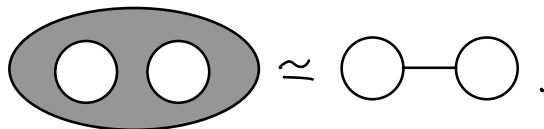


Lec 1. Homotopy

Homotopy theory: Study spaces and maps up to continuous deformation.

Ex:



Def. A **homotopy** is a continuous function $F: X \times I \rightarrow Y$, where X, Y topological spaces and $I = [0, 1]$. For $t \in [0, 1]$, the map $F_t: X \rightarrow Y$ is defined s.t.

$$F_t(x) = F(x, t).$$

$F_0, F_1: X \rightarrow Y$ are **homotopic maps** if there is a homotopy F_t connecting them. Write: $F_0 \simeq F_1$.

(Note: $\simeq$ homotopic, $\cong$ homeomorphic, $\approx$ isomorphic)

We say f is **null homotopic** if $f \simeq$ a constant map.

Ex: (1) Let $f, g: X \rightarrow \mathbb{R}^n$. Define the straight line homotopy

$$F: X \times I \rightarrow \mathbb{R}^n$$

$$F(x, t) = t f(x) + (1-t) g(x).$$

Then F is continuous (check). So $f \simeq g$.

(2) Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Given $n \in \mathbb{Z}$, let

$$f_n: S^1 \rightarrow S^1 \quad f_n(z) = z^n.$$

Later we shall see that $f_m \simeq f_n \Leftrightarrow m = n$.

Def. If $A \subseteq X$, and $\forall t \in I$, $F_t|_A = 1_A$, we say F is a **homotopy relative to A** , or homotopy rel A . We write

$$F_0 \simeq F_1 \text{ rel } A.$$

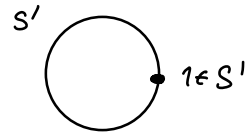
→ map = continuous map in topology!

Def. Let $A \subseteq X$. A **map** $r: X \rightarrow X$ is a **retraction** onto A if $r(X) = A$ and $r|_A = 1_A$. ($\Leftrightarrow r: X \rightarrow X$, $r^2 = r$, $r(x) \in A$). Further, r is a **deformation retraction** onto A if in addition

$$r \simeq 1_X.$$

i.e. $\exists F_t: X \rightarrow X$ homotopy s.t. $F_0 = 1_X$, $F_1 = r$.

Ex: (1) $r: S^1 \rightarrow \{1\}$ constant map
is a retraction but not a
deformation retraction.



(2) $r: \mathbb{C} - \{0\} \rightarrow S^1$ by
 $r(xe^{i\theta}) = e^{i\theta} \quad (x > 0)$
is a deformation retraction: let
 $F(xe^{i\theta}, t) = ((1-t)x + t)e^{i\theta}$,
Then $F_0 = 1$, $F_1 = r$.

(3) There is no retraction $r: [0,1] \rightarrow \{0,1\}$ (r cannot be continuous).
Since image of a connected space under a continuous function
is connected.

Def. Let $f: X \rightarrow Y$, $g: Y \rightarrow X$. Then g is **homotopy inverse** of f
if $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Then f is a **homotopy equivalence**.
 X and Y are **homotopy equivalent spaces**. Write $X \simeq Y$.

Def. X is **contractible** if $X \simeq \{\text{one point}\}$.

lem 1.1. Every deformation retraction is a homotopy equivalence.

pf. Let $A \subseteq X$ and r is a d.r. onto A . Then $\exists$ homotopy $F_t: X \rightarrow X$
s.t. $F_0 = 1_X$, $F_1 = r$. Let $i: A \hookrightarrow X$. Then $r \circ i = 1_A$ and
 $i \circ r = F_1 \simeq F_0 = 1_X$. □

lem 1.2. homotopy ($\simeq$) is an equivalence relation on $C(X, Y) = \{f: X \xrightarrow{cts} Y\}$.
pf. Exercise.

Prop 1.3. X is contractible $\iff 1_X \simeq$ a constant map.

pf. ($\Rightarrow$): X contractible implies $X \simeq Y$ ^{a pt.}. Thus $\exists g: Y \rightarrow X$ & $f: X \rightarrow Y$
s.t. $g \circ f \simeq 1_X$. But $g \circ f$ is a constant map.

($\Leftarrow$): Conversely suppose $1_X \simeq f$. Let $f: X \rightarrow Y = \{p\}$, i.e. $p = f(x)$.
Then $\exists F: X \times I \rightarrow Y$ s.t. $F_0 = 1_X$ and $F_1 = f$.

let $g: Y \rightarrow X$ be $g(p) = p$. i.e. $g = 1_Y$. Then
 $f \circ g = 1_Y$. And
 $g \circ f = f \simeq 1_X$ via F .
 Thus f is htpy equivalent. $\square$

Def. let $f: X \rightarrow Y$. Define the **mapping cylinder** M_f of f by
 $M_f = (X \times I) \sqcup Y / \sim$ where $(x, 1) \sim f(x)$.

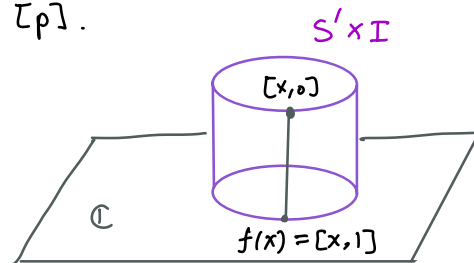
The associated quotient map:

$$\pi: (X \times I) \sqcup Y \rightarrow Y$$

$$\pi(p) = [p].$$

Ex: $X = S^1 \xrightarrow{f} \mathbb{C}$.

lem 1.4. There is a deformation retraction $r: M_f \rightarrow Y$.



pf. "Squash the cylinder flat": let $Z = (X \times I) \sqcup Y$. Define homotopy

$$\tilde{F}: Z \times I \rightarrow Z$$

$$\tilde{F}((x, t'), t) = (x, (1-t)t' + t) \quad (x, t') \in X \times I$$

$$\tilde{F}(y, t) = y. \quad y \in Y.$$

Then $\tilde{F}_0 = 1_Z$. $\tilde{F}_1 = \tilde{r}$ rel $(X \times 1) \sqcup Y$

where $\tilde{r}: Z \rightarrow (X \times 1) \sqcup Y$

$$\tilde{r}(x, t') = (x, 1) \quad \tilde{r}(y) = y.$$

Then $\tilde{r}$ is a deformation retraction. We have the following commutative diagram:

$$\begin{array}{ccc} Z \times I & \xrightarrow{\tilde{F}} & Z \\ \pi \times 1_I \downarrow & & \downarrow \pi \\ M_f \times I & \xrightarrow{\exists F} & M_f \end{array}$$

$$\pi \circ \tilde{F} = F \circ (\pi \times 1_I). \quad (*)$$

(*) implies F is continuous. So F is continuous and $F_0 = 1_{M_f}$, $F_1 = r$. Thus r is a deformation retraction. $\square$

→ Given a pt $\bar{x} \in M_f \times I$, pick a pt $x \in Z \times I$ s.t. $\pi \times 1_I(x) = \bar{x}$.
 Define $F(\bar{x}) = \pi \circ \tilde{F}(x)$. Check this is well-defined.

Remark on Lem 1.4 :

let $O \subseteq M_f$ be open. Then

$$(\pi \times 1_I)^{-1}(F^{-1}(O)) = (\pi \circ \tilde{F})^{-1}(O)$$

is open since $\pi, \tilde{F}$ are continuous. Then since $\pi \times 1_I$ is a quotient map (Hatcher Prop A.4), $(\pi \times 1_I)^{-1}(F^{-1}(O))$ open iff $F^{-1}(O)$ open.

$\tilde{F}$ open: $\tilde{F}((x, t'), t) = (x, (1-t)t' + t) = (f_1((x, t'), t), f_2((x, t'), t))$.

$$f_1((x, t'), t) = x. \quad f_1: Z \times I \rightarrow X.$$

$$\begin{aligned} \text{let } U \subseteq X \text{ open. } f_1^{-1}(U) &= \{(x, t'), t) \in Z \times I : x \in U\} \\ &= \{(x, t') \in X \times I : x \in U\} \times I \\ &= (U \times I) \times I \text{ open} \end{aligned}$$

$$f_2((x, t'), t) = (1-t)t' + t \quad (x, t') \in X \times I \quad f_2: Z \times I \rightarrow I$$

let $U \subseteq I$ be an open set: $U = (a, b)$ wlog.

$$f_2^{-1}(U) = \{(x, t'), t) \in Z \times I : (1-t)t' + t \in (a, b)\}.$$

$$= X \times \{(t', t) \in I \times I : (t', t) \in \eta^{-1}(a, b)\}$$

$$\text{where } \eta: I \times I \rightarrow I \quad \eta(t', t) = (1-t)t' + t$$

$$= X \times \text{open set} = \text{open}.$$

Thus by universal property of the product, $\tilde{F}: (X \times I) \times I \rightarrow Z$

is continuous. Also, $\tilde{F}: Y \times I \rightarrow Z$ continuous, where

$$\tilde{F}(y, t) = y \in (X \times I) \sqcup Y. \text{ let } \Omega \subseteq Y \text{ be an open set,}$$

$$\tilde{F}^{-1}(\Omega) = \Omega \times I \text{ open in } Y \times I. \text{ Thus } \tilde{F} \text{ is cts on}$$

two open sets: $(X \times I) \times I$ and $Y \times I$. So $\tilde{F}$ is cts.

Note: $A \sqcup B = (A \times \{0\}) \cup (B \times \{1\})$

$$\begin{aligned} (A \sqcup B) \times I &= (A \times \{0\} \times I) \cup (B \times \{1\} \times I) \\ &= (A \times I) \sqcup (B \times I). \end{aligned}$$

Open set in $(A \sqcup B) \times I$ is of the form $(O_A \sqcup O_B) \times U_I$.

Lec 2. Cell Complexes

Notation:

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

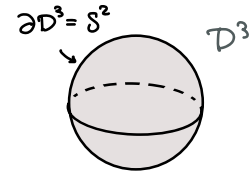
The unit n -sphere.

$$D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

The unit n -disk.

$$\partial D^n = S^{n-1}$$

$$n\text{-cell: } e^n = D^n - \partial D^n.$$



Def (CW complex):

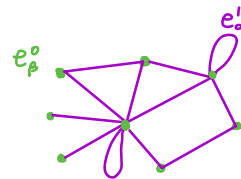
- (1) Start with a discrete set X^0 with discrete topology.
- (2) Inductively form the n -skeleton X^n from X^{n-1} by attaching n -cells e_α^n via the attaching maps $\varphi_\alpha^n: \partial D_\alpha^n \rightarrow X^{n-1}$. This means

$$X^n = (X^{n-1} \amalg_\alpha D_\alpha^n) / \sim \quad \text{where } x \sim \varphi_\alpha^n(x) \text{ for } x \in \partial D_\alpha^n.$$

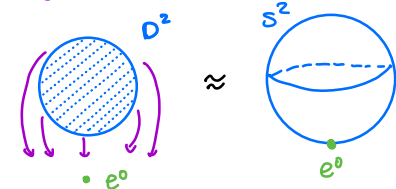
Thus as a set, $X^n = X^{n-1} \amalg_\alpha e_\alpha^n$, where each e_α^n is an open n -disk.

- (3) If $X = X^n$ for some $n < \infty$, then call $n = \dim X$. Otherwise $\dim X = \infty$ and $X = \bigcup_n X^n$. In this case X is given the weak topology: A set $A \subseteq X$ is open iff $A \cap X^n$ is open in X^n , for all n .

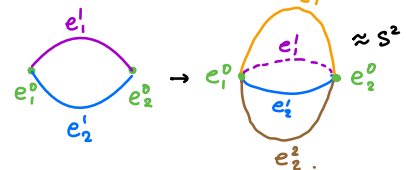
Ex: (1) A dim 1 CW complex $X = X^1$ is called a graph.



- (2) S^n can be obtained by just two cells: e^0 and e^n . $\varphi^n: \partial D^n \rightarrow e^0$ is the constant attaching map.



S^n can also be obtained from attaching two k -cells for $k=0,1,\dots,n$. An example of S^2 is given.



We can also define $S^\infty = \bigcup_{n=0}^\infty S^n$.

(3) $\mathbb{R}P^n = \{ \text{all lines through origin of } \mathbb{R}^{n+1} \}$.

Each line in $\mathbb{R}P^n$ is determined by a nonzero vector in $\mathbb{R}^{n+1}$, unique up to scalar multiplication. So topologize as follows:

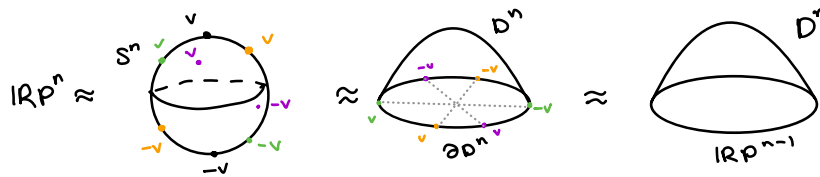
$$\mathbb{R}P^n = (\mathbb{R}^{n+1} - \{0\}) / v \sim \lambda v, \lambda \neq 0.$$

We can restrict to unit vectors, so

$$\mathbb{R}P^n = S^n / (v \sim -v).$$

This is the sphere with antipodal pts identified.

$\Leftrightarrow \mathbb{R}P^n = \text{quotient space of a hemisphere } \approx D^n \text{ with antipodal pts of } \partial D^n \text{ identified.}$



But $\partial D^n / (\text{antipodal pts } \sim) \approx \mathbb{R}P^{n-1}$. So $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n -cell e^n , with $\varphi^n: \partial D^n \rightarrow \mathbb{R}P^{n-1}$ being the quotient map that identifies antipodal pts.

It follows by induction that $\mathbb{R}P^n$ has a cell decomposition structure $e^0 \cup e^1 \cup \dots \cup e^n$ with one cell e^i in each dim $i \leq n$.

Def. Each cell e_α^n in a CW complex X has a **characteristic map** $\Phi_\alpha: D_\alpha^n \rightarrow X$ which extends the attaching map φ_α^n and is a homeomorphism from $\text{int}(D_\alpha^n)$ onto $e_\alpha^n = D_\alpha^n - \partial D_\alpha^n$.

Namely,

$$\Phi_\alpha^n: D_\alpha^n \hookrightarrow X^{n-1} \amalg_\alpha D_\alpha^n \xrightarrow{\pi} X^n \hookrightarrow X.$$

Def. A **subcomplex** of a CW complex X is a closed subspace $A \subseteq X$ that is a union of cells and (X, A) is a **CW pair**.

prop 2.1. The n -skeleton X^n is closed in X .

pf 221A midterm Q1 + lemma 3.13 in topology with uncountably many tear notes. □

Def. If X is a finite-dim CW complex, define the Euler Characteristic of X by

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i n_i \quad \text{where } n_i = \# \text{ of cells of dim } i \text{ in } X.$$

Thm. If X, Y are finite CW complex, then $X \simeq Y \Rightarrow \chi(X) = \chi(Y)$.
i.e. Euler char is a "homology invariant".

pf. Later.

□

Operations on Spaces

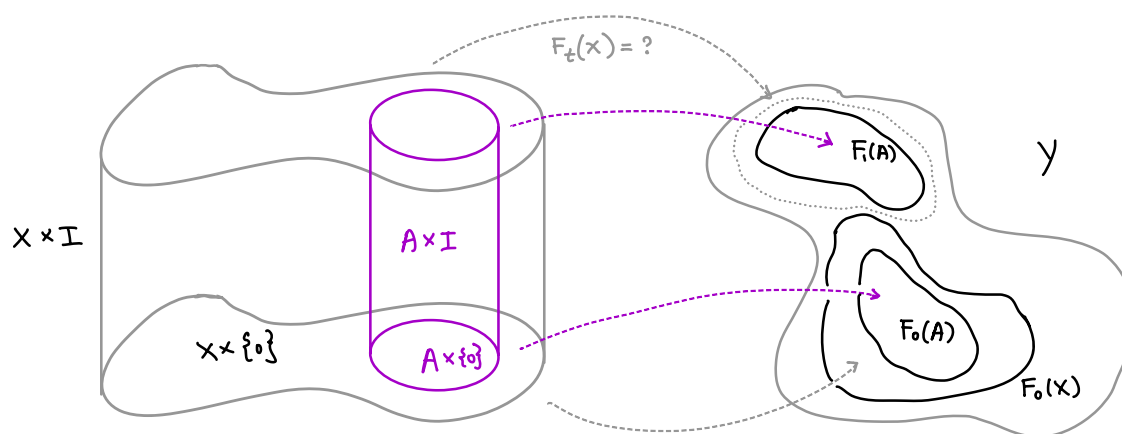
(TO BE DONE)

Lec 3. The Homotopy Extension Property.

Def. Let X be a space and $A \subseteq X$. The pair (X, A) is said to have the homotopy extension property if for any space Y , given a homotopy $f_t: A \rightarrow Y$, a map $F_0: X \rightarrow Y$, s.t. $F_0|_A = f_0: A \rightarrow Y$, one can always find a homotopy $F_t: X \rightarrow Y$ s.t. $F_t|_A = f_t, \forall t$.

(OR, since two maps agree on intersection, just the map $X \times \{0\} \cup A \times I \rightarrow Y$)

Equivalently, every pair of maps $X \times \{0\} \rightarrow Y, A \times I \rightarrow Y$ can be extended to a map $X \times I \rightarrow Y$.



lem. A pair (X, A) has the homotopy extension property iff $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

pf ($\Rightarrow$): Let (X, A) have HEP. Then consider $1_{A \times I}: A \times I \rightarrow A \times I$ and $1_{X \times \{0\}}: X \times \{0\} \rightarrow X \times \{0\}$. Then this pair amounts to a map $1: X \times \{0\} \cup A \times I \rightarrow X \times \{0\} \cup A \times I$, which extends to a map $r: X \times I \rightarrow X \times \{0\} \cup A \times I$ that is identity on $X \times \{0\} \cup A \times I$ by HEP. This r is the desired retraction.

($\Leftarrow$): Given $f: X \times \{0\} \cup A \times I \rightarrow Y$, we want to extend it ctsly. By assumption $\exists$ retraction $r: X \times I \rightarrow X \times \{0\} \cup A \times I$. So:
for $: X \times I \xrightarrow{r} X \times \{0\} \cup A \times I \xrightarrow{f} Y$
is the desired extension. $\square$

prop [H, prop 0.16]. If (X, A) is a CW pair, then it has the homotopy extension property.

pf. WTS: (by previous lem) $\exists$ retraction $r: X \times I \rightarrow X \times \{0\} \cup A \times I$.

(I) There is a retraction $\sigma: D^n \times I \rightarrow D^n \times \{0\} \cup \partial D^n \times I$.

Done by radial projection: For every $(x, s) \in D^n \times I$, there is a straight line connecting $(0, 2)$ and (x, s) that hits either $\partial D^n \times I$ or $D^n \times \{0\}$. Let $\sigma(x, s)$ be st. $\sigma(x, s)$ is where the line hits either $\partial D^n \times I$ or $D^n \times \{0\}$.

Specifically, let $(x, s) \in D^n \times I$ be given, $\lambda \in \mathbb{R}$, write

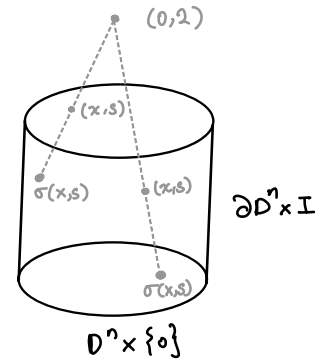
$$\sigma(x, s) = \lambda(x, s) + (1 - \lambda)(0, 2) = (\lambda x, 2 + \lambda(s - 2)).$$

So when $\lambda = 0$, this is $(0, 2)$. When $\lambda = 1$, this is (x, s) . We should solve for some $\lambda \geq 1$ st. either $\|\lambda x\| = 1$ (on $\partial D^n \times I$) or $2 + \lambda(s - 2) = 0$ (on $D^n \times \{0\}$). We get:

$$\|\lambda x\| = 1 \Rightarrow \sigma(x, s) = \left(\frac{x}{\|x\|}, 2 + \frac{s-2}{\|x\|} \right) \text{ if } \|x\| > \frac{2-s}{2}$$

$$2 + \lambda(s - 2) = 0 \Rightarrow \sigma(x, s) = \left(\frac{2-s}{2} x, 0 \right) \text{ if } \|x\| \leq \frac{2-s}{2}.$$

Check: σ is continuous on both domains and agree on the intersection.



(II) We consider the special case where X is just A with one more cell attached: $X = A \cup e^n$. Denote $Z = (X \times \{0\}) \cup (A \times I)$.

We want to find a retraction $r: X \times I \rightarrow Z$. Denote

$\varphi: \partial D^n \rightarrow A$ attaching map

$\Phi: D^n \rightarrow X$ characteristic map

$\Psi: D^n \times I \rightarrow X \times I$ by $\Psi(x, t) = (\Phi(x), t)$.

Then we can view

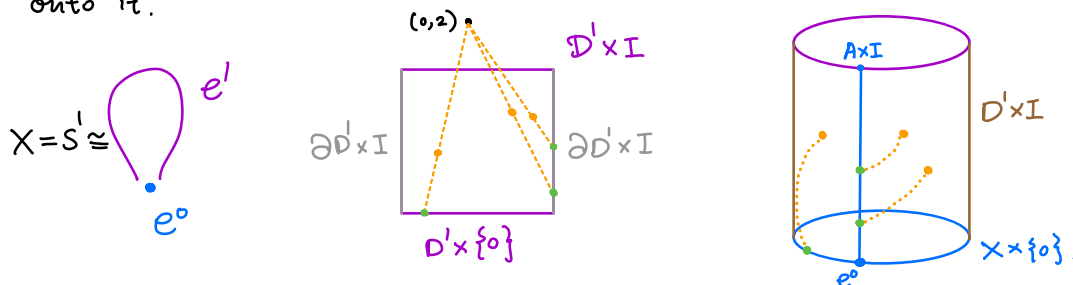
$$X \times I = Z \sqcup (D^n \times I) / \begin{matrix} (x, t) \sim \Psi(x, t) \\ \text{if } (x, t) \in (D^n \times \{0\}) \cup (\partial D^n \times I) \end{matrix}.$$

Then our retraction $r: X \times I \rightarrow X \times \{0\} \cup A \times I = Z$ is

$$r(p, t) = \begin{cases} (p, t) & \text{if } (p, t) \in Z \\ \Psi(\sigma(x, t)) & \text{if } (p, t) = \Psi(x, t). \end{cases}$$

Again, check this is a continuous map.

Rmk. We can consider the following example: $X=S^1$, $A=e^0$ and $e^n=e^1$ is just a 1-cell. We want to find a retraction from $X \times I$ to $X \times \{0\} \cup A \times I$. We know how to retract $e^1 \times I \cong D^1 \times I$ to $D^1 \times \{0\} \cup \partial D^1 \times I$ by projection. Thus we fix points on $X \times \{0\} \cup A \times I$ and we retract everything else onto it.



(III) $X = A \cup_{\alpha} e^n_{\alpha}$: just repeat the steps in (II) simultaneously.

(IV) Suppose inductively, there exists a retraction

$$r_n: (X^n \cup A) \times I \rightarrow (X^n \times \{0\}) \cup (A \times I)$$

st. r_n is an extension of r_{n-1} . This is the case $X = X^n \cup A$, and note that

$$\begin{aligned} (X \times \{0\}) \cup (A \times I) &= (X^n \cup A) \times \{0\} \cup (A \times I) \\ &= (X^n \times \{0\}) \cup (A \times \{0\}) \cup (A \times I) \\ &= (X^n \times \{0\}) \cup (A \times I). \end{aligned}$$

Such an r_n exists, for $X^n = \{\text{all cells in } X \text{ with } \dim \leq n\}$ and each retraction $(e^n \cup A) \times I \rightarrow (e^n \times \{0\}) \cup (A \times I)$ can be constructed by (II), and doing (II) simultaneously for all cells gives r_n by (III). Finally define the retraction $r: X \times I \rightarrow X \times \{0\} \cup A \times I$ by

$$r = \bigcup r_n.$$

By this, we mean whenever $(x,t) \in X \times I$, we must have $(x,t) \in (X^n \cup A) \times I$ for some n . Then $r(x,t) = r_n(x,t)$. Since on the intersections r_n s agree, r is well-defined. Then

$$\text{dom}(r) = \text{dom}(\bigcup r_n) = \bigcup \text{dom}(r_n) = \bigcup (X^n \cup A) \times I = X \times I.$$

It remains to check r is continuous. Note that $(X^n \cup A) \times I$ is closed for each n , since X^n closed in X , any subcomplex A is closed in X , and $I \subseteq I$ closed. So $(X^n \cup A) \times I \subseteq X \times I$ is closed. And $X = \bigcup_n (X^n \cup A) \times I$.

And $r|_{(X^n \cup A) \times I} = r_n$ is continuous, so r is continuous. $\square$

Prop. (H0.7). If the pair (X, A) has the HEP and A is contractible, then the quotient map $q: X \rightarrow X/A$ is a homotopy equivalence.

pf. To show the claim holds, we need to find a map $g: X/A \rightarrow X$ s.t. $qg \simeq 1_{X/A}$ and $gq \simeq 1_X$. First, since A is homotopic to a point, we can find a homotopy $g_t: A \rightarrow A$, with $g_0 = 1_A$ and $g_1(A) = a_0 \in A$. By the hEP, we can extend the pair $(g_t: A \rightarrow A, f_0 = 1_X: X \rightarrow X)$ to a homotopy $f_t: X \rightarrow X$ with $f_0 = 1_X$, $f_1(A) = a_0 \in A$, and $f_t|_A = g_t: A \rightarrow A$, i.e. $f_t(A) \subseteq A$.

Step 1. We claim that $\exists \bar{f}_t: X/A \rightarrow X/A$ a homotopy, s.t. the following diagram commutes (TFDC):

$$\begin{array}{ccc} X & \xrightarrow{f_t} & X \\ q \downarrow & & \downarrow q \\ X/A & \xrightarrow{\bar{f}_t} & X/A \end{array} \quad q \circ f_t = \bar{f}_t \circ q.$$

We define $\bar{f}_t([x]) = [f_t(x)]$. This map is well-defined, for if $[x] = [a]$ with $a \in A$, since $f_t(A) \subseteq A$, we have $q f_t(a) = [a]$ for any $a \in A$. Now we show $\bar{f}_t: X/A \rightarrow X/A$ is continuous. Let

$\mathcal{O} \subseteq X/A$ be an open set, then

$$(q \circ f_t)^{-1}(\mathcal{O}) = (\bar{f}_t \circ q)^{-1}(\mathcal{O}) = q^{-1}(\bar{f}_t^{-1}(\mathcal{O})).$$

open by continuity, which forces $\bar{f}_t^{-1}(\mathcal{O})$ to be open by quotient topology.

Step 2. Now we find the inverse of q . Claim: $\exists g: X/A \rightarrow X$ s.t.

TFDC:

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X \\ q \downarrow & \nearrow g & \downarrow q \\ X/A & \xrightarrow{\bar{f}_1} & X/A \end{array} \quad \dots \text{which implies} \quad \begin{array}{l} q \circ g = \bar{f}_1 \simeq \bar{f}_0 = 1_{X/A} \\ g \circ q = f_1 \simeq f_0 = 1_X. \end{array}$$

Define $g([x]) = \begin{cases} a_0 & \text{for } x \in A \\ f_1(x) & \text{for } x \in X \setminus A. \end{cases}$

Clearly g makes the diagram commute. Remains to show g cts.

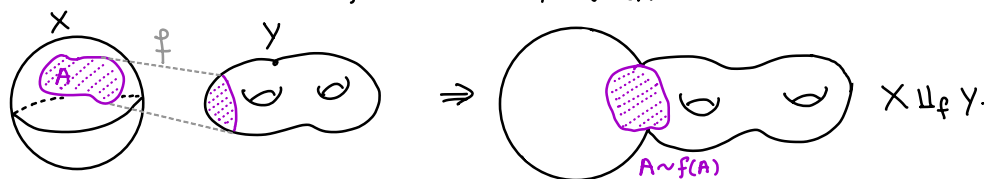
Let $\mathcal{O} \subseteq X$ be open. Then by commutativity

$$f_1^{-1}(\mathcal{O}) = (g \circ q)^{-1}(\mathcal{O}) = q^{-1}(g^{-1}(\mathcal{O})).$$

LHS is open, so $g^{-1}(\mathcal{O})$ is open by quotient topology. $\square$

Def. If $A \subseteq X$ and $f: A \rightarrow Y$ is a cts map, define

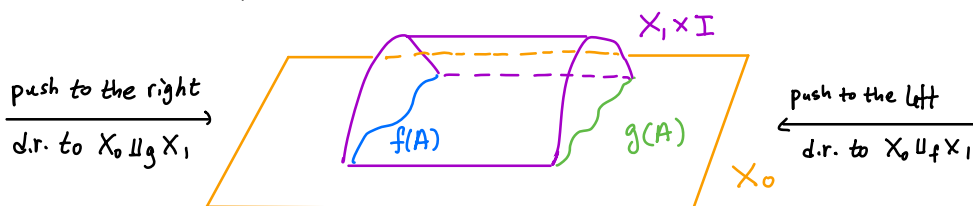
$$X \sqcup_f Y = (X \sqcup Y) / \sim \text{ where } f(a) \sim a \text{ for } a \in A.$$



Prop [H0.18]. "homotopic attaching maps give homotopy equivalent spaces".

If (X_1, A) is a CW pair, $f, g: A \rightarrow X_0$ s.t. $f \simeq g$. Then we have $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0 .

pf. If $F: A \times I \rightarrow X_0$ is a homotopy from f to g , i.e. $F_0 = f$, $F_1 = g$, then consider the space $X_0 \sqcup_F (X_1 \times I)$. This contains both $X_0 \sqcup_f (X_1 \times I)$ and $X_0 \sqcup_g (X_1 \times I)$ as subspaces. Then by [H0.16], $\exists$ a deformation retraction $X_1 \times I \rightarrow X_1 \times \{0\} \cup A \times I$, which induces a deformation retraction $X_0 \sqcup_F (X_1 \times I) \rightarrow X_0 \sqcup_f X_1$. Similarly, $X_0 \sqcup_F (X_1 \times I)$ deformation retracts to $X_0 \sqcup_g X_1$. So $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_F (X_1 \times I) \simeq X_0 \sqcup_g X_1$ and these deformation retractions restricted to X_0 is identity. $\square$



Rmk. Given the deformation retraction $X_1 \times I \xrightarrow{\eta} X_1 \times \{0\} \cup A \times I$, we can extend η over X_0 by identity 1_{X_0} . Then let $r: X_0 \sqcup_F (X_1 \times I) \rightarrow X_0 \sqcup_f X_1$ by

$$r \circ \mu_t(x_0) = x_0, \text{ if } x_0 \in X_0.$$

$$r \circ \mu_t(x_1, s) = \eta_t(x_1, s) \text{ if } (x_1, s) \in X_1 \times I.$$

Lec 4. More on HEP.

Prop [H0.19]. Suppose (X, A) , (Y, A) has HEP and $f: X \rightarrow Y$ is a homotopy equivalence and $f|_A = 1_A$. Then f is a homotopy equivalence rel A .

pf Let $g: Y \rightarrow X$ the homotopy inverse of f . Then $gf \simeq 1_X$. Let $h_t: X \rightarrow X$ s.t. $h_0 = gf$, $h_1 = 1_X$. Since $f|_A = 1_A$, view $h_t|_A$ as a homotopy from $g|_A$ to 1_A . Extend the pair $(h_t: A \rightarrow X, h_0 = g: Y \rightarrow X)$ to a homotopy $g_t: Y \rightarrow X$, $g_0 = g$, $g_t|_A = 1_A$, for (Y, A) has HEP.

Now we show that

$$(I) \quad g \circ f \simeq 1 \text{ rel } A;$$

$$(II) \quad f \circ g \simeq 1 \text{ rel } A.$$

To show (I), we construct a "homotopy of the homotopy". A homotopy from $g \circ f$ to 1_X is

$$k_t = \begin{cases} g_{1-2t} \circ f & 0 \leq t \leq \frac{1}{2} \\ h_{2t-1} & \frac{1}{2} \leq t \leq 1. \end{cases}$$

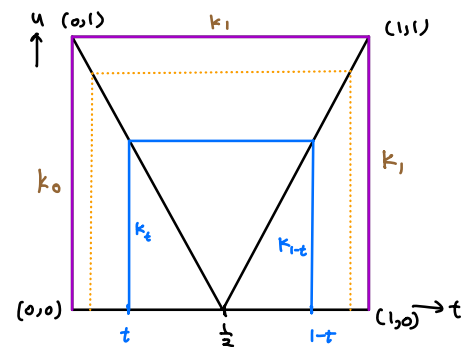
This map is well-defined, and $k_t = k_{1-t}$ on A . Also, $k_0|_A = k_1|_A = 1_A$.

Define a homotopy of the homotopy k_t as follows: Define $k_{tu}: A \rightarrow X$

by the figure on the right. For every pt $(t, u) \in I^2$, $\exists$ a square like the blue one, containing (t, u) . Then on this square, let $k_{tu} = k_t$. If we move continuously on I^2 , then the t coordinate moves continuously, so k_{tu} is varying continuously. Thus the map

$$K: A \times I \times I \rightarrow X$$

$$(a, t, u) \mapsto k_{tu}(a)$$



is continuous. Let $J \subseteq \partial I^2$ denote the purple line. Then $J \cong [0, 1]$:

Thus $k_t = k_{tu}|_{[0,1] \times \{0\}} \simeq k_{tu}|_J$. And $k_{tu}|_J = 1_A$ on A . Viewing k_{tu} as a homotopy of $k_t|_A$, and note (X, A) has HEP $\Rightarrow (X \times I, A \times I)$ has

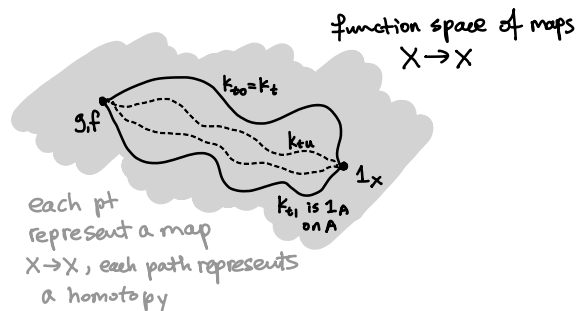
HEP (use H prop on pg 14), extend $(k_{tu}: A \rightarrow X, k_{t0} = k_t: X \rightarrow X)$

to a map $k_{tu}: X \rightarrow X$. Thus if we restrict $k_{tu}: X \rightarrow X$ to J ,

this is a homotopy $k_{t0} = k_t \simeq k_{t1}$ s.t. $k_{t1}|_A = 1_A$. Then we

have a homotopy $g \circ f \simeq 1$ rel A as the drawing suggests.

(II) Now $f g_1 \simeq 1$, so we can use a similar argument to (I) and show that we can deform the homotopy $\eta_t : Y \rightarrow Y$, $\eta_0 = f g_1$, $\eta_1 = 1$ to a homotopy fixing A . Thus $f g_1 \simeq 1 \text{ rel } A$.



This proves the claim. $\square$

Cor [H 0.20]. If (X, A) has HEP and $i: A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X .

pf. $i = 1_A: A \rightarrow X$. By previous prop applied to (A, A) , (X, A) , $i: A \hookrightarrow X$, $\exists$ a homotopy inverse $r: X \rightarrow A$ s.t. $ir \simeq 1_X \text{ rel } A$. This homotopy starts with 1_X , ends with $ir = r: X \rightarrow A$. Thus this is a deformation retraction of X onto A . $\square$

Cor (H 0.21). $f: X \rightarrow Y$ is a homotopy equivalence $\Leftrightarrow X$ is a deformation retract of M_f .

pf. Let $i: X \hookrightarrow M_f$, $j: Y \hookrightarrow M_f$ be two inclusion maps. Since M_f deformation retract to Y , j is a hom. eqv. And $i \simeq j \circ f$ via $h_t(x) = [(x, t)]$.

Since also if we let $r: M_f \rightarrow Y$ be the def. retraction, then $f = r \circ i$, we have:

f is hom. eqv $\Leftrightarrow i$ is hom. eqv. shown below.

($\Rightarrow$): Suppose i is hom. eqv. Then by Hatcher Ex 0.15, (M_f, X) has the HEP. Thus by previous Cor, M_f def retract to X .

($\Leftarrow$): If $s: M_f \rightarrow X$ a def. retraction, then s is the homotopy inverse of i . Thus i is a hom. eqv $\Rightarrow$ so is f . $\square$

Now we prove Hatcher Ex 0.15.

identified with subspace $X \times \{0\}$.

Prop. Let $f: X \rightarrow Y$ cts. Then (M_f, X) has HEP.

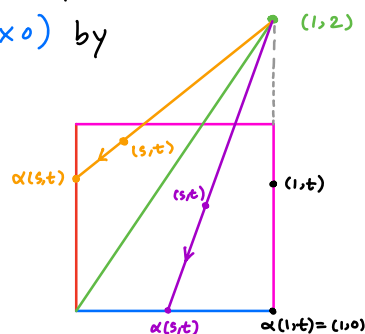
pf. We want to show $\exists$ a retraction $\beta: M_f \times I \rightarrow (M_f \times 0) \cup ((X \times 0) \times I)$.

Step 1. There is a retraction $\alpha: I^2 \rightarrow (0 \times I) \cup (I \times 0)$ by

$$\alpha(s,t) = \begin{cases} (0, (t-2s)/(1-s)) & t \geq 2s \\ ((2s-t)/2-t, 0) & t \leq 2s \end{cases}$$

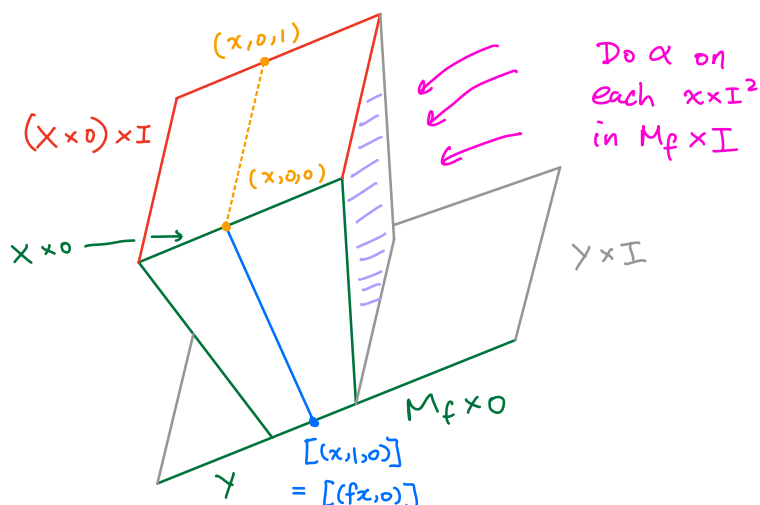
Note that $\alpha(1,t) = (1,0)$.

The picture is shown in the left. This is really just the radial projection from $(1,2)$.



Step 2. This retraction α induces a retraction

$$\beta: M_f \times I \rightarrow (M_f \times 0) \cup ((X \times 0) \times I).$$



The retraction goes as follows: Let $p \in M_f \times I = \frac{(X \times I \cup Y) \times I}{\sim}$.

- If $p = (y,t) \in Y \times I$, then $\beta(p) = y \times 0$.
- If $p = [(x,s),t] \in M_f \times I$, Let $\beta(p) = [(x, \alpha(s,t))]$.

These two cases agree when $p \in [(X \times 1) \times I] \in M_f \times I$: Let $x \in X$, $f(x) = y$.

Then we have

$$\begin{aligned} \beta([(x,1),t]) &= [(x, \alpha(1,t))] = [(x, (1,0))] = [(x,1),0] \\ &= [(fx,0)] = \beta(y,t). \end{aligned}$$

And we have β a cts map on two subspaces $[Y \times I]$ and $[X \times I^2]$ of $M_f \times I$, which agrees on the intersection. Thus β is cts. But the existence of the retraction $\beta: M_f \times I \rightarrow (M_f \times 0) \cup ((X \times 0) \times I)$ implies the pair (M_f, X) has the HEP.

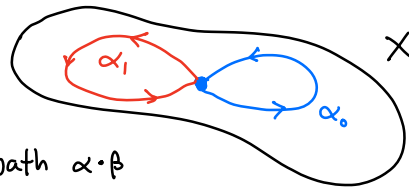
□

Chap 1. The Fundamental Groups

Lec 5. Introduction

Def. Let X be a topological space.

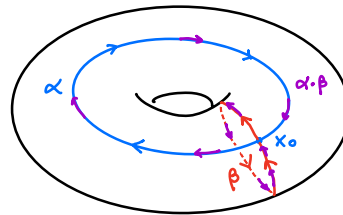
- A **loop** based at $x_0 \in X$ is a cts map $\alpha: I \rightarrow X$ s.t. $\alpha(\partial I) = x_0$.
- Loops α_0, α_1 are **homotopic** rel the base pt x_0 if $\exists$ a homotopy $\alpha_t: I \rightarrow X$ s.t. $\alpha_t(\partial I) = x_0, \forall t \in I$.
- If $\alpha \simeq \beta$, then they are in the same **homotopy class**. Denote $[\alpha]$ the homotopy class of the based loop α (i.e. all based loops homotopic to α).



Def. Let $\alpha, \beta: I \rightarrow X$ be two loops based at $x_0 \in X$. The composition or product path $\alpha \cdot \beta$ is the loop

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s) & s \leq \frac{1}{2} \\ \beta(2s-1) & s \geq \frac{1}{2} \end{cases}$$

If $\alpha_t, \beta_t: I, \partial I \rightarrow X, x_0$ are homotopies. Then $\alpha_t \cdot \beta_t$ is a homotopy from $\alpha_0 \cdot \beta_0$ to $\alpha_1 \cdot \beta_1$.



Def. The **fundamental group** of X at the base pt $x_0 \in X$, denoted by $\pi_1(X, x_0)$, is the set

$$\pi_1(X, x_0) = \{ [\alpha]: \alpha: I, \partial I \xrightarrow{\text{cts}} X, x_0 \}$$

together with the product $\cdot: \pi_1(X, x_0) \times \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$ by $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$.

Thm. $(\pi_1(X, x_0), \cdot)$ defined above is a group.

(LONG PROOF!)

pf. We verify the axioms of a gp individually:

(I): Show $\cdot$ is well-defined.

Let $\alpha \simeq \eta, \beta \simeq \tau$. Then let δ_t be a homotopy connecting α and η , let ϵ_t be the homotopy connecting β to τ . Then $\delta_t \epsilon_t$ connects $\alpha \cdot \beta$ to $\eta \cdot \tau$. So

$$[\alpha] \cdot [\beta] = [\alpha \cdot \beta] = [\eta \cdot \tau] = [\eta] \cdot [\tau]. \quad \checkmark$$

NOTE: Define a **reparametrization** of a path α to be a composition $\alpha \circ \varphi$ where $\varphi: I \rightarrow I$ is cts s.t. $\varphi(0)=0$, $\varphi(1)=1$. Then $\alpha \circ \varphi \simeq \alpha$ by the homotopy $\alpha \circ \varphi_t$ where $\varphi_t(s) = (1-t)\varphi(s) + ts$. So $\varphi_0 = \varphi$, $\varphi_1(s) = s$. Note $(1-t)\varphi(s) + ts \in [\varphi(s), s] \subseteq [0, 1]$. So φ_t is well-defined.

(II) Show $\cdot$ is associative.

If we can show $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$, then

$$([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha \cdot \beta] \cdot [\gamma] = [(\alpha \cdot \beta) \cdot \gamma] = [\alpha \cdot (\beta \cdot \gamma)] = [\alpha] \cdot [\beta \cdot \gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$$

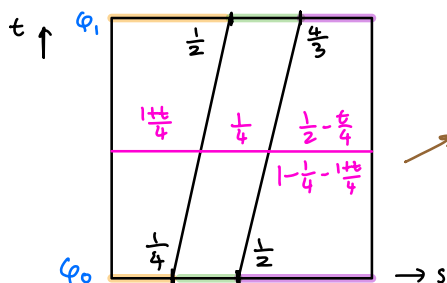
and we are done. So we wts $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$. Now:

$$(\alpha \cdot \beta) \cdot \gamma(s) = \begin{cases} \alpha(4s) & s \in [0, \frac{1}{4}] \\ \beta(4s-1) & s \in [\frac{1}{4}, \frac{1}{2}] \\ \gamma(2s-1) & s \in [\frac{1}{2}, 1] \end{cases} \quad (\alpha \cdot (\beta \cdot \gamma))(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}] \\ \beta(4s-2) & s \in [\frac{1}{2}, \frac{3}{4}] \\ \gamma(4s-3) & s \in [\frac{3}{4}, 1] \end{cases}$$

Consider the reparametrization

$$\varphi_t(s) = \begin{cases} (1+t)s & s \in [0, \frac{1}{4}] \\ s + t/4 & s \in [\frac{1}{4}, \frac{1}{2}] \\ 1 - (1-\frac{t}{2})(1-s) = \frac{1}{2} + (1-\frac{t}{2})s & s \in [\frac{1}{2}, 1] \end{cases}$$

Then φ_t shifts the interval $I = [0, 1]$ as follows:



the scale factor is $(\frac{1}{2} - t/4) / (\frac{1}{2}) = 1 - t/2$

Let $s \in [\frac{1}{2}, 1]$ and fix t . Then s is $(1-s)$ away from 1 at $t=0$. This distance is scaled by the factor $(1-t/2)$. So at time t , this pt is $(1-t/2)(1-s)$ away from 1. Thus its position is $1 - (1-t/2)(1-s)$.

Let our homotopy of loops be $F_t = (\alpha \cdot (\beta \cdot \gamma)) \circ \varphi_t$.

Then one can check: $\varphi_0 = 1_I \Rightarrow F_0 = \alpha \cdot (\beta \cdot \gamma)$. Writting out φ_1 , we

see:
$$\varphi_1(s) = \begin{cases} 2s & s \in [0, \frac{1}{4}] \\ s + \frac{1}{4} & s \in [\frac{1}{4}, \frac{1}{2}] \\ \frac{1}{2}s + \frac{1}{2} & s \in [\frac{1}{2}, 1] \end{cases}$$

So
$$F_1(s) = \begin{cases} \alpha(2(2s)) & s \in [0, \frac{1}{4}] \\ \beta(4(s+\frac{1}{4})-2) & s \in [\frac{1}{4}, \frac{1}{2}] \\ \gamma(4(\frac{1}{2}s+\frac{1}{2})-3) & s \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} \alpha(4s) & s \in [0, \frac{1}{4}] \\ \beta(4s-1) & s \in [\frac{1}{4}, \frac{1}{2}] \\ \gamma(2s-1) & s \in [\frac{1}{2}, 1] \end{cases} = ((\alpha \cdot \beta) \cdot \gamma)(s).$$

Thus we have $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$. Hence this proves the associativity of $\cdot$.

✓

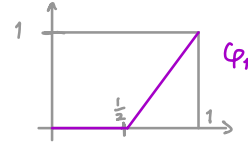
(III) Existence of identity element:

Define the constant loop $c: I, \partial I \rightarrow X, x_0$ by $c(s) = x_0, \forall s \in I$.

WTS: $c \cdot \alpha \simeq \alpha \simeq \alpha \cdot c$ So $[c] \cdot [\alpha] = [\alpha] = [\alpha] \cdot [c]$.

To show $c \cdot \alpha \simeq \alpha$, consider the reparametrization

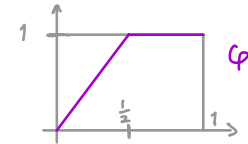
$$\varphi_t(s) = \begin{cases} 0 & s \in [0, t/2] \\ \frac{s-t/2}{1-t/2} & s \in [t/2, 1] \end{cases}$$



Then $\varphi_t(0) = 0, \varphi_t(1) = 1, \forall t$. And $\varphi_0(s) = s, \varphi_1(s) = \begin{cases} 0 & s \leq \frac{1}{2} \\ 2s-1 & s \geq \frac{1}{2} \end{cases}$.

Thus $\alpha \circ \varphi_0 = \alpha, \alpha \circ \varphi_1 = c \cdot \alpha$. So $c \cdot \alpha \simeq \alpha$.

Similarly, $\alpha \cdot c$ is a reparametrization of α by $\alpha \cdot c = \alpha \circ \varphi_1$ of the function on the right.



So $\alpha \cdot c \simeq \alpha$.

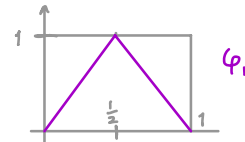
(IV) Existence of inverse.

Given $\alpha: I, \partial I \rightarrow X, x_0$. Define $\beta(s) = \alpha(1-s)$. So β is just a loop that goes on the loop α backwards. WTS

$\alpha \cdot \beta \simeq c \simeq \beta \cdot \alpha$ so that $[\alpha] \cdot [\beta] = [c] = [\beta] \cdot [\alpha]$.

Consider the map

$$\varphi_t(s) = \begin{cases} 2ts & s \in [0, \frac{1}{2}] \\ 2t(1-s) & s \in [\frac{1}{2}, 1] \end{cases}$$



Then $\varphi_0(s) = 0$. And $\alpha \circ \varphi_0(s) = x_0$. So $\alpha \circ \varphi_0 = c$.

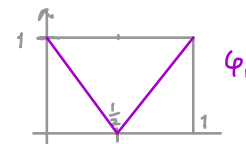
$$\varphi_1(s) = \begin{cases} 2s & s \in [0, \frac{1}{2}] \\ 2-2s & s \in [\frac{1}{2}, 1] \end{cases} \Rightarrow \alpha \circ \varphi_1(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}] \\ \alpha(2(1-s)) & s \in [\frac{1}{2}, 1] \end{cases} = \alpha \cdot \beta$$

Thus $\alpha \cdot \beta \simeq c$. Similarly $\beta \cdot \alpha \simeq c$ can be proven by considering the map

$$\varphi_t(s) = \begin{cases} t(1-2s) & s \in [0, \frac{1}{2}] \\ t(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

Then $\varphi_0(s) = 0, \alpha \circ \varphi_0 = c$.

$$\varphi_1(s) = \begin{cases} 1-2s & s \in [0, \frac{1}{2}] \\ 2s-1 & s \in [\frac{1}{2}, 1] \end{cases}$$



$$\Rightarrow \alpha \circ \varphi_1(s) = \begin{cases} \alpha(1-2s) & s \in [0, \frac{1}{2}] \\ \alpha(2s-1) & s \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} \beta(2s) & s \in [0, \frac{1}{2}] \\ \beta(2s-1) & s \in [\frac{1}{2}, 1] \end{cases} = \beta \cdot \alpha$$

Thus $\beta \cdot \alpha \simeq c$.

This proves the claim.

✓

□

Lec 6. More on Fundamental Groups & Covering Spaces.

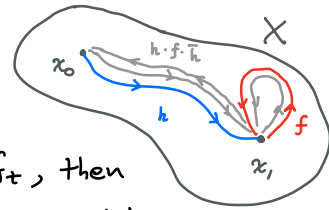
• Dependence of Fundamental Group on basepoint:

Prop 1.5. Let X be a path-connected topological space and $x_0, x_1 \in X$. Then $\pi_1(X, x_0) \approx \pi_1(X, x_1)$.

pf. Let $h: [0, 1] \rightarrow X$ be a path connecting x_0, x_1 . Let $\bar{h} = h(1-t)$, $t \in [0, 1]$ denote the "reverse" of h by traveling backwards. Let $f: I \rightarrow X$ be a loop based at x_1 . Then $h \cdot f \cdot \bar{h}$ is a loop based at x_0 :

This gives a map

$$\begin{aligned} \beta_h: \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [h \cdot f \cdot \bar{h}]. \end{aligned}$$



This map is well-defined, for if $f_0 \simeq f_1$ via f_t , then $h \cdot f_0 \cdot \bar{h} \simeq h \cdot f_1 \cdot \bar{h}$ via $h \cdot f_t \cdot \bar{h}$. This is a homomorphism, for $\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}] = \beta_h[f] \beta_h[g]$.

This is a bijection with inverse $\beta_{\bar{h}}$, for

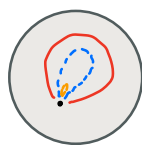
$$\beta_h \beta_{\bar{h}}[f] = \beta_h[\bar{h} \cdot f \cdot h] = [h \cdot \bar{h} \cdot f \cdot h \cdot \bar{h}] = [f]$$

and similarly $\beta_{\bar{h}} \beta_h[f] = [f]$. □

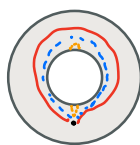
Rmk. By above, if X is path-connected, the fundamental group is the same up to isomorphism. So usually we do not make distinction of base pts and just write $\pi_1(X)$.

Def. A space is **simply-connected** if it is path-connected and has trivial fundamental group. I.e. every loop can be deformed ctsly into the constant loop around the base point.

Ex. Simply Connected?

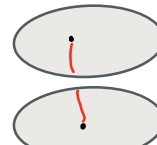


Yes.



No.

← DOES NOT PRESERVE CONTINUITY!



No.

(NOT EVEN PATH CONNECTED!)

Prop 1.6. X is simply connected $\Leftrightarrow \exists!$ homotopy class of paths connecting any two pts in X .

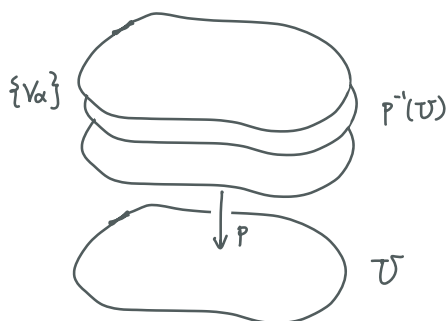
pf. $(\Rightarrow)$ Then X path-connected and $\pi_1(X) = \{0\}$. If f, g paths joining $x_0, x_1 \in X$, then $f \simeq f \cdot (\bar{g} \cdot g) \simeq (f \cdot \bar{g}) \cdot g \simeq g$ for $\bar{g} \cdot g$ a loop at x_0 and $f \cdot \bar{g}$ also loop at x_0 . Since $\pi_1(X) = \{0\}$, they are both homotopic to the constant loop.

$(\Leftarrow)$: Then there is only one homotopy class of loops based at any $x_0 \in X$. So all loops are homotopic to constant loop $\Rightarrow \pi_1(X) = 0$.

□

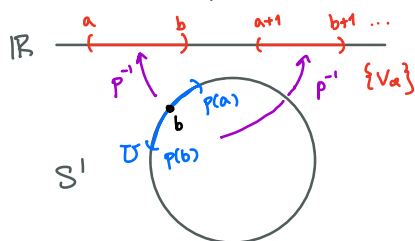
• Covering Spaces and Map Lifting.

Def. Let $p: E \rightarrow B$ be a cts surjective map between top. spaces. The open set $U \subseteq B$ is said to be evenly covered by p if the inverse image $p^{-1}(U)$ is a union of disjoint open sets $V_\alpha \subseteq E$ st. for each α , $p|_{V_\alpha}$ maps V_α homeomorphically onto U . The set $\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into slices.

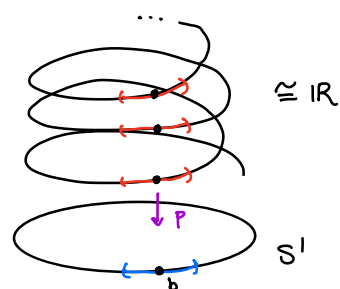


If every pt $b \in B$ has a nbh U that is evenly covered by p , then p is called a covering map, and E a covering space of B .

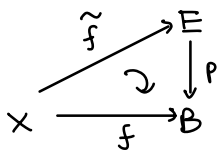
Ex. $p: \mathbb{R} \rightarrow S^1$ by $p(s) = (\cos 2\pi s, \sin 2\pi s)$.



OR



Def. Let $p: E \rightarrow B$ be a map. If $f: X \rightarrow B$ is a cts map, a lifting of f is a map $\tilde{f}: X \rightarrow E$ s.t. $p \circ \tilde{f} = f$.



[Munkres
lem 54.1]

lem 6.1. Let $p: E \rightarrow B$ be a covering map. Let $p(e_0) = b_0$. Any path $f: [0,1] \rightarrow B$ s.t. $f(0) = b_0$ has a unique lifting to a path $\tilde{f}: [0,1] \rightarrow E$ s.t. $\tilde{f}(0) = e_0$.

pf. Cover B by open sets U that are evenly covered by p . Find a partition of $[0,1]$, say $0 = s_0 < s_1 < \dots < s_n = 1$ s.t. for each i , $f([s_i, s_{i+1}])$ lies inside such an open set U (by Lebesgue number lemma). Define $\tilde{f}$ inductively:

- Define $\tilde{f}(0) = e_0$.
- Suppose $\tilde{f}$ defined on $[0, s_i]$. The set $f([s_i, s_{i+1}])$ lies in some U that is evenly covered by p . Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. $\forall \alpha, V_\alpha \xrightarrow{p|_{V_\alpha}} U$. Now $f(s_i)$ lies in one of these sets, say in V_0 . Define

$$\tilde{f}(s) = (p|_{V_0})^{-1}(f(s)) \quad \text{for } s \in [s_i, s_{i+1}].$$

This is cts (pasting lemma).

- Continue with finitely many steps and get a map $\tilde{f}: [0,1] \rightarrow E$. Continuity of $\tilde{f}$ by pasting lemma and $p \circ \tilde{f} = f$ is clear from def.

To prove uniqueness. Suppose $\tilde{\eta}: [0,1] \rightarrow E$ is another lifting s.t. $\tilde{\eta}(0) = e_0$, $p \circ \tilde{\eta} = f$. Then $\tilde{\eta}(0) = e_0 = \tilde{f}(0)$. Suppose $\tilde{\eta} = \tilde{f}$ on $[0, s_i]$. Let V_0 be the same as above. Since $\tilde{\eta}$ is a lifting of f ,

$$\tilde{\eta}^{-1}(U) = \bigcup_\alpha V_\alpha.$$

$\{V_\alpha\}$ s are open and disjoint. $[s_i, s_{i+1}]$ connected $\Rightarrow \tilde{\eta}([s_i, s_{i+1}])$ must lie entirely in one of V_α s. But $\tilde{\eta}(s_i) = \tilde{f}(s_i) \in V_0 \Rightarrow$ it lies in V_0 .

For $s \in [s_i, s_{i+1}]$, $\tilde{\eta}(s)$ must equal some $p^{-1}y$ of V_0 lying in $p^{-1}(f(s))$. But $\exists!$ such pt, i.e. $(p|_{V_0})^{-1}(f(s))$. So $\tilde{\eta} = \tilde{f}$ on $[s_i, s_{i+1}]$.

□

lem 6.2 [M L 54.2]. let $p: E \rightarrow B$ be a covering map. let $p(e_0) = b_0$. let $F: I \times I \rightarrow B$ be cts w/ $F(0,0) = b_0$. Then $\exists!$ lifting of F to a cts map $\tilde{F}: I \times I \rightarrow E$ st. $\tilde{F}(0,0) = e_0$. If F is a path homotopy, so is $\tilde{F}$.

pf. Use lem 6.1 to extend $\tilde{F}$ to the edge $0 \times I$ and $I \times 0$ of $I \times I$. Then choose partitions

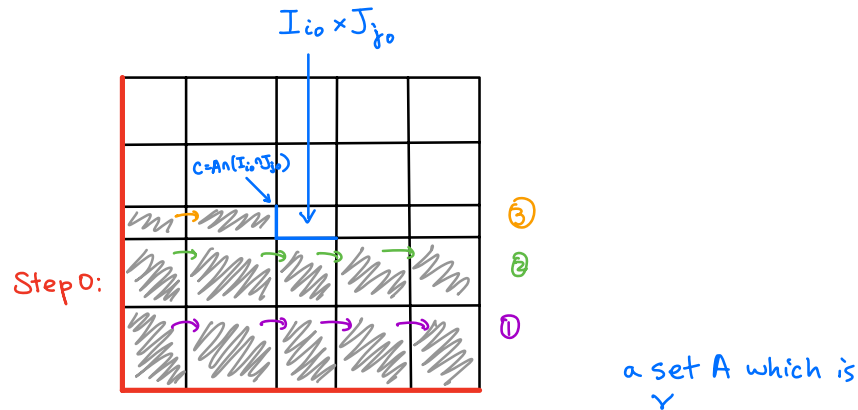
$$s_0 < s_1 < \dots < s_m$$

$$t_0 < t_1 < \dots < t_n$$

s.t. each rectangle

$$I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$$

is mapped by F into an open set of B evenly covered by p (Lebesgue $\#$ lem). Define lifting $\tilde{F}: I^2 \rightarrow E$ step by step in the following order:



In general, given i_0, j_0 . Suppose $\tilde{F}$ is defined on all the previous $I_i \times J_j$ for $j < j_0$ and $i < i_0$, and on $0 \times I \cup I \times 0$. Choose an open set U of B evenly covered by p containing $F(I_{i_0} \times J_{j_0})$. let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. $V_\alpha \xrightarrow{p} U$. Now $\tilde{F}$ already defined on $C = A \cap (I_{i_0} \times J_{j_0})$. Note C is connected. So $\tilde{F}(C)$ is connected and must lie entirely in one of V_α , say V_0 . Then let $p_0 := p|_{V_0}: V_0 \rightarrow U$. Since $\tilde{F}$ is a lift of $F|_A$, we know for $x \in C$:

$$p_0(\tilde{F}(x)) = p(\tilde{F}(x)) = F(x) \Rightarrow \tilde{F}(x) = p_0^{-1}(F(x)).$$

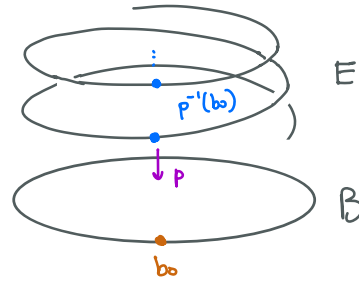
Hence extend $\tilde{F}$ by defining

$$\tilde{F}(x) = p_0^{-1}(F(x)) \quad x \in I_{i_0} \times J_{j_0}.$$

This $\tilde{F}$ extension is cts by pasting. Continuing this way, we get a lift $\tilde{F}: I^2 \rightarrow E$ of $F: I^2 \rightarrow B$.

Uniqueness: Since we first extend $0 \times I \cup I \times 0$ of I^2 uniquely by lem 6.1, and then by the order of our extension (as above drawing shows), once $\tilde{F}(0,0)$ is fixed, $\tilde{F}$ is uniquely determined.

Now, if F is a path hmtpy, then $F(0 \times I)$ is a single pt $b_0 \in B$. So the lifting $\tilde{F}$ takes $0 \times I$ to $p^{-1}(b_0) \subseteq E$, with discrete topology as a subset of E . Since $0 \times I$ is connected, so is $\tilde{F}(0 \times I)$. Thus $\tilde{F}(0 \times I)$ is a one point set. Similarly, $\tilde{F}(1 \times I)$ is a one point set. Thus $\tilde{F}$ is a path homotopy.



This proves the lemma. □

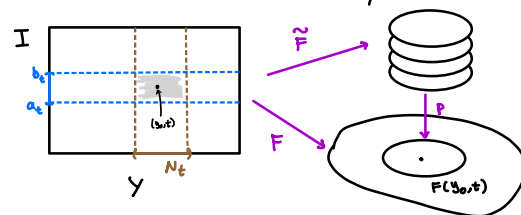
These are all special cases to the following

Thm 6.3 [H pg 30-(c)]. Given a map $F: Y \times I \rightarrow X$ and a map $\tilde{F}: Y \times \{0\} \rightarrow \tilde{X}$ lifting $F|_{Y \times \{0\}}$, $\exists!$ map $\tilde{F}: Y \times I \rightarrow \tilde{X}$ lifting F and extending $\tilde{F}|_{Y \times \{0\}}$.

pf: First, construct a lift $\tilde{F}: N \times I \rightarrow \tilde{X}$ for some nbh N of a pt $y_0 \in Y$.

Since F is cts, every pt $(y_0, t) \in Y \times I$ has a product nbh $N_t \times (a_t, b_t)$ s.t. $F(N_t \times (a_t, b_t))$ is contained in an evenly covered nbh of $F(y_0, t)$. By compactness of $\{y_0\} \times I (\cong I)$, we can choose a partition

$$0 = t_0 < t_1 < \dots < t_m = 1$$



of I and a single nbh N of y_0 (intersection), s.t. for each i , $F(N_t \times [t_i, t_{i+1}]) \subseteq U_i$, where U_i is some evenly covered nbh. Assume inductively $\tilde{F}$ has been constructed on $N \times [0, t_i]$, starting with the given $\tilde{F}$ on $N \times \{0\}$. We have $F(N \times [t_i, t_{i+1}]) \subseteq U_i$, and U_i is some evenly covered nbh, $\tilde{F}(y_0, t_i)$ must be contained in a slice $\tilde{U}_i$ s.t. $p: \tilde{U}_i \rightarrow U_i$ is a homeomorphism. WLOG assume $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$, otherwise replace $N \times \{t_i\}$ by $(N \times \{t_i\}) \cap (\tilde{F}|_{N \times \{t_i\}})^{-1}(\tilde{U}_i)$. Then we define:

$$\tilde{F} = (p|_{\tilde{U}_i})^{-1} \circ F \quad \text{on } N \times [t_i, t_{i+1}].$$

After a finite number of steps we get a lift $\tilde{F}: N \times I \rightarrow \tilde{X}$, as before. Since y_0, N are arbitrary, this gives a lift $\tilde{F}: Y \times I \rightarrow \tilde{X}$.

Uniqueness: Step 1. Case when $Y = \{\text{a point}\}$. Then suppose $\tilde{F}, \tilde{F}': I \rightarrow X$ be two liftings of F s.t. $\tilde{F}(0) = \tilde{F}'(0)$. Then by Lebesgue # Lem choose a partition $0 = t_0 < t_1 < \dots < t_m = 1$ s.t. for each i , $F([t_i, t_{i+1}])$ is in some evenly covered nbh $U_i \subseteq X$. Assume inductively $\tilde{F} = \tilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, so is $\tilde{F}([t_i, t_{i+1}])$, which must then lie in a single slice $\tilde{U}_i$ of U_i in $\tilde{X}$. Similarly for $\tilde{F}'([t_i, t_{i+1}])$. But $\tilde{F}(t_i) = \tilde{F}'(t_i)$ implies $\tilde{F}'([t_i, t_{i+1}]) \subseteq \tilde{U}_i$ as well. Since $p|_{\tilde{U}_i}$ is injective and $p\tilde{F} = p\tilde{F}' \Rightarrow \tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$.

Step 2. Since $\tilde{F}$ is unique when restricted to each segment $\{y\} \times I$, two liftings $\tilde{F}: N \times I \rightarrow \tilde{X}$, $\tilde{F}': N' \times I \rightarrow \tilde{X}$ must agree on $(N \times I) \cap (N' \times I)$. Thus we have a unique and well-defined map $\tilde{F}: Y \times I \rightarrow \tilde{X}$.

□

Lec 7. Fundamental Group of a Circle.

Recall that since S^1 is path-connected, we can just write $\pi_1(S^1)$.

Thm. There is an isomorphism $\phi: \mathbb{Z} \rightarrow \pi_1(S^1)$ given by $\phi(n) = [\omega_n]$ where $\omega_n: I \rightarrow S^1$ is given by $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$.

pf. ϕ surjective: Let $f: I \rightarrow S^1$ be a loop based at $x_0 = (1, 0) \in S^1$, representing an element of $\pi_1(S^1, x_0)$. Let $p: \mathbb{R} \rightarrow S^1$ be $p(s) = (\cos 2\pi s, \sin 2\pi s)$ as in the last section, and by Lem 6.1, $\exists$ a lift $\tilde{f}: I \rightarrow \mathbb{R}$ st. $\tilde{f}(0) = 0$. This lifted path ends at some integer n since $p\tilde{f}(1) = f(1) = x_0$ and $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$. Now consider $\tilde{\omega}_n: I \rightarrow \mathbb{R}$ given by the lift of ω_n : $\tilde{\omega}_n(s) = ns$. Then $\tilde{\omega}_n(0) = 0$ and $\tilde{\omega}_n(1) = n$. Also, $\tilde{f} \simeq \tilde{\omega}_n$ via $\eta_t = (1-t)\tilde{f} + t\tilde{\omega}_n$. So $\tilde{f} \simeq \tilde{\omega}_n$. Thus $f = p\tilde{f} \simeq p\tilde{\omega}_n = \omega_n$. So $[f] = [\omega_n]$.

ϕ injective: Suppose $\phi(n) = \phi(m) \Rightarrow [\omega_n] = [\omega_m] \Rightarrow \omega_n \simeq \omega_m$.

Let f_t be a homotopy from $\omega_m = f_0$ and $\omega_n = f_1$. By Lem 6.2 this homotopy f_t lifts to a homotopy $\tilde{f}_t$. By the uniqueness of this lifting, we have $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths, its endpoint is independent of t : Thus $m = \tilde{f}_0(1) = \tilde{\omega}_m(1) = \tilde{\omega}_n(1) = \tilde{f}_1(1) = n$.

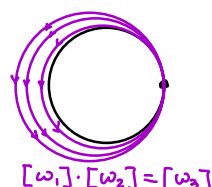
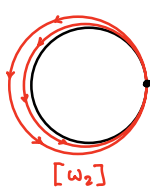
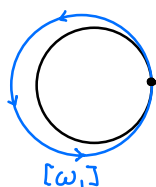
Thus ϕ is a bijection.

Grp Hom: Easy to check:

$$\phi(n+m) = [\omega_{n+m}] = [\omega_n \cdot \omega_m] = [\omega_n] \cdot [\omega_m] = \phi(n) \cdot \phi(m).$$

□

Rmk. The result that $\pi_1(S^1) \simeq \mathbb{Z}$ should be obvious, since given a loop at a base pt on S^1 , the only distinction is how many times it wraps around S^1 .



Cor. S^1 does not deformation retract to a point.

pf. Otherwise, $\exists x_0 \in S^1$, $f_t: S^1 \rightarrow S^1$ cts s.t. $f_0 = 1_{S^1}$, $f_1 = x_0$, $f_t(x_0) = x_0$ $\forall t \in [0,1]$. Then let $[\omega_1] \in \pi_1(S^1, x_0)$ be a nonzero elmnt. Then $\omega_1 = f_0 \circ \omega_1 \simeq f_1 \circ \omega_1 \equiv x_0 \Rightarrow [\omega_1] = 0$.

Contradiction. $\square$

Prop. $\nexists$ retraction $r: D^2 \rightarrow S^1$.

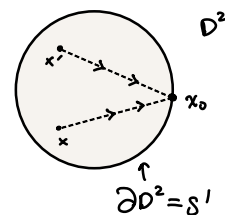
pf. There is a deformation retraction of D^2 to a pt $x_0 \in S^1$, by

$$F_t(x) = tx_0 + (1-t)x \quad x \in D^2.$$

Then if r is a retraction $r: D^2 \rightarrow S^1$, then $(F_t \circ r)|_{S^1}$ is the map

$$S^1 \xrightarrow{r=1_{S^1}} S^1 \xrightarrow{F_t|_{S^1}} \{x_0\}$$

which is a deformation retraction, contradicting the Corollary. $\square$

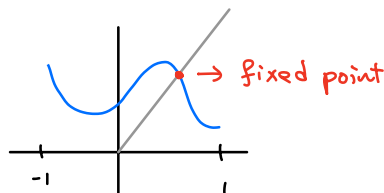


Def. A **fixed point** of $f: X \rightarrow X$ is a pt $p \in X$ s.t. $f(p) = p$.

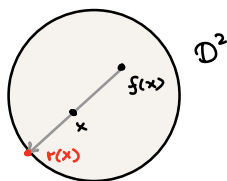
Thm (Brouwer Fixed Point Thm). Every cts $f: D^n \rightarrow D^n$ has a fixed point.

pf. ($n=1$): $D^1 = [-1,1]$. $f: [-1,1] \rightarrow [-1,1]$.

This function $f(x)$ has an intersection with the line $y=x$.



($n=2$): Suppose otherwise $f(x) \neq x$, $\forall x \in D^2$. Then define $r(x)$, for each $x \in D^2$ to be the intersection of the ray starting at $f(x)$, crossing x , and the boundary S^1 . Clearly r is cts, for f is cts, then the ray has the form $tf(x) + (1-t)x$, for some $t \geq 1$ satisfying $\|tf(x) + (1-t)x\| = 1$. Also, note that $r(x) = x \quad \forall x \in S^1$.



Thus r is a retraction of D^2 to S^1 . But by the previous prop, no such retraction exists. Contradiction.

($n > 2$): Not now QwQ...

$\square$

Rmk. Fun fact as an application of Brouwer's Fixed Point Thm? Consider a cup of coffee in a cup. Then $\text{Coffee} \cong D^3$ a sphere. Now stir the coffee in any way you want. Given an arbitrary particle starting at position x in the cup, let $f(x): D^3 \rightarrow D^3$ denote its position after stirring. Then f is cts, for if x and y are close, then they experience similar currents + flows... So they end up not far away from each other. By the Thm, $\exists$ a particle that end up the same as where it started when you finish stirring!



• Induced Homomorphisms.

Let $\varphi: (X, x_0) \rightarrow (Y, y_0)$ be a map. Then φ induces an homomorphism

$$\begin{aligned} \varphi_*: \pi_1(X, x_0) &\longrightarrow \pi_1(Y, y_0) \\ [\alpha] &\longmapsto [\varphi\alpha]. \end{aligned}$$

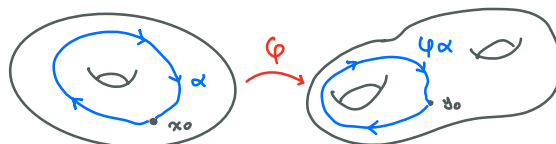
pf. φ_* well-defined: If $\alpha \simeq \beta$ then

$$\varphi\alpha \simeq \varphi\beta. \text{ So } \varphi_*[\alpha] = \varphi_*[\beta].$$

φ_* is a homomorphism since

$$\varphi(f \cdot g) = (\varphi f) \cdot (\varphi g),$$

$$\text{So } \varphi_*([f] \cdot [g]) = \varphi_*[f] \cdot \varphi_*[g]. \quad \square$$



Prop. • $(\varphi\psi)_* = \varphi_*\psi_*$ for a composition $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$.
• $1_* = 1$.

pf. Exercise. □

Ex. Let $f: S^1 \rightarrow S^1$, where $f: z \mapsto z^n$. Think of $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi)\}$.
Let $\omega_n: [0, 1] \rightarrow S^1$ be as before. In this context $\omega_n = e^{2\pi i n t}$. Then

$$\begin{array}{ccc} \pi_1(S^1, 1) & \xrightarrow{f_*} & \pi_1(S^1, 1) \\ \phi \downarrow \cong & \wr & \cong \downarrow \phi \\ \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} \end{array} \quad \begin{array}{l} \text{for } f_*[\omega_1] = [f\omega_1] \\ = [(e^{2\pi i t})^n] \\ = [e^{2\pi i n t}] = [\omega_n]. \end{array}$$

Thm (Fundamental Thm of Algebra). If $p(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$, $a_i \in \mathbb{C}$, then $\exists \omega \in \mathbb{C}$ s.t. $p(\omega) = 0$.

pf. WLOG assume $a_n = 1$. I.e. $p(z) = z^n + \dots + a_0$. Assume $p(z) \neq 0$, $\forall z \in \mathbb{C}$.

For every $r \geq 0$,

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

is a loop in $S^1 \subset \mathbb{C}$ based at 1. As r varies, f_r is a homotopy of loops at 1. Since $f_0 = 1 \forall s$, $[f_r] = 0 \in \pi_1(S^1) \forall r$. Now fix r large s.t.

$$r > \max(1, |a_0| + \dots + |a_{n-1}|).$$

Then for $|z| = r$, we have

$$\begin{aligned} |z^n| &> (|a_0| + \dots + |a_{n-1}|) |z|^{n-1} > |a_{n-1} z^{n-1}| + \dots + |a_0| \\ &\geq |a_{n-1} z^{n-1} + \dots + a_0|. \end{aligned}$$

It follows that $P_t(z) = z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$ has no roots when $t \in [0, 1]$. Then let

$$\eta_r^t(s) = \frac{P_t(re^{2\pi i s})/P_t(r)}{|P_t(re^{2\pi i s})/P_t(r)|}.$$

This gives a homotopy

$$\omega_n = \eta_r^0 \simeq \eta_r^1 = f_r \simeq f_0 = \omega_0.$$

Thus $[\omega_n] = 0 \in \pi_1(S^1) \Rightarrow n = 0$. So only constant polynomials have no roots in $\mathbb{C}$. $\square$

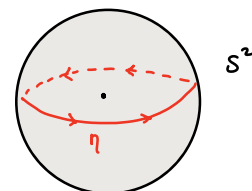
Thm (Borsuk-Ulam). For every cts map $f: S^2 \rightarrow \mathbb{R}^2$, $\exists$ a pair of antipodal points x and $-x$ in S^2 s.t. $f(x) = f(-x)$.

pf. Otherwise, $f(x) \neq f(-x) \forall x \in S^1$. Then let $g: S^2 \rightarrow S^1$ be the map

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

Define a loop $\eta: I \rightarrow S^2$ be

$$\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0) \in \mathbb{R}^3.$$



Then the map $h: I \rightarrow S^1$ by $h = g \circ \eta$ is a loop on S^1 . Since η is nullhomotopic (i.e. $\eta \simeq c$ where c is a constant map, just "lift" η ctsly to north pole), then $h = g \circ \eta \simeq g \circ c$ is also nullhomotopic.

This shows that $h \in [\omega_0] \in \pi_1(S')$.

On the other hand, $g(-x) = x$ means g takes antipodal pts to antipodal pts. We can denote this as $g(\text{Anti}(x)) = \text{Anti}(g(x))$. Then we get $h(s + \frac{1}{2}) = \text{Anti}(h(s))$ or $h(s + \frac{1}{2}) = -h(s)$. This is true for all $s \in [0, \frac{1}{2}]$. Now

let $\tilde{h}: I \rightarrow \mathbb{R}$ be a lift of

h under $p: \mathbb{R} \rightarrow S'$ by

$$p(s) = (\cos 2\pi s, \sin 2\pi s).$$

Then we can use the fact

$$p^{-1}(\text{Anti}(x)) = p^{-1}(x) + \frac{1}{2}$$

as sets. This is shown on the right. This implies that

$$\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + \frac{q}{2} \quad \text{for } q \in \mathbb{Z} \text{ odd}$$

This is simply because $\tilde{h}(s) \in p^{-1}(x)$ and $\tilde{h}(s + \frac{1}{2}) \in p^{-1}(\text{Anti}(x))$.

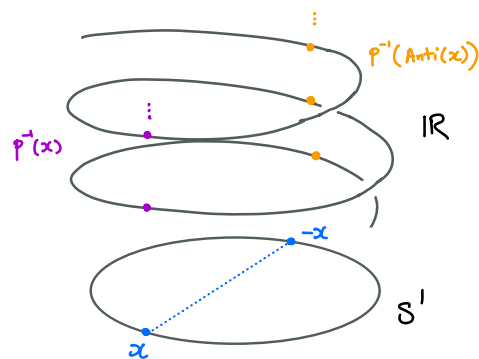
Then plugging in $s=0$ and $s=\frac{1}{2}$:

$$\tilde{h}(\frac{1}{2}) = \tilde{h}(0) + \frac{q}{2}$$

$$\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q.$$

But recall that $\tilde{h}(1) - \tilde{h}(0)$ represents how many times $h: I \rightarrow S'$ wraps around S' . And this is an odd number.

But this implies that $h = [\omega_q] \in \pi_1(S')$. But this is a contradiction for 0 is an even number and q is an odd number. So $\omega_0 \neq \omega_q$. □



Rmk. Fun fact: From the thm above, let $f: S^2 \rightarrow \mathbb{R}^2$ be a cts function defined by

$$f(\vec{x}) = (T(\vec{x}), H(\vec{x})).$$

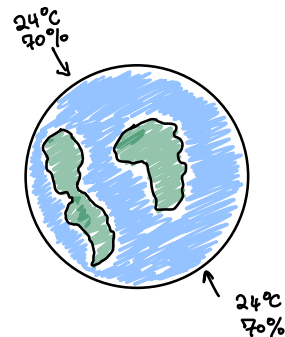
a pt on Earth

temperature at $\vec{x}$

humidity at $\vec{x}$

f is cts (clearly).

Then $\exists$ antipodal pt on earth at any time that has the same temperature and humidity!



Cor. Whenever S^2 is expressed as the union of three closed sets A_1, A_2, A_3 , then at least one of these sets must contain a pair of antipodal pts.

pf. Let $d_i: S^2 \rightarrow \mathbb{R}$ measure distances to A_i , that is,
$$d_i(x) = \inf_{y \in A_i} |x - y| \quad \text{where } |\cdot| \text{ is dist on } S^2.$$

Apply Borsuk-Ulam Thm to the cts function $\eta: S^2 \rightarrow \mathbb{R}^2$
$$\eta(x) = (d_1(x), d_2(x)) \quad , \quad x \in S^2.$$

This gives a pair of antipodal pts $x, -x$, s.t.

$$d_1(x) = d_1(-x) \quad d_2(x) = d_2(-x).$$

If either of these two distances is zero, then $x, -x$ lies in the corresponding set, for A_i are all closed. If none of them is zero, then $\{x, -x\} \in A_3$.

□