let P_M be a deleted projective resolution of M .

$$\text{Define } R^n T(M) = H_n(T P_M)$$

lec 23.

Dedicated to Nick Sun :)

shift of index position in right derived functors:

$$B : \cdots \rightarrow B_{n-1} \xrightarrow{d_n} B_n \xrightarrow{d_{n+1}} B_{n+1} \rightarrow \cdots$$

n^{th} homology:

$$H^n(B) = \frac{\text{Ker } d_{n+1}}{\text{Im } d_n}.$$

Chapter IV. The Functors Tor & Ext, and 1st Application.

§1. Def and Balanceness of Tor, Connection to Flatness.

Defⁿ. ${}_S X_R$ a bimodule. $T: X \otimes_R - : R\text{-Mod} \rightarrow S\text{-Mod}$.

$$\text{Tor}_n^R(X, Y) = H_n(X \otimes IP_Y) = L_n T(Y).$$

where IP_Y a deleted proj. resol. of Y .

$$\text{Tor}_n^R(X, f) = L_n T(f)$$

for $f \in \text{Hom}_R(Y, Y')$.

Comment: 1) T is right exact. Hence $\text{Tor}_0^R(X, Y) = X \otimes_R Y$.

2) If $0 \rightarrow Y \xrightarrow{f} Y' \xrightarrow{f'} Y'' \rightarrow 0$ exact in $R\text{-Mod}$, we get an exact seq in $S\text{-Mod}$

$$\xrightarrow{\partial_3} \text{Tor}_1^R(X, Y) \rightarrow \text{Tor}_1^R(X, Y') \rightarrow \text{Tor}_1^R(X, Y'') \rightarrow X \otimes Y \rightarrow X \otimes Y' \rightarrow X \otimes Y'' \rightarrow 0.$$

3) let $Y \cong P/K$, P projective, i.e.

$$0 \xrightarrow{\iota} K \rightarrow P \rightarrow Y \rightarrow 0$$

is exact. Then

$$\text{Tor}_n^R(X, P) = 0 \quad \forall n \geq 1.$$

So

$$0 \rightarrow \text{Tor}_1^R(X, Y) \rightarrow X \otimes K \rightarrow X \otimes P \rightarrow X \otimes Y \rightarrow 0 \quad \text{is exact.}$$

$$\text{Tor}_1^R(X, Y) = \text{Ker}(X \otimes \iota).$$

Define analogously ($\check{T} = - \otimes_R Y$):

$$\text{Tor}_n^R(-, Y)_{(X)} = H_n(\mathbb{P}_X \otimes_R Y) = L_n \check{T}(X).$$

Note: $\text{Tor}_n^R(X, Y)$ has two defs.

Q: How are $H_n(X \otimes_R \mathbb{P}_Y)$ and $H_n(\mathbb{P}_X \otimes_R Y)$ related, for deleted proj. resl. $\mathbb{P}_X, \mathbb{P}_Y$ of X, Y ?

Thm 39. For $X \in \text{Mod-}R$, $Y \in R\text{-Mod}$, we have

$$H_n(X \otimes_R \mathbb{P}_Y) \cong H_n(\mathbb{P}_X \otimes_R Y)$$

in $S\text{-Mod}$.

pf. See Rotman : (

□

Comment: 1) Tor commutes with arbitrary direct sums. Reason: $X \otimes_R -$,

H_n , commutes with direct sums.

2) Tor also commutes with $\varinjlim$.

Thm 40 (Connection to Flatness).

A) For $X \in \text{Mod-}R$, TFAE:

(i) X_R is flat

(ii) $\text{Tor}_n^R(X, Y) = 0 \quad \forall n \geq 1.$

(iii) $\text{Tor}_1^R(X, Y) = 0.$

B) Analogous statement for ${}_R Y$.

pf. (i) $\Rightarrow$ (ii): Assume (i). Let $Y \in R\text{-Mod}$,

$$\rightarrow P_1 \rightarrow P_0 \rightarrow Y \rightarrow 0$$

a proj. resl. By flatness of X_R ,

$$\cdots \rightarrow X \otimes P_2 \rightarrow X \otimes P_1 \rightarrow X \otimes P_0 \rightarrow 0$$

is exact at least in the purple part. Thus $H_n(X \otimes \mathbb{P}_Y) = 0 \quad \forall n \geq 1.$

(ii) $\Rightarrow$ (iii) : trivial.

(iii) $\Rightarrow$ (i): Assume (iii). Let $0 \rightarrow Y \rightarrow Y' \rightarrow Y'' \rightarrow 0$ be exact in $R\text{-Mod}$.

Then this is 0 by (iii)

$$\cdots \rightarrow \text{Tor}_1^R(X, Y'') \rightarrow X \otimes Y \rightarrow X \otimes Y' \rightarrow X \otimes Y'' \rightarrow 0$$

is exact. Thus $0 \rightarrow X \otimes Y \rightarrow X \otimes Y' \rightarrow X \otimes Y'' \rightarrow 0$ is exact. Hence X is flat. $\square$

§2. Tor for $R = \mathbb{Z}$. [Analogous Results for $R = \text{PID}$].

Comment: If R is commutative, $X \otimes_R Y \cong Y \otimes_R X$. Hence

$$\text{Tor}_n^R(X, Y) \cong \text{Tor}_n^R(Y, X).$$

Prop 41. $\text{Tor}_n^R(X, Y) = 0$ for $n \geq 2$ and $X, Y \in \mathbb{Z}\text{-Mod}$.

Pf. $\text{pdim}_{\mathbb{Z}} Y = 1$. i.e., any first syzygy K_1 of Y is projective.

So $\text{Tor}_2^{\mathbb{Z}}(X, Y) \cong \text{Tor}_1^{\mathbb{Z}}(X, K_1) = 0$ by dim shifting. Rest by induction.

For $n \geq 3$. $\text{Tor}_n^{\mathbb{Z}}(X, Y) \cong \text{Tor}_{n-1}^{\mathbb{Z}}(X, K_1) = 0$.

E.g. $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/6\mathbb{Z}) = 0$ because $\mathbb{Q}$ is torsion-free, thus flat by Hw2.