Note: Throughout this note, an open & connected set will be called a region. A disc $D(z,r)=\{w\in\mathbb{C} : |w-z|<r\}$.

Chap 2 & 3: Complex Differentiability. Power Series.

Def. A complex valued function $f(z)$ is said to be complex differentiable at z if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists, along all paths $h\to 0$.

Prop 3.1 If $f=u+iv$ is differentiable at z, f_x, f_y exists at z and satisfy the Cauchy-Riemann equations

$$f_y = i f_x$$

or equivalently

$$u_x = v_y$$
$$u_y = -v_x.$$

pf. Suppose $h\to 0$ through real values. Then

$$L_1=\frac{f(z+h)-f(z)}{h}=\frac{f(x+h,y)-f(x,y)}{h}\to f_x.$$

Now suppose $h\to 0$ along imaginary axis, $h=i\eta$ and

$$L_2=\frac{f(z+h)-f(z)}{h}=\frac{f(x,y+\eta)-f(x,y)}{i\eta}\to\frac{f_y}{i}.$$

$L_1=L_2 \Rightarrow f_y = i f_x.$

$\Rightarrow (u_y+iv_y)=i(u_x+iv_x)$

$\Rightarrow$ C.R. eq. $\square$

The converse is not true!

Some differential forms:

Define the operators:

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\qquad \frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Then since $z=x+iy$, $\bar z = x-iy$, we have

$$\frac{\partial}{\partial z}(z)=\frac{\partial}{\partial \bar z}(\bar z)=1\qquad \frac{\partial}{\partial z}(\bar z)=\frac{\partial}{\partial \bar z}(z)=0.$$

Prop. The CR equation is equivalent to the expression
$$\frac{\partial}{\partial \bar{z}}(f) = 0.$$

pf. By assuming CR:
$$\frac{\partial}{\partial \bar{z}}(f) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(f_x + if_y) \overset{\text{CR}}{=} \frac{1}{2}(f_x - f_x) = 0.$$

By assuming $\frac{\partial}{\partial \bar{z}}(f) = 0$, we see that
$$\frac{1}{2}(f_x + if_y) = 0 \Rightarrow f_x = -if_y \Rightarrow f_y = if_x \quad \text{CR} \qquad \square$$

Now, let $U \subseteq \mathbb{C}$ an open set, $f : U \to \mathbb{C}$. View $f = u + iv$, where $u, v : \underset{\mathbb{R}^2}{\overset{\cap}{U}} \to \mathbb{R}^2$. Suppose, as real function, f is of the class C^1.

View f as a zero form on U: $f \in \Lambda^{(0)}(U)$. Take the exterior derivative in the coordinate of $\mathbb{R}^2$, which gives:

$$\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\
&= \frac{\partial u}{\partial x}dx + i\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\frac{\partial v}{\partial y}dy \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv)(dx+idy) + \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)(dx-idy) \\
&= \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.
\end{aligned}$$

Now,
$$d\bar{z} \wedge dz = (dx - idy) \wedge (dx + idy) = 2i\, dx \wedge dy.$$

Consider $f dz \in \Lambda^{(1)}(U)$. We have
$$d(f dz) = df \wedge dz = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = \frac{\partial f}{\partial \bar{z}}(2i\, dx \wedge dy).$$

Thus, if f is complex differentiable, by CR:
$$d(f dz) = 0.$$

Thus, we have:

Thm. Let $f : U \to \mathbb{C}$ be of the class C^1. Then f is complex differentiable $\Leftrightarrow d(f dz) = 0$.

Def. A complex function differentiable everywhere on $\mathbb{C}$ is called an entire function.

Prop. Let $f = u+iv$ be analytic in a region D. If $u(x,y) = \text{constant}$, then $f = \text{constant}$.

pf. $u(x,y)$ const $\Rightarrow u_x = u_y = 0 \xRightarrow{CR} v_x = v_y = 0 \Rightarrow v$ const $\Rightarrow f$ const. $\square$

prop. If f analytic in region D, and $|f|$ constant there, then f is constant.

pf.
$$|f|^2 = f\bar{f} = (u+iv)(u-iv) = u^2+v^2. \qquad (*)$$

If $|f| = 0 \Rightarrow u = v = 0 \Rightarrow f$ const. If $|f| \neq 0$, then take ∂x and ∂y of $(*)$:

$$u u_x + v v_x = 0 \qquad ①$$
$$u u_y + v v_y = 0 \xRightarrow{CR} -u v_x + v u_x = 0 \qquad ②$$

$$u \times ① + v \times ② \Rightarrow \underbrace{(u^2+v^2)}_{=\text{const} \neq 0} u_x = 0 \Rightarrow u_x = 0$$

Similarly $v_x = 0 \xRightarrow{CR} u_x = u_y = v_x = v_y = 0$. $\square$

Power Series

A power series is the function of the form $f(z) = \sum_{k=0}^{\infty} c_k z^k$.

Thm (Cauchy–Hadamard). Suppose $L = \limsup_k |c_k|^{1/k}$.

(1) If $L = 0$, $\sum c_k z^k$ converges $\forall z$.

(2) If $L = \infty$, $\sum c_k z^k$ converges for $z = 0$ only.

(3) If $0 < L < \infty$, set $R = \frac{1}{L}$. Then $\sum c_k z^k$ converges for $|z| < R$ and diverges for $|z| > R$.

pf. Basic Analysis. See Bak & Newmann Thm 2.8. $\square$

Thm 2.9. A power series $\sum c_n z^n$ is infinitely differentiable within its radius of convergence $0 < |z| < R$.

Thm 2.12 (Uniqueness). Suppose $\sum c_n z^n$ is zero at all pts of a nonzero sequence $\{z_k\}$ s.t. $z_k \to 0$. Then $\sum c_n z^n \equiv 0$ identically.

pf. Let $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$. Then $c_0 = f(0) = \lim_{z\to 0} f(z) = \lim_{k\to\infty} f(z_k) = 0$ by contionuity of f. But then

$$g(z) = \frac{f(z)}{z} = c_1 + c_2 z + c_3 z^2 + \cdots$$

is also continous at the origin and similarly $c_1 = 0$... So induction shows that $c_k = 0 \ \forall k$. $\square$

Cor 2.13 & 2.14.

(1) If $\sum c_n z^n = 0$ at all pts of a set with a limit pt at $z=0$, then $\sum c_n z^n \equiv 0$ identically.

(2) If $\sum a_n z^n$ & $\sum b_n z^n$ converge and $\sum a_n z^n = \sum b_n z^n$ on a set with a limit pt at $z=0$, then $a_n = b_n \ \forall n$.

Chapter 4. Line integrals and Entire Functions.

Thm 4.14. If f is entire and $\Gamma = \partial(\text{rectangle})$, then

$$\int_\Gamma f(z)\,dz = 0.$$

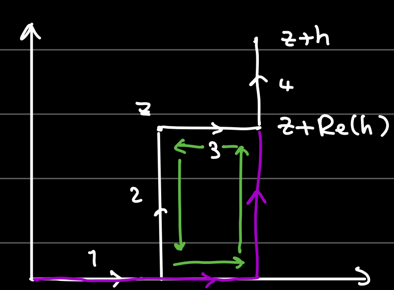
pf. f entire $\Rightarrow$ f holomorphic on $\mathbb{C}$. So $f(z)dz$ is well-defined and $d[f(z)dz] = 0$. Then

$$\int_\Gamma f(z)dz = \int_R d[f(z)dz] = \int_R 0 = 0. \qquad \square$$

Thm 4.15. f entire $\Rightarrow$ $\exists$ entire F st. $F'(z) = f(z)$, $\forall z \in \mathbb{C}$.

pf. Define $F(z) = \int_0^z f(\xi)\,d\xi$ where the integral takes the path $1 \to 2$.

Then $F(z+h) = \int_0^{z+h} f(\xi)\,d\xi$ takes the path that is purple. But this is also equal to the path $1 \to 2 \to 3, 4$:



the difference of these two paths is integral on green path which is zero. So: $F(z+h) = F(z) + \int_z^{z+h} f(\xi)\,d\xi$ (path $3 \to 4$).

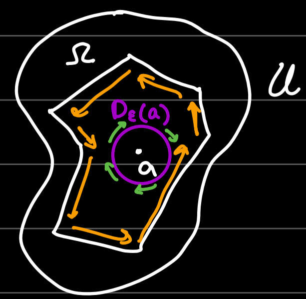
$$\frac{1}{h}\int_z^{z+h} f(z)\,d\xi = f(z)\cdot \frac{1}{h}(z+h-z) = f(z) \Rightarrow \frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} [f(\xi) - f(z)]\,d\xi$$

$$\Rightarrow \left| \frac{F(z+h)-F(z)}{h} - f(z) \right| \le \frac{1}{|h|} 2|h| \cdot \varepsilon = 2\varepsilon \xrightarrow{h\to 0} 0 \qquad \square$$

Chap 5. Properties of Entire Function.

Thm (Cauchy's integral formula). Let $U \subseteq \mathbb{C}$ be open, $f: U \to \mathbb{C}$ complex differentiable. Let $\Omega \subseteq U$ be a relative compact set (i.e. $\bar{\Omega}$ compact), and $\partial\Omega$ is piecewise smooth. Then for any $a \in \Omega$, we have

$$f(a) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-a}\, dz.$$



pf. Let $\Omega_\varepsilon = \Omega \setminus D_\varepsilon(a)$, where $D_\varepsilon(a) = \{z \in \mathbb{C} : |z-a| \le \varepsilon\}$.
Then the one form $\frac{f}{z-a} \in \Lambda^{(1)}(\Omega_\varepsilon)$. Then by Stoke's Thm:

$$\int_{\partial\Omega_\varepsilon} \frac{f\,dz}{z-a} = \int_{\Omega_\varepsilon} d\left(\frac{f\,dz}{z-a}\right) = 0.$$

Since both f and $1/(z-a)$ are complex differentiable for $\partial_{\bar{z}}$ of them is zero. Then the integral along $\partial\Omega_\varepsilon$ gives

$$\int_{\partial\Omega_\varepsilon} \frac{f\,dz}{z-a} = \int_{\partial\Omega} \frac{f\,dz}{z-a} - \int_{\partial D_\varepsilon(a)} \frac{f\,dz}{z-a} = 0.$$

Now, take $\varepsilon \to 0$, note:

$$\int_{\partial D_\varepsilon(a)} \frac{f\,dz}{z-a} = \int_{\partial D_\varepsilon(a)} \frac{f(z)-f(a)}{z-a}\, dz + \int_{\partial D_\varepsilon(a)} \frac{f(a)}{z-a}\, dz.$$

Since f is complex differentiable at a, hence

$$\left| \frac{f(z)-f(a)}{z-a} \right| \le M \quad \text{for some constant } M < \infty.$$

Then
$$\left| \int_{\partial D_\varepsilon(a)} \frac{f(z)-f(a)}{z-a}\, dz \right| \le M \left| \int_{\partial D_\varepsilon(a)} dz \right| = M \cdot 2\pi\varepsilon \to 0$$

as $\varepsilon \to 0$. For the second integral, let

$$z = a + \varepsilon e^{i\theta} \quad \theta \in [0, 2\pi] \quad dz = i\varepsilon e^{i\theta}\, d\theta$$

Then
$$\int_{\partial D_\varepsilon(a)} \frac{f(a)}{z-a}\, dz = f(a) \int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\, d\theta}{\varepsilon e^{i\theta}} = 2\pi i f(a).$$

This gives (as $\varepsilon \to 0$):

$$\int_{\partial\Omega} \frac{f\,dz}{z-a} = 2\pi i f(a).$$

$\square$

Thm. Let $U \subseteq \mathbb{C}$ be open, $f: U \to \mathbb{C}$ complex differentiable. Let $a \in U$ and $r < \mathrm{dist}(a, \partial U)$. Then $\exists$ a convergent power series

$$\varphi(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad |z-a| < r$$

s.t. $\varphi(z) = f(z)$ when $|z-a| < r$.

pf. Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta .$$

Note:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z-a)} = \frac{1}{\zeta - a}\left(1 - \frac{z-a}{\zeta - a}\right)^{-1} = \frac{1}{\zeta - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta - a}\right)^n .$$

Since $|z-a| < r$, $|\zeta - a| = r$, we have

$$\left| \frac{z-a}{\zeta - a} \right| \le \eta^n \quad \text{for some } \eta < 1 .$$

Since $\sum \eta^n$ converges, $\sum \left(\frac{z-a}{\zeta - a}\right)^n$ converge uniformly by M-test. Then we have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{z-a} \, d\zeta = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \right)}_{\equiv c_n} (z-a)^n$$

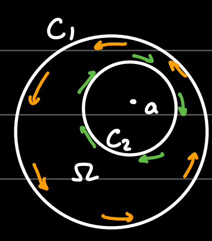
This gives the desired result. $\square$

Cor. $U \subseteq \mathbb{C}$ open and $f: U \to \mathbb{C}$ complex diff. Then f is infinitely differentiable: $f \in C^\infty(U)$. And $f^{(n)} = \left(\frac{d}{dz}\right)^n f$ exists. If $C \subseteq U$ is a circle whose interior is also contained in U, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \, dz .$$

pf. by the last result, we know

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \, dz . \qquad \square$$



Remark: Note that if f is analytic in Ω, then by Stoke's Thm

$$\int_{C_1} f\,dz - \int_{C_2} f\,dz = \int_{\partial\Omega} f\,dz = \int_{\Omega} d[f\,dz] = 0 \implies \int_{C_1} f\,dz = \int_{C_2} f\,dz .$$

More results from Chapter 5:

Thm. If f is entire, then

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

is entire.

pf. When $z \neq a$, do a power series expansion of f:

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

$$\Rightarrow g(z) = \frac{f(z)-f(a)}{z-a} = f'(a) + \frac{f''(a)}{2}(z-a) + \frac{f'''(a)}{3!}(z-a)^2 + \cdots$$

plug in $z = a$, the expression above is also valid. So g is equal everywhere to a convergent power series. Thus g is entire.

Cor. By the previous thm, if $\Gamma = \partial(\text{rectangle})$, then if f entire,

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a, \end{cases}$$

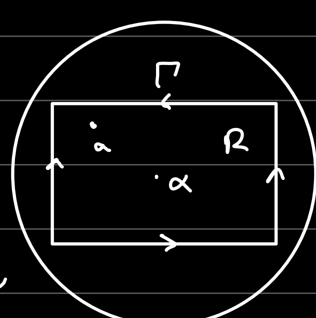
then $\int_\Gamma g(z)\,dz = 0$.

Thm. If f analytic in $D = D(\alpha, r)$, and $\alpha \in R \subseteq D$ a rectangle, and $\Gamma = \partial R$. Then

$$\int_\Gamma f(z)\,dz = \int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz = 0.$$

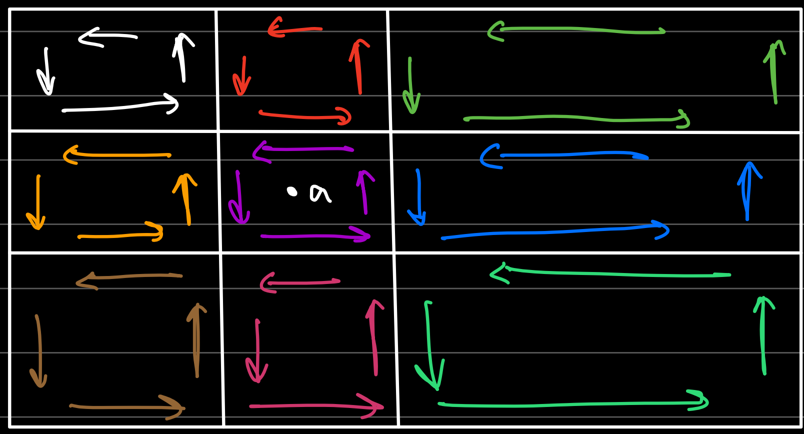
pf. f analytic in R. So by the previous Thm,

$$\int_\Gamma f(z)\,dz = 0.$$



Now, for the second integral, if $a \in \operatorname{int}(R)$, Consider the division;

(for $a \in \partial R$ the same thing).



Let Γ_1 be the boundary of the rectangle containing a. Let Γ_n, $n=2\cdots 9$ be the other rectangles. Then

$$\int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz = \sum_{n=1}^{9} \int_{\Gamma_n} \frac{f(z)-f(a)}{z-a}\,dz$$

$$= \int_{\Gamma_1} \frac{f(z)-f(a)}{z-a}\,dz \qquad \left(\text{all others are } 0 \text{ since } \frac{f(z)-f(a)}{z-a} \text{ analytic in } R_n.\right)$$

Since $\frac{f(z)-f(a)}{z-a} \to f'(a)$ as $z \to a$, So $\exists\, \delta > 0$ s.t.

$$\left|\frac{f(z)-f(a)}{z-a}\right| < M_\delta < \infty \quad \text{for } z \in B_\delta(a),$$

and $\frac{f(z)-f(a)}{z-a}$ continuous on $R_1 \setminus B_\delta(a)$ is closed and bounded, i.e. compact, $\left|\frac{f(z)-f(a)}{z-a}\right| < M' < \infty$ for $z \in R_1 \setminus B_\delta(a)$. So

$$\left|\frac{f(z)-f(a)}{z-a}\right| < \max(M_\delta, M') = M < \infty, \quad \text{for } z \in R_1.$$

Then

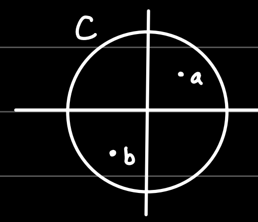
$$\left|\int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz\right| = \left|\int_{\Gamma_1} \frac{f(z)-f(a)}{z-a}\,dz\right| \le M \left|\int_{\Gamma_1} dz\right|$$

$$= M \cdot \text{Length}(\Gamma_1) \le M\varepsilon \longrightarrow 0.$$

Since Length(Γ_1) can be made arbitrarily small.

$\square$

Thm 5.10 (Louisville's Thm). A bounded entire function is constant.



pf. let $a, b \in \mathbb{C}$, let C be the circle shown on the right with $R > \max(|a|, |b|)$.
Say $|f| < M < \infty$. Since $|z-a| + |a| \le R$ and similarly for $|z-b|$, we have from Cauchy:

$$|f(b)-f(a)| = \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z-b} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz \right|$$

$$= \left| \frac{1}{2\pi i} \int_C \frac{f(z)(b-a)}{(z-a)(z-b)} dz \right|$$

$$\le \frac{1}{2\pi} \frac{M \cdot |b-a|}{(R-|a|)(R-|b|)} \left| \int_C dz \right| = \frac{M \cdot |b-a| \cdot R}{(R-|a|)(R-|b|)}.$$

We may take $R \to \infty$, thus $|f(b)-f(a)| < \varepsilon$ $\forall \varepsilon > 0$, provided R big enough. Then $f(b) = f(a)$ follows. $\square$

Thm 5.11. If f entire and for some integer $k \ge 0$, $\exists A, B > 0$ s.t.

$$|f(z)| \le A + B|z|^k,$$

then f is a polynomial of deg at most k.

pf. Induction: $k=0$ is Louisville. When $k>0$, consider

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z} & z \neq 0 \\ f'(0) & z = 0. \end{cases}$$

By previous thm, g is entire. And when $z \neq 0$,

$$|g(z)| = \frac{|f(z)-f(0)|}{|z|} \le \frac{|f(z)|+|f(0)|}{|z|} \le \frac{2A + B|z|^k}{|z|}$$

$$\le \frac{2A}{|z|} + B|z|^{k-1} \le 2A + B|z|^{k-1}$$

when $|z| > 1$. When $|z| \le 1$, $g(z)$ is entire, hence continuous on a compact set: $|g(z)| \le M$ for $|z| \le 1$. Let $C = \max(2A, M)$ and $D = B$, we see

$$|g(z)| \le C + D|z|^{k-1} \xRightarrow{\text{induction}} g \text{ a polynomial of deg} \le k-1$$

$$\Rightarrow f \text{ a polynomial of deg} \le k. \quad \square$$

Cor 5.12 (Fundamental Thm of Algebra). Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

pf. Otherwise, let $P(z)$ be a nonconstant poly. If $P(z) \neq 0 \ \forall z \in \mathbb{C}$, then $f(z) = 1/P(z)$ is entire. As $z \to \infty$, $f(z) = 1/P(z) \to 0$. Thus $f(z)$ is entire & bounded $\Rightarrow$ constant. Contradiction. $\square$

Chap 6. Properties of Analytic Functions.

Thm 6.9 (Uniqueness Thm). Suppose f is analytic in a region D and that $f(z_n) = 0$ where z_n is a seq of distinct pts s.t. $z_n \to z_0 \in D$. Then $f \equiv 0$ in D.

pf. Since f analytic in D, it has a power series expansion around z_0. Then by uniqueness thm of power series, $f = 0$ in some disc containing z_0. To show $f \equiv 0$ throughout D, we let

$$A = \{ z \in D : \ z \text{ is a limit of zeros of } f \}$$
$$B = \{ z \in D : \ z \notin A \}.$$

Then $A \cup B = D$, $A \cap B = \phi$. A is open: if z is a limit of zeros of f, $f \equiv 0$ in some disc around z, which is contained in A.
B is open: for each $z \in B$, since z is not a limit pt of zeros of f, $\exists \delta > 0$ s.t. $f(w) \neq 0$ for $0 < |z - w| < \delta$ by continuity of f. Then the disc $D(z, \delta) \subseteq B$.
Since A, B are open, $A \cup B = D$, and D connected, one of them is ϕ. Since $z_0 \in A$, $B = \phi$ and $D = A$. Thus $f \equiv 0$ in D. $\square$

Cor 6.10. If f, g analytic in a region D, and $f = g$ at a set of pts with a limit pt in D, then $f \equiv g$ through D.

pf. Consider $f - g$. $\square$

Thm 6.12 (Mean Value Thm). Let f be analytic in D and $\alpha \in D$. Then $f(\alpha)$ is equal to the mean value of f taken around the boundary of any disc contained in D. i.e.

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta$$

whenever $D(\alpha, r) \subseteq D$.

pf. by Cauchy: $f(\alpha) = \frac{1}{2\pi i} \int_{\partial D(\alpha,r)} \frac{f(z)}{z-\alpha} dz$. Now let $z = \alpha + re^{i\theta}$.
Done. □

Thm 6.13 (Max-Modulus Thm). A nonconstant analytic function in a region D doesn't have any interior maximum points: For each $z \in D$ and $\delta > 0$, $\exists\, w \in D(z, \delta) \cap D$ s.t. $|f(w)| > |f(z)|$.

pf. Since for some $r > 0$ s.t. $D(z, r) \subseteq D$ we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta,$$

it follows that

$$|f(z)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{i\theta})|\, d\theta \le \max_{\theta} \underbrace{|f(z + re^{i\theta})|}_{|f(w)|}.$$

In fact, this is a strict equality: $\exists\, w \in D(z, r)$ for some r s.t. $|f(w)| > |f(z)|$. If not, then we must have equality:

$$\int_0^{2\pi} |f(z + re^{i\theta})|\, d\theta = \max_{\theta} |f(z + re^{i\theta})| \Rightarrow f \text{ constant on } \partial D(z, r),$$
$\forall r > 0$ sufficiently small.

Then if $f(z')$ is constant on a circle and $f(z)$ is the average of $f(z')$ over the circle, then $f(z') = f(z)$. Since $f(z') = f(z)$ holds for all z' on a circle with small enough radius, f is constant on a disc. Then uniqueness thm implies f constant on D. □

Rmk. Equivalently, if f is analytic on region D, then either f is constant or max of $|f|$ occurs only on ∂D.

Thm 6.14 (Min-Modulus Thm). f nonconstant analytic function in a region D. Then no pts $z \in D$ can be a relative minimum unless $f(z)=0$.

pf. Suppose $f(z) \neq 0$ and let $g = 1/f$. If z were a minimum of f, then it is a max of g. Hence $g = \text{const}$. Contradiction. $\square$

Chap 7. Further Properties of Analytic Functions.

Thm 7.1 (Open Mapping Thm). The image of an open set under a nonconstant analytic mapping is an open set.

pf. Assume $f: U \to \mathbb{C}$ nonconst, analytic. Let $a \in U$.

- Case 1: $f'(a) \neq 0$.
  Then by inverse function thm, $\exists V \ni a$ open s.t. $W = f(V)$ is open, and $f|_V : V \to W$ is bijective with an continous inverse. That is, $\exists \varepsilon > 0$ s.t. $D(f(a), \varepsilon)$ is contained in the image of f.
- Case 2: $f'(a) = 0$.
  Then consider the power series of f around a. $\exists r > 0$ s.t.

$$f(z) = (z-a)^m h(z)$$

with $h(z)$ analytic in $|z-a| < r$ s.t. $h(a) \neq 0$. Since $h(z)$ analytic, say $h(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$ with $c_0 \neq 0$. Then we can write

$$h(z)^{1/m} = c_0^{1/m}\left[1 + \frac{c_1}{c_0}(z-a) + \cdots\right]^{1/m}.$$

Now, $f(z) = \left[(z-a)h(z)^{1/m}\right]^m \equiv [\eta(z)]^m$. Then $\eta(z)$ is analytic, $\eta(a) = 0$, and $\eta'(a) \neq 0$, since the power series of η starts with $c_0^{1/m}(z-a)$, i.e. $\eta'(z) = c_0^{1/m} + \cdots \neq 0$ when $z = a$. Then back to case 1, η is a local homeomorphism via inverse function thm. Thus $f = \eta^m$ is also a local homeomorphism: i.e. $f|_V : V \to W$ is a homeomorphism for some open set $V \ni a$. Thus $\exists \varepsilon > 0$ s.t. $D(f(a), \varepsilon)$ is contained in $f(U)$. $\square$

Thm 7.2 (Schwarz Lemma). Suppose f analytic in the unit disc, $|f| \le 1$ there and $f(0)=0$. Then

(i). $|f(z)| \le |z|$

(ii). $|f'(0)| \le 1$

with equality in either of the above iff $f(z) = e^{i\theta} z$.

pf. Apply Max Modulus Thm to the analytic function

$$g(z) = \begin{cases} f(z)/z & 0<|z|<1 \\ f'(0) & z=0 \end{cases}.$$

Since $|g| \le 1/r$ on the circle of radius $r>0$, let $r \to 1$ and apply the Max Modulus Thm, $|g| \le 1$ throughout the unit disc. This proves (i) and (ii). If $|g(z_0)|=1$ for some $|z_0|<1$, then by the Max Modulus Thm, $g=$ const and $|g|=1 \Rightarrow f = e^{i\theta} z$. □

Prop. Let D be the unit disc. If $f: D \to D$ analytic, and $a \in D$ with $f(a)=0$, then

$$|f(z)| \le \left| \frac{z-a}{1-\bar{a}z} \right| \quad \text{for } |z|<1.$$

pf. Let $a \in D$, $f(a)=0$. Then $w = \frac{z-a}{1-\bar{a}z}$ has inverse $z = \frac{w+a}{1+\bar{a}w}$. And we see w is biholomorphic, takes D to D, and ∂D to ∂D. Also, we see $\left|\frac{z-a}{1-\bar{a}z}\right| = 1$ for $|z|=1$. So $f \circ w^{-1}$ maps D to D and $f(w^{-1}(0)) = f(a) = 0$. By Schwarz lemma, $|f\circ w^{-1}(z)| \le |z|$ and this gives $|f(z)| \le |w(z)|$ for $|z| \le 1$. □

Thm 7.4 (Morera's Thm). Let f be continous on an open set D. If $\int_\Gamma f(z)\,dz = 0$ whenever $\Gamma = \partial$(closed rectangle in D), then f is analytic. (Converse of closed curve thm.)

pf. In a small disc around z_0, define the primitive

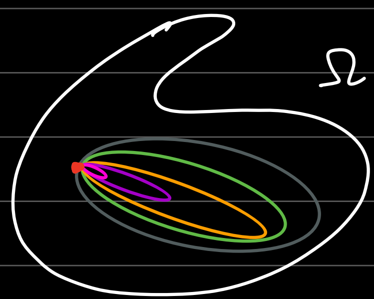
$$F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$$

where the path is shown on the right. Then do the same as the pf of thm 4.15 $\Rightarrow F'(z) = f(z)$ & F analytic in the disc. Since z_0 arbitrary, f is analytic in D. □

Chap 8. Simply Connected Domains.

Def. A domain Ω is simply connected if it is path-connected, and every simple closed curve can be shrunk to a point continously in Ω.

Picture:



simple closed curve shrunk continously to a point.

Thm. (General Closed Curve Thm). Let D be a region and $C = \partial D$. Let f be analytic in D. Then

$$\int_C f(z)\,dz = 0.$$

pf. f analytic in $D \Rightarrow d[f(z)dz] = 0$. By Stoke's Thm:

$$0 = \int_D d[f(z)dz] = \int_{\partial D} f(z)\,dz = \int_C f(z)\,dz.$$

$\square$

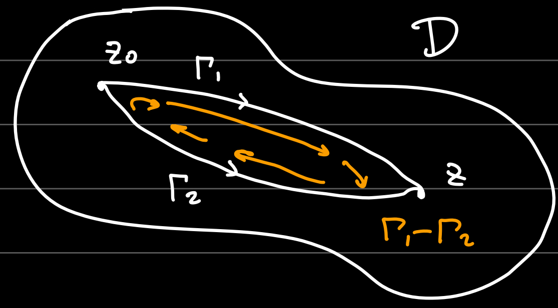
Thm. If f is analytic in a simply connected region D, there exists a "primitive" F, analytic in D s.t. $F' = f$.

pf. Choose $z_0 \in D$, define

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$$

where the line integral takes any polygonal path contained in D. This F is well-defined, for if $\exists$ another polygonal path from $z_0 \to z$, then

$$\int_{\Gamma_1} f(\zeta)\,d\zeta - \int_{\Gamma_2} f(\zeta)\,d\zeta = \int_{\Gamma} f(\zeta)\,d\zeta = 0.$$



Also,

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(\xi)\,d\xi$$

$$\Rightarrow \left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \left|\frac{1}{h}\right| \cdot \int_z^{z+h} |f(\xi)-f(z)|\,d\xi.$$

when $h \to 0$, $|f(\xi)-f(z)| < \varepsilon$,

$$\le \left|\frac{1}{h}\right| \cdot |h| \cdot |f(\xi)-f(z)|$$
$$\le \varepsilon.$$

$\square$

Logarithmic Functions

Complex exponentials: $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$.

Note $e^z \neq 0 \ \forall z \in \mathbb{C}$.

Problem: unlike real exponentials, $e^z = a$ has infinitely many solutions: if $z=\theta$ is a solution, so is $z = \theta + 2k\pi$, $k \in \mathbb{Z}$.

Remark: We define

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz}+e^{-iz}}{2}.$$

Observations:

(1) $\sin^2(z) + \cos^2(z) = 1$

(2) $\sin'(z) = \cos(z)$

(3) $\sin(2z) = 2\sin(z)\cos(z)$.

(4) UNLIKE the real case, complex $\sin$ & $\cos$ are NOT bounded by modulus, for $|e^{iz}|$ not bounded by modulus.

Def. We say f is an analytic branch of $\log z$ in a domain D if

(1) f is analytic in D.

(2) $\exp(f(z)) = z$. Of course if f is an analytic branch of $\log(z)$ then

so is $g(z) = f(z) + 2\pi k i$, $\forall k \in \mathbb{Z}$.

Rmk. if we set $\log z = u(z) + iv(z)$, then (2) becomes

$$\exp(f(z)) = e^{u(z)} e^{iv(z)} = Re^{i\theta} \equiv z.$$

This is possible iff

$$e^{u(z)} = |z| = R \qquad v(z) = \operatorname{Arg} z = \theta + 2k\pi.$$

Hence a function f satisfying (2) can always be found by setting

$$f(z) = u(z) + iv(z) = \log|z| + i\operatorname{Arg}(z). \qquad (*)$$

But $\operatorname{Arg}(z)$ is not a well-defined function. Even if a convention is chosen, it is not clear that $(*)$ is analytic or even continous.

But if D is simply connected and $0 \notin D$, we can define an analytic branch of $\log z$ there. Recall that if an analytic inverse of e^z exists, by (Bak & Newmann Thm 3.5) its derivative must be $1/z$. Thus we proceed as follows:

Thm 8.8. Suppose D is simply connected and $0 \notin D$. Choose $z_0 \in D$. Fix a value $\log z_0$ and set

$$f(z) = \int_{z_0}^{z} \frac{d\xi}{\xi} + \log(z_0).$$

Then $f(z)$ is an analytic branch of $\log z$ in D.

pf. f well defined since $1/\xi$ is analytic in D and use closed curve thm. By FT of calculus $f'(z) = 1/z$. So f is analytic there.

Consider: $\qquad g(z) = ze^{-f(z)}$.

$$g'(z) = e^{-f(z)} - zf'(z)e^{-f(z)} = 0.$$

So g is constant and $g(z) = g(z_0) = z_0 e^{-f(z_0)} = z_0 e^{-\log z_0} = z_0 \cdot 1/z_0 = 1$.

Thus

$$e^{f(z)} = z. \qquad \square$$

Chap 9. Isolated Singularities.

Def. A deleted neighborhood of z_0 is a set of the form
$$\{z : 0 < |z - z_0| < d\}.$$

Def. (1) f is said to have an isolated singularity at z_0 if f is analytic in a deleted neighborhood of z_0 but not analytic at z_0.
(Requires f to be discontinous at z_0 by B&N Thm 7.7)

Now suppose f has an isolated singularity at z_0.

(2) If $\exists g$ analytic at z_0 s.t. $f(z) = g(z)$ for all z in some deleted neighborhood of z_0, say f has a removable singularity at z_0.

(3) If, for $z \neq z_0$, $f(z) = A(z)/B(z)$, where $A(z), B(z)$ analytic at z_0, $A(z_0) \neq 0$, $B(z_0) = 0$, say f has a pole at z_0.
If B has a zero of order k at z_0, say f has a pole of order k at z_0.

(4) If f has neither a removable singularity nor a pole at z_0, say f has an essential singularity at z_0.

Thm 9.3 (Riemann's Principle of Removable Singularities).
If f has an isolated singularity at z_0 and if $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then the singularity is removable.

pf. let
$$h(z) = \begin{cases} (z - z_0) f(z) & z \neq z_0 \\ 0 & z = z_0. \end{cases}$$

Then h cts at z_0 and h, like f, is analytic in a deleted nbh of z_0. By B&N Thm 7.7 $\Rightarrow$ h analytic at z_0. Since $h(z_0) = 0$, $g(z) = h(z)/(z - z_0)$ is likewise analytic & $g(z) = f(z)$ for $z \neq z_0$. $\square$

Thm. f has a removable singularity at $z_0 \iff f$ is bounded on a deleted neighborhood of z_0.

pf. ($\Rightarrow$) If f has a removable singularity at z, then $\exists F$ analytic at z_0 s.t. $F(z) = f(z)$ $\forall z \in$ a deleted nbh Ω^* of z_0.

Then take a closed disc $\overline{D}(z_0,\varepsilon)$ s.t. $\overline{D}(z_0,\varepsilon)-\{z_0\}\in\Omega^*$.
Then $\overline{D}(z_0,\varepsilon)$ is compact on which F is continous. Hence F bounded on $\overline{D}(z_0,\varepsilon)$. Say $|F(z)|\le M$ there. Then f is bounded on the deleted nbh $\{z:\ 0<|z-z_0|<\varepsilon\}$ by M.

($\Leftarrow$): Say $|f|\le M$ on Ω^*. Then

$$\lim_{z\to z_0}|(z-z_0)f(z)|\le\lim_{z\to z_0}|z-z_0|M=0.$$

Thus by Thm 9.3 done. $\square$

Thm 9.5. If f analytic in a deleted nbh of z_0 and $\exists k\in\mathbb{Z}^+$ s.t.

$$\lim_{z\to z_0}(z-z_0)^k f(z)\neq 0 \quad\text{but}\quad \lim_{z\to z_0}(z-z_0)^{k+1}f(z)=0$$

then f has a pole of order k at z_0.

pf. let
$$g(z)=\begin{cases}(z-z_0)^{k+1}f(z) & z\neq z_0\\ 0 & z=z_0\end{cases}$$
then by B&N Thm 7.7 g is analytic at z_0. And $g(z_0)=0$.
So
$$A(z)=\frac{g(z)}{z-z_0}=(z-z_0)^k f(z) \qquad (z\neq z_0)$$
is likewise analytic at z_0. By hypothesis $A(z_0)\neq 0$.
Then
$$f(z)=\frac{A(z)}{(z-z_0)^k}\qquad z\neq z_0.$$
$\square$

Thm 9.6. If f has an essential singularity at z_0 and if D is a deleted nbh of z_0, then $f(D)$ is dense in $\mathbb{C}$.

pf. Assume $w\in\mathbb{C}$, $\delta>0$ s.t. $D(w,\delta)\cap f(D)=\phi$. Then $\forall z$, $|f(z)-w|>\delta \Rightarrow |\frac{1}{f(z)-w}|<\delta$ throughout D. By Thm 9.3, $\frac{1}{f(z)-w}$ has (at most) a removable singularity at z_0.
Hence $\frac{1}{f(z)-w}=g(z)$ where g is analytic at z_0. But then
$$f(z)=w+\frac{1}{g(z)}$$
has either a pole ($g(z_0)=0$) or a removable singularity. $\square$

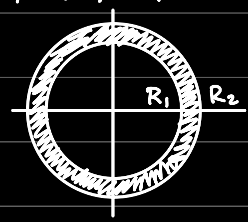
Laurant Expansion.

Def. We say $\sum_{k=-\infty}^{\infty} \mu_k = L$ if both $\sum_{k=1}^{\infty} \mu_k$ and $\sum_{k=1}^{\infty} \mu_{-k}$ converges and their sum is equal to L.

Thm 9.8. $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ is convergent in the domain $R_1 < |z| < R_2$ where

$$R_2 = 1 / \limsup_{k\to\infty} |a_k|^{1/k}$$

$$R_1 = \limsup_{k\to\infty} |a_{-k}|^{1/k}$$



If $R_1 < R_2$, then f is analytic in D.

pf. Clearly R_2 is the radius of convergence of the series

$$f_1(z) = \sum_0^{\infty} a_k z^k.$$

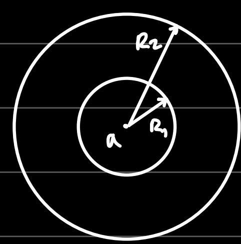
Similarly, when $\left|\frac{1}{z}\right| < \frac{1}{R_1}$ or $|z| > R_1$, the series

$$f_2(z) = \sum_{-\infty}^{-1} a_k z^k = \sum_1^{\infty} a_{-k} \left(\frac{1}{z}\right)^k$$

converges. Hence $\sum_{-\infty}^{\infty} a_k z^k$ converges in the intersection, on which $\sum_{-\infty}^{\infty} a_k z^k = f_1 + f_2$ is analytic, since f_1 & f_2 are analytic in the region where they converges. $\square$

Thm. If f is analytic in the annulus $A: R_1 < |z-a| < R_2$, then f has a unique Laurant expansion

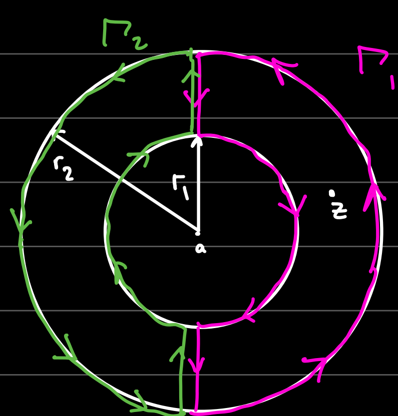
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \quad \text{in } A.$$



pf. Take $R_1 < r_1 < r_2 < R_2$, and for $r_2 < |z-a| < r_1$, define

$$f_1(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r_2} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta - a| = r_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$



Claim: $f(z) = f_1(z) + f_2(z)$ on $r_2 < |z-a| < r_1$.

Divide $\Omega = \{z : r_1 < |z-a| < r_2\}$ into 2 regions with boundary Γ_1 and Γ_2, WLOG say z is contained in the set with boundary Γ_1. Then by Cauchy's integral formula:

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\xi)}{\xi-z}\,d\xi = f(z).$$

And by Cauchy theorem (closed curve thm),

$$\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\xi)}{\xi-z}\,d\xi = 0.$$

Thus

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\xi)}{\xi-z}\,d\xi + \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\xi)}{\xi-z}\,d\xi \\ &= \frac{1}{2\pi i}\int_{\partial B(a,r_2)}\frac{f(\xi)}{\xi-z}\,d\xi - \frac{1}{2\pi i}\int_{\partial B(a,r_1)}\frac{f(\xi)}{\xi-z}\,d\xi \\ &= f_1(z)+f_2(z). \end{aligned}$$

Let C_1, C_2 denote $\partial B(a,r_1)$ and $\partial B(a,r_2)$. Then

$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(\xi)}{\xi-z}\,d\xi - \frac{1}{2\pi i}\int_{C_1}\frac{f(\xi)}{\xi-z}\,d\xi.$$

On C_2, $|\xi|>|z|$, so that

$$\begin{aligned} \frac{1}{\xi-z} = \frac{1}{(\xi-a)-(z-a)} &= \frac{1}{\xi-a}\,\frac{1}{1-\left(\frac{z-a}{\xi-a}\right)} \\ &= \frac{1}{\xi-a}+\frac{z-a}{(\xi-a)^2}+\frac{(z-a)^2}{(\xi-a)^3}+\cdots \end{aligned}$$

On C_1, $|\xi|<|z|$, so

$$\begin{aligned} \frac{1}{\xi-z} = \frac{1}{(\xi-a)-(z-a)} &= \frac{-1}{z-a}\,\frac{1}{1-\frac{\xi-a}{z-a}} \\ &= \frac{-1}{z-a}\left(1+\frac{\xi-a}{z-a}+\frac{(\xi-a)^2}{(z-a)^2}+\cdots\right) \\ &= -\frac{1}{z-a}-\frac{\xi-a}{(z-a)^2}-\frac{(\xi-a)^2}{(z-a)^3}\cdots \end{aligned}$$

The convergence are uniform, so

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_2}\left(\sum_{k=0}^{\infty}\frac{f(\xi)(z-a)^k}{(\xi-a)^{k+1}}\right)d\xi - \frac{1}{2\pi i}\int_{C_1}\left(\sum_{k=-1}^{-\infty}\frac{f(\xi)(z-a)^k}{(\xi-a)^{k+1}}\right)d\xi \\ &= \sum_{k=-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_C\frac{f(\xi)}{(\xi-a)^{k+1}}\right)(z-a)^k. \end{aligned}$$

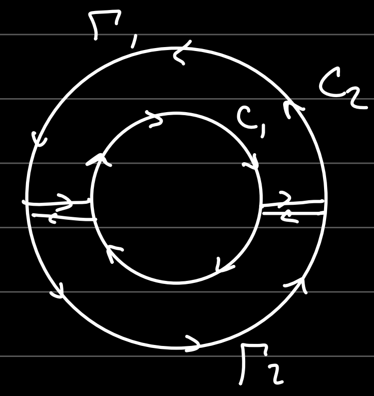
where

$C=C_1$ when $k<0$ and $C=C_2$ when $k\geq 0$.

But actually C can be any circle in A centered at a. Since $\frac{f(\xi)}{\xi^{k+1}}$ is analytic in A and Cauchy's closed curve thm gives

$$\int_{C_2} \frac{f(\xi)}{\xi^{k+1}}\,d\xi - \int_{C_1} \frac{f(\xi)}{\xi^{k+1}}\,d\xi$$

$$= \int_{\Gamma_2} \frac{f(\xi)}{\xi^{k+1}}\,d\xi - \int_{\Gamma_1} \frac{f(\xi)}{\xi^{k+1}}\,d\xi = 0-0=0.$$



□.

Def. If $f(z)=\sum_{-\infty}^{\infty} a_k (z-z_0)^k$ is the Laurent expansion of f about an isolated singularity z_0, $\sum_{-\infty}^{-1} a_k(z-z_0)^k$ is called the principal part of f at z_0, and $\sum_0^{\infty} a_k(z-z_0)^k$ the analytic part.

Cor. Let $\sum_{-\infty}^{\infty} a_k(z-z_0)^k$ be the Laurent expansion of f about z_0.

(1) If f has a removable singularity at z_0, then $a_k=0$ $\forall k<0$.

(2) If f has a pole of order k at z_0, then $a_{-k}\neq 0$ but $a_{-N}=0$ $\forall N>k$.

(3) If f has an essential singularity at z_0, it must have infinitely many nonzero terms in its principal part.

pf. (1) $f(z)=g(z)$ for $z\neq z_0$ and $g(z)$ analytic at z_0, the Laurent expansion of f agrees with the Taylor expansion for g around z_0.

(2) $f(z)=A(z)/B(z)$ where $A(z_0)\neq 0$ and B has zero of ord k at z_0. Then $f(z)=Q(z)/(z-z_0)^k$ where $Q(z_0)\neq 0$ and Q analytic at z_0. Hence if $Q(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then

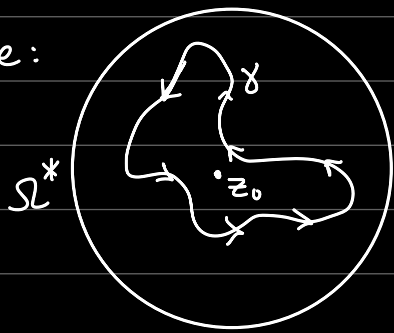
$$f(z)=\sum_{n=0}^{\infty} a_n \frac{(z-z_0)^n}{(z-z_0)^k} = \sum_{j=-k}^{\infty} a_j (z-z_0)^j.$$

(3) Follows immediately from (2). □

Chap 10. The Residue Theorem.

<u>Def.</u> If $f(z) = \sum_{-\infty}^{\infty} C_k (z-z_0)^k$ in a deleted nbh of z_0, C_{-1} is called the residue of f at z_0, denote $C_{-1} = \mathrm{Res}(f, z_0)$.

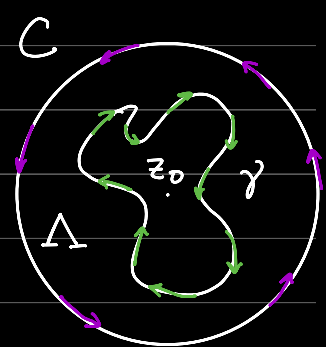
Note:



If γ is a curve sorrounding an isolated singularity z_0 of f that is contained in a deleted nbh Ω^* of z_0, and $f = \sum_{-\infty}^{\infty} C_k (z-z_0)^k$, then
$$\int_\gamma f = 2\pi i\, C_{-1}.$$

<u>Rmk.</u> This is because if C is a circle s.t. γ is sorrounded by C and $C \subseteq \Omega^*$, then
$$\int_C (z-z_0)^k dz = \begin{cases} 2\pi i & k=-1 \\ 0 & k \neq -1 \end{cases},$$
(when $k \geq 0$ by closed curve thm. $k<0$, use periodicity of $e^{-ik\theta}$.)

And since $(z-z_0)^k$ is analytic in the region Λ, by Stokes Thm:
$$0 = \int_\Lambda d[(z-z_0)^k dz] = \int_{\partial\Lambda} (z-z_0)^k dz = \int_C (z-z_0)^k dz - \int_\gamma (z-z_0)^k dz.$$



<u>Prop</u> (Some simple residues).

(1) If f has a simple pole, i.e. $f(z) = \frac{A(z)}{B(z)}$, where A, B analytic at z_0, $A(z_0) \neq 0$ and B has a simple zero at z_0, then
$$C_{-1} = \lim_{z \to z_0} (z-z_0) f(z) = \frac{A(z_0)}{B'(z_0)}.$$

(2) If f has a pole of order k at z_0, then
$$C_{-1} = \frac{1}{(k-1)!} \left(\frac{d}{dz}\right)^{k-1} \left[(z-z_0)^k f(z)\right] \Big|_{z=z_0}.$$

<u>pf.</u> Trivial. Exercise. $\square$

<u>Some examples:</u>

(1) Since $\csc(z) = \frac{1}{\sin(z)}$, $\mathrm{Res}(\csc(z), 0) = \frac{1}{\sin'(z)}\Big|_{z=0} = \frac{1}{\cos 0} = 1$.

(2) Consider $\frac{1}{z^4-1} = \frac{1}{(z+1)(z-1)(z+i)(z-i)}$, we have $\frac{1}{(i+1)(i-1)2i} = \frac{i}{4}$ and $(z-i)' = 1$ at $z=i$, we have $\mathrm{Res}\left(\frac{1}{z^4-1}, i\right) = \frac{i}{4}$.

(3) What about $\mathrm{Res}\left(\frac{1}{z^3}, 0\right)$? By prop above, this is

$$\frac{1}{(3-1)!}\frac{d^2}{dz^2}\left[z^3 \cdot \frac{1}{z^3}\right]\Big|_{z=0} = 0.$$

(4) Consider $\mathrm{Res}(\sin\frac{1}{z-1}, 1)$. Since the Laurant expansion reads:

$$\sin\frac{1}{z-1} = \frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - \cdots$$

$$= \sum_{k=1}^{\infty} \frac{1}{(2k+1)!}(z-1)^{-(2k+1)}$$

In this (and most) case, the easiest way to find the residue is to do a Laurant expansion at z_0 directly.

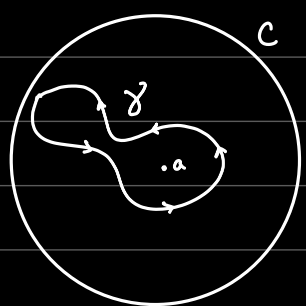
We have $C_{-1} = \frac{1}{(2\cdot 0+1)!} = 1$. Thus

Def. Suppose γ is a closed curve and $a \notin \gamma$. Then

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

is called the **winding number** of γ around a.

Note. If γ circles around a only once, then $n(\gamma, a) = 1$, for



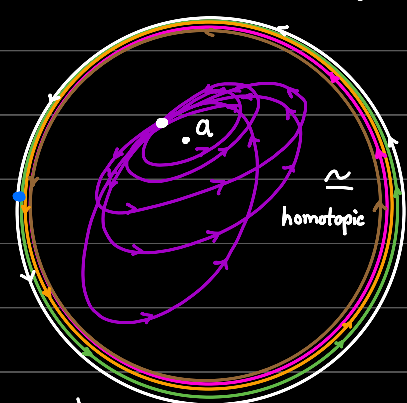
$$\int_\gamma \frac{dz}{z-a} = \int_C \frac{dz}{z-a} = 2\pi i$$

as we just saw. If a is "outside" of the "interior" of γ, then by closed curve thm and analyticity of $\frac{1}{z-a}$ "inside" γ, we see that

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 0.$$

If γ circles around a k times, then γ is "homotopic" to the parametrized circle $\gamma(\theta) = a + re^{i\theta}$, $0 \le \theta \le 2k\pi$. And winding number is homotopic invariant. So

$$n(\gamma, a) = \frac{1}{2\pi i}\int_0^{2k\pi} i\, d\theta = k.$$



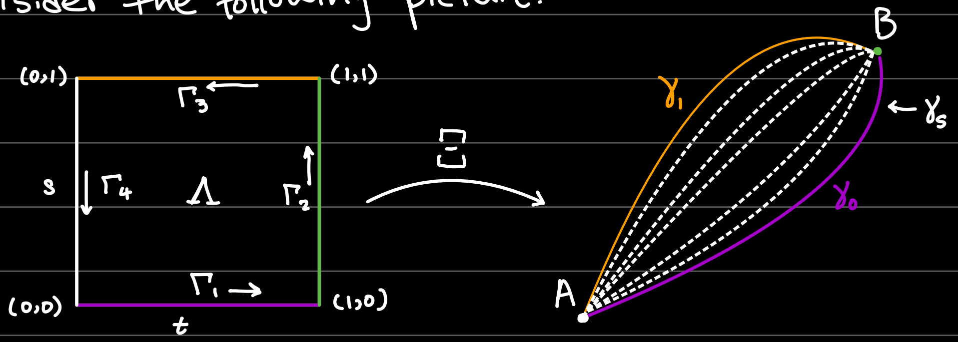
Cor. For any closed curve γ, $n(\gamma, a)$ is an integer.

Now, we use Stoke's Thm to prove that winding number is indeed a homotopy invariant. Two curves $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ are homotopic if $\exists$ a continous map $\Xi : [0,1]\times[0,1] \to \mathbb{C}$ s.t. $\Xi(t,0) = \gamma_0(t)$, $\Xi(t,1) = \gamma_1(t)$, $t \in [0,1]$. We write $\gamma_0 \simeq \gamma_1$. That is, γ_0 can be "continously deformed" to γ_1.

Thm. Let γ_t be a family of differentiable curves connected by the homotopy $\Xi : [0,1]\times[0,1] \to \mathbb{C}$ s.t. $\Xi(t,s) = \gamma_s(t)$. Let $\gamma_0(0)=\gamma_1(0)$ and $\gamma_0(1)=\gamma_1(1)$. Let ω be a closed form ($d\omega = 0$) defined everywhere on the range of Ξ.

Then
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega .$$

pf. Denote $\Xi^*\omega$ the "pull back" of ω by Ξ, i.e. $\omega \circ \Xi$, whose domain is now $[0,1]\times[0,1] = \Lambda$. We use the fact from differential geometry that exterior derivative commutes with pull back: $d(\Xi^*\omega) = \Xi^*(d\omega)$. Now, consider the following picture:



The vertical lines are crushed into 2 pts by Ξ while horizontal lines were deformed into γ_0, γ_1, on Λ. Stoke's thm gives
$$\int_{\partial\Lambda} \Xi^*\omega = \int_{\Lambda} d(\Xi^*\omega) = \int_{\Lambda} \Xi^*(d\omega) = 0.$$

Now
$$\int_{\partial\Lambda} \Xi^*\omega = \sum_{i=1}^{4} \int_{\Gamma_i} \Xi^*\omega = \int_{\Gamma_1} \Xi^*\omega + \int_{\{A\}} dz + \int_{\Gamma_3} \Xi^*\omega + \int_{\{B\}} dz$$
$$= \int_{\gamma_0} \omega - \int_{\gamma_1} \omega = 0.$$

This is just the change of variable formula: $\int_\Gamma \Xi^*\omega = \int_{\Xi(\Gamma)} \omega$. And integrate on a pt B just zero. Done. $\square$

Cor. The winding number is homotopy invariant.

pf. Let $\omega = dz/(z-a)$. Then
$$d\omega = \frac{\partial}{\partial z}\left(\frac{1}{z-a}\right) dz \wedge dz + \frac{\partial}{\partial \bar z}\left(\frac{1}{z-a}\right) d\bar z \wedge dz$$
$$= 0.$$

So ω is closed. This proves the claim. $\square$

Thm 10.5 (Cauchy's Residue Thm). Let f be analytic in a simply connected domain D except for isolated singularities at $z_1, \cdots, z_m$. Let γ be a closed curve not intersecting any of $z_1, \cdots, z_m$. Then

$$\int_\gamma f = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\, \mathrm{Res}(f, z_k).$$

Pf. The idea is beautiful. Let us Laurant expand f around z_1 and write

$$f(z) = \underbrace{P_1\left(\frac{1}{z-z_1}\right)}_{\text{principal}} + \underbrace{A_1(z)}_{\text{analytic}}.$$

Then

$$A_1(z) = f(z) - P_1\left(\frac{1}{z-z_1}\right)$$

has a removable singularity at z_1 and has the same principal parts as $f(z)$ at $z_2, z_3, \cdots, z_m$: for $P\left(\frac{1}{z-z_1}\right)$ is analytic in a small disc containing $z_2, \ldots, z_m$, so it does not contribute any terms for the principal part. Inductively,

$$g(z) = f(z) - P_1\left(\frac{1}{z-z_1}\right) - P_2\left(\frac{1}{z-z_2}\right) - \cdots - P_m\left(\frac{1}{z-z_m}\right)$$

with appropriate definitions at $z_1, \cdots, z_m$, is analytic in D, for $g(z)$ has only finitely many removable singularities. Then by closed curve thm,

$$\int_\gamma g = 0 \Rightarrow \int_\gamma f = \sum_{k=1}^{m} \int_\gamma P_k\left(\frac{1}{z-z_k}\right)$$

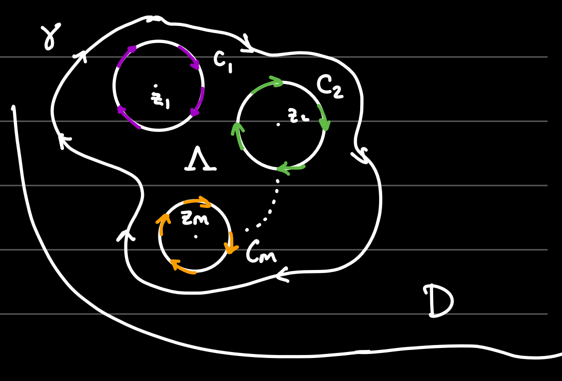
Now, for any k, we have $\int_\gamma \frac{1}{(z-z_k)^n} = 0$, if $n \neq 1$.

Hence if $P_k\left(\frac{1}{z-z_k}\right) = \frac{C_{-1}}{z-z_k} + \frac{C_{-2}}{(z-z_k)^2} + \cdots$, then

$$\int_\gamma P_k\left(\frac{1}{z-z_k}\right) = C_{-1}\int_\gamma \frac{dz}{z-z_k} = 2\pi i\, n(\gamma, z_k)\, \mathrm{Res}(f, z_k). \qquad \square$$

pf (II). Enclose each singularities by small circles $C_1, \ldots, C_m$ and apply Stoke's Thm to Λ, where f is analytic:

$$\int_\gamma f = \sum_{k=1}^{m} \int_{C_m} f\, dz = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f, z_k).$$



The case where γ goes around more than once introduces the factor $n(\gamma, z_k)$.

$\square$

Applications of Residue Thm:

Def. (1) γ is called a **regular closed curve** if $n(\gamma, a) = 0$ or 1 $\forall a \notin \gamma$. We call $\{a : n(\gamma, a) = 1\}$ the **inside** of γ and $\{a : n(\gamma, a) = 0\}$ **outside** of γ.

(2) We say f is **meromorphic** in a domain D if f is analytic there except at isolated poles.

Thm 10.8. Let γ be a regular closed curve, f meromorphic inside and on γ and contains no zeros or poles on γ. Denote:

- Z = # of zeros of f inside γ counted by multiplicity;
- P = # of poles of f inside γ counted by multiplicity.

Then:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = Z - P.$$

Pf. Note that f'/f is analytic except at the zeros or poles of f.

If f has a zero of order k at $z = a$, i.e.

$$f(z) = (z-a)^k g(z) \qquad \text{with } g(a) \neq 0.$$

then $$f'(z) = (z-a)^{k-1} \left[k g(z) + (z-a) g'(z) \right]$$

has a zero of order $k-1$ at $z = a$. And

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}.$$

Since in a deleted nbh of $z = a$, $g'(z)/g(z)$ is analytic, thus it contributes no principal part to f'/f. And

$$\operatorname{Res}(f'/f, a) = \operatorname{Res}\left(\tfrac{k}{z-a}, a\right) = k.$$

Similarly, if f has a pole of order k at $z = a$:

$$f(z) = (z-a)^{-k} g(z) \qquad g(z) \text{ analytic at } a.$$

Then $$\frac{f'(z)}{f(z)} = \frac{-k}{z-a} + \frac{g'(z)}{g(z)} \Rightarrow \operatorname{Res}\left(\frac{f'}{f}, a\right) = \operatorname{Res}\left(\frac{-k}{z-a}, a\right) = -k.$$

Thus by Residue thm:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \sum \operatorname{Res}\left(\frac{f'}{f}\right) = Z - P. \qquad \square$$

Cor 10.9 (Argument Principle). If f is analytic on a closed curve γ and is nonzero on γ, then let $Z(f)$ denote the # of zeros of f inside γ, we have:

$$Z(f) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}.$$

Rmk. We may view $\int_\gamma f'/f$ as the winding number of the curve $f(\gamma(t))$ around $z=0$ (see Def of winding number). It follows that the number of zeros of f inside γ equals the number of times $f(\gamma(t))$ winds around the origin. Similarly, if we consider $f(z)-a$, it follows that the number of times $f(z)=a$ inside γ equals the number of times $f(\gamma(t))$ winds around the complex number a.

$$\begin{tikzcd} \mathbb{C} \supseteq U \arrow[r, "f"] & \mathbb{C} \\ {[0,1]} \arrow[u, "\gamma"] \arrow[ur, "f\circ\gamma"'] & \end{tikzcd}$$

Thm 10.10 (Rouché's Thm). Let f, g be analytic inside and on a regular closed curve γ and $|f(z)| > |g(z)|$, $\forall z \in \gamma$. Then

$$Z(f+g) = Z(f) \quad \text{inside } \gamma.$$

pf. First note that if $f(z) = A(z)B(z)$, then

$$\frac{f'}{f} = \frac{A'}{A} + \frac{B'}{B} \Rightarrow \int_\gamma \frac{f'}{f} = \int_\gamma \frac{A'}{A} + \int_\gamma \frac{B'}{B}.$$

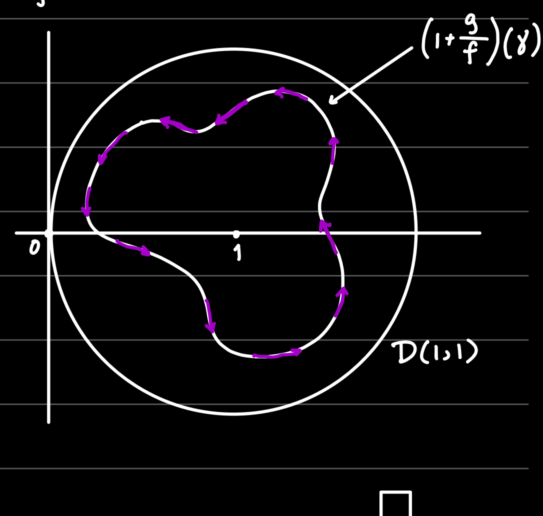
Thus, write $f+g = f(1+\frac{g}{f})$. Then

$$Z(f+g) = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} + \frac{1}{2\pi i}\int_\gamma \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}}$$

$$= Z(f) + \frac{1}{2\pi i}\int_\gamma \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}}.$$

But since $|f| > |g|$ on γ, the curve $(1+\frac{g}{f})(\gamma)$ remains within a disc of radius 1 around $z=1$. Hence the winding number of $(1+\frac{g}{f})(\gamma)$ around $z=0$ is 0. So by Rmk above, the last integral above is zero. Thus

$$Z(f+g) = Z(f).$$



□