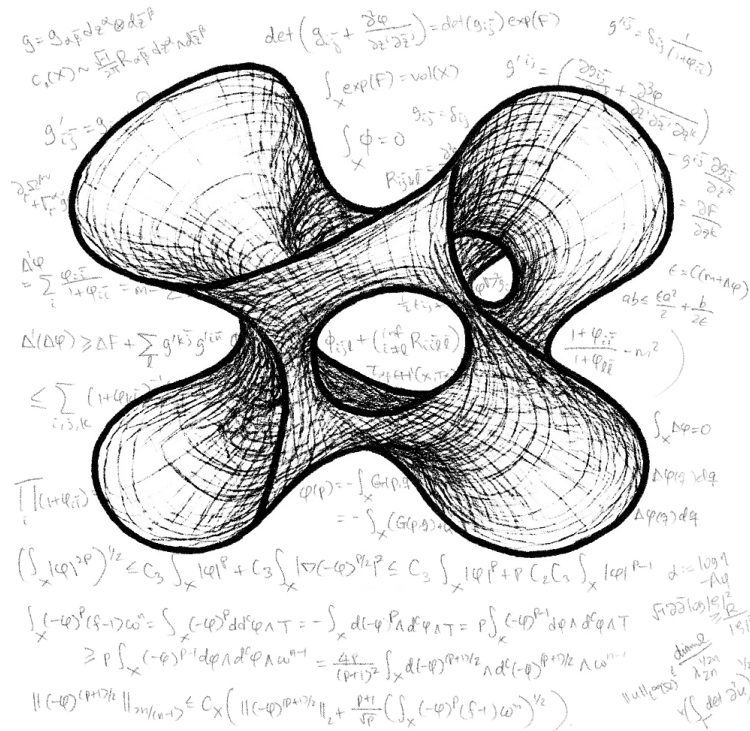


NOTES ON COMPLEX ALGEBRAIC GEOMETRY

WRITTEN BY
Tianyi Wang



PUBLISHED
IN THE WILD

Preface

This book originated from a set of notes I wrote for a year long course on complex algebraic geometry at the University of Pennsylvania, taught by Prof. Ron Donagi.

I assume the reader is familiar with sheaves and sheaf cohomology. This is quite thoroughly covered in Lee's excellent textbook [1], which greatly influenced the writing of this book. Also, I assume that the readers are comfortable with smooth manifolds, and de Rham cohomology. All of these materials can be found in my previous book [7].

I am quite aware of several shortcomings of this book. Throughout the book there are several results I used without proof, most notably in section 2.1 (complex structures on covering and quotient manifolds), the Nirenberg-Newlander theorem, the Fredholm theorem, Riemann-Roch theorem for vector bundles, etc. Several proofs are sketched and some details are left out as well. If all of these results were fully and appropriately dealt with, then the book will be undeniably more complete. But I think it is more important to use these results when needed, and the proofs of these results are not so important by contrast.

The book should really end at Chapter 6 (Hodge theory). However, I decided to add a little bit of curve theory and one chapter on Kodaira vanishing theorem, because these are quite important topics. But due to limited time I had, Chapter 7 on curves is sketchy at best. I only presented some of the most well-known results, and the goal of the chapter is merely to prove these results as quickly and as efficiently as possible. For readers interested in curve theory, the classic by Griffiths and Harris [3] is an excellent source. Chapter 8 is mostly fine, except that I completely left out the Kodaira embedding theorem, because I again think it is more important to know the result than to know the proof, plus the proof is too long for me to type out. However, it is a great result to know, because it demonstrates the motivation for us to study the associated map $\phi_L : H^0(X, L) \dashrightarrow \mathbb{P}^N$ of a holomorphic line bundle $L \rightarrow X$: Because if L is positive then $\phi_{L^{\otimes k}}$ will actually be an embedding for some $k \geq 0$ and X will be a projective manifold.

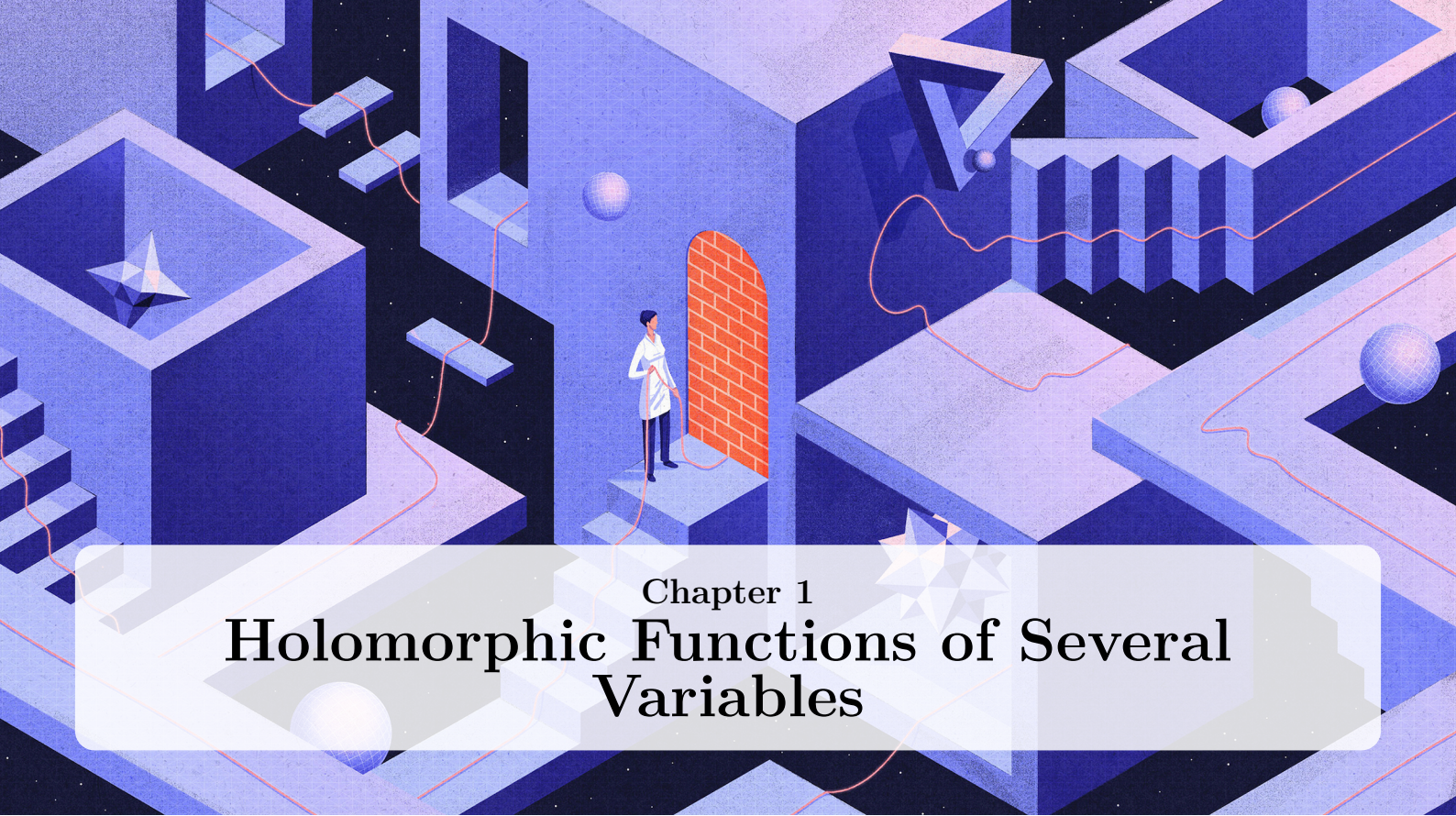
In short, I think this is a fun but highly incomplete book. The primary purpose of this book is that it serves as my review notes in my preparation for my oral exam on complex algebraic geometry. However, I hope the readers can also extract some values of it, regardless of how small they are.

Tianyi Wang
Department of Mathematics
University of Pennsylvania, PA 19104, USA
2026/02/22

Contents

Preface	1
1 Holomorphic Functions of Several Variables	5
1.1 Basic Definitions	5
1.2 Basic Properties	7
1.3 Hartog's Extension Theorem	8
2 Complex Manifolds	10
2.1 Definitions and Examples	10
2.2 The Complexified Tangent and Cotangent Bundles	15
2.3 Almost Complex Structure	19
2.4 The Dolbeault Complex	23
2.5 Bundle-Valued Forms	27
3 Holomorphic Vector Bundles	29
3.1 Holomorphic Vector Bundles	29
3.2 Divisors and Line Bundles	35
3.3 Basic Results in Complex Projective Space	38
3.4 Blow-Ups	43
4 Connections and Characteristic Classes	47
4.1 Connections and Curvatures	47
4.2 Chern Classes	52
4.3 Chern Connection	56
4.4 Other Characteristic Classes: Thom and Euler Class	60
4.5 First Chern Class and Intersection Theory	64
5 Kähler Manifolds	68
5.1 Hermitian and Kähler Metrics	68
5.2 Examples and Nonexamples of Kähler Manifolds	72
6 Hodge Theory	80
6.1 Hodge Theory on Riemannian Manifolds	80
6.2 Hodge Theory on Complex Manifolds	86
6.3 Bundle-Valued Forms and Serre Duality	88
6.4 Hodge Theory on Kähler Manifolds	90
6.5 Lefschetz Theorems on Kähler Manifolds	93
6.6 Applications of Hodge Theory	98

7	Curves	102
7.1	Basic Computational Tools	102
7.1.1	Degree Genus Formula	102
7.1.2	Riemann-Roch Theorem	103
7.1.3	Riemann-Hurwitz Formula	105
7.2	Residue Theorem	108
7.3	Linear Systems	110
7.4	Applications of Riemann-Roch Theorem	113
7.4.1	The Canonical Morphism and Geometric Riemann-Roch	113
7.4.2	Clifford's Theorem	116
7.5	Abel-Jacobi Theory	117
7.6	Classification of Curves of Low Genus	120
7.6.1	$g = 0, 1$: Projective Line and Complex Torus	120
7.6.2	$g = 2$: Hyperelliptic Curves	120
7.6.3	$g = 3$: Generically Quartic	122
7.6.4	$g = 4$: Intersection of Quadric and Cubic	123
7.6.5	$g = 5$: Generically Triple Intersection of Quadrics	124
8	Kodaira Vanishing Theorem	126
8.1	Proof of the Kodaira Vanishing Theorem	126
8.2	Applications of Kodaira Vanishing Theorem	130



Chapter 1

Holomorphic Functions of Several Variables

Introduction

One of the most common objects of study in complex algebraic geometry is complex manifold. Just like real manifolds are spaces that are locally diffeomorphic to $\mathbb{R}^n$, complex manifolds are spaces that are locally biholomorphic to $\mathbb{C}^n$. One of the big advantages of such objects of study is that we can do calculus locally on these spaces. Not surprisingly, a huge part of the study of complex manifolds is doing complex analysis on them. Therefore, our goal in this first chapter is to develop multivariable complex analysis on $\mathbb{C}^n$, which we can then use to study complex manifolds.

1.1 Basic Definitions

We begin with the basic definitions of holomorphic functions in several variables:

Definition 1.1. Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \rightarrow \mathbb{C}$. For $p = (p_1, \dots, p_n) \in U$ and $1 \leq j \leq n$, we say that f has a complex partial derivative at p with respect to z_j if the following limit exists:

$$\frac{\partial f}{\partial z_j}(p) = \lim_{h \rightarrow 0} \frac{f(p_1, \dots, p_j + h, \dots, p_n) - f(p_1, \dots, p_n)}{h}.$$

We say that f is **holomorphic** if it is continuous and has a complex partial derivative with respect to each variable $z_1, \dots, z_n$ at each point of U . More generally, a complex function $F : U \rightarrow \mathbb{C}^k$ is said to be holomorphic if each of its component functions is holomorphic.

Remark 1.2. Recall that in one variable complex analysis, our definition of holomorphicity only assumes existence of complex derivative, and continuity follows from it. In fact, in our definition above we could have just assumed partial derivatives exists as well: Hartog proved existence of partial derivatives implies continuity as well in his paper [6] in 1906. However, the proof is very difficult, so it is perhaps best for our convenience to assume continuity in our definition, despite the fact that it is unnecessary.

In single variable complex analysis, we established several equivalent definitions for holomorphicity. Our first goal is to generalize these results to higher dimensions;

Theorem 1.3. *Let $U \subset \mathbb{C}^n$ be open and $f : U \rightarrow \mathbb{C}^n$. The following statements are equivalent:*

- (a) f is holomorphic.
- (b) f is smooth, and if $f(z) = u(z) + iv(z)$ where $u, v : U \rightarrow \mathbb{R}$, with $z_j = x_j + iy_j$, then f satisfies the following Cauchy Riemann equations:

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \quad \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j} \quad j = 1, \dots, n.$$

- (c) For each $p = (p_1, \dots, p_n) \in U$, there exists a neighborhood of p in U on which f is equal to the sum of an absolutely convergent power series of the form

$$f(z) = \sum_{k_1, \dots, k_n=0}^{\infty} a_{k_1 \dots k_n} (z_1 - p_1)^{k_1} \dots (z_n - p_n)^{k_n}.$$

Proof. We prove that (b) and (c) are both equivalent to (a) as follows:

- (b) $\Rightarrow$ (a): If (b) holds, then f is smooth so it is certainly continuous. Then single variable theory implies that f has complex derivative with respect to each variable, so it implies (a).
- (a) $\Rightarrow$ (b): If f is holomorphic, then by single variable theory it satisfies Cauchy Riemann equation in each variable. Hence it only remains to prove that f is smooth. To this end, let $p = (p_1, \dots, p_n) \in U$, choose $r > 0$ such that the *closed* polydisk

$$\bar{D}^n(p, r) := \prod_{j=1}^n \bar{D}(p_j, r)$$

is contained in U . Since f is holomorphic in each variable separately, we can apply the single variable Cauchy integral formula repeatedly to show that for all $z \in \bar{D}^n(p, r)$, we have

$$\begin{aligned} f(z_1, \dots, z_n) &= \frac{1}{2\pi i} \int_{|\zeta_n - p_n|=r} \frac{f(z_1, \dots, z_{n-1}, \zeta_n)}{\zeta_n - z_n} d\zeta_n \\ &= \frac{1}{(2\pi i)^n} \int_{|\zeta_n - p_n|=r} \dots \int_{|\zeta_1 - p_1|=r} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_n - z_n) \dots (\zeta_1 - z_1)} d\zeta_1 \dots d\zeta_n. \end{aligned}$$

Note that the domain of integration is compact, and the integrand is continuous in all variables and smooth as a function of (real and imaginary parts of) $z_1, \dots, z_n$, we can differentiate under the integral as many times as we like with respect to whatever variables we want. This shows that f is smooth, proving (b).

- (a) $\Rightarrow$ (c): Assuming (a), we already derived the integral expression of f above. Now note that

$$\frac{1}{\zeta_j - z_j} = \frac{1}{(\zeta_j - p_j) - (z_j - p_j)} = \frac{1}{\zeta_j - p_j} \frac{1}{1 - \left(\frac{z_j - p_j}{\zeta_j - p_j}\right)}.$$

Since $|z_j - p_j|/|\zeta_j - p_j| < 1$ on the domain of integration above, we may expand the right hand side as a power series:

$$\frac{1}{\zeta_j - z_j} = \frac{1}{\zeta_j - p_j} \sum_{k=0}^{\infty} \left(\frac{z_j - p_j}{\zeta_j - p_j} \right)^k.$$

The series above converges uniformly and absolutely for $z_j \in D(p_j, \rho)$ for any $0 < \rho < r$. Inserting this formula back into the integral equation and comparing coefficients, we see that the coefficients for the power series should be

$$a_{k_1 \dots k_n} = \int_{|\zeta_n - p_n|=r} \dots \int_{|\zeta_1 - p_1|=r} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_n - p_n)^{k_n+1} \dots (\zeta_1 - p_1)^{k_1+1}} d\zeta_1 \dots d\zeta_n.$$

- (c) $\Rightarrow$ (a): Assuming f satisfies (c) and choose $p \in U$ arbitrary. Choose some closed polydisk $\bar{D}^n(p, r)$ contained in U and centered at p such that the series in (c) converges absolutely. Because the series converges at $z_0 = (p_1 + r, \dots, p_n + r)$, the terms in the series for $f(z_0)$ are all uniformly bounded: there exists a constant C such that

$$|a_{k_1 \dots k_n}| r^{k_1} \dots r^{k_n} \leq C.$$

On a polydisk $\bar{D}^n(p, \rho)$ for any $0 < \rho < r$, the terms of the series is bounded by:

$$\sum_{k_1, \dots, k_n=0}^{\infty} |a_{k_1 \dots k_n} (z_1 - p_1)^{k_1} \dots (z_n - p_n)^{k_n}| \leq \sum_{k_1, \dots, k_n=0}^{\infty} C \left(\frac{\rho}{r} \right)^{k_1} \dots \left(\frac{\rho}{r} \right)^{k_n}.$$

The series on the right is geometric, so it converges uniformly and absolutely on $\bar{D}^n(p, \rho)$. Thus by the Wierstrass M-test, the series for f converges uniformly and absolutely on $\bar{D}^n(p, \rho)$. So f is continuous there, in particular at p . If we fix all the variables but z_j , we obtain a convergent power series in z_j , which is holomorphic in z_j , proving (a).

This completes our proof of the theorem. $\square$

1.2 Basic Properties

Having established the definition(s) of holomorphicity of a multivariable function, we now quickly generalizes the single variable analysis results. Most of these proofs are straightforward, so I will omit certain details and leave the readers to fill them in if they wish.

Proposition 1.4. Let $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ be open subsets and $f : U \rightarrow V$ and $g : V \rightarrow \mathbb{C}^k$ are holomorphic. Then $g \circ f : U \rightarrow \mathbb{C}^k$ is holomorphic.

Proof. Use chain rule on $g \circ f$ to check that $g \circ f$ satisfies the CR equations as in the statement of Theorem 1.3, part (b). $\square$

Corollary 1.5. If f is a holomorphic function given by an absolutely convergent power series of the form stated in Theorem 1.3 part (c) on a polydisk $\bar{D}^n(p, r) \subset \mathbb{C}^n$, then the power series is given explicitly by

$$f(z) = \sum_{k_1 \dots k_n=0}^{\infty} \frac{1}{k_1! \dots k_n!} \frac{\partial^{k_1+\dots+k_n} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} \Big|_p (z_1 - p_1)^{k_1} \dots (z_n - p_n)^{k_n}.$$

Proof. Just differentiate the expression in Theorem 1.3 (c) repeatedly term by term and evaluate at $z = p$ to determine the coefficients $a_{k_1 \dots k_n}$. $\square$

Proposition 1.6 (Identity Theorem). Suppose $U \subset \mathbb{C}^n$ is a connected open subset, and $f, g : U \rightarrow \mathbb{C}$ are holomorphic functions that agree on a nonempty open subset of U , then $f \equiv g$ on U .

Proof. Set $h = f - g$, and by assumption on some nonempty open set $U_0 \subset U$, we have $h|_{U_0} \equiv 0$. Now let $V(h) \subset U$ denote the subset of U on which h and its complex partial derivatives of all order vanishes.

If $p \in V(h)$, then h is equal to a convergent power series in a neighborhood of p . By Corollary 1.5, every term of the series is zero, so h is identically zero on that neighborhood, thus $V(h)$ is open.

Now let $p \in U$ such that there exists a sequence $(p_j)_{j=1}^\infty \subset V(h)$ converging to p . Then by continuity, all partial derivatives must vanish at p since they vanish at all p_j . Thus $p \in V(h)$. So $V(h)$ is closed.

Since $V(h) \neq \emptyset$ by assumption, we have $V(h) = U$, concluding the proof. $\square$

Proposition 1.7 (Liouville's Theorem). Every holomorphic function that is defined on all of $\mathbb{C}^n$ and bounded is constant.

Proof. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be as given in the statement. Let $z \in \mathbb{C}^n$ be fixed. Then define $g : \mathbb{C} \rightarrow \mathbb{C}$ via $g(\lambda) = f(\lambda z)$. Then g is a bounded holomorphic function. By Liouville's theorem, g is constant. Since λ is arbitrary, f must also be constant. $\square$

Proposition 1.8 (Maximum Modulus Principle). Suppose $f : U \rightarrow \mathbb{C}$ is holomorphic on a connected open subset $U \subset \mathbb{C}^n$. If $|f(z)|$ obtains a maximum at some point of U , then f is constant.

Proof. Let $z_0 \in U$ be the point where $|f(z)|$ obtains a maximum. Set $c = f(z_0)$ and consider the set $W = \{z \in U : f(z) = c\}$. Then $W \neq \emptyset$, and is closed by definition and continuity of f . Now suppose $z \in W$, choose $\epsilon > 0$ such that the open ball $B(z, \epsilon) \subset U$. Then for each $\lambda \in \mathbb{C}^n$ with $|\lambda| = 1$, the function $g(\zeta) = f(z + \zeta\lambda)$ is a holomorphic function on $B(0, \epsilon) \subset \mathbb{C}$ and achieves its maximum modulus at $\zeta = 0$. By single variable maximum modulus principle, g is constant. Since λ is arbitrary, it follows that f is constant on $B(z, \epsilon)$. Thus W is open. So $W = U$ by connectedness of U . $\square$

1.3 Hartog's Extension Theorem

So far, we have been generalizing results in one variable theory. We conclude this writing by proving something radically different from the one variable theory, the Hartog's Extension theorem. It was proven by Friedrich Hartogs in 1906 in his paper [6].

Theorem 1.9 (Hartog's Extension). Let $n \geq 2$, and $\Omega \subset \mathbb{C}^n$ be an open set of the form $\Omega = D^n(p, R) - \bar{D}^n(p, r)$ for some $p \in \mathbb{C}^n$ and $0 < r < R$. Here $D^n(p, R) = \prod_j D(p_j, R)$ means open polydisk whereas $\bar{D}^n(p, r)$ means closed polydisk.

Then every holomorphic function $f : \Omega \rightarrow \mathbb{C}$ has a unique extension to a holomorphic function on all of $D^n(p, R)$.

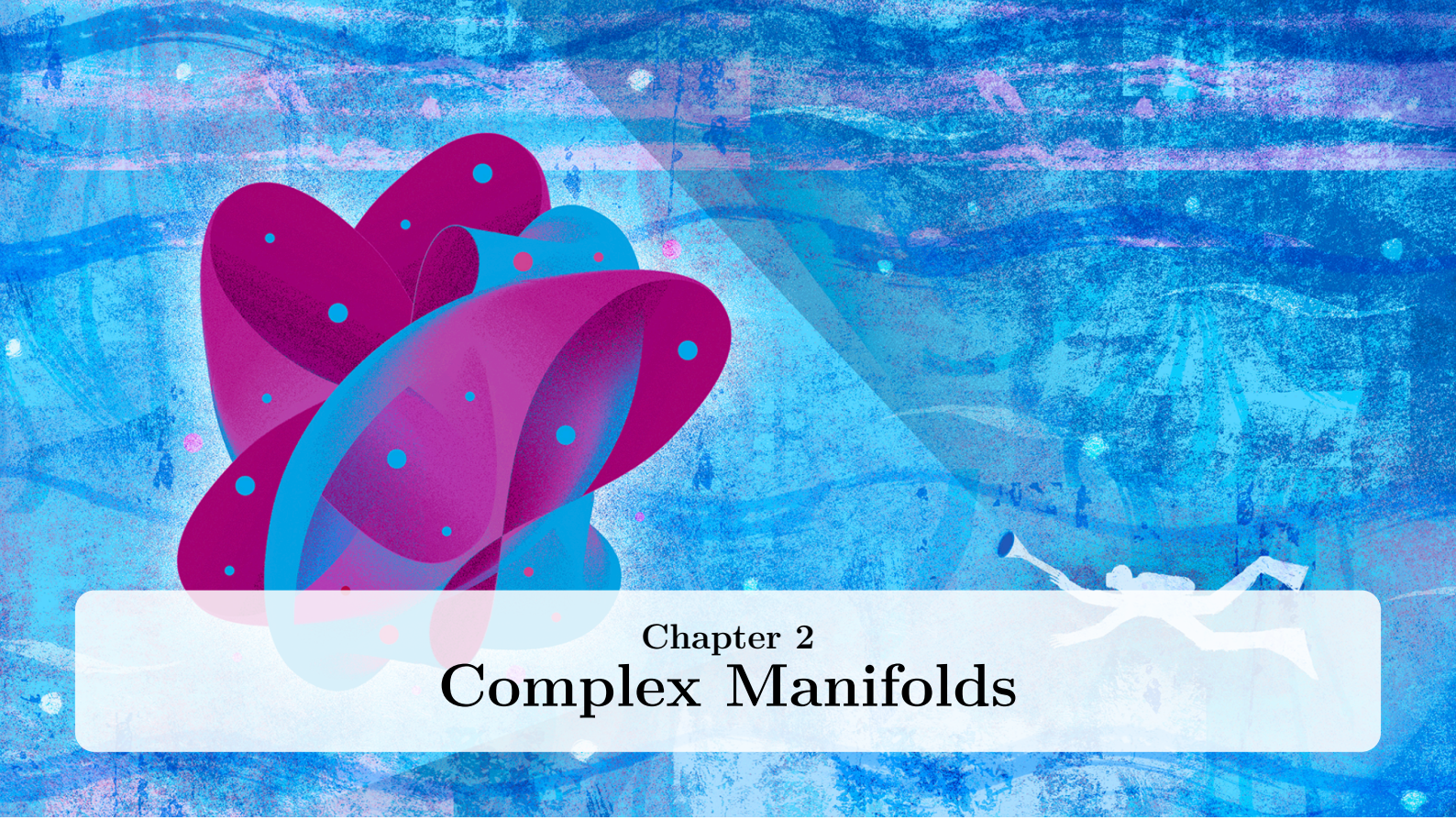
Proof. After a translation, we may assume that $p = 0$. We choose ρ such that $r < \rho < R$. Note that the function $z_1 \mapsto f(z_1, z_2, \dots, z_n)$ is holomorphic provided that $r < |z_2| < R$. Then by Theorem 1.3, we have the Cauchy integral formula representation

$$f(z_1, \dots, z_n) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta.$$

But note that this formula makes sense on the entire $D^n(0, \rho)$, because the integration contour is contained in Ω , and the denominator is never zero. Thus the formula above actually defines a holomorphic function F on $D^n(0, \rho)$: It is holomorphic because we can differentiate under the integral sign.

Now note that $F = f$ on the open subset of $D^n(0, \rho)$ where $r < |z_2| < \rho$. Hence Theorem 1.6 shows that it agrees on the entire connected set $D^n(0, \rho) - \bar{D}^n(0, r)$. Thus we can define a holomorphic function on all of $D^n(0, R)$ by letting it equal to f on Ω and to F on $D^n(0, \rho)$. Uniqueness follows from Theorem 1.6.

This concludes the proof. □



Chapter 2

Complex Manifolds

Introduction

In this chapter, we begin our study of complex manifolds. We will first give their definitions, and lots of examples of complex manifolds. After that, we establish the machinery of differential calculus (and complex analysis) on complex manifolds: We will first develop the theory of complexified tangent bundles over a complex manifold. Then we deviate a little bit to study almost complex structure on the tangent bundles, and almost complex manifolds, which are manifolds with an almost complex structure. The reason for this is that every complex manifold has a natural almost complex structure (Theorem 2.47), and moreover these structures give a nice decomposition of the complexified tangent spaces into holomorphic and antiholomorphic pieces. Next, we take the dual perspective and study differential forms on complex manifolds. We see that the complexified space of differential forms also decompose nicely into specific types, on which there are derivative operators ($\bar{\partial}$ and $\bar{\partial}_E$) defined. Finally, we will see that we can obtain algebraic holomorphic invariants, namely the Dolbeault cohomology groups. The proof of some results in this chapter are omitted, because the proof is either too deep to cover in an introductory text like this, or the proof are rather technical (and tedious), and they might distract us from the main themes that complex algebraic geometers care most about.

2.1 Definitions and Examples

Definition 2.1. A *holomorphic atlas* on a C^∞ -manifold X is an open cover $X = \bigcup_\alpha U_\alpha$ of X , together with a homeomorphism $\varphi_\alpha : U_\alpha \rightarrow \varphi(U_\alpha) \subset \mathbb{C}^n$ such that the chart transition maps

$$\tau_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic. Two holomorphic atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ of X are called *equivalent* if all maps

$$\varphi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap V_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap V_\beta)$$

are holomorphic. A **complex structure** on X is an equivalence class of holomorphic atlas.

Definition 2.2. A **complex manifold** X of (complex) **dimension** n is a real C^∞ -manifold of real dimension $2n$, with a complex structure.

Definition 2.3. A **holomorphic** function f on a complex manifold X is a function $f : X \rightarrow \mathbb{C}$, such that $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \rightarrow \mathbb{C}$ is holomorphic for any chart $(\varphi_\alpha, U_\alpha)$ of a holomorphic atlas that is a representative of the complex structure on X .

Notation 2.4. Let X be a complex manifold. We use $\mathcal{O}_X(U)$ to denote the space of holomorphic functions on $U \subset X$. Sometimes we will also use the notation $\Gamma(U, \mathcal{O}_X)$. The reason is that $\mathcal{O}_X$ is actually a sheaf, and Γ is the functor of taking global sections.

Theorem 2.5 (Liouville's theorem). *Let X be a compact connected complex manifold. Then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. That is, any global holomorphic function on X is constant.*

Proof. The continuous function $|f| : X \rightarrow \mathbb{R}$ is defined on a compact set, hence attains a maximum at some point $p \in X$. Then consider the set

$$M = \{x \in X : f(x) = f(p)\} \subset X.$$

Clearly this set is closed. However, for every $x \in M$, we can choose a holomorphic chart (U, φ) around x . Then $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C}$ is holomorphic and attains maximum modulus at $\varphi(x)$. By Proposition 1.8, we conclude that $f \circ \varphi^{-1}$ is constant. Hence f is constant on U , an open neighborhood containing x . Therefore M is open. Since X is connected, it follows that $M = X$ and f is globally constant. $\square$

Definition 2.6. Let X and Y be two complex manifolds. A continuous map $f : X \rightarrow Y$ is a holomorphic map if for any holomorphic charts (U, φ) of X and (V, ψ) of Y , the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \rightarrow \psi(V)$$

is holomorphic. Two complex manifolds are called **isomorphic** or **biholomorphic** if there exists a holomorphism homeomorphism $f : X \rightarrow Y$.

Remark 2.7. Actually, a biholomorphism $f : X \rightarrow Y$ means a holomorphic map with holomorphic inverse. But in fact only requiring that $f : X \rightarrow Y$ is a holomorphic homeomorphism is enough (without requiring the inverse $f^{-1} : Y \rightarrow X$ is also holomorphic). This is a consequence of the holomorphic inverse function theorem. See Theorem 2.38.

Definition 2.8 (Meromorphic Functions). Let $(X, \mathcal{O}_X)$ be a complex manifold. We define a presheaf $\mathcal{K}_X^{\text{pre}}$ on X as follows:

$$\mathcal{K}_X^{\text{pre}} : U \mapsto \left\{ \frac{f}{g} : f, g \in \mathcal{O}_X(U), g \text{ not identically zero on } U \right\}.$$

Then the sheaf $\mathcal{K}_X$ of **meromorphic functions** on X is the sheafification of $\mathcal{K}_X^{\text{pre}}$. One can define $\mathcal{K}_X$ equivalently as follows: A meromorphic function φ on a complex manifold is a map

$$\varphi : X \rightarrow \coprod_{p \in X} \text{Frac}(\mathcal{O}_{X,p})$$

which associates for any $p \in X$ an element $\varphi_p \in \text{Frac}(\mathcal{O}_{X,p})$ such that for any $p_0 \in X$ there exists an open neighborhood $U \subset X$ and two holomorphic functions $f, g : U \rightarrow \mathbb{C}$ with $\varphi = f/g$ for all $p \in U$. The global section of this sheaf is denoted by $K(X) := \Gamma(X, \mathcal{K}_X)$, called the **function field** on X .

It is easy to check that $K(X)$ is a field, and it contains $\mathbb{C}$ (the globally constant meromorphic functions) as a subfield.

Next, we give some examples of complex manifolds.

Example 2.9. $\mathbb{C}^n$ and open subsets of $\mathbb{C}^n$ are complex manifolds for obvious reason.

Example 2.10 (The Complex Projective Space). We define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$:

$$(z_0, \dots, z_n) \sim (w_0, \dots, w_n)$$

if and only if we can find a non-zero $\lambda \in \mathbb{C}$ (write $\lambda \in \mathbb{C}^*$ for short) such that $w_i = \lambda z_i$ for all $i = 0, \dots, n$. The equivalence class of $(z_0, \dots, z_n)$ is denoted by $[z_0 : \dots : z_n]$.

The n -dimensional **complex projective space** $\mathbb{CP}^n$ (since we will only be talking about projective space over the complex numbers, we will simply denote them as $\mathbb{P}^n$ from now on) is defined to be the space of all equivalence classes, endowed with quotient topology. Then it is a compact Hausdorff space. We choose the holomorphic coordinate charts as follows: Define

$$U_i := \{[z_0 : \dots : z_n] \in \mathbb{P}^n : z_i \neq 0\}, \quad i = 0, \dots, n.$$

These are open sets, and we define

$$\varphi_i : U_i \rightarrow \mathbb{C}^n, \quad \varphi_i([z_0 : \dots : z_n]) = \left(\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i} \right).$$

The checking of compatibility is left to readers. Also it is easy to check that $\mathbb{P}^1$ is homeomorphic to S^2 .

Example 2.11 (Projectivization of a Vector Space). For some purposes, it is useful to construct projective spaces starting with different complex vector spaces in place of $\mathbb{C}^{n+1}$ itself. Suppose V is an n -dimensional complex vector space with $n > 0$. The **projectivization** of V , denoted by $\mathbb{P}(V)$, is the set of 1-dimensional complex subspaces of V , endowed with the quotient topology obtained from the equivalence relation on $V \setminus \{0\}$ given by $v_1 \sim v_2$ if $v_2 = \lambda v_1$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

A choice of basis for V yields an isomorphism $V \cong \mathbb{C}^n$ that descends to a bijection $\mathbb{P}(V) \rightarrow \mathbb{P}^{n-1}$, which we can use to give $\mathbb{P}(V)$ the structure of a complex manifold. One can check that the holomorphic structure obtained in this way is independent of the choice of basis.

Example 2.12 (Complex Grassmannians). Suppose V is an n -dimensional complex vector space with $n > 0$, and k is a nonnegative integer less than or equal to n . Let $G_k(V)$ be the set of k -dimensional complex-linear subspaces of V , called a **complex Grassmannian**. The case $k = 1$ is exactly the projective space $\mathbb{P}(V)$.

We can construct complex coordinates on $G_k(V)$ as follows. Choose a subspace $P \subseteq V$ of dimension k and a complementary $(n-k)$ -dimensional subspace Q , and write $V = P \oplus Q$. Then the graph of each complex-linear map $X : P \rightarrow Q$ is a k -dimensional subspace $\Gamma(X) \subseteq V$, and every subspace whose intersection with Q is trivial is the graph of a unique such map. Let $U_Q \subseteq G_k(V)$ denote the set of such subspaces.

By choosing bases for P and Q , we obtain a bijection from U_Q to the vector space $M((n-k) \times k, \mathbb{C})$ of complex $(n-k) \times k$ matrices, whose matrix entries we can use as coordinates on U_Q . The argument in the real Grassmannian case (adapted in an obvious way to the complex case) shows that when two such charts overlap, the matrix entries in the new chart are rational functions of the original ones, so any two such charts overlap holomorphically. Therefore, $G_k(V)$ is a complex manifold of dimension $(n-k)k$. In fact, it is compact.

Here are two useful theorems that will give more examples of complex manifolds. The proofs are not very interesting, so we omit them:

Definition 2.13. Suppose Γ is a **discrete Lie group** (i.e., a countable group with the discrete topology). An action of Γ on a manifold M is **free** if $g \cdot x = x$ for some $g \in \Gamma$ and $x \in M$ implies g is the identity; and it is **proper** if the map $\Gamma \times M \rightarrow M \times M$ given by $(g, x) \mapsto (g \cdot x, x)$ is a proper map, meaning that the preimage of every compact set is compact. If M is a complex manifold, the action is **holomorphic** if the map $x \mapsto g \cdot x$ is holomorphic for each $g \in \Gamma$.

Theorem 2.14 (Holomorphic Quotient Manifold Theorem). *Suppose Γ is a discrete Lie group acting holomorphically, freely, and properly on a complex manifold M . Then the quotient space M/Γ has a unique complex manifold structure such that the quotient map $q : M \rightarrow M/\Gamma$ is a holomorphic normal covering map.*

Example 2.15 (Complex Torus). Let $\{e_1, \dots, e_{2n}\}$ be any fixed $\mathbb{R}$ -basis of $\mathbb{C}^n$, and let

$$\Lambda := \{m_1 e_1 + \dots + m_{2n} e_{2n} : m_i \in \mathbb{Z}\}$$

be a lattice of rank $2n$. Then we can define the quotient space $Q = \mathbb{C}^n/\Lambda$; it is a compact Hausdorff space equipped with quotient topology. There is a natural complex manifold structure on $Q = \mathbb{C}^n/\Lambda$ by Theorem 2.14. We call this complex manifold Q a **complex torus**.

Example 2.16 (Hopf Manifolds). Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an ordered n -tuple of real numbers with $0 < \lambda_j < 1$, and define an action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ by

$$k \cdot z = (\lambda_1^k z_1, \dots, \lambda_n^k z_n).$$

This action is holomorphic, free, and proper, so the quotient $H_\lambda = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is an n -dimensional complex manifold called a **Hopf manifold**. Regarding S^{2n-1} as the set of unit vectors in $\mathbb{C}^n$, we define a smooth map

$$A : S^{2n-1} \times \mathbb{R} \rightarrow \mathbb{C}^n \setminus \{0\}$$

by

$$A(z, t) = (\lambda_1^t z_1, \dots, \lambda_n^t z_n).$$

It is easy to show that A is surjective. If $\pi : \mathbb{C}^n \setminus \{0\} \rightarrow H_\lambda$ is the quotient map, then it follows that $\pi \circ A : S^{2n-1} \times \mathbb{R} \rightarrow H_\lambda$ is surjective, and moreover one can check that $\pi \circ A$ makes the same identifications as the quotient map from $S^{2n-1} \times \mathbb{R}$ to $S^{2n-1} \times (\mathbb{R}/\mathbb{Z}) \cong S^{2n-1} \times S^1$, so all Hopf manifolds are diffeomorphic to $S^{2n-1} \times S^1$.

The next theorem shows that covering spaces of complex manifolds are complex manifolds.

Theorem 2.17. *Suppose M is a connected complex manifold and $\pi : E \rightarrow M$ is a (topological) covering map. Then E is a topological manifold and has a unique holomorphic structure such that π is a holomorphic covering map.*

Our last (huge) class of examples will be complex submanifolds.

Definition 2.18. If M is a complex manifold, an (embedded) **complex submanifold** of M is a subset $S \subset M$ that is a topological manifold in the subspace topology, and is endowed with a holomorphic structure such that the inclusion $i : S \rightarrow M$ is a holomorphic embedding. A **complex hypersurface** is a complex submanifold of complex codimension 1 (so real codimension 2).

Here is one way to verify a subset of a complex manifold is a complex submanifold. Once again, we omit the proof.

Definition 2.19 (*k*-slice). Suppose (U, φ) is a holomorphic coordinate chart on M and $K \subset \mathbb{C}^n$ is a complex linear subspace of dimension k . Then $\varphi^{-1}(K)$ is called a holomorphic *k*-**slice** of U .

Theorem 2.20 (Slice Criterion). *Let M be a complex manifold and $S \subset M$ a subset with subset topology. Then S has a unique complex manifold structure making it into an embedded complex submanifold of M iff for some fixed k , each $p \in S$ is contained in the domain of a holomorphic coordinate chart (U, φ) in which $S \cap U$ is a holomorphic k -slice of U .*

Example 2.21 (Open Submanifolds). If M is a complex manifold, every open subset of M is an embedded complex submanifold of codimension 0.

Example 2.22 (Images of Holomorphic Embeddings). If $F : N \rightarrow M$ is a holomorphic embedding, then the image $F(N)$ is an embedded complex submanifold, directly from the definition.

Example 2.23 (Graphs of Holomorphic Maps). Suppose $f : M \rightarrow N$ is a holomorphic map. The **graph** of f is the set $\Gamma(f) = \{(p, f(p)) \in M \times N : p \in M\}$. This is a complex submanifold of $M \times N$, biholomorphic to M , because the map from M to $M \times N$ given by $p \mapsto (p, f(p))$ is a holomorphic embedding.

Example 2.24 (Regular Level Sets). Suppose M and N are complex manifolds, $\Phi : M \rightarrow N$ is a holomorphic map, and $c \in N$ is a **regular value** of Φ , meaning that $\phi_*(p)$ is surjective for each $p \in \Phi^{-1}(c)$. Then the level set $\Phi^{-1}(c)$ is a complex submanifold of M whose codimension is equal to the dimension of N . The proof is a simple application of the holomorphic rank theorem and the holomorphic slice criterion.

Example 2.25 (Transverse Preimages). Suppose M and N are complex manifolds and $S \subseteq N$ is a complex submanifold. A holomorphic map $F : M \rightarrow N$ is said to be **transverse to S** if for every $x \in F^{-1}(S)$, the subspaces $F_*(x)(T_x M)$ and $T_{F(x)} S$ together span $T_{F(x)} N$. If this is the case, then $F^{-1}(S)$ is a complex submanifold whose codimension in M is equal to the codimension of S in N .

Example 2.26 (Complex Submanifolds of $\mathbb{C}^n$). It is important to observe, to begin with, that there are no compact complex submanifolds of $\mathbb{C}^n$ except 0-dimensional ones. To verify this, note that if $M \subseteq \mathbb{C}^n$ is a compact complex submanifold, then each of the coordinate functions z^j restricts to a global holomorphic function on M , which must therefore be constant on each connected component of M by Liouville's theorem (Theorem 2.5).

Example 2.27 (Affine Algebraic Variety). A particularly important class of examples is defined as follows: a subset $V \subseteq \mathbb{C}^n$ is called an **affine algebraic variety** if it is the common zero set of a finite collection $p_1, \dots, p_k$ of holomorphic polynomials. (Some authors require that a variety be irreducible, meaning that it is not the union of two or more nonempty varieties, but we do not require this.) A point $z_0 \in V$ is a **regular point** of V if it has a neighborhood $U \subseteq \mathbb{C}^n$ such that $V \cap U$ is the common zero set of some of the polynomials $p_{i_1}, \dots, p_{i_r}$ with the property that the differentials $dp_{i_1}, \dots, dp_{i_r}$ are linearly independent at every point of $V \cap U$; points that are not regular are called **singular points** of V . An affine algebraic variety is said to be **nonsingular** or **smooth** if it has no singular points; in this case V is a complex submanifold of $\mathbb{C}^n$.

Example 2.28 (Complex Lie Subgroups). Suppose G is a complex Lie group. An (embedded) complex submanifold of G that is also a subgroup is called a **complex Lie subgroup**; it inherits a complex Lie group structure of its own because the group operations are holomorphic by restriction. Some specific examples:

- The group $SL(n, \mathbb{C})$ of complex matrices with determinant 1 is a complex Lie subgroup of $GL(n, \mathbb{C})$ because it is a level set of the holomorphic submersion $\det : GL(n, \mathbb{C}) \rightarrow \mathbb{C}$.
- $O(n, \mathbb{C})$ denotes the group of $n \times n$ complex matrices A satisfying $A^T A = \text{Id}$. It is a regular level set of the holomorphic polynomial function $A \mapsto A^T A$, so it is a complex Lie subgroup of $GL(n, \mathbb{C})$.
- $SO(n, \mathbb{C})$ is the subgroup of $O(n, \mathbb{C})$ consisting of matrices of determinant 1; because it is a regular level set of $\det : O(n, \mathbb{C}) \rightarrow \mathbb{C}$, it is a complex Lie subgroup.

It should be noted that the unitary group $U(n) \subseteq GL(n, \mathbb{C})$ is *not* a complex Lie group, because it does not have a holomorphic defining function, even locally.

Example 2.29 (Projective Manifold and Projective Algebraic Variety). A complex manifold M is called a **projective manifold** if it is biholomorphic to a closed complex submanifold of $\mathbb{P}^n$.

Suppose $p : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic polynomial function (that is, a polynomial function of the variables $w^0, \dots, w^n$). It is said to be **homogeneous** of degree d for a nonnegative integer d if $p(\lambda w) = \lambda^d p(w)$ for all $\lambda \in \mathbb{C}$ and $w \in \mathbb{C}^{n+1}$. This just means that p can be written as a sum of monomial terms each of which has the same total degree d . Although such a polynomial does not produce a well-defined function from $\mathbb{P}^n$ to $\mathbb{C}$, nonetheless its zero set is a well-defined subset of $\mathbb{P}^n$, because if $p(w) = 0$, then $p(\lambda w) = 0$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

A **projective algebraic variety** is a subset $V \subseteq \mathbb{P}^n$ of the form

$$V = \{[w] \in \mathbb{P}^n : p_1(w) = \dots = p_k(w) = 0\},$$

where $p_1, \dots, p_k : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ are homogeneous holomorphic polynomials of various degrees. A projective variety is said to be **nonsingular** or **smooth** if the affine variety $\tilde{V} \subseteq \mathbb{C}^{n+1}$ defined by $p_1, \dots, p_k$ has no singular points other than perhaps the origin. It follows from the next lemma that every nonsingular projective algebraic variety is actually a complex submanifold of $\mathbb{P}^n$. Projective algebraic varieties are the main objects of study in complex algebraic geometry.

A powerful theorem by Wei-Liang Chow shows that studying closed complex submanifold of $\mathbb{P}^n$ is the same as doing algebraic geometry:

Theorem 2.30 (Chow). *Every closed complex submanifold of $\mathbb{P}^n$ is algebraic.*

This is a difficult theorem, so we will omit the proof. However, we will provide a proof for the case of hypersurfaces using Hodge theory (Theorem 6.55).

2.2 The Complexified Tangent and Cotangent Bundles

Let M be a complex manifold of dimension n . Then it is also a real smooth manifold of dimension $2n$. We first introduce a definition:

Definition 2.31 (Complexification). Let $E \rightarrow M$ be a real vector bundle of rank r . Let $\underline{\mathbb{C}} = M \times \mathbb{C}$ be the trivial complex line bundle over M , viewed as a real rank 2 vector bundle. The **complexification** $E_{\mathbb{C}}$ of E , is the rank $2k$ real vector bundle

$$E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \underline{\mathbb{C}}.$$

With the obvious projection $E_{\mathbb{C}} \rightarrow M$, this makes $E_{\mathbb{C}}$ into a rank r complex vector bundle over M .

Remark 2.32. Fiberwise at $p \in M$, this is nothing but complexification of the real vector space E_p , which is simply $E_p \otimes_{\mathbb{R}} \mathbb{C}$.

Let TM be the tangent bundle of M , and T^*M the cotangent bundle of M , where M is viewed as a real $2n$ dimensional manifold. We denote $T_{\mathbb{C}}M$ and $T_{\mathbb{C}}^*M$ be their respective complexification. A section of $T_{\mathbb{C}}M$, called a **complex vector field**, will be of the form $\chi = X + iY$, where $X, Y \in \Gamma(TM)$ are real vector fields. Similarly, we have complex covector fields of the form $\omega = \alpha + i\beta$, where $\alpha, \beta \in \Gamma(T^*M)$.

In our case, M is a complex manifold of dimension n . So locally at some open set $U \subset M$, we can find holomorphic coordinates $z^\mu = x^\mu + iy^\mu$ for $\mu = 1, \dots, n$. Then x^μ and y^μ are local C^∞ -coordinates on M , then a real local frame of the bundle TM is given by $\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\mu}$. After complexifying, $T_{\mathbb{C}}M$ is now a real vector bundle of rank $4n$, with real basis $\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\mu}, i\frac{\partial}{\partial x^\mu}, i\frac{\partial}{\partial y^\mu}$, where $\mu = 1, \dots, n$. Similar description holds for $T_{\mathbb{C}}^*M$, where a real basis would now be $dx^\mu, dy^\mu, idx^\mu, idy^\mu$ for $\mu = 1, \dots, n$.

But we also know that $T_{\mathbb{C}}M$ is a complex vector bundle of rank $2n$ over M . So we should be able to find $2n$ linearly independent basis element of $\Gamma(U, T_{\mathbb{C}}M)$ over $\mathbb{C}$. Indeed, one can verify that one such basis over $\mathbb{C}$ is given by

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2} \left(\frac{\partial}{\partial x^\mu} - i \frac{\partial}{\partial y^\mu} \right) \quad \frac{\partial}{\partial \bar{z}^\mu} = \frac{1}{2} \left(\frac{\partial}{\partial x^\mu} + i \frac{\partial}{\partial y^\mu} \right). \quad (2.1)$$

Similarly, one such basis of $\Gamma(U, T_{\mathbb{C}}^*M)$ over $\mathbb{C}$ is given by

$$dz^\mu = dx^\mu + idy^\mu \quad d\bar{z}^\mu = dx^\mu - idy^\mu. \quad (2.2)$$

A simple computation shows that

Proposition 2.33. The local complex vector and covector fields defined by (2.1) and (2.2) satisfies

$$\begin{aligned} dz^\mu \left(\frac{\partial}{\partial z^\nu} \right) &= \delta_\nu^\mu, & dz^\mu \left(\frac{\partial}{\partial \bar{z}^\nu} \right) &= 0, \\ d\bar{z}^\mu \left(\frac{\partial}{\partial \bar{z}^\nu} \right) &= \delta_\nu^\mu, & d\bar{z}^\mu \left(\frac{\partial}{\partial z^\nu} \right) &= 0 \end{aligned}$$

for all $\mu, \nu = 1, \dots, n$. Therefore, $\{dz^\mu, d\bar{z}^\mu\}$ are local frame of $\Gamma(T_{\mathbb{C}}^*M)$ over $\mathbb{C}$ dual to $\left\{ \frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\mu} \right\}$, which are local frame of $\Gamma(U, T_{\mathbb{C}}M)$ over $\mathbb{C}$.

Theorem 2.34 (Cauchy-Riemann). Let $U \subset \mathbb{C}^n$ be an open set, and let $z^\mu = x^\mu + iy^\mu$ for $\mu = 1, \dots, n$ be the standard coordinate on $\mathbb{C}^n$. Let $f : U \rightarrow \mathbb{C}$ be any smooth function, and let $\left\{ \frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\mu} \right\}$ be the complex vector field defined by (2.1). Then f is holomorphic iff

$$\frac{\partial f}{\partial \bar{z}^\mu} = 0$$

for all $\mu = 1, \dots, n$.

Proof. Write $f = u + iv$ as the real and imaginary part, substitute into the above equation using definition (2.1), separating the real and imaginary parts, we recover the Cauchy-Riemann equation in each component. Then the result follows from Theorem 1.3. $\square$

In fact, it is now an easy corollary that

Corollary 2.35. *Let M be a complex manifold and $f : M \rightarrow \mathbb{C}$ is a smooth function. If $z^1, \dots, z^n$ are local holomorphic coordinates on U and $\{\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\mu}\}$ are defined by (2.1), then f is holomorphic on U iff*

$$\frac{\partial f}{\partial \bar{z}^\mu} = 0$$

for all $\mu = 1, \dots, n$.

Proof. This follows from the previous result, and Definition 2.3. Note that in the statement above, by $\frac{\partial f}{\partial \bar{z}^\mu} = 0$ we really mean $\frac{\partial(f \circ \varphi^{-1})}{\partial \bar{z}^\mu} = 0$ where $\varphi : U \rightarrow \mathbb{C}^n$ is the holomorphic chart. $\square$

Now if $F : M \rightarrow N$ is a smooth map between complex manifolds, we can define its pushforward or differential $F_* : TM \rightarrow TN$ as in the theory of smooth manifolds [7]. We can also define its complexification, and we will abuse the notation a bit and still denote it by

$$F_* : T_{\mathbb{C}}M \rightarrow T_{\mathbb{C}}N.$$

Proposition 2.36. *Let M, N be complex manifolds and $F : M \rightarrow N$ be a smooth map. Let $z^\alpha = x^\alpha + iy^\alpha$ be holomorphic coordinates on M and $w^\beta = u^\beta + iv^\beta$ be holomorphic coordinates on N . Then in terms of complex local frames $\{\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\alpha}\}$ for M and complex local frame $\{\frac{\partial}{\partial w^\beta}, \frac{\partial}{\partial \bar{w}^\beta}\}$ for N , the complexified differential has the form*

$$F_*\left(\frac{\partial}{\partial z^\alpha}\right) = \frac{\partial F^\beta}{\partial z^\alpha} \frac{\partial}{\partial w^\beta} + \frac{\partial \bar{F}^\beta}{\partial z^\alpha} \frac{\partial}{\partial \bar{w}^\beta} \quad (2.3)$$

$$F_*\left(\frac{\partial}{\partial \bar{z}^\alpha}\right) = \frac{\partial F^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial w^\beta} + \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial \bar{w}^\beta}. \quad (2.4)$$

Proof. Write F as real and imaginary parts: $F = U + iV$. Considering M and N as smooth manifolds, the usual coordinate formula for differential gives

$$F_*\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial U^\beta}{\partial x^\alpha} \frac{\partial}{\partial u^\beta} + \frac{\partial V^\beta}{\partial x^\alpha} \frac{\partial}{\partial v^\beta} \quad F_*\left(\frac{\partial}{\partial y^\alpha}\right) = \frac{\partial U^\beta}{\partial y^\alpha} \frac{\partial}{\partial u^\beta} + \frac{\partial V^\beta}{\partial y^\alpha} \frac{\partial}{\partial v^\beta}.$$

Now we use the definition (2.1), the above equations, and complex linearity of F_* to obtain

$$F_*\left(\frac{\partial}{\partial z^\alpha}\right) = \frac{\partial U^\beta}{\partial z^\alpha} \frac{\partial}{\partial u^\beta} + \frac{\partial V^\beta}{\partial z^\alpha} \frac{\partial}{\partial v^\beta} \quad F_*\left(\frac{\partial}{\partial \bar{z}^\alpha}\right) = \frac{\partial U^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial u^\beta} + \frac{\partial V^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial v^\beta}.$$

Now we substitute $\partial/\partial u^\beta = \partial/\partial w^\beta + \partial/\partial \bar{w}^\beta$ and $\partial/\partial v^\beta = i(\partial/\partial w^\beta - \partial/\partial \bar{w}^\beta)$ and collect terms, we get

$$F_*\left(\frac{\partial}{\partial z^\alpha}\right) = \left(\frac{\partial U^\beta}{\partial z^\alpha} + i \frac{\partial V^\beta}{\partial z^\alpha}\right) \frac{\partial}{\partial w^\beta} + \left(\frac{\partial U^\beta}{\partial z^\alpha} - i \frac{\partial V^\beta}{\partial z^\alpha}\right) \frac{\partial}{\partial \bar{w}^\beta},$$

which proves (2.3). The proof of (2.4) is similar. $\square$

Corollary 2.37. *Assuming the assumptions in Proposition 2.36, assume in addition that F is holomorphic. Let $J_F = (\partial F^\beta / \partial z^\alpha)$ denote the matrix of **holomorphic Jacobian** of F . Then in terms of complex local frames $\{\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\alpha}\}$ for M and complex local frame $\{\frac{\partial}{\partial w^\beta}, \frac{\partial}{\partial \bar{w}^\beta}\}$ for N , the complexified differential F_* has the form*

$$F_* = \begin{pmatrix} J_F & 0 \\ 0 & \overline{J_F} \end{pmatrix}.$$

Thus F_* is invertible iff the holomorphic Jacobian J_F is invertible in this case. In particular, every local biholomorphism is orientation preserving.

Proof. The expression of F_* follows from Corollary 2.35, and the expressions (2.3) and (2.4). The last two statements immediately follows because $\det F_* = |J_F|^2$, and $\det F_*$ is positive if F_* is a biholomorphism, i.e., if $|J_F| \neq 0$. $\square$

Theorem 2.38 (holomorphic inverse function theorem). *Suppose M, N are complex n -manifolds, $F : M \rightarrow N$ is a holomorphic map, and the holomorphic Jacobian $J_F(p)$ is nonsingular for some $p \in M$. Then there exists connected neighborhoods U of p and V of $F(p)$ such that $F|_U : U \rightarrow V$ is a biholomorphism.*

Proof. Since $J_F(p)$ is nonsingular, we know F_* is nonsingular by Corollary 2.37. So by the ordinary inverse function theorem we can find U, V such that $F|_U : U \rightarrow V$ is a diffeomorphism. We already know that F is holomorphic, so we just need to show that $G = F^{-1}$ is also holomorphic.

By choosing local holomorphic coordinates on U, V (shrinking them if possible) and replacing F by its coordinate representation, we can reduce the problem to the case where F is a holomorphic diffeomorphism between open subsets of $\mathbb{C}^n$. Use coordinates $z^1, \dots, z^n$ for both the domain and codomain, and consider the μ -th coordinate function G^μ . It satisfies $G^\mu \circ F = z^\mu$. Using (2.3) and (2.4) we can verify that the chain rule holds:

$$0 = \frac{\partial z^\mu}{\partial \bar{z}^\alpha} = \frac{\partial (G^\mu \circ F)}{\partial \bar{z}^\alpha} = \frac{\partial G^\mu}{\partial z^\beta} \frac{\partial F^\beta}{\partial \bar{z}^\alpha} + \frac{\partial G^\mu}{\partial \bar{z}^\beta} \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha}.$$

Now the first term on the right hand side vanish by Corollary 2.35 because F is holomorphic. Now $(\partial \bar{F}^\beta / \partial \bar{z}^\alpha) = \overline{J_F}$ which is invertible. Hence this suggests that $\partial G^\mu / \partial \bar{z}^\beta = 0$, which proves that G is holomorphic. $\square$

We conclude this section by claiming that every complex manifold is orientable. The proof, however, uses further techniques, so the reader may skip the proof until they study the corresponding chapters.

Theorem 2.39. *Every complex manifold is orientable (as a real manifold).*

Proof. This follows from the fact that every complex manifold admits a Hermitian metric h (Proposition 5.8), and the fundamental 2-form $\omega = -\text{Im } h$ of this Hermitian metric gives a top degree form

$$d \text{ vol} = \frac{\omega^n}{n!}.$$

This form is nowhere vanishing because

$$\omega^n = \left(\frac{i}{2}\right)^n \det(h_{\alpha\beta}) dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n$$

by (5.2). Because h is positive definite we have $\det(h_{\alpha\beta}) > 0$. Therefore, ω^n is nowhere vanishing. Moreover, Theorem 5.11 shows that $\omega^n/n!$ is indeed the volume form on M . $\square$

2.3 Almost Complex Structure

Definition 2.40. Let V be a real vector space of dimension $2n$. An *almost complex structure* on V is a real linear endomorphism $J : V \rightarrow V$ such that $J^2 = -1$. A pair (V, J) is called a *almost complex vector space*.

Proposition 2.41. An almost complex structure J on V makes V into a complex vector space.

Proof. One can check that the multiplication defined by $(a + bi)v := av + bJv$ makes V into a complex vector space. $\square$

Now let us look at eigenvalues of J . Since $J^2 = -1$, the eigenvalue λ of J must satisfy $\lambda^2 = -1$. Therefore J has no real eigenvalue and the complex eigenvalues are $\lambda = \pm i$. Therefore, we must complexify $J : V \rightarrow V$. By a slight abuse of notation, we still use J to denote the complexification

$$J : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}.$$

Theorem 2.42. Let (V, J) be an almost complex vector space. Then the complexification $V_{\mathbb{C}}$ decomposes

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

into the $(+i)$ -eigenspaces $V^{1,0}$ and $(-i)$ -eigenspace $V^{0,1}$ of $J : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$. If V is finite dimensional, then $V^{1,0}$ and $V^{0,1}$ have the same complex dimension.

Proof. Indeed, for any $v \in V_{\mathbb{C}}$, one can check that the eigenspace decomposition is given by $v = v^{1,0} + v^{0,1}$, where

$$v^{1,0} = \frac{1}{2}(v - iJv) \quad v^{0,1} = \frac{1}{2}(v + iJv). \quad (2.5)$$

By the expression above, one can see that conjugation on $V_{\mathbb{C}}$ exchanges $V^{1,0}$ and $V^{0,1}$, hence $V^{1,0} \cong V^{0,1}$ as real vector spaces, therefore they must also have the same complex dimension. $\square$

Corollary 2.43. If V admits an almost complex structure, then it must be even dimensional.

Definition 2.44. A smooth manifold M is said to be an *almost complex manifold* if there is a bundle morphism $J : TM \rightarrow TM$ such that $J^2 = -1$.

Example 2.45 (Almost Complex Structure on $\mathbb{C}^n$). On $\mathbb{C}^n$, there is a natural almost complex structure J , which is multiplication by the complex number i . Let us consider vectors $X_j = (0, \dots, 1, \dots, 0)$ with only a 1 in the j -th place, and $Y_j = (0, \dots, i, \dots, 0)$ with only an i in the j -th place. Then we see that

$$JX_j = Y_j \quad JY_j = -X_j.$$

Now consider $\mathbb{C}^n$ as a smooth manifold. The standard identification of vector field identifies X_j with $\partial/\partial x^j$, and Y_j with $\partial/\partial y^j$, where $z^j = x^j + iy^j$ are the coordinates on $\mathbb{C}^n$. Then we see that $T\mathbb{C}^n$ has a canonical almost complex structure

$$J_{\mathbb{C}^n} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j} \quad J_{\mathbb{C}^n} \frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}.$$

The complexified tangent bundle $T_{\mathbb{C}}\mathbb{C}^n$ splits, according to Theorem 2.42, into

$$T_{\mathbb{C}}\mathbb{C}^n = T^{1,0}\mathbb{C}^n \oplus T^{0,1}\mathbb{C}^n.$$

By Theorem 2.42, each summand has the same complex dimension n . Using the expressions (2.42), one can check that $T^{1,0}\mathbb{C}^n$ is spanned the vector fields $\{\frac{\partial}{\partial z^j}\}$, and $T^{0,1}\mathbb{C}^n$ is spanned the vector fields $\{\frac{\partial}{\partial \bar{z}^j}\}$: Indeed, one can check that these are eigenvectors of $J_{\mathbb{C}^n}$; They are certainly spanning because from linear combinations of these vectors recovers $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$. Moreover, linear independence of $\{\frac{\partial}{\partial z^j}\}$ follows from linear independence of $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$ over $\mathbb{C}$.

Lemma 2.46. *Let $U \subset \mathbb{C}^n$ be open. A smooth function $F : U \rightarrow \mathbb{C}^m$ is holomorphic iff at every point, the complexified differential F_* satisfies*

$$F_* \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ F_*. \quad (2.6)$$

Proof. Suppose (2.6) holds at every point. Then using (2.4), we have that

$$\begin{aligned} 0 &= F_* \left(J_{\mathbb{C}^n} \frac{\partial}{\partial \bar{z}^\alpha} \right) - J_{\mathbb{C}^m} \left(F_* \frac{\partial}{\partial \bar{z}^\alpha} \right) \\ &= F_* \left(-i \frac{\partial}{\partial \bar{z}^\alpha} \right) - J_{\mathbb{C}^m} \left(F_* \frac{\partial}{\partial \bar{z}^\alpha} \right) \\ &= -i \frac{\partial F^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial w^\beta} - i \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial \bar{w}^k} - J_{\mathbb{C}^m} \frac{\partial F^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial w^\beta} - J_{\mathbb{C}^m} \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial \bar{w}^\beta} \\ &= -2i \frac{\partial F^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial w^\beta}. \end{aligned}$$

This shows that $\partial F^\beta / \partial \bar{z}^\alpha = 0$, hence F is holomorphic. Conversely, suppose F is holomorphic. Then the same computation as above shows that when both sides of (2.6) are applied to the vector fields $\{\frac{\partial}{\partial \bar{z}^\alpha}\}$, one has that the difference is precisely

$$-i \frac{\partial F^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial w^\beta} - i \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial \bar{w}^k} - J_{\mathbb{C}^m} \frac{\partial F^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial w^\beta} - J_{\mathbb{C}^m} \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial \bar{w}^\beta} = -i \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial \bar{w}^k} - J_{\mathbb{C}^m} \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} \frac{\partial}{\partial \bar{w}^\beta}$$

which is zero because $J_{\mathbb{C}^m} \frac{\partial}{\partial \bar{w}^\beta} = -i \frac{\partial}{\partial \bar{w}^\beta}$. Similarly, one can show that both sides of (2.6) when applied to the vector fields $\{\frac{\partial}{\partial z^\alpha}\}$ gives the same result, proving (2.6). $\square$

It is then not too hard to generalize this construction of almost complex structure to complex manifolds:

Theorem 2.47. *For every complex manifold M , there is a canonical almost complex structure, denoted by $J_M : TM \rightarrow TM$. If N is another complex manifold and $F : M \rightarrow N$ is a smooth map, then F is holomorphic iff at every point we have*

$$F_* \circ J_M = J_N \circ F_*. \quad (2.7)$$

Proof. Let $\dim M = n$ and (U, φ) be a holomorphic chart on M . Inside U , we define $J_M : TM|_U \rightarrow TM|_U$ by

$$J_M = \varphi_*^{-1} \circ J_{\mathbb{C}^n} \circ \varphi_*.$$

Now we need to check that this makes sense globally. Suppose we have intersecting charts $(U, \varphi), (V, \psi)$ on M with $U \cap V \neq \emptyset$. Then the transition function $\psi \circ \varphi^{-1}$ is holomorphic, so its differential commutes with $J_{\mathbb{C}^n}$ by Lemma 2.46. Therefore,

$$\begin{aligned}\psi_*^{-1} \circ J_{\mathbb{C}^n} \circ \psi_* &= \psi_*^{-1} \circ J_{\mathbb{C}^n} \circ (\psi_* \circ \varphi_*^{-1}) \circ \varphi_* \\ &= \psi_*^{-1} \circ (\psi_* \circ \varphi_*^{-1}) \circ J_{\mathbb{C}^n} \circ \varphi_* \\ &= \varphi_*^{-1} \circ J_{\mathbb{C}^n} \circ \varphi_*.\end{aligned}$$

Hence J_M is well-defined. Clearly $J_M^2 = -1$ because $J_{\mathbb{C}^n}^2 = -1$. Next, let N be a complex m -manifold and $F : M \rightarrow N$ be a smooth map. Since (2.7) is a local statement, it suffices to choose holomorphic local charts (U, φ) on M and (V, ψ) on N such that $F(U) \subset V$, and prove that (2.7) is satisfied there. By definition, F is holomorphic iff $G = \psi \circ F \circ \varphi^{-1}$ is holomorphic, which is true iff $G_* \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ G_*$ by Lemma 2.46. Hence we have

$$\begin{aligned}G_* \circ J_{\mathbb{C}^n} - J_{\mathbb{C}^m} \circ G_* &= \psi_* \circ F_* \circ \varphi_*^{-1} \circ J_{\mathbb{C}^n} - J_{\mathbb{C}^m} \circ \psi_* \circ F_* \circ \varphi_*^{-1} \\ &= \psi_* \circ F_* \circ J_M \circ \varphi_*^{-1} - \psi_* \circ J_N \circ F_* \circ \varphi_*^{-1} \\ &= \psi_* \circ (F_* \circ J_M - J_N \circ F_*) \circ \varphi_*^{-1}.\end{aligned}$$

Since ψ_* and φ_*^{-1} are bundle isomorphisms, the last expression is zero iff (2.7) holds. $\square$

By Theorem 2.47, for any complex manifold M , there is a canonical almost complex structure $J : TM \rightarrow TM$ induced from the holomorphic structure of M . After complexification we have an almost complex structure

$$J : T_{\mathbb{C}}M \rightarrow T_{\mathbb{C}}M.$$

By Theorem 2.42, the complexified tangent bundle decomposes as

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

The subbundle $T^{1,0}M$ is called the **holomorphic tangent bundle**, and $T^{0,1}M$ the **anti-holomorphic tangent bundle**. In terms of local holomorphic coordinates $z^\alpha = x^\alpha + iy^\alpha$ on M , a local frame of $T^{1,0}M$ is $\{\frac{\partial}{\partial z^\alpha}\}$, and a local frame of $T^{0,1}M$ is $\{\frac{\partial}{\partial \bar{z}^\alpha}\}$. By Proposition 2.36, if $F : M \rightarrow N$ is holomorphic, then $F_*(T^{1,0}M) \subset T^{1,0}N$.

Proposition 2.48. Let M be a manifold with an almost complex structure J . Then as real vector bundles, we have $TM \cong T^{1,0}M$.

Proof. Following (2.5), one can construct the isomorphisms easily. On one hand, we have a map $TM \rightarrow T^{1,0}M$ given by $\chi \mapsto \frac{1}{2}(\chi - iJ\chi)$. On the other hand, the map $T^{1,0}M \rightarrow TM$ is given by $\Psi \mapsto 2\operatorname{Re}(\Psi)$. $\square$

Remark 2.49. Local smooth sections of $T^{1,0}M$ is of the form $\chi = f^\alpha \frac{\partial}{\partial z^\alpha}$, where the coefficients f^α a priori are only required to be smooth: they need not be holomorphic. However, sometimes complex algebraic geometers only want to consider holomorphic sections of $T^{1,0}M$, i.e., sections of the same local form but f^α 's are required to be holomorphic.

In such cases, complex algebraic geometers identify holomorphic vector bundles over a complex manifold M with locally free sheaves of $\mathcal{O}_M$ -modules. They define the sheaf of holomorphic sections

$$T_M : U \mapsto \left\{ f^\alpha \frac{\partial}{\partial z^\alpha} \in \Gamma(U, T^{1,0}M) : f^\alpha \in \Gamma(U, \mathcal{O}_M) \right\}.$$

This construction T_M will be ubiquitous throughout the rest of the book.

Finally, applying Proposition 2.41 to TM , we define yet another tangent bundle:

Definition 2.50. We define $T_J M$ to be a vector bundle over M which has the same total space as TM , but endowed with the complex vector space structure on each fiber according to 2.41.

One of the most confusing things in complex algebraic geometry is that there are so many tangent bundles. They are summarized in Table 2.1.

Notation	Definition	$\mathbb{R}$ -rk	$\mathbb{C}$ -rk	Local frame
TM	Ordinary tangent bundle	$2n$	–	$\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}$ over real.
$T_{\mathbb{C}} M$	Complexified tangent bundle	$4n$	$2n$	$\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}, i \frac{\partial}{\partial x^\alpha}, i \frac{\partial}{\partial y^\alpha}$ over real.
$T^{1,0} M$	Holomorphic tangent bundle	$2n$	n	$\frac{\partial}{\partial z^\alpha}$ over complex.
$T^{0,1} M$	Antiholomorphic tangent bundle	$2n$	n	$\frac{\partial}{\partial \bar{z}^\alpha}$ over complex.
$T_J M$	Ordinary tangent bundle equipped with an almost complex structure J	$2n$	n	Same as that of TM . Over $\mathbb{C}$, a local frame may be taken to be $\frac{\partial}{\partial x^\alpha}$, since $i \frac{\partial}{\partial x^\alpha} := J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha}$.
T_M	Sheaf of holomorphic sections of $T^{1,0} M$	$2n$	n	Same as that of $T^{1,0} M$, but the coefficients are required to be holomorphic functions rather than smooth ones.

Table 2.1: Tangent bundles associated with a complex manifold M of dimension n .

Definition 2.51. An almost complex structure $J : TM \rightarrow TM$ is said to be *integrable* if $[T^{1,0} M, T^{1,0} M] \subset T^{1,0} M$.

Lemma 2.52. *Every almost complex structure on a two-dimensional real manifold is integrable.*

Proof. Since M has real dimension 2, it has complex dimension 1. Hence $T^{1,0} M$ will be a rank 1 complex vector bundle. Now let χ be a local frame of $T^{1,0} M$. Take two arbitrary local sections of $T^{1,0} M$, say $\chi_1 = f\chi$ and $\chi_2 = g\chi$, where $f, g \in C^\infty(M, \mathbb{C})$. Their Lie bracket is

$$[\chi_1, \chi_2] = [f\chi, g\chi] = f\chi(g)\chi - g\chi(f)\chi = (f\chi(g) - g\chi(f))\chi.$$

This is again a smooth complex function times χ , hence a section of $T^{1,0} M$. Therefore $[T^{1,0} M, T^{1,0} M] \subset T^{1,0} M$. $\square$

One of the most natural question to ask is that suppose we have a smooth manifold M with an almost complex structure $J : TM \rightarrow TM$, then is this almost complex structure induced by a complex structure on M (in the sense of Theorem 2.47)? August Newlander and Louis Nirenberg showed the following powerful result:

Theorem 2.53 (Newlander-Nirenberg). *An almost complex structure on a smooth manifold is induced from a complex structure iff it is integrable.*

Example 2.54 (Riemann Surfaces). Suppose M is an orientable real 2-manifold. We can always endow M with a Riemannian metric and an orientation. With that data, we can define an almost complex structure on M by letting J be “counterclockwise rotation by 90 degree.” More precisely, for each nonzero $v \in T_p M$, we let Jv be the unique vector w such that $\langle v, w \rangle = 0$, $|w| = |v|$, and (v, w) is an oriented basis for $T_p M$. If (b_1, b_2) is any smooth oriented orthonormal local frame, then we have $Jb_1 = b_2$ and $Jb_2 = -b_1$, which shows that J is smooth. This almost complex structure is integrable by the result of Lemma 2.52, so it arises from a holomorphic structure by the Newlander-Nirenberg theorem. Thus every orientable smooth real 2-manifold can be given a holomorphic structure. A real 2-manifold endowed with a particular holomorphic structure is called a **Riemann surface**. Be careful about the terminology: a Riemann surface is a complex curve (1-dimensional complex manifold), while a complex surface is a 2-dimensional complex manifold.

2.4 The Dolbeault Complex

Let (X, J) be an almost complex manifold. Then the decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ also induces a dual decomposition

$$T_{\mathbb{C}}^*M = T^{1,0*}M \oplus T^{0,1*}M.$$

Locally, the frames are given by $\{dz^\alpha\}$ and $\{d\bar{z}^\alpha\}$. This also induces a direct sum decomposition on the space of complex-valued differential k -forms $\mathcal{A}^k(M)$ on M :

$$\mathcal{A}^k(M) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M),$$

where $\mathcal{A}^{p,q}(M)$ is the space of (p, q) -forms on M whose local sections is given by

$$\omega = f_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}.$$

where $\alpha_1 < \dots < \alpha_p$ and $\beta_1 < \dots < \beta_q$ and $f_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q} \in C^\infty(M; \mathbb{C})$. More generally, we can have sheaves

$$\mathcal{A}^{p,q} : U \mapsto \mathcal{A}^{p,q}(U),$$

where $\mathcal{A}^{p,q}(U)$ is the space of smooth (p, q) -forms defined on U . We also define the sheaf of **holomorphic p -forms**

$$\Omega^p : U \mapsto \{\omega \in \Gamma(U, \mathcal{A}^{p,0}) : \omega \text{ holomorphic section of } \mathcal{A}^{p,0}\}.$$

That is, locally in holomorphic coordinates we have $\omega = f_{\alpha_1 \dots \alpha_p} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p}$, but now the coefficients $f_{\alpha_1 \dots \alpha_p}$ are required to be holomorphic (instead of just smooth).

We can extend the exterior differentiation operator by complex linearity and get an operator $d : \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$.

Proposition 2.55. Let M be a complex manifold. Then $d(\mathcal{A}^{p,q}) \subset \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}$.

Proof. Locally in holomorphic coordinate $z^\alpha = x^\alpha + iy^\alpha$ on M , one can check that for $f \in \mathcal{A}^{0,0}$, locally we have

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha + \frac{\partial f}{\partial y^\alpha} dy^\alpha = \frac{\partial f}{\partial z^\alpha} dz^\alpha + \frac{\partial f}{\partial \bar{z}^\alpha} d\bar{z}^\alpha \quad (2.8)$$

using the definition (2.1), (2.2). Hence $d(\mathcal{A}^{0,0}) \subset \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$. The rest follows from induction and the observation that if $\omega \in \mathcal{A}^{p,q}$ is locally given by $\omega = f_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}$, then

$$df = df_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q} \wedge dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}.$$

This concludes the proof. $\square$

The above Proposition motivates the following definition:

Definition 2.56. Let $\pi^{p,q} : \mathcal{A}^k \rightarrow \mathcal{A}^{p,q}$ for $p+q=k$. We define the *Dolbeault operator* $\bar{\partial}^{p,q} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}$ and its conjugate $\partial^{p,q} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q}$ for every pair of p, q , by

$$\bar{\partial}^{p,q} = \pi^{p,q+1} \circ d \quad \partial^{p,q} = \pi^{p+1,q} \circ d.$$

More generally, for every $\omega \in \mathcal{A}^k$, it admits a direct sum decomposition $\omega = \sum_{p+q=k} \omega^{p,q}$. We define

$$\bar{\partial}\omega = \sum_{p+q=k} \bar{\partial}^{p,q} \omega^{p,q} \quad \partial\omega = \sum_{p+q=k} \partial^{p,q} \omega^{p,q}.$$

Let us study the Dolbeault operator and its conjugate in coordinates. If $\omega \in \mathcal{A}^k$, locally it is given by the expression

$$\omega = \sum_{p+q=k} f_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}.$$

Then it is easy to see that the local expression for $\partial, \bar{\partial}$ are given by

$$\partial\omega = \sum_{p+q=k} \frac{\partial f_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q}}{\partial z^\rho} dz^\rho \wedge dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}, \quad (2.9)$$

$$\bar{\partial}\omega = \sum_{p+q=k} \frac{\partial f_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q}}{\partial \bar{z}^\rho} d\bar{z}^\rho \wedge dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}. \quad (2.10)$$

Corollary 2.35 immediately translate to the following statement:

Corollary 2.57. *On a complex manifold M , a smooth function $f \in C^\infty(M; \mathbb{C})$ is holomorphic iff $\bar{\partial}f = 0$.*

From the local expressions (2.9), (2.10), and the calculation in (2.8) it is easy to see that

Proposition 2.58. Let M be a complex manifold. For every complex differential form α on M , we have

$$d\alpha = \partial\alpha + \bar{\partial}\alpha \quad \overline{\partial\alpha} = \bar{\partial}(\bar{\alpha}).$$

Proposition 2.59. On a complex manifold M , if $\alpha \in \mathcal{A}^k(M)$ and $\beta \in \mathcal{A}^l(M)$, then

$$\begin{aligned} \partial(\alpha \wedge \beta) &= \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta, \\ \bar{\partial}(\alpha \wedge \beta) &= \bar{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \bar{\partial}\beta. \end{aligned}$$

Proof. Both sides of the equations above are clearly bilinear in α, β , so it suffices to assume that $\alpha \in \mathcal{A}^{p,q}$ and $\beta \in \mathcal{A}^{p',q'}$. This is a straightforward computation following the expressions (2.9) and (2.10). $\square$

Proposition 2.60. Let M be a complex manifold. For every complex differential form α on M , we have

$$\partial^2 \alpha = \bar{\partial}^2 \alpha = 0 \quad \partial \bar{\partial} \alpha = -\bar{\partial} \partial \alpha.$$

Proof. Since both equations are bilinear in α , it suffices to assume $\alpha \in \mathcal{A}^{p,q}$. We note that

$$0 = d^2 = (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial^2 + \bar{\partial}^2 + \partial \bar{\partial} + \bar{\partial} \partial.$$

Now for $\alpha \in \mathcal{A}^{p,q}$, we have $\partial^2 \alpha \in \mathcal{A}^{p+2,q}$, $\bar{\partial}^2 \alpha \in \mathcal{A}^{p,q+2}$, and $\partial \bar{\partial} \alpha + \bar{\partial} \partial \alpha \in \mathcal{A}^{p+1,q+1}$. Hence, for the equation above to be true, each of these pieces must be zero individually, which proves the two desired equations. $\square$

The importance of Dolbeault operators stem from the fact that they are preserved by holomorphic maps.

Proposition 2.61. Suppose M, N are complex manifolds and $F : M \rightarrow N$ is a holomorphic map. Then for all p, q we have $F^*(\mathcal{A}^{p,q}(N)) \subset \mathcal{A}^{p,q}(M)$. Moreover, for all $\alpha \in \mathcal{A}^{p,q}(N)$, we have

$$F^*(\partial \alpha) = \partial(F^* \alpha) \quad F^*(\bar{\partial} \alpha) = \bar{\partial}(F^* \alpha).$$

Proof. Choose local holomorphic coordinates z^α of M and w^β of N . Then since F is holomorphic, one can verify by direct computation that

$$F^* dw^\beta = \frac{\partial F^\beta}{\partial z^\alpha} dz^\alpha \quad F^* d\bar{w}^\beta = \frac{\partial \bar{F}^\beta}{\partial \bar{z}^\alpha} d\bar{z}^\alpha.$$

Then it follows that $F^*(\mathcal{A}^{p,q}(N)) \subset \mathcal{A}^{p,q}(M)$. From this it follows that $\pi^{p,q} \circ F^* = F^* \circ \pi^{p,q}$. Then the last two equality follows becomes F^* commutes with d and the projections $\pi^{p,q}$. $\square$

By Proposition 2.60, we know $\bar{\partial}^2 = 0$. Therefore, it follows that for every p , the following sequence

$$0 \rightarrow \mathcal{A}^{p,0}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2}(M) \xrightarrow{\bar{\partial}} \dots$$

is a chain complex. We define the **Dolbeault cohomology group** $H^{p,q}(M)$ of M to be

$$H^{p,q}(M) = \frac{\ker(\bar{\partial} : \mathcal{A}^{p,q}(M) \rightarrow \mathcal{A}^{p,q+1}(M))}{\text{im}(\bar{\partial} : \mathcal{A}^{p,q-1}(M) \rightarrow \mathcal{A}^{p,q}(M))}.$$

Clearly they are zero outside of $0 \leq p, q \leq n$ because otherwise $\mathcal{A}^{p,q}(M) = 0$.

Theorem 2.62 (Functoriality). *If $F : M \rightarrow N$ is a holomorphic map, then for each p, q , the pullback $F^* : \mathcal{A}^{p,q}(N) \rightarrow \mathcal{A}^{p,q}(M)$ descends to a linear map $F^* : H^{p,q}(N) \rightarrow H^{p,q}(M)$. It satisfies*

$$(\text{id}_M)^* = \text{id}_{H^{p,q}(M)} \quad (G \circ F)^* = F^* \circ G^*$$

if $F : M \rightarrow N$ and $G : N \rightarrow Q$ are holomorphic. Moreover, the Dolbeault cohomology groups are biholomorphism invariants: If $F : M \rightarrow N$ is a biholomorphism, then for all p, q we have isomorphisms $F^ : H^{p,q}(N) \cong H^{p,q}(M)$.*

Proof. F^* descends to cohomology because we already showed $\bar{\partial} \circ F^* = F^* \circ \bar{\partial}$. Then the two equality holds on cohomology groups because they were already true on the cochain level. The last statement follows from the observation that if $F : M \rightarrow N$ is a biholomorphism, then $(F^{-1})^*$ is an inverse to F^* . $\square$

The cohomology groups $H^{p,q}(M)$ measures the failure of a $\bar{\partial}$ -closed form to be $\bar{\partial}$ -exact. The next theorem shows that locally every $\bar{\partial}$ -closed form is $\bar{\partial}$ -exact. Its proof is quite technical, and will distract us from the main theme of this book. So we will simply present the result. Interested readers can refer to [1], p.109 for a detailed proof.

Theorem 2.63 (The $\bar{\partial}$ -Poincaré Lemma). *Let M be a complex manifold and $\omega \in \Omega^{p,q}(M)$ such that $\bar{\partial}\omega = 0$, with $q \geq 1$. Then in a neighborhood of each point there is a smooth $(p, q-1)$ -form η with $\bar{\partial}\eta = \omega$.*

The next result tells us how to compute the Dolbeault cohomology from sheaf cohomology:

Theorem 2.64. *Suppose M is a complex manifold, and Ω^p its sheaf of holomorphic p -forms. Then for each $q \geq 0$, we have*

$$H^{p,q}(M) \cong H^q(M, \Omega^p).$$

Proof. We note that there is a morphism $\Omega^p \rightarrow \mathcal{A}^{p,0}$ of sheaves induced by inclusions. By the $\bar{\partial}$ -Poincaré lemma (Lemma 2.63), we have an exact sequence of sheaves associated with the Dolbeault complex:

$$0 \rightarrow \Omega^p \rightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \dots$$

Moreover, $\mathcal{A}^{p,q}$ are acyclic sheaves because of the existence of partition of unity. Therefore, the sheaf cohomology is given by $H^q(M, \Omega^p) \cong R^q\Gamma(M; \mathcal{A}^{p,q}) = H^{p,q}(M)$. $\square$

The $\bar{\partial}$ -Poincaré Lemma (Theorem 2.63) has another corollary:

Corollary 2.65 ($\partial\bar{\partial}$ -lemma). *Suppose θ is a smooth closed (p,q) -form on a complex manifold M , with $p, q > 0$. In a neighborhood of each point of M , there exists a smooth $(p-1, q-1)$ -form α such that $\theta = i\partial\bar{\partial}\alpha$.*

Proof. Since θ is a closed $(p+q)$ -form, the ordinary Poincaré lemma shows that locally we can find $(p+q-1)$ -form η such that $d\eta = \theta$. Since the only part of η that can contribute to the (p,q) -part of η are $(p-1, q)$ -part and $(p, q-1)$ -part, we might as well assume that $\eta = \eta^{p-1,q} + \eta^{p,q-1}$.

Using $d = \partial + \bar{\partial}$, we can decompose the equation $d\eta = \theta$ as three parts:

$$\bar{\partial}\eta^{p-1,q} = 0 \quad \bar{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q} = \theta \quad \partial\eta^{p,q-1} = 0.$$

Now apply Theorem 2.63, shrinking the neighborhood if necessary, we can conclude there exists a $\beta \in \mathcal{A}^{p-1,q-1}$ such that $\bar{\partial}\beta = \eta^{p-1,q}$. Similarly, since $\bar{\partial}\eta^{p,q-1}$ is $\bar{\partial}$ -closed, there exists $\gamma \in \mathcal{A}^{q-1,p-1}$ such that $\bar{\partial}\gamma = \eta^{p,q-1}$.

Now set $\alpha = i\bar{\gamma} - i\beta \in \mathcal{A}^{p-1,q-1}$. We compute

$$i\partial\bar{\partial}\alpha = i\partial\bar{\partial}(i\bar{\gamma}) - i\partial\bar{\partial}(i\beta) = \bar{\partial}\partial\bar{\gamma} + \partial\bar{\partial}\beta = \bar{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q} = \theta.$$

This concludes the proof. $\square$

2.5 Bundle-Valued Forms

The formalism we saw in the previous section can be generalized one step further to *bundle-valued forms*. Let E be a complex vector bundle over a complex manifold M .

Definition 2.66. By $\mathcal{A}^{p,q}(E)$ we denote the sheaf

$$\mathcal{A}^{p,q}(E) : U \mapsto \Gamma(U, \mathcal{A}_M^{p,q} \otimes E).$$

This is called the sheaf of E -valued (p, q) -forms. By $\mathcal{A}^{p,q}(M; E)$ we mean the space $\Gamma(M, \mathcal{A}^{p,q}(E))$.

Now suppose E is a holomorphic vector bundle (see Definition 3.1), we can extend the Dolbeault operator $\bar{\partial} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}$ to the space of bundle-valued (p, q) -forms.

Theorem 2.67. *Suppose M is a complex manifold and $E \rightarrow M$ is a holomorphic vector bundle. There are operators $\bar{\partial}_E : \mathcal{A}^{p,q}(M; E) \rightarrow \mathcal{A}^{p,q+1}(M; E)$ satisfying the following properties:*

1. For $\sigma \in \mathcal{A}^{0,0}(M; E) = \Gamma(E)$, we have $\bar{\partial}_E \sigma = 0$ iff σ is a holomorphic section.
2. For $\alpha \in \mathcal{A}^{p,q}(M)$ and $\beta \in \mathcal{A}^{p',q'}(M; E)$, we have

$$\bar{\partial}_E(\alpha \wedge \beta) = \bar{\partial} \alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \bar{\partial}_E \beta.$$

3. For $\gamma \in \mathcal{A}^{p,q}(M; E^*)$ and $\beta \in \mathcal{A}^{p',q'}(M; E)$, we have

$$\bar{\partial}(\gamma \wedge \beta) = \bar{\partial}_{E^*} \gamma \wedge \beta + (-1)^{p+q} \gamma \wedge \bar{\partial}_E \beta.$$

4. $\bar{\partial}_E^2 = 0$.

5. If $\alpha \in \mathcal{A}^{p,q}(E)$ such that $\bar{\partial}_E \alpha = 0$, then in a neighborhood of each point there exists $\beta \in \mathcal{A}^{p,q-1}(M; E)$ such that $\bar{\partial}_E \beta = \alpha$.

Proof. We first need to define $\bar{\partial}_E$. In any open set $U \subset M$ over which there are holomorphic local frame $\{s_1, \dots, s_r\}$ of E , write $\sigma|_U = \sigma^\mu \otimes s_\mu$ for some $\sigma^\mu \in \mathcal{A}^{p,q}(U)$. We define $\bar{\partial}_E \sigma$ by setting

$$\bar{\partial}_E \sigma|_U = \bar{\partial} \sigma^\mu \otimes s_\mu.$$

To check that this definition makes sense globally, let $\{t_1, \dots, t_r\}$ be another set of local frame of E . Then there exists some holomorphic transition functions τ_ν^μ such that $t_\beta = \tau_\beta^\alpha s_\alpha$. Then if $\sigma = \tilde{\sigma}^\mu t_\mu$, we must have $\tau_\nu^\mu \tilde{\sigma}^\mu = \sigma^\nu$. Since $\bar{\partial} \tau_\rho^\mu = 0$ because τ_ρ^μ is holomorphic, we have

$$\bar{\partial} \sigma^\mu \otimes s_\mu = \bar{\partial}(\tau_\rho^\mu \tilde{\sigma}^\rho) \otimes s_\mu = \bar{\partial} \tilde{\sigma}^\rho \otimes (\tau_\rho^\mu s_\mu) = \bar{\partial} \tilde{\sigma}^\rho \otimes t_\rho.$$

Hence $\bar{\partial}_E$ is well-defined globally.

To prove (1), write $\sigma = \sigma^\mu s_\mu$ locally. Then $\bar{\partial}_E \sigma = (\bar{\partial} \sigma^\mu) \otimes s_\mu = (\bar{\partial} \sigma^\mu) s_\mu$ in this case. Then since s_μ 's are linearly independent, the claim follows from Theorem 2.35.

To prove (2), we can write $\beta = \beta^\mu \otimes s_\mu$. Then

$$\bar{\partial}_E(\alpha \wedge \beta) = \bar{\partial}_E((\alpha \wedge \beta^\mu) \otimes s_\mu) = (\bar{\partial} \alpha \wedge \beta^\mu + (-1)^{p+q} \alpha \wedge \bar{\partial} \beta^\mu) \otimes s_\mu$$

which gives the desired equation.

To prove (3), write $\beta = \beta^\mu \otimes s_\mu$ and let ϵ^ν be the dual frame to s_μ . Then we can write $\gamma = \gamma_\nu \otimes \epsilon^\nu$. Then we have

$$\begin{aligned}\bar{\partial}_{E^*} \gamma \wedge \beta + (-1)^{p+q} \gamma \wedge \bar{\partial}_E \beta &= (\bar{\partial} \gamma_\nu \otimes \epsilon^\nu) \wedge (\beta^\mu \otimes s_\mu) + (-1)^{p+q} (\gamma_\nu \otimes \epsilon^\nu) \wedge (\bar{\partial} \beta^\mu \otimes s_\mu) \\ &= \bar{\partial} \gamma_\nu \wedge \beta^\mu + (-1)^{p+q} \gamma_\nu \wedge \bar{\partial} \beta^\mu \\ &= \bar{\partial}(\gamma_\nu \wedge \beta^\mu) \\ &= \bar{\partial}(\gamma \wedge \beta),\end{aligned}$$

as desired.

To prove (4), we check locally: write $\sigma = \sigma^\mu \otimes s_\mu$. Then

$$\bar{\partial}_E^2 \sigma = (\bar{\partial}^2 \sigma^\mu) \otimes s_\mu = 0$$

by Proposition 2.60.

To prove (5), we write $\alpha = \alpha^\mu \otimes s_\mu$. Then $\bar{\partial}_E \alpha = 0$ implies $\bar{\partial} \alpha^\mu \otimes s_\mu = 0$. By linear independence of s_μ this implies $\bar{\partial} \alpha^\mu = 0$ for all μ . It follows from Theorem 2.63 that there exists $\beta^\mu \in \mathcal{A}^{p,q-1}$ locally such that $\bar{\partial}_E \beta^\mu = \alpha^\mu$. Thus we can set $\beta = \beta^\mu \otimes s_\mu$. $\square$

Thanks to part (4) of Theorem 2.67, we know that for each p we again have that

$$0 \rightarrow \mathcal{A}^{p,0}(M; E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,1}(M; E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,2}(M; E) \xrightarrow{\bar{\partial}_E} \dots$$

is a chain complex. We define the **Dolbeault cohomology group of vector bundle E** , denoted by $H^{p,q}(M; E)$, to be

$$H^{p,q}(M; E) = \frac{\ker(\bar{\partial}_E : \mathcal{A}^{p,q}(M; E) \rightarrow \mathcal{A}^{p,q+1}(M; E))}{\operatorname{im}(\bar{\partial}_E : \mathcal{A}^{p,q-1}(M; E) \rightarrow \mathcal{A}^{p,q}(M; E))}.$$

Similarly, these groups can be computed using the machinery of sheaf cohomology:

Theorem 2.68. *Suppose M is a complex manifold, and Ω^p its sheaf of holomorphic p -forms. Let $E \rightarrow M$ be a holomorphic vector bundle. Then for each $q \geq 0$, we have*

$$H^{p,q}(M; E) \cong H^q(M, \Omega^p \otimes E).$$

Proof. We note that there is a morphism $\Omega^p \otimes E \rightarrow \mathcal{A}^{p,0}(E)$ of sheaves induced by inclusions. By Theorem 2.67, we have an exact sequence of sheaves associated with the Dolbeault complex:

$$0 \rightarrow \Omega^p \otimes E \rightarrow \mathcal{A}^{p,0}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,1}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,2}(E) \xrightarrow{\bar{\partial}_E} \dots$$

Moreover, $\mathcal{A}^{p,q}(E)$ are acyclic sheaves because of the existence of partition of unity. Therefore, the sheaf cohomology is given by

$$H^q(M, \Omega^p \otimes E) \cong R^q \Gamma(M; \mathcal{A}^{p,q}(E)) = H^{p,q}(M; E).$$

$\square$



Chapter 3 Holomorphic Vector Bundles

Introduction

In this chapter, we begin our study of complex manifolds in depth. The main objects of our study are vector bundles on complex manifolds in the holomorphic category. With the tool of sheaf cohomology at hand, we will discover relations between divisors and line bundles on complex manifolds, and we will study two of the most important examples in great detail: complex projective space and blow ups.

3.1 Holomorphic Vector Bundles

Definition 3.1. Let X be a complex manifold. A **holomorphic vector bundle** $\pi : E \rightarrow X$ of rank r is a vector bundle with fiber $\mathbb{C}^r$, such that the maps π and the trivializations $\varphi_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$ are holomorphic maps between complex manifolds.

Example 3.2. The two typical example of holomorphic vector bundle we have already seen in the previous chapter is T_X (the sheaf of holomorphic sections of $T^{1,0}X$, identified with a vector bundle), and Ω_X^p (identified with a vector bundle similarly).

Now let $E \rightarrow X$ be a holomorphic vector bundle, with trivializations $\varphi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{C}^r$, where $X = \bigcup_\alpha U_\alpha$. Then for any $p \in U_{\alpha\beta} := U_\alpha \cap U_\beta$, there exists a nowhere vanishing holomorphic function $\psi_{\alpha\beta} \in GL_r(U_{\alpha\beta}; \mathbb{C})$, holomorphic in each entry, such that

$$\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = (p, \psi_{\alpha\beta}(p)v).$$

These are called **transition functions** of E . They satisfy the cocycle condition by construction:

$$\psi_{\alpha\beta}\psi_{\beta\gamma} = \psi_{\alpha\gamma}$$

on triple intersections $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$.

Theorem 3.3. *Moreover, Transition functions $\psi_{\alpha\beta}$ completely determines E up to isomorphism.*

Proof. Let $X = \bigcup_{\alpha} U_{\alpha}$ be a trivializing cover of E with transition functions $\{\psi_{\alpha\beta}\}$. Then we can construct a holomorphic vector bundle E_{ψ} defined by $E_{\psi} = \coprod_{\alpha} U_{\alpha} \times \mathbb{C}^r / \sim$, where $(p, v) \times \mathbb{C}^r \sim (q, w)$ iff $p = q$ and there exists some α, β such that $w = \psi_{\alpha\beta}(p)v$. It is easy to check that $E_{\psi} \cong E$. $\square$

Theorem 3.3 is very important because it gives a guideline that we shall use countless times throughout the rest of this book without mentioning:

GUIDELINE: If we want to prove that two holomorphic vector bundles E, E' are isomorphic, then all we need to do is to find a trivializing open cover $X = \bigcup_{\alpha} U_{\alpha}$ for E, E' simultaneously (which we can always do by taking common intersections), and check that their transition functions $\psi_{\alpha\beta}, \psi'_{\alpha\beta}$ with respect to this cover agree. Similarly, we can define a holomorphic vector bundles by its transition functions, if all we care about is the vector bundle up to isomorphism.

Definition 3.4. The **Picard group** $\text{Pic } X$ of a complex manifold X is the isomorphism class of all holomorphic line bundles on X . This is an abelian group with multiplications given by tensor products $L \otimes L'$, the inverse of L given its dual L^* , and the identity element given by the trivial bundle $\mathcal{O}_X$.

Theorem 3.5. *We have an isomorphism $\text{Pic } X \cong H^1(X, \mathcal{O}_X^*)$.*

Proof. By Theorem 3.3, a holomorphic line bundle is completely determined by its transition functions $\{\psi_{\alpha\beta} \in \mathcal{O}_X^*(U_{\alpha\beta})\}$ up to isomorphism. And by the above discussion, $\psi_{\alpha\beta}\psi_{\beta\gamma} = \psi_{\alpha\gamma}$. Hence, the transition function defines a Čech 1-cocycle in $H^1(X, \mathcal{O}_X^*)$.

If $\{\psi_{\alpha\beta}\}$ is zero in $H^1(X, \mathcal{O}_X^*)$, then it means that there exists a Čech 0-chain $f_{\alpha} \in \mathcal{O}_X^*(U_{\alpha})$ such that $\psi_{\alpha\beta} = f_{\alpha}f_{\beta}^{-1}$. Then we can use these to define a nowhere vanishing section $s : X \rightarrow L_{\psi}$ such that $s|_{U_{\alpha}} = f_{\alpha}$. This is well-defined, as $s_{\alpha} = \psi_{\alpha\beta}s_{\beta} = f_{\alpha}f_{\beta}^{-1}s_{\beta}$. This nowhere vanishing section trivializes L_{ψ} . $\square$

On X , there is an important short exact sequence of sheaves:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0$$

called the **exponential sequence**, where $\mathcal{O}_X \rightarrow \mathcal{O}_X^*$ is given by the map $f \mapsto \exp(2\pi i f)$. Taking the long exact sequence of cohomology we get a boundary map

$$c_1 : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X; \mathbb{Z})$$

called the **Chern map**. By Theorem 3.5, we can associate for each line bundle L an element $c_1(L) \in H^2(X; \mathbb{Z})$, called the **first Chern class** of the line bundle L . If X is a complex curve (Riemann surface), we define the **degree** of the line bundle L by

$$\deg L = \int_X c_1(L).$$

We conclude this section with a bunch of examples of holomorphic vector bundles.

Example 3.6. For a complex manifold X , its holomorphic tangent bundle T_X and holomorphic cotangent bundle Ω_X^1 are both holomorphic vector bundles.

Example 3.7. Any operation of vector space in linear algebra give rise to vector bundles in a fiberwise fashion. That is, given two holomorphic vector bundles E, E' , we can define their *direct sum* $E \oplus E'$, their *tensor product* $E \otimes E'$, their *hom bundle* $\text{Hom}(E, E')$. We can also define the *wedge bundle* $\Lambda^k E$, the *dual bundle* E^* .

Definition 3.8. Let r be the rank of a holomorphic vector bundle E . The *determinant bundle* $\det E$ is the line bundle given by the highest wedge power of E . That is, $\det E = \Lambda^r E$.

Definition 3.9. The *canonical bundle* K_X of a complex manifold is given by $K_X = \Lambda^n \Omega_X^1$, where $n = \dim X$.

Now we want to study transition maps of the determinant bundle. Suppose $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL_n(\mathbb{C})$ is a transition function between two local trivializations U_α, U_β of V . Recall this means that if e_μ is a local frame of V on U_α and f_ν a local frame of V on U_β , for $\mu = 1, \dots, n$, and if $v = u^\mu e_\mu = w^\nu f_\nu$, we have that

$$(\varphi_{\alpha\beta})_\mu^\nu u^\mu = w^\nu.$$

Taking the special case where $v = e_\sigma$ is the standard basis vector, i.e., taking $u^\mu = \delta_\sigma^\mu$, we find that under the transition from $U_\alpha \rightarrow U_\beta$, we have

$$e_\sigma \rightarrow (\varphi_{\alpha\beta}) e_\sigma = (\varphi_{\alpha\beta})_\sigma^\nu f_\nu.$$

The equation above can be viewed as an equivalent definition of transition function.

Proposition 3.10. The determinant bundle $\det V = \Lambda^n V$ has the same local trivializations as that of V , with chart transition map given by $\det(\varphi_{\alpha\beta})$.

Proof. This is the same as proving the transformation rule for a differential form in different coordinate charts. Let the transition function from U_α to U_β of $\Lambda^n V$ be $\psi_{\alpha\beta}$. Since we can choose the frame $e_1 \wedge \dots \wedge e_n$ on U_α and $f_1 \wedge \dots \wedge f_n$ on U_β . Since under transition, $e_1 \wedge \dots \wedge e_n$ becomes

$$\begin{aligned} (\varphi_{\alpha\beta})_1^{\nu_1} f_{\nu_1} \wedge \dots \wedge (\varphi_{\alpha\beta})_n^{\nu_n} f_{\nu_n} &= \sum_{\nu_1, \dots, \nu_n} \text{sign}(\nu_1 \dots \nu_n) (\varphi_{\alpha\beta})_1^{\nu_1} \dots (\varphi_{\alpha\beta})_n^{\nu_n} (f_1 \wedge \dots \wedge f_n) \\ &= \det(\varphi_{\alpha\beta}) f_1 \wedge \dots \wedge f_n. \end{aligned}$$

Hence, we conclude that $\psi_{\alpha\beta} = \det(\varphi_{\alpha\beta})$. $\square$

Lemma 3.11. Let $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ be an exact sequence of vector bundles over M . Then we have that $\det F \cong \det E \otimes \det G$.

Proof. By exactness, the transition functions for F are a block upper triangular matrix with top left block the transition functions for E and bottom right block the transition functions for G , while the top right is some garbage we don't care about. It is a linear algebra fact that the determinant of such a matrix is the product of the determinants of its diagonal blocks. Since the transition function of a tensor product of line bundles is the product of their transition functions, and identical transition function implies isomorphism between bundles, this proves the claim. $\square$

Definition 3.12. Let $f : X \rightarrow Y$ be a smooth (or holomorphic) map between real (or complex) manifolds, and let $\pi : E \rightarrow Y$ be a smooth (or holomorphic) vector bundle. The *pullback bundle* f^*E is defined to be the set

$$f^*E = \{(x, v) \in X \times E : f(x) = \pi(v)\}.$$

The pullback bundle f^*E comes with two natural projection maps to X and to E , since it is a subset of $X \times E$. In fact, we have a Cartesian square

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

It is not hard to show that the natural projection map $f^*E \rightarrow X$ makes f^*E a bundle over X . Indeed, if $Y = \bigcup_{\alpha} U_{\alpha}$ is a trivializing cover of E , then it is not hard to show that $X = \bigcup_{\alpha} f^{-1}(U_{\alpha})$ is a trivializing cover of f^*E , and the fiber $(f^*E)_x$ is canonically isomorphic to $E_{f(y)}$. Thus f^* takes bundle over Y to bundle over X . The pullback definition can be generalized to arbitrary fiber bundle with fiber F . Then in this general setting, the above statements are made precise by the following theorem:

Theorem 3.13. *Let $p : E \rightarrow B$ be a fiber bundle with fiber F , and $f : B \rightarrow B'$ be a continuous map. Then:*

- (a) *If $p : E \rightarrow B$ is a trivial bundle, then the pullback $f^*E \rightarrow B'$ is similarly a trivial bundle.*
- (b) *$f^*E \rightarrow B'$ is an F -bundle over B' .*
- (c) *If $B = \bigcup_{\alpha} U_{\alpha}$ is a trivializing cover of $E \rightarrow B$, and the transition function is $\psi_{\alpha\beta}$, then the cover $B' = \bigcup_{\alpha} f^{-1}(U_{\alpha})$ trivializes f^*E , and f^*E has transition functions $f^*\psi_{\alpha\beta} = \psi_{\alpha\beta} \circ f$, in the case that E is a vector bundle (in fact, the same result holds for general fiber bundles).*

Proof.

- (a) Let $E = B \times F$ be trivial. Then by definition

$$f^*E = \{((b, v), b') : v \in F, f(b') = b\} \cong_{\text{bijection}} \{(v, b') \in F \times B'\} = F \times B'.$$

In other words, we have a bijection $f^*E \rightarrow F \times B'$ given by simply forgetting b . This is a continuous map because this is the restriction (up to permutation of variables) to projection $B \times F \times B' \rightarrow B' \times F$ which is continuous, and f^*E is given by subspace topology of $B \times F$. It has an inverse

$$F \times B' \rightarrow f^*E \quad (b', v) \mapsto ((b', f(b)), v)$$

which is again continuous for the same reason. Therefore, we conclude that f^*E is trivial.

- (b) Let $B = \bigcup_i U_i$ be an open cover for which $p^{-1}(U_i) \rightarrow U_i$ is a trivial F -bundle. Then $B' = \bigcup_i f^{-1}(U_i)$ is an open cover of B' . We claim that $f^{-1}(U_i)$ gives local trivialization of f^*E with fiber F . This is because we have the following commutative diagram induced by the commutative square of the base, where all the unlabeled maps are inclusions of subspaces:

$$\begin{array}{ccccc} & & f^*E|_{f^{-1}(U_i)} & \xrightarrow{p_E} & p^{-1}(U_i) \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ f^*E & \xleftarrow{\quad} & & \xrightarrow{p_E} & E \\ & \downarrow & & \downarrow & \downarrow \\ & & f^{-1}(U_i) & \xrightarrow{f} & U_i \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ B' & \xleftarrow{\quad} & & \xrightarrow{f} & B \end{array}$$

Now we claim that $f^*E|_{f^{-1}(U_i)}$ is a trivial F -bundle: Indeed, $f^*E|_{f^{-1}(U_i)}$ is, by definition, the pullback along the map $f^{-1}(U_i) \rightarrow B' \rightarrow B$ of E . Note that this map is a path from $f^{-1}(U_i)$ to B in the base of our commutative diagram. On the other hand, since the diagram commutes, $f^*E|_{f^{-1}(U_i)}$ is also the pullback along the composition $f^{-1}(U_i) \rightarrow U_i \rightarrow B$ of E . Since $p^{-1}(U_i) \rightarrow U_i$ is trivial, we conclude from part (a) that $f^*E|_{f^{-1}(U_i)} \rightarrow f^{-1}(U_i)$ is also trivial.

- (c) By (b), the open cover $B' = \bigcup_{\alpha} f^{-1}(U_{\alpha})$ is a local trivializing cover of f^*E . We now prove the claim about the transition function $\psi'_{\alpha\beta} = \psi_{\alpha\beta} \circ f$.

Let $(s_1, \dots, s_n)$ be a local frame of $E \rightarrow B$. Then a local frame of $f^*E \rightarrow B'$ would be given by $(\rho_1, \dots, \rho_n)$ where ρ_i 's are defined by

$$\rho_i(p) = s_i(f(p))$$

for $p \in B'$. Now if $(s'_1, \dots, s'_n)$ are overlapping frames for E such that $s'_{\alpha} = \psi_{\alpha\beta} s_{\beta}$, then the corresponding ρ'_i will be given by

$$\rho'_{\alpha}(p) = s'_{\alpha} \circ f(p) = \psi_{\alpha\beta}(f(p)) s_{\beta}(f(p)) = f^* \psi_{\alpha\beta}(p) \rho_{\beta}(p).$$

□

Lemma 3.14. *Let $f : X \rightarrow Y$ be a surjective map between complex manifolds. Then there is an exact sequence of sheaves on X given by*

$$0 \rightarrow T_{X/Y} \rightarrow T_X \xrightarrow{f_*} f^*T_Y \rightarrow 0,$$

where f^*T_Y is the pullback of the tangent bundle T_Y along the morphism f , the map $f_* : T_X \rightarrow f^*T_Y$ is given by the pushforward, and $T_{X/Y}$ is the tangent space to the fiber of f . See Fig 3.1 for an illustration.

Proof. Recall that f^*T_Y is a vector bundle (locally free sheaf) on X whose fiber at $x \in X$ is given by $T_Y|_{f(x)}$. Hence the morphism $f_* : T_X \rightarrow f^*T_Y$ is defined by $v \mapsto f_*(v)$ for $v \in T_X$. Then clearly the tangent to the fiber $T_{X/Y}$ is a subspace of T_X , and lies in the kernel of f_* , and surjectivity of f_* is due to the surjectivity of f . □

We conclude with a construction that might not come familiar. Given a vector bundle $p : E \rightarrow B$, which could be a holomorphic vector bundle over a complex manifold, or a C^{∞} -vector bundle over a smooth manifold. We can always projectivize its fiber to construct a projective bundle $\mathbb{P}(E)$ with fiber canonically isomorphic to $\mathbb{P}(E_b)$ over any $b \in B$. This will no longer be a vector bundle over B , but it will be a fiber bundle over B with fiber $\mathbb{P}^{n-1}$. To be extremely concise, the construction is as follows: There is a k^* -action on the fibers of E given by scalar multiplication. We take out the zero section, and mod out by this action. That is:

$$\mathbb{P}(E) := (E \setminus B)/k^*,$$

where we identify B with the image of the zero section $\sigma_0 : B \rightarrow E$. The construction is made precise by the following theorem:

Theorem 3.15. *Let $k = \mathbb{R}$ or $\mathbb{C}$ and $p : E \rightarrow B$ be a rank n vector bundle over a manifold B with fiber k^n . There exists a fiber bundle $\mathbb{P}(E) \rightarrow B$ such that the fiber over each point $b \in B$, its fiber is canonically isomorphic to $\mathbb{P}(E_b)$.*

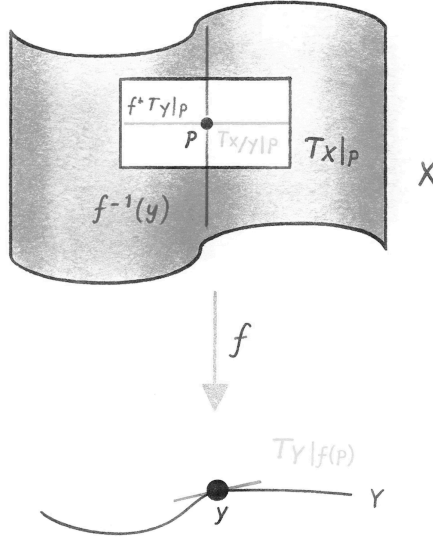


Figure 3.1: The picture to have in mind is the following: f is a surjective map from X to Y , hence it is sort of a projection. Then $T_{X/Y}$ is the relative tangent bundle, i.e., the direction which we are projecting down to (in other words it is the tangent space of the fiber over f), so the tangent vector in this direction is killed by f_* , while the surviving part f^*T_Y is naturally identified with T_Y at the appropriate points (more specifically we identify $T_Y|_{f(p)} = T_Y|_y$ with $f^*T_Y|_p$), and this identification is necessary since exact sequence of sheaves only make sense over the same space.

Proof. We note that for every fiber E_b of E , there is a k^* -action on E_b by scalar multiplication. To see this, we can choose a trivializing cover U near b , such that $\varphi : E|_U \cong U \times k^n$. Then restricted to each fiber this gives an isomorphism $\varphi_b : E_b \cong k^n$, on which k^* acts. We claim that this fiberwise k^* -action actually give rise to a well-defined global action of k^* on the total space E . To see this, let $x \in E$ be any points. Under any trivialization $\varphi : E|_U \cong U \times k^n$ containing x , we define

$$\lambda \cdot x := \varphi^{-1}(p(x), \lambda \cdot \text{pr}_2 \varphi(x)),$$

where the latter action is the standard k^* -action on k^n . To see that this action is well-defined, suppose $\psi : E|_V \cong V \times k^n$ is another trivialization around x , then we have another action

$$\lambda \odot x := \psi^{-1}(p(x), \lambda \cdot \text{pr}_2 \psi(x)).$$

But note that by definition

$$\psi \circ \varphi^{-1}(p(x), \lambda \cdot \text{pr}_2 \varphi(x)) = (p(x), \lambda \cdot \text{pr}_2 \psi(x)).$$

Hence we conclude that $\lambda \cdot x = \lambda \odot x$. In other words, the action of k^* on E does not depend on the trivialization. Hence k^* acts on E .

Now, note that $B \subset E$ naturally as the zero section of E . We define the projectivization $\mathbb{P}(E)$ to be the space $E \setminus B$ quotient out by the k^* -action on E . If $p : E \rightarrow B$ is the bundle map, then we have an induced map $q : \mathbb{P}(E) \rightarrow B$, defined by

$$q : \mathbb{P}(E) \rightarrow B \quad q([x]) = p(x).$$

Note that this map is well-defined and surjective, since the k^* -action is defined fiberwise, so $[x] = [y]$ iff x, y are in the same fiber of E over B and they differ by an element in k^* .

Moreover, for any $b \in B$, we have

$$q^{-1}(b) = \{[x] \in \mathbb{P}(E) : x \in E_b\} \cong \mathbb{P}(E_b).$$

It remains to show that $q : \mathbb{P}(E) \rightarrow B$ admits local trivializations. For any $[x] \in \mathbb{P}(E)$, let $b = p(x)$ and $U \subset B$ a trivializing neighborhood of E near b with trivializations $\varphi : E|_U \cong U \times k^n$. Then we define a trivialization $\Phi : \mathbb{P}(E)|_U \rightarrow U \times \mathbb{P}^{n-1}$ as follows:

$$\begin{aligned} \Phi : \mathbb{P}(E)|_U &\rightarrow U \times \mathbb{P}^{n-1} \\ [x] &\mapsto (q([x]), [\text{pr}_2 \varphi(x)]) \end{aligned}$$

where $\text{pr}_2 \varphi(x) \in k^n$ and $[\text{pr}_2 \varphi(x)] \in \mathbb{P}(k^n) = \mathbb{P}^{n-1}$ is the equivalence class under the projectivization. We note that this is well-defined, for if $[x] = [y]$, then x, y are over the same fiber in E so $q([x]) = q([y])$, and they must differ by an element in k^* , so after projectivization in k^n , the second component also coincide.

Finally, we construct the inverse map

$$\begin{aligned} \Phi^{-1} : U \times \mathbb{P}^{n-1} &\rightarrow \mathbb{P}(E)|_U \\ (b, [v]) &\mapsto [\varphi^{-1}(b, v)] \end{aligned}$$

where $\varphi^{-1}(b, v) \in E$ and $[\varphi^{-1}(b, v)]$ is its equivalence class in $\mathbb{P}(E)$. Again, this is well-defined, for if $[v] = [w]$ then $v = \lambda w$ in k^n for some $\lambda \in k^*$ and by definition $\varphi^{-1}(b, v) = \lambda \cdot \varphi^{-1}(b, w)$, so after quotienting the k^* -action we have $[\varphi^{-1}(b, v)] = [\varphi^{-1}(b, w)]$.

This shows that $\mathbb{P}(E) \rightarrow B$ is a $\mathbb{P}^{n-1}$ -bundle over B whose fiber is canonically isomorphic to $\mathbb{P}(E_b)$ at every $b \in B$. $\square$

3.2 Divisors and Line Bundles

We will first start with a bunch of definitions. Let X be a complex manifold of dimension n . A **hypersurface** $Y \subset X$ is an analytic subvariety of (complex) codimension 1. In other words, Y is locally given as the zero set of a nontrivial holomorphic function.

Definition 3.16. A **divisor** D on X is a formal $\mathbb{Z}$ -linear combination

$$D = \sum_i a_i Y_i$$

where $Y_i \subset X$ are irreducible hypersurfaces. The **divisor group** $\text{Div } X$ is the set of all divisors endowed with the natural abelian group structure. The **degree** of a divisor $D = \sum_i a_i Y_i$ is defined by

$$\deg D = \sum_i a_i.$$

We say that a divisor D is **effective** if $a_i \geq 0$ for all i . In this case, one writes $D \geq 0$.

Let $Y \subset X$ be a hypersurface and $p \in Y$. Then locally near p , the hypersurface Y is given as the zero set of a holomorphic function g . Since $\mathcal{O}_{X,p}$ is a UFD of dimension 1, and f is meromorphic, in the local ring we have a unique factorization $f = g^k h$ for some integer k and some holomorphic h with $h(p) \neq 0$. Then we define the **order** of f along Y at p be

$$\text{ord}_{Y,p}(f) = k.$$

Moreover, one can check that such definition does not depend on p . Therefore, there is a well-defined integer $\text{ord}_Y(f)$ called the **order** of f along Y . We say that a meromorphic function $f \in H^0(X, \mathcal{K}_X)$ has zeros or poles along Y of order $d \geq 0$ if $\text{ord}_Y(f) = d$ or $\text{ord}_Y(f) = -d$.

Definition 3.17. Let $f \in H^0(X, \mathcal{K}_X)$. The *divisor associated to f* is

$$(f) = \sum_Y \text{ord}_Y(f) Y$$

where the sum is over all hypersurfaces $Y \subset X$. A divisor of this form is called *principal*.

Our next result gives an alternative description of the group $\text{Div } X$. Recall that $\mathcal{K}_X^*$ is the sheaf of invertible elements in $\mathcal{K}_X$, i.e., over each open set U , the space $\mathcal{K}_X^*(U)$ is the set of nontrivial (not identically zero) meromorphic functions over U . The sheaf $\mathcal{O}_X^*$ is similarly defined to be the sheaf of invertible holomorphic functions. Note that this definition is equivalent to saying that $\mathcal{O}_X^*(U)$ is the space of nowhere vanishing (not just nontrivial!) holomorphic functions on U .

Lemma 3.18. *We have an isomorphism $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \text{Div } X$.*

Proof. Let $X = \bigcup_i U_i$ be a good open cover. An element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is given by $f_i \in \mathcal{K}_X^*(U_i)$ on U_i such that $f_i f_j^{-1}$ are nonvanishing holomorphic functions for all i, j . Therefore, for any irreducible hypersurface Y we have $\text{ord}_Y(f_i) = \text{ord}_Y(f_j)$. Hence we get an element

$$(f) = \sum_Y \text{ord}_Y(f) Y \in \text{Div } X.$$

Conversely, let $D = \sum_i a_i Y_i \in \text{Div } X$ be given, and let $Y_i \cap U_j$ be defined by $g_{ij} \in \mathcal{O}_X(U_j)$. Then let $f_j = \prod_i g_{ij}^{a_i} \in \mathcal{K}_X^*(U_j)$. Since g_{ij} and g_{ik} on $U_j \cap U_k$ define the same irreducible hypersurface, they differ by an element in $\mathcal{O}_X^*(U_j \cap U_k)$. Thus the functions f_j glue to an element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. It is easy to check that these two constructions are inverses to each other. $\square$

Now let X be a complex manifold and $D \in \text{Div } X$ be a divisor on X . There is a way to associate a line bundle $\mathcal{O}_X(D)$ on X to the divisor D . The construction is easiest when $\dim X = 1$. In this case X is a Riemann surface. Let $\mathcal{K}_X$ denote the sheaf of meromorphic functions on X . We define

$$\mathcal{O}_X(D) : U \mapsto \{f \in \mathcal{K}_X(U) : \text{div } f|_U + D|_U \geq 0\}. \quad (3.1)$$

We claim that $\mathcal{O}_X(D)$ is a line bundle. To see this, suppose $D = n_1 p_1 + \dots + n_m p_m$. We choose open covers $U_0 = X - \{p_1, \dots, p_m\}$, and $U_j = B_\epsilon(p_j)$ be a small ball around p_j with radius $\epsilon > 0$, for $j = 1, \dots, m$. We choose ϵ so small such that $U_i \cap U_j = \emptyset$ for all $i, j \neq 0$. Then locally,

$$\begin{aligned} \Gamma(U_0, \mathcal{O}_X(D)) &= \Gamma(U_0, \mathcal{O}_X) = \mathcal{O}_X(U_0) \cdot \langle 1 \rangle. \\ \Gamma(U_j, \mathcal{O}_X(D)) &= \mathcal{O}_X(U_j) \cdot \langle (z - p_j)^{-n_j} \rangle. \end{aligned}$$

Therefore, we can define local basis

$$e_0 = 1 \quad e_j = (z - p_j)^{-n_j}.$$

The transition functions ψ_{ij} from the U_i chart to the U_j chart is given by

$$\psi_{j0} = (z - p_j)^{n_j}.$$

Note that this is nowhere vanishing on $U_0 \cap U_j = B_\epsilon(p_j) - \{p_j\}$. And there are no more transition functions to calculate because $U_i \cap U_j = \emptyset$.

In the case where X is not a Riemann surface, but a complex manifold of higher dimension, the definition for $\mathcal{O}_X(D)$ is still given by (3.1). However, the definition of $\text{div } f$ deserves a bit of remark. Locally on some open set U , f is defined by the quotient of two holomorphic functions $f = g/h$, so $\text{div } f|_U = Z(g) - Z(h)$, where both $Z(g)$ and $Z(h)$ are codimension 1 submanifold. But in fact, we can also define $\mathcal{O}_X(D)$ by its transition functions:

Definition 3.19. Let X be a complex manifold and D a divisor on X . By Lemma 3.18, D corresponds to an element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, which is a Čech 0-cocycle $f_i \in \mathcal{K}_X(U_i)$ on an open covering $X = \bigcup_i U_i$. Then we define $\mathcal{O}_X(D) \in \text{Pic } X$ as the line bundle whose transition function is given by $\psi_{ij} = f_i f_j^{-1} \in H^0(U_{ij}, \mathcal{O}_X^*)$. By definition, $f_i f_j^{-1}$ satisfies the cocycle condition, hence we have a well defined element in $H^1(X, \mathcal{O}_X^*) = \text{Pic } X$.

Proposition 3.20. There exists a natural group homomorphism $\rho : \text{Div } X \rightarrow \text{Pic } X$, given by the assignment $D \mapsto \mathcal{O}_X(D)$.

Proof. If $D, D' \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ are given by $\{f_i\}, \{f'_i\}$, then $D + D'$ corresponds to $\{f_i f'_i\}$. Thus $\mathcal{O}_X(D + D')$ is defined by $\{(f_i f'_i)(f_j f'_j)^{-1}\} = \{\psi_{ij} \psi'_{ij}\}$. Hence $\mathcal{O}_X(D + D') = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$. Clearly, $\mathcal{O}_X(0) \cong \mathcal{O}_X$, hence $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^*$. $\square$

Definition 3.21. Let $f : X \rightarrow Y$ be a holomorphic map between complex manifolds that is **dominant**, meaning it has dense image. Let $D \in \text{Div } Y$ be defined by the cocycle $\{U_i, f_i \in \mathcal{K}_X^*(U_i)\}$. Then the **pullback divisor** $f^*D \in \text{Div } X$ is defined by the cocycle $\{f^{-1}(U_i), f_i \circ f \in \mathcal{K}_X^*(f^{-1}(U_i))\}$.

We call that we also have a notion of pullback for vector bundles. Thus we have a well-defined homomorphism $f^* : \text{Pic } Y \rightarrow \text{Pic } X$. It is easy to check that the homomorphism in Proposition 3.20 is natural in the following sense:

Corollary 3.22. Let $f : X \rightarrow Y$ be a dominant holomorphic map. If $D \in \text{Pic } Y$, then we have $\mathcal{O}_X(f^*D) \cong f^*\mathcal{O}_Y(D)$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Div } Y & \xrightarrow{\rho} & \text{Pic } Y \\ f^* \downarrow & & \downarrow f^* \\ \text{Div } X & \xrightarrow{\rho} & \text{Pic } X \end{array}$$

Moreover, we will now show that the homomorphism in Proposition 3.20 is actually injective when we don't care about principal divisors

Proposition 3.23. A divisor $D \in \text{Div } X$ is principal iff $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

Proof. Suppose $D = (f)$ is principal. Then the corresponding element in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is the image of f under the natural map $K(X)^* = H^0(X, \mathcal{K}_X^*) \rightarrow H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. But then the cocycle $\psi_{ij} = f_i f_j^{-1}$ is trivial. Hence $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

Conversely, suppose $\mathcal{O}_X(D)$ is trivial. Then we may find open cover $X = \bigcup_i U_i$ such that D corresponds to $\{f_i \in \mathcal{K}_X^*(U_i)\}$ and holomorphic units $g_i \in \mathcal{O}_X^*(U_i)$ with $f_i f_j^{-1} = g_i g_j^{-1}$. Hence $f_i g_i^{-1} = f_j g_j^{-1}$ and we get a well-defined meromorphic function $f \in K(X)$ with $f|_{U_i} = f_i g_i^{-1}$. Since g_i are nowhere vanishing, one finds $(f) = D$. $\square$

Definition 3.24. The **divisor class group** $\text{Cl } X$ on X is the quotient group $\text{Div } X / \sim$ where $D \sim D'$ iff $D - D'$ is principal.

Then Proposition 3.23 gives the following easy corollary:

Corollary 3.25. *The homomorphism $\rho : \text{Div } X \rightarrow \text{Pic } X$ in Proposition 3.20 descends to an injective homomorphism $\rho : \text{Cl } X \rightarrow \text{Pic } X$.*

Near the end of this book, we will prove that $\rho : \text{Cl } X \rightarrow \text{Pic } X$ is actually an isomorphism, if X is a compact Riemann surface (Theorem 7.6).

Theorem 3.26. *Let $s : X \rightarrow L$ be a nontrivial section of a holomorphic line bundle, or more compactly $s \in H^0(X, L)$. Let $Z(s)$ denote the divisor on X where the section is zero. Then the line bundle $\mathcal{O}_X(Z(s))$ is isomorphic to L .*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be a trivializing cover for both $\mathcal{O}_X(Z(s))$ and L . Let the trivialization of L be $\psi_\alpha : L|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$. Then under this trivialization we have $\psi_\alpha \circ s(z) = (z, f_\alpha(z))$ for some holomorphic function $f_\alpha \in \mathcal{O}_X(U_\alpha)$. Moreover, $s|_{U_\alpha}(p) = 0$ iff $f_\alpha(p) = 0$. So the transition functions of $\mathcal{O}_X(Z(s))$ is given by $f_\alpha f_\beta^{-1}$ by construction. But this is exactly the transition functions L . Hence we conclude that $\mathcal{O}_X(Z(s)) \cong L$. $\square$

Theorem 3.27. *For any effective divisor $D \in \text{Div } X$ on a complex manifold X , there exists a nontrivial section $s \in H^0(X, \mathcal{O}_X(D))$ with $Z(s) = D$.*

Proof. Let $D \in \text{Div } X$ be given by $\{U_i, f_i \in \mathcal{K}_X^*(U_i)\}$. Since D is effective, the meromorphic functions $f_i \in \mathcal{K}_X^*(U_i)$ are actually holomorphic. On the other hand, the line bundle $\mathcal{O}_X(D)$ is associated to the cocycle $\psi_{ij} = f_i f_j^{-1}$. The holomorphic functions $f_i \in \mathcal{O}_X(U_i)$ define a global holomorphic section $s : X \rightarrow \mathcal{O}_X(D)$ because $\psi_{ij} f_j = f_i$. Moreover, $Z(s)|_{U_i} = Z(s|_{U_i}) = Z(f_i) = D \cap U_i$. Hence $Z(s) = D$. $\square$

Theorem 3.28. *Let $Y \subset X$ be an irreducible hypersurface. Then there is a short exact sequence of sheaves*

$$0 \rightarrow \mathcal{O}_X(-Y) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0.$$

This is a exact sequence of sheaves on X . Hence by $\mathcal{O}_Y$ what we mean is $i_ \mathcal{O}_Y$ where $i : Y \rightarrow X$ is the inclusion map. In other words, $i_* \mathcal{O}_Y(U) = \mathcal{O}_Y(U \cap Y)$. The map $\mathcal{O}_X(-Y) \rightarrow \mathcal{O}_X$ is locally defined by multiplying the (local) defining equation of Y .*

Proof. We need to prove exactness on stalks. Let $p \in X$ be a point. If $p \notin Y$, then the sequence is just $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \rightarrow 0 \rightarrow 0$, which is clearly exact. Hence let's assume $p \in Y$. Since $\alpha : \mathcal{O}_X(-Y) \rightarrow \mathcal{O}_X$ is given by multiplying the defining equation of Y locally, it is clearly injective on stalks, and $\text{im } \alpha_p \subset \ker \beta_p$. The map $\beta : \mathcal{O}_X \rightarrow \mathcal{O}_Y$ is given by restriction, hence is surjective on stalks because Y is a submanifold of X , so we can always pick a slice chart such that $Y = Z(z_n) \subset X$ locally. Then every holomorphic function $f(z_1, \dots, z_{n-1})$ on Y arises locally from the holomorphic function on X which is also defined by $f(z_1, \dots, z_{n-1})$, but now on all of the local slice chart in X . This local description also shows that if $f \in \mathcal{O}_{X,p}$ vanishes on Y , then locally it must be given by $z_n g$ for some $g \in \mathcal{O}_{X,p}$, hence $\ker \beta_p \subset \text{im } \alpha_p$. $\square$

3.3 Basic Results in Complex Projective Space

In this section, we will present explicit computations in complex projective space $\mathbb{P}^n$. We will first aim to classify all holomorphic line bundles on $\mathbb{P}^n$. We first define the most important line bundle on $\mathbb{P}^n$, the **tautological bundle**:

Definition 3.29. The **tautological bundle** $\mathcal{O}(-1)$ on $\mathbb{P}^n$ is a holomorphic vector bundle defined by

$$\mathcal{O}(-1) = \{([z], v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in [z]\}$$

viewed as a subset of $\mathbb{P}^n \times \mathbb{C}^{n+1}$ (with the subset topology). There is an obvious projection map $\mathcal{O}(-1) \rightarrow \mathbb{P}^n$, and it is easy to see that the fiber of any point $[z] \in \mathbb{P}^n$ (which is a complex line in $\mathbb{C}^{n+1}$) is precisely $[z]$.

Usually we denote the tautological bundle by $\mathcal{O}_{\mathbb{P}^n}(-1)$. When there is no danger of confusion, we simply write $\mathcal{O}(-1)$.

Definition 3.30. We now define the holomorphic vector bundles $\mathcal{O}(k)$ for any integer k on $\mathbb{P}^n$. If $n < 0$, then $\mathcal{O}(k) := \mathcal{O}(-1)^{\otimes k}$. If $k > 0$, we define the dual bundle $\mathcal{O}(1) := \mathcal{O}(-1)^*$ of the tautological bundle, and we define $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$. Finally, $\mathcal{O}(0) := \mathcal{O}_{\mathbb{P}^n} = \mathcal{O}$ is defined to be the trivial bundle $\mathbb{P}^n \times \mathbb{C}$.

Our first results determines global sections of $\mathcal{O}(k)$ for $k \geq 0$.

Theorem 3.31. *For $k \geq 0$, the space $H^0(\mathbb{P}^n, \mathcal{O}(k))$ is isomorphic to the space $\mathbb{C}[z_0, \dots, z_n]_k$ of homogeneous polynomials of degree k .*

Proof. Since $\mathcal{O}(0) = \mathcal{O}$ and $H^0(\mathbb{P}^n, \mathcal{O}) = \mathbb{C}$, it suffices to consider $k > 0$. Let $f \in \mathbb{C}[z_0, \dots, z_n]_k$ be any degree k homogeneous polynomial in $\mathbb{C}^{n+1}$. Then there is a natural way to construct a rough section $s_f \in H^0(\mathbb{P}^n, \mathcal{O}(k))$, which is simply a map $s_f : \mathbb{P}^n \rightarrow \mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$, via

$$s_f([z]) = f|_{[z]}.$$

Note that this is a section of $\mathcal{O}(1)^{\otimes k}$ because a degree 1 polynomial in $\mathbb{C}[z_0, \dots, z_n]_1$ can be viewed as a section of $\mathcal{O}(1)$. It is easy to check that this section s_f is holomorphic by going to local charts. Moreover, it is easy to see that the assignment $f \mapsto s_f$ is injective, for if $s_f = s_g$, then $f(v) = g(v)$ for all $v \in \mathbb{C}^{n+1}$, which implies $f = g$. Hence it suffices to show that $f \mapsto s_f$ is a surjective map onto $H^0(\mathbb{P}^n, \mathcal{O}(k))$.

Now let $\sigma \in H^0(\mathbb{P}^n, \mathcal{O}(k))$ be any section. We can define a holomorphic function $f : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}$ via

$$f(z) = \sigma([z])(z),$$

where we are viewing $\sigma([z]) \in \mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$, which is a homogeneous function from the line $[z]$ to $\mathbb{C}$ of degree k . Hence f is homogeneous of degree k by construction. By Hartog's theorem (Theorem 1.9), f extends to a holomorphic function on all of $\mathbb{C}^{n+1}$.

We now show that f is actually a polynomial. For nonzero $z \in \mathbb{C}^{n+1}$, homogeneity implies

$$|f(z)| = |z|^k \cdot |f(z/|z|)| \leq \sup_{|w|=1} |f(w)| \cdot |z|^k = C|z|^k.$$

Hence the Taylor expansion of f at the origin has no terms of order less than k . Let p be the polynomial function obtained by considering only the terms of degree d in the Taylor series of f at the origin, and let $R(z) = f(z) - p(z)$. Then R is a homogenous function of degree k because f and p are. Since the Taylor series of R starts with terms of order $k+1$, there is some constant M such that $|R(z)| \leq M|z|^{k+1}$ for all z in the closed unit ball. For any $z \in \mathbb{C}^{n+1}$ and $\epsilon > 0$ small enough such that $|\epsilon z| \leq 1$, we have

$$|R(z)| = \epsilon^{-k} |R(\epsilon z)| \leq \epsilon^{-k} M |\epsilon z|^{k+1} = \epsilon M |z|^{k+1}.$$

Now taking the limit as $\epsilon \rightarrow 0$, we find that $R(z) \equiv 0$. Hence f is equal to the polynomial p , and $\sigma = s_p$. This concludes the proof. $\square$

Then it follows quite easily that there is no global section for $\mathcal{O}(-k)$ for $k > 0$. In other words,

Corollary 3.32. *We have $H^0(\mathbb{P}^n, \mathcal{O}(-k)) = 0$ for $k > 0$.*

Proof. We will see that $\mathcal{O}(-1)$ has no global section. Now if $\mathcal{O}(-k)$ has global section σ for $k > 0$, we have already seen that for any degree $k-1$ polynomial f we have a global section $s_f \in H^0(\mathbb{P}^n, \mathcal{O}(k-1))$. Then $\sigma \otimes s_f$ would be a global section of $\mathcal{O}(-1)$, which is a contradiction. $\square$

Our next result is an important short exact sequence, called the **Euler sequence**, on $\mathbb{P}^n$:

Theorem 3.33. *On $\mathbb{P}^n$, there exists a short exact sequence of holomorphic vector bundles*

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus(n+1)} \rightarrow T_{\mathbb{P}^n} \rightarrow 0.$$

By taking the dual, this is equivalent to

$$0 \rightarrow \Omega_{\mathbb{P}^n}^1 \rightarrow \mathcal{O}(-1)^{\oplus(n+1)} \rightarrow \mathcal{O} \rightarrow 0.$$

Proof. Let $X = \mathbb{C}^{n+1} \setminus \{0\}$ and $\pi : X \rightarrow \mathbb{P}^n$ be the quotient map. Let $X_0, \dots, X_n$ be linear coordinates on $\mathbb{C}^{n+1}$, and let $x_i = X_i/X_0$ be the corresponding affine coordinates on $\mathbb{P}^n$ (on the open set $X_0 \neq 0$). Then we have

$$\pi^* dx_i = d(x_i \circ \pi) = d(X_i/X_0) = \frac{X_0 dX_i - X_i dX_0}{X_0^2}.$$

Since the pullback $\pi^* : T_{\mathbb{P}^n} \rightarrow TX$ and $\pi_* : TX \rightarrow T_{\mathbb{P}^n}$ are dual maps on fibers, we can use the calculation of $\pi^* dx_i$ to solve for $\pi_* \frac{\partial}{\partial X_i}$. For example, for $i \geq 1$, we have

$$\pi^* dx_i \left(\frac{\partial}{\partial X_i} \right) = \frac{X_0 \cdot 1 - X_i \cdot 0}{X_0^2} = \frac{1}{X_0},$$

hence using $\pi^* dx_i(v) = dx_i(\pi_* v)$, we see that

$$\pi_* \frac{\partial}{\partial X_i} = \frac{1}{X_0} \frac{\partial}{\partial x_i} \quad i = 1, \dots, n, \quad (3.2)$$

and similar calculation shows

$$\pi_* \frac{\partial}{\partial X_0} = - \sum_{i \neq 0} \frac{X_i}{X_0^2} \frac{\partial}{\partial X_i}. \quad (3.3)$$

Moreover, we make two observations: 1. If ρ is any linear functional on $\mathbb{C}^{n+1}$, the vector field $v(x) = \rho(x) \frac{\partial}{\partial X_i}$ descends to $\mathbb{P}^n$: That is, $\pi_* v(x) = \pi_* v(\lambda x)$ for any $x \in \mathbb{C}^{n+1}$, $\lambda \in \mathbb{C}$, which is easy to check given (3.2) and (3.3); 2. The holomorphic tangent space $T_{\mathbb{P}^n}$ at each point is spanned by the vectors $\pi_* \frac{\partial}{\partial X_i}$ with the single relation $\pi_* \left(\sum_{i=0}^n X_i \frac{\partial}{\partial X_i} \right) = 0$.

Therefore, we may define a map

$$\epsilon : \mathcal{O}(1)^{\oplus(n+1)} \rightarrow T_{\mathbb{P}^n}$$

by setting, for $\sigma = (\sigma_0, \dots, \sigma_n)$ a section of $\mathcal{O}(1)^{\oplus(n+1)}$,

$$\epsilon(\sigma)|_{\pi(x)} = \pi_* \left(\sum_i \sigma_i(x) \frac{\partial}{\partial X_i} \right) \Big|_x.$$

Then by the second observation, ϵ is surjective, and the kernel of ϵ is the trivial bundle (isomorphic to $\mathcal{O}$) spanned by the section $(X_0, \dots, X_n) \in \mathcal{O}(1)^{\oplus(n+1)}$. This gives the desired exact sequence. $\square$

With the Euler sequence, we can identify the canonical bundle of $\mathbb{P}^n$:

Corollary 3.34. *The canonical bundle $K_{\mathbb{P}^n}$ is isomorphic to $\mathcal{O}(-n-1)$.*

Proof. The dualized version of the Euler sequence is

$$0 \rightarrow \Omega_{\mathbb{P}^n}^1 \rightarrow \mathcal{O}(-1)^{\oplus(n+1)} \rightarrow \mathcal{O} \rightarrow 0.$$

Taking the determinant bundle gives

$$\det \Omega_{\mathbb{P}^n}^1 \otimes \det \mathcal{O} = \det \mathcal{O}(-1)^{\oplus(n+1)} = \mathcal{O}(-n-1),$$

where the last equality can be checked by calculating the transition functions, as the transition functions of $\det \mathcal{O}(-1)^{\oplus(n+1)}$ is the $(n+1)$ -fold product of the transition functions of $\mathcal{O}(-1)$. Finally, since $\det \mathcal{O} = \mathcal{O}$ and $\det \Omega_{\mathbb{P}^n}^1 = K_{\mathbb{P}^n}$, the desired result follows immediately. $\square$

Our next big result, the adjunction formula, is more general. We first need a few preliminary results:

Lemma 3.35. *Let $\pi : E \rightarrow M$ be any vector bundle and Z the zero locus of a transversal section. Then Z is a submanifold of M and its normal bundle in M is $N_{Z/M} \cong E|_Z$.*

Proof. Let $s : M \rightarrow E$ be a generic section that intersects with the zero section $S_0 \cong M$ transversely. We denote S the image of $s : M \rightarrow E$. Then $M \cong S$. Since S is transverse to S_0 , we know $S \cap S_0$ is a submanifold of S , hence also a submanifold of M via the identification $S \cong M$.

Since E is locally trivial, its tangent bundle on S_0 has the canonical decomposition $T_E|_{S_0} = E|_{S_0} \oplus T_{S_0}$ (See Figure 7.1). By transversality of $S \cap S_0$, we have $T_S + T_{S_0} = T_E = E \oplus T_{S_0}$. Therefore, the projection $T_S \rightarrow E$ is surjective, and the kernel is precisely $T_{S_0} \cap T_S = T_{S \cap S_0}$. Using the identification $Z \cong S \cap S_0$, we have an exact sequence

$$0 \rightarrow T_Z \rightarrow T_S|_Z \rightarrow E|_Z \rightarrow 0.$$

Hence we conclude that $E|_Z \cong N_{Z/S} \cong N_{Z/M}$ because $S \cong M$. $\square$

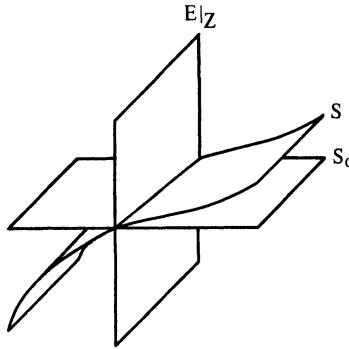


Figure 3.2: We have a decomposition $T_E|_{S_0} = E|_{S_0} \oplus T_{S_0}$.

Remark 3.36. Lemma 3.35 is also used frequently in algebraic geometry. Let X be a variety and H be a hypersurface (divisor) in X . If H corresponds to an effective divisor, we conclude that $N_{H/X} \cong \mathcal{O}_X(H)|_H$ using Lemma 3.35. In fact, in the case where X is a complex manifold, a hypersurface is always locally the zero locus of one holomorphic function, hence H is always an effective divisor.

Theorem 3.37 (Adjunction Formula). *Let H be a smooth hypersurface in a variety X , then we have*

$$K_X|_H = K_H \otimes \mathcal{O}_X(-H)|_H.$$

Proof. By Lemma 3.35 and Remark 3.36, we know that $N_{H/X} \cong \mathcal{O}_X(H)$. Now, dualizing the normal bundle sequence $0 \rightarrow T_H \rightarrow T_X|_H \rightarrow N_{H/X} \rightarrow 0$ gives

$$0 \rightarrow \mathcal{O}_X(-H)|_H \rightarrow \Omega_X^1 \rightarrow \Omega_H^1 \rightarrow 0.$$

Taking the determinant of this exact sequence gives the desired adjunction formula. $\square$

We conclude this section by introducing an important construction:

Definition 3.38. Let $L \rightarrow X$ be a holomorphic line bundle. A point $p \in X$ is a **base point** of L if $s(p) = 0$ for all $p \in H^0(X, L)$. The **base locus** $\text{Bs}(L)$ is the set of all base points of L .

Now let $s_0, \dots, s_N \in H^0(X, L)$ be a basis. Then we consider the map

$$\phi_L : X \setminus \text{Bs}(L) \rightarrow \mathbb{P}^N$$

defined by

$$\phi_L(p) = (s_0(p) : \dots : s_N(p)),$$

where by $(s_0(p) : \dots : s_N(p))$ we mean $(\varphi \circ s_0(p) : \dots : \varphi \circ s_N(p))$ under some trivializations φ of L . But clearly ϕ_L does not depend on the trivialization, because any two trivializations differ by a transition function, which becomes the same after we projectivize to $\mathbb{P}^N$. Also, one can see that ϕ_L does not depend on the choice of the basis $s_0, \dots, s_N$, for any two basis of $H^0(X, L)$ differ by a change of basis matrix, which again goes away when we projectivize to $\mathbb{P}^N$.

Theorem 3.39. *We have $\phi_L^* \mathcal{O}_{\mathbb{P}^N}(1) \cong L|_{X \setminus \text{Bs}(L)}$.*

Proof. Let $z_0 \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. Let $Y = Z(z_0) \subset \mathbb{P}^N$ be the associated divisor. Then $\phi_L^* Y$ is the divisor $Z(s_0) \subset X$. Then by Theorem 3.26, we have

$$L \cong \mathcal{O}_X(Z(s_0)) = \mathcal{O}_X(\phi_L^* Y) = \phi_L^* \mathcal{O}_X(Y) \cong \phi_L^* \mathcal{O}_{\mathbb{P}^N}(1)$$

on $X \setminus \text{Bs}(L)$. $\square$

This construction will be important when we study Kodaira theorems, because what we would like to do is find an embedding $X \rightarrow \mathbb{P}^N$ for some large N given a complex manifold. If we have line bundles $L \rightarrow X$ with empty base locus, then we will see that if the situation is nice enough, the map $\phi_L : X \rightarrow \mathbb{P}^N$ will be an embedding. But for now, we will see how this construction gives two important maps in projective spaces.

Definition 3.40. Let $N = \dim H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) - 1$. Then the degree d (for $d > 0$) **veronese map** $v_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$ is simply the map $\phi_{\mathcal{O}_{\mathbb{P}^n}(d)}$. Explicitly, a basis of $\mathcal{O}_{\mathbb{P}^n}(d)$ is given by homogeneous polynomials $z_0^d, z_0^d z_1, \dots, z_n^d$. Therefore,

$$v_d(z_0 : \dots : z_n) = (z_0^d : z_0^d z_1 : \dots : z_n^d) = (z^I)$$

where $I = (i_0, \dots, i_n)$ is a multi-index with $|I| = \sum i_k = d$, and $z^I = z_0^{i_0} z_1^{i_1} \dots z_n^{i_n}$.

Definition 3.41. For $m, n \geq 0$, let $p_1 : \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^m$ and $p_2 : \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ be the projection maps. Let $N = (m+1)(n+1) - 1$. Then the **Segre map** $\sigma : \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^N$ is simply the map $\phi_{p_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^n}(1)}$. Explicitly, this is given by

$$\sigma((z_0 : \dots : z_m), (z'_0 : \dots : z'_n)) = (z_0 z'_0 : z_0 z'_1 : \dots : z_m z'_n).$$

In fact, both $v_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$ and $\sigma : \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^N$ are embeddings.

3.4 Blow-Ups

Let X be a complex manifold and $Y \subset X$ a complex submanifold of codimension k . Then locally along Y , there exist holomorphic functions $f_1, \dots, f_k$ with independent differentials, such that $Y = Z(f_1, \dots, f_k)$. These equations are not unique, but we have the following:

Lemma 3.42. *If $g_1, \dots, g_k$ form another system of local equations for Y , then locally in the neighborhood of Y , there exists a matrix M_{ij} of holomorphic functions, such that*

$$g_i = \sum_j M_{ji} f_j. \quad (3.4)$$

Moreover, the matrix M_{ij} is invertible along Y , and its restriction to Y is uniquely determined by f_i, g_j .

Proof. It suffices to prove the lemma in the case where $f_i(z) = z_i$ are the first k coordinates. Then the functions g_i have the property of vanishing on $Z(z_1, \dots, z_k)$. Taking power series expansion we claim that we must have

$$g_i = \sum_{j \leq k} M_{ji} z_j. \quad (3.5)$$

In other words, if we write $g_i = \sum_{\alpha} c_{\alpha} z^{\alpha}$ for multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $z^{\alpha} = z_1^{\alpha_1} \dots z_n^{\alpha_n}$, then any term $c_{\alpha} z^{\alpha}$ that does not contain at least one of $z_1, \dots, z_k$ must be zero. To see this, let us write $u = (z_1, \dots, z_k), v = (z_{k+1}, \dots, z_n), \alpha = (\beta, \gamma)$ with $\beta = (\alpha_1, \dots, \alpha_k)$ and $\gamma = (\alpha_{k+1}, \dots, \alpha_n)$. Then we have $c_{\alpha} z^{\alpha} = c_{(\beta, \gamma)} u^{\beta} v^{\gamma}$. The condition that $g_i(z) = g_i(u, v) = 0$ for all v when $u = 0$ implies that

$$0 = g(0, v) = \sum_{\beta, \gamma} c_{(\beta, \gamma)} (0)^{\beta} v^{\gamma} = \sum_{\gamma} c_{(0, \gamma)} v^{\gamma}.$$

Since this is true for all v , by uniqueness principle of power series, we must have that $c_{(0, \gamma)} = 0$ for all γ . This proves (3.5).

(3.5) shows existence of M_{ij} . Taking the differential of (3.4) along Y , we have that

$$dg_i = \sum_{j \leq k} M_{ji} df_j$$

since along Y we have $f_j \equiv 0$. Since df_j 's are independent, it follows that given dg_i , the coefficients M_{ji} are uniquely determined. To find inverse of M_{ji} along Y (hence also in a neighborhood of Y), we interchange the role of g_i, f_j and repeat the argument. $\square$

If we cover a neighborhood of Y by open sets U in which Y is defined by k holomorphic equations, we obtain transition matrices M^{UV} which are invertible matrices with holomorphic coefficients defined in the neighborhood of Y . These matrices satisfies the condition

$$f_i^U = \sum_j M_{ji}^{UV} f_j^V,$$

where the f_i^U are equations for $Y \cap U$. The restrictions of these matrices to Y are uniquely determined by these equations. These matrices are the transition matrices for the conormal bundle $N_{Y/X}^*$, defined by taking the dual of the normal bundle sequence

$$0 \rightarrow N_{Y/X}^* \rightarrow \Omega_X^1 \rightarrow \Omega_Y^1 \rightarrow 0,$$

whose fiber at y consists of complex linear forms on $T_y X$ which vanishes on $T_y Y$. Indeed, this follows from differentiating the equation above, which yields

$$df_i^U = \sum_j M_{ji}^{UV} df_j^V.$$

Now let U be an open set of X , on which there exists functions $f_1, \dots, f_k$ with independent differentials such that $Y \cap U = Z(f_1, \dots, f_k)$. Now define

$$\hat{U} = \text{Bl}_Y U := \{(Z, z) \in \mathbb{P}^{k-1} \times U; Z_i f_j(z) = Z_j f_i(z), 1 \leq i, j \leq k\}.$$

Here, $Z = (Z_1, \dots, Z_k) \in \mathbb{P}^{k-1}$. We easily check that $\hat{U}$ is smooth complex submanifold of $\mathbb{P}^{k-1} \times U$, and we have a map $\tau : \hat{U} \rightarrow U$ which is projection onto the second coordinate, which is an isomorphism over $U - Y$. Above $Y \cap U$, the fiber of τ is equal to $\mathbb{P}^{k-1}$. We will now use Lemma 3.42 to glue up open sets $\hat{U}$ together to construct the **blow up** $\hat{X} = \text{Bl}_Y X$ of X along Y .

Lemma 3.43. *Let U, V be two open sets of X , in which Y is defined by equations $f_1^U, \dots, f_k^U, f_1^V, \dots, f_k^V$ respectively, with independent differentials. Then if $\tau_U : \hat{U} \rightarrow U$ and $\tau_V : \hat{V} \rightarrow V$ are blowups of U and V along $U \cap Y$ and $V \cap Y$ respectively, there exists a natural isomorphism*

$$\phi_{UV} : \tau_U^{-1}(U \cap V) \cong \tau_V^{-1}(U \cap V)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \tau_U^{-1}(U \cap V) & \xrightarrow{\phi_{UV}} & \tau_V^{-1}(U \cap V) \\ & \searrow \tau_U \quad \swarrow \tau_V & \\ & U \cap V & \end{array}$$

In other words, we can glue manifolds $\hat{U}, \hat{V}$ along the intersections $U \cap V$, hence we get a well-defined manifold $\hat{X} = \text{Bl}_Y X$.

Proof. Recall that the defining relations are related by $f_i^U = \sum_j M_{ji}^{UV} f_j^V$. Now let $P_{UV} = (M_{UV})^{-1}$. We define the holomorphic diffeomorphism $\varphi_{UV} : \mathbb{P}^{k-1} \times U \cap V \rightarrow \mathbb{P}^{k-1} \times U \cap V$ by

$$\phi_{UV}(Z, z) = (P_{UV}(z) \cdot Z, z).$$

Then it is easy to check that ϕ_{UV} sends $\tau_U^{-1}(U \cap V)$ to $\tau_V^{-1}(U \cap V)$, and the inverse map is given by the inverse diffeomorphism. $\square$

Definition 3.44. The manifold $\hat{X} = \text{Bl}_Y X$ obtained by gluing the manifolds $\hat{U}$ above the intersections $U \cap V$, using the identifications given by the previous result, is called the **blow up** of X along Y . The map $\tau : \hat{X} \rightarrow X$ is sometimes called the **blow-down** map.

Theorem 3.45. *The blow-up $\tau : \hat{X} \rightarrow X$ of X along Y has the property that*

- (i) $\tau : \hat{X} \setminus \tau^{-1}(Y) \rightarrow X \setminus Y$ is an isomorphism;
- (ii) $\tau : \tau^{-1}(Y) \rightarrow Y$ is isomorphic to $\mathbb{P}(N_{Y/X}) \rightarrow Y$.

Proof. (i) follows immediately by construction. To prove (ii), we note that the matrices P_{UV} which give transition morphisms for the projective bundle $\tau^{-1}(Y)$ are the transition matrices of the normal bundle $N_{X/Y}$, which is the dual of the conormal bundle (recall the discussion after the proof of Lemma 3.42). Thus $\tau^{-1}(Y) \cong \mathbb{P}(N_{Y/X})$. $\square$

The most elementary example, which we will focus on the remainder of this section, is when $Y = p$ is just a single point in X . The set $\tau^{-1}(p)$ is called the **exceptional divisor**, usually denoted by E .

We will first study the canonical bundle of this simple model.

Theorem 3.46. *Let $\hat{X} = \text{Bl}_p X$ be the blow up of a complex manifold X of dimension n at a point p , and let $E = \tau^{-1}(p)$ be the exceptional divisor. Then we have*

$$K_{\hat{X}} \cong \tau^* K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E).$$

Proof. We prove the statement by showing the transition functions of both sides are the same. We introduce some notations for future convenience: Let the transition function of $K_{\hat{X}}$ be labeled by ϕ , that of K_X be labeled by ψ , and that of $\mathcal{O}_{\hat{X}}(E)$ be labeled by ρ . We must first show that these bundles share the same trivializing neighborhoods on $\hat{X}$.

We first choose a holomorphic coordinate chart W_0 with coordinates $v^1, \dots, v^n$ centered at p , whose image is a ball $B(0, \epsilon) \subset \mathbb{C}^n$. Let $\{W_c\}_{c \in C}$ be a cover of $X \setminus \{p\}$ by holomorphic coordinate domains, with coordinates $\zeta_c^1, \dots, \zeta_c^n$. In each such open set there is a nonvanishing holomorphic section of K_X , namely $s_c = d\zeta_c^1 \wedge \dots \wedge d\zeta_c^n$ in W_c and $s_0 = dv^1 \wedge \dots \wedge dv^n$ in W_0 . The corresponding transition functions are ψ_{cd} on $W_c \cap W_d$ and ψ_{0d} on $W_0 \cap W_d$.

Next we construct a trivializing cover of $\hat{X}$ for $K_{\hat{X}}$. The map $\tau : \hat{X} \rightarrow X$ restricts to biholomorphism away from p , so we can define $\hat{W}_c = \tau^{-1}(W_c)$ for $c \in C$, and for each such set there is a holomorphic local frame $\hat{s}_c = \tau^* s_c$ for $K_{\hat{X}}$. We let $\hat{W}_0 = \tau^{-1}(W_0)$. We can cover $\hat{W}_0$ by open sets $U_\alpha = \{([w], v) : w^\alpha \neq 0\}$. On each such set there are holomorphic coordinates $z_\alpha^1, \dots, z_\alpha^n$ defined by

$$z_\alpha^j = \begin{cases} w^j / w^\alpha & j \neq \alpha \\ v^\alpha & j = \alpha. \end{cases}$$

On each U_α , there is a holomorphic local frame for $K_{\hat{X}}$ given by $\hat{s}_\alpha = dz_\alpha^1 \wedge \dots \wedge dz_\alpha^n$. To compute the transition functions for these frames, it is easiest to express them in terms of the functions $v^1, \dots, v^n$ on $\hat{W}_0$. On the complement of E in U_α , we have $w^j / w^\alpha = v^j / v^\alpha$. So

$$\begin{aligned} \hat{s}_\alpha &= d\left(\frac{v^1}{v^\alpha}\right) \wedge \dots \wedge dv^\alpha \wedge \dots \wedge d\left(\frac{v^n}{v^\alpha}\right) \\ &= \left(\frac{1}{v^\alpha}\right)^{n-1} dv^1 \wedge \dots \wedge dv^\alpha \wedge \dots \wedge dv^n \\ &= \left(\frac{1}{v^\alpha}\right)^{n-1} \pi^* s_0. \end{aligned}$$

Therefore, we see that

$$\hat{s}_\beta = \left(\frac{v^\alpha}{v^\beta}\right)^{n-1} \hat{s}_\alpha = \left(\frac{w^\alpha}{w^\beta}\right)^{n-1} \hat{s}_\alpha$$

on $(U_\alpha \cap U_\beta) \setminus E$. Hence by continuity this equation must hold on all of $U_\alpha \cap U_\beta$. The calculation above shows that (using the convention that for transition functions we have $\sigma_\beta = g_{\alpha\beta} \sigma_\alpha$ where $\sigma_\alpha, \sigma_\beta$ are local frames of some vector bundle and $g_{\alpha\beta}$ the transition functions)

$$\begin{aligned} \phi_{cd} &= \pi^* \psi_{cd} && \text{on } \hat{W}_c \cap \hat{W}_d \\ \phi_{\alpha d} &= (v^\alpha)^{n-1} \pi^* \psi_{0d} && \text{on } U_\alpha \cap \hat{W}_d \\ \phi_{\alpha\beta} &= (w^\alpha / w^\beta)^{n-1} && \text{on } U_\alpha \cap U_\beta. \end{aligned}$$

This same cover is a trivializing cover for π^*K_X . Since K_X is trivial on W_0 , it is trivial over each U_α , and it has transition functions

$$\begin{aligned} \pi^*\psi_{cd} & \quad \text{on } \hat{W}_c \cap \hat{W}_d \\ \pi^*\psi_{0d} & \quad \text{on } U_\alpha \cap \hat{W}_d \\ 1 & \quad \text{on } U_\alpha \cap U_\beta. \end{aligned}$$

Finally, the divisor E is given by $Z(1)$ in $\hat{W}_c$, and is defined as the vanishing locus of all functions $v^1, \dots, v^n$ inside U_α . Using the relations $w^\alpha v^\beta = w^\beta v^\alpha$ on U_α , we can describe $E \cap U_\alpha$ by a single equation, namely $v^\alpha = 0$. Therefore, it follows that $\{U_\alpha, W_c : c \in C\}$ is also a trivializing cover for $\mathcal{O}_{\hat{X}}(E)$, whose transition functions are

$$\begin{aligned} \rho_{cd} &= 1 & \text{on } \hat{W}_c \cap \hat{W}_d \\ \rho_{\alpha d} &= v^\alpha & \text{on } U_\alpha \cap \hat{W}_d \\ \rho_{\alpha\beta} &= w^\alpha/w^\beta & \text{on } U_\alpha \cap U_\beta. \end{aligned}$$

Therefore, we see that $\phi_{\mu\nu} = (\pi^*\psi_{\mu\nu})(\rho_{\mu\nu})^{n-1}$ for all indices μ, ν . This shows that we have isomorphism $K_{\hat{X}} \cong \tau^*K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E)$, as desired. $\square$

We conclude this section by pointing out that the self-intersection number of the exceptional divisor is negative:

Proposition 3.47. For the exceptional divisor $E \cong \mathbb{P}^{n-1}$ inside the blow up $\tau : \hat{X} \rightarrow X$, one has $\mathcal{O}_{\hat{X}}(E)|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

Proof. Indeed, by Corollary 3.34 and adjunction (Theorem 3.37), one has

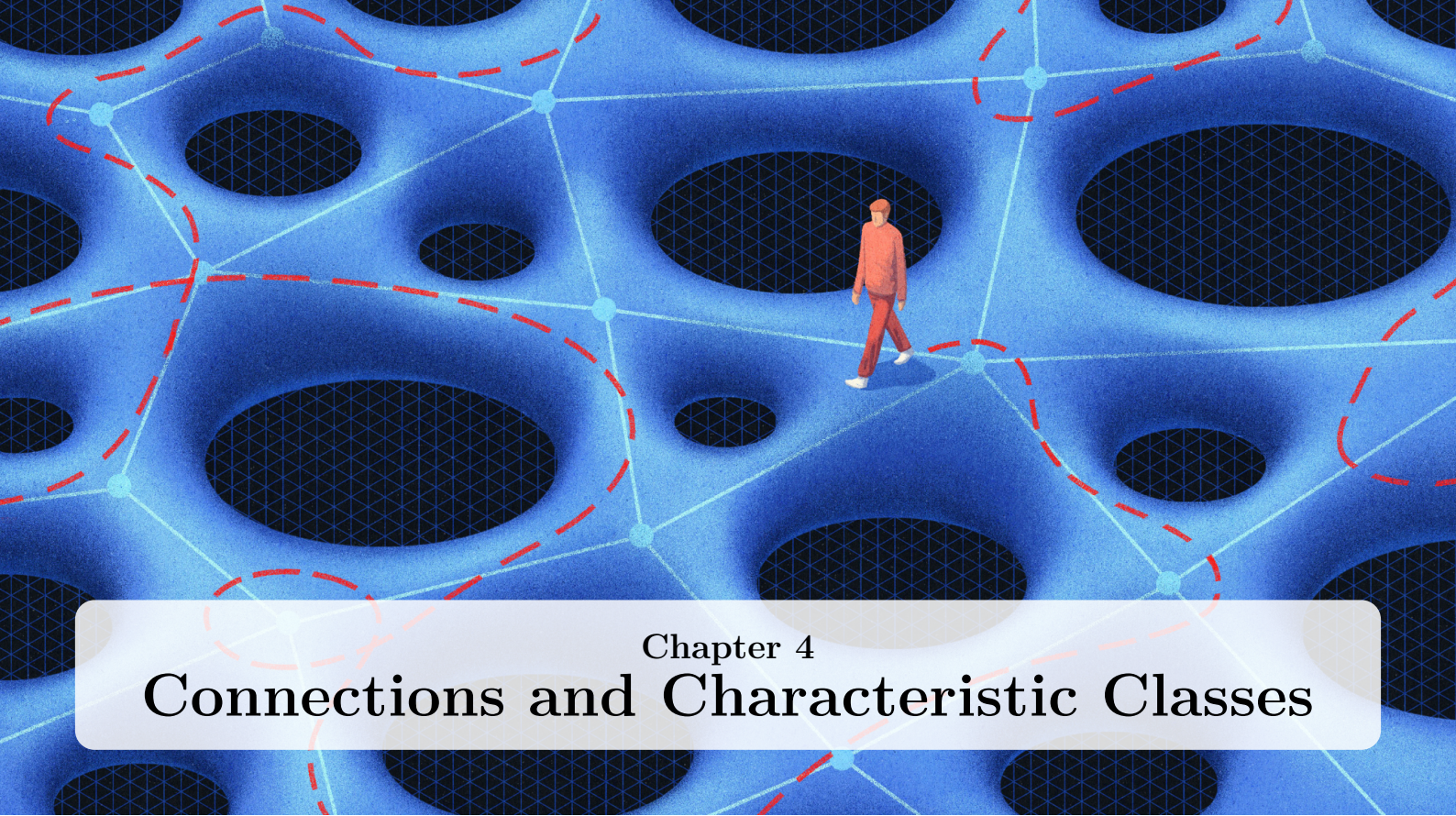
$$\mathcal{O}_{\mathbb{P}^{n-1}}(-n) \cong K_{\mathbb{P}^{n-1}} \cong (K_{\hat{X}} \otimes \mathcal{O}_{\hat{X}}(E))|_E.$$

By Theorem 3.46, we get that

$$\mathcal{O}_{\hat{X}}(nE)|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-n),$$

because $\tau^*K_X|_E = 0$. Then the desired statement follows from the fact that $\text{Pic}(\mathbb{P}^{n-1}) \cong \mathbb{Z}$ is torsion-free. We will prove this fact later using the machinery of Hodge theory (Theorem 6.54). $\square$

In fact, this result can be generalized to the case where Y is an embedded submanifold. See Lemma 5.22.



Chapter 4

Connections and Characteristic Classes

Introduction

Given a smooth or holomorphic vector bundle $E \rightarrow X$, a **characteristic class** $\alpha(E)$ of E is a cohomology class in $H^*(X; R)$ with coefficients in some ring R . These classes are topological invariants, and they tell us important topological information about the structure of E . In this chapter, we will see three important examples of such classes: Thom and Euler classes $\tau(E)$ and $e(E)$ of E , which are defined for both real vector bundles (for complex vector bundles of rank r , we view them as real vector bundles of rank $2r$). Finally, we will study the characteristic classes that are most relevant in complex algebraic geometry: the Chern classes of a holomorphic vector bundle. We have already seen that the 1st Chern class is defined via the boundary map $c_1 : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X; \mathbb{Z})$. We will see that there are two more equivalent definitions of c_1 , and in many situations one is more useful than the others. To do this, we will first need the machinery of connection and curvature on a vector bundle. We will prove the other two definitions of Chern classes are indeed equivalent (this is essentially Theorem 4.23). The proof that they are equivalent to the image of the sheaf theoretic map (actually they differ by a minus sign, but this is not important) defined above is tedious and is just a checking of definitions, so we omit it. A nicely written proof of this result can be found in Theorem 7.14 of [1] or Theorem 4.4.12 of [2].

4.1 Connections and Curvatures

Definition 4.1. Let E be a smooth complex vector bundle of rank m on a smooth manifold M . A **connection** is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E) \quad (X, \sigma) \mapsto \nabla_X \sigma,$$

that is linear over $C^\infty(M)$ in X , linear over $\mathbb{C}$ in σ , and satisfies the product rule

$$\nabla_X(f\sigma) = f\nabla_X\sigma + (Xf)\sigma$$

for all smooth complex-valued function f . The expression $\nabla_X \sigma$ is called the **covariant derivative** of σ in the direction X .

Proposition 4.2. The value of $\nabla_X \sigma$ at a point $p \in M$ depends only on the value of X at p and the value of σ in an arbitrarily small neighborhood of p .

Proof. For the first claim, let Y be a vector field such that $X(p) = Y(p)$. Let $U = X - Y$. Then $\nabla_X \sigma - \nabla_Y \sigma = \nabla_U \sigma = U^\mu \nabla_\mu \sigma$. Since $U^\mu(p) = 0$ for all μ , evaluating at p gives $\nabla_X \sigma(p) = \nabla_Y \sigma(p)$. Next, let $\sigma, \tau \in \Gamma(E)$ be such that $\sigma|_W = \tau|_W$ on some open set $W \subset M$. Then let φ be a bump function supported on W . Hence $\varphi(\sigma - \tau) \equiv 0$ on M . Hence

$$0 = \nabla_X \varphi(\sigma - \tau) = \varphi(\nabla_X \sigma - \nabla_X \tau) + (X\varphi)(\sigma - \tau).$$

Evaluating at p using $\varphi(p) = 1$ and $\sigma(p) - \tau(p) = 0$ gives $\nabla_X \sigma = \nabla_X \tau$, as desired. $\square$

The previous result immediately gives

Corollary 4.3. Any connection ∇ on E determines a connection on the restriction of E to any open subset of M , which we still denote by ∇ .

In addition to determining covariant derivatives with respect to specific vector fields, a connection also determines a total covariant derivative of each smooth section σ , defined by

$$\sigma \mapsto (X \mapsto \nabla_X \sigma)$$

Since $\text{Hom}(TM, E) \cong T^*M \otimes E = \mathcal{A}^1(E)$, one can equally define a connection ∇ as a map

$$\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E).$$

Note that here, we use $\mathcal{A}^p$ to denote p -forms on M , and we reserve Ω^p to denote the holomorphic p -forms on M . The product rule of the connection then becomes

$$\nabla(f\sigma) = f\nabla\sigma + df \otimes \sigma.$$

We extend our connections by complex linearity to accept complex vector fields as well as the real ones: If $Z = X + iY$ is a smooth complex vector field on M and ∇ a connection on E , we set

$$\nabla_Z \sigma := \nabla_X \sigma + i\nabla_Y \sigma.$$

Now let us write down local expressions of connections ∇ on E . We pick a smooth local frame $s_1, \dots, s_m$ of E over an open set $U \subset M$. For a given vector field X on U , we can express each covariant derivative locally as

$$\nabla_X s_\nu = \theta_\nu^\mu(X) s_\mu$$

for some coefficients $\theta_\nu^\mu(X)$. Then for an arbitrary section, we have

$$\nabla_X(\sigma^\alpha s_\alpha) = (X\sigma^\alpha) s_\alpha + \sigma^\alpha \theta_\alpha^\beta(X) s_\beta.$$

The coefficients $\theta_\nu^\mu \in \mathcal{A}^1(M)$ define a matrix of complex 1-forms, called the **connection form** with respect to this frame. Therefore, we can write the total covariant derivative as

$$\begin{aligned} \nabla s_\nu &= \theta_\nu^\mu \otimes s_\mu. \\ \nabla(\sigma^\alpha s_\alpha) &= d\sigma^\alpha \otimes s_\alpha + \sigma^\alpha \theta_\alpha^\mu \otimes s_\mu. \end{aligned} \tag{4.1}$$

If we replace the indices in the second term of (4.1) we get $\sigma^\alpha \theta_\alpha^\mu \otimes s_\mu = \sigma^\rho \theta_\rho^\alpha s_\alpha$. Collecting the terms, we see that $\nabla : \sigma^\alpha \mapsto d\sigma^\alpha + \theta_\rho^\alpha \sigma^\rho$. That is, if we view $\sigma^\alpha s_\alpha$ as a vector $(\sigma^1, \dots, \sigma^m)$ in the local frame, then the connection ∇ can be decomposed into an exterior differentiation operator and multiplication by the matrix of connection form θ acting on this vector. That is,

$$\nabla = d + \theta. \quad (4.2)$$

Definition 4.4. Let ∇ be a connection on a smooth complex vector bundle $E \rightarrow M$. We define a map $F : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E)$, called the **curvature** of ∇ , by

$$F(X, Y)\sigma = [\nabla_X, \nabla_Y]\sigma - \nabla_{[X, Y]}\sigma.$$

Here, by TM we mean the complexified tangent bundle, where F is first defined on the real tangent bundle and then extended by complex linearity.

Let θ_ν^μ be the connection 1-forms of ∇ . We can also write down local expressions of F as follows:

$$\begin{aligned} F(X, Y)s_\nu &= \nabla_X(\theta_\nu^\mu(Y)s_\mu) - \nabla_Y(\theta_\nu^\mu(X)s_\mu) - \theta_\nu^\mu([X, Y])s_\mu \\ &= X(\theta_\nu^\mu(Y))s_\mu + \theta_\nu^\mu(Y)\theta_\mu^\rho(X)s_\rho - Y(\theta_\nu^\mu(X))s_\mu - \theta_\nu^\mu(X)\theta_\mu^\rho(Y)s_\rho - \theta_\nu^\mu([X, Y])s_\mu \\ &= d\theta_\nu^\mu(X, Y)s_\mu + (\theta_\mu^\rho \wedge \theta_\nu^\mu)(X, Y)s_\rho \end{aligned}$$

where we used the formula of exterior derivative for 1-forms $\omega \in \mathcal{A}^1(M)$:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Relabeling the indices, we have that

$$F(X, Y)s_\nu = \Theta_\nu^\mu(X, Y)s_\mu,$$

where

$$\Theta_\nu^\mu = d\theta_\nu^\mu + \theta_\rho^\mu \wedge \theta_\nu^\rho \quad (4.3)$$

is a matrix of 2-forms. This is called the **curvature form** of ∇ . (4.3) in matrix form is sometimes written more compactly as

$$\Theta = d\theta + \theta \wedge \theta. \quad (4.4)$$

Note that the curvature $F : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E)$ is equivalently given by a map $F : \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E)$ defined by

$$F : \sigma \mapsto ((X, Y) \mapsto F(X, Y)\sigma).$$

Recall that the curvature is given by a map $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$. In fact, there is a sequence of maps on bundle-valued differential forms

$$\mathcal{A}^0(E) \xrightarrow{D} \mathcal{A}^1(E) \xrightarrow{D} \mathcal{A}^2(E) \rightarrow \dots$$

such that $D : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$ is precisely ∇ and $D^2 : \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E)$ is F , in the following sense:

Theorem 4.5. Let M be a complex manifold and $E \rightarrow M$ a smooth complex vector bundle, and ∇ is a connection on E . Then there are operators $D : \mathcal{A}^q(E) \rightarrow \mathcal{A}^{q+1}(E)$ for each $q \geq 0$ such that:

1. For $\sigma \in \mathcal{A}^0(E)$ we have $D\sigma = \nabla\sigma$.
2. For $\alpha \in \mathcal{A}^q(E)$ and $\beta \in \mathcal{A}^{q'}(E)$, we have

$$D(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^q \alpha \wedge D\beta.$$

3. $D^2\alpha = \Theta \wedge \alpha$. More specifically, if $\alpha = \alpha^\nu s_\nu$ in terms of a local frame of E , then $D^2\alpha = \Theta \wedge \alpha = (\Theta_\nu^\mu \wedge \alpha^\nu) s_\mu$.

Proof. We can simply define $D : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$ to be $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$ so that (1) is automatically satisfied. To extend this definition, we choose a smooth local frame (s_ν) for E . For a section $\alpha \in \mathcal{A}^q(E)$, expressed locally as $\alpha = \alpha^\mu s_\mu$, where $\alpha \in \mathcal{A}^q(M)$, we define $D\alpha$ to be the element in $\mathcal{A}^{q+1}(E)$ whose local expression is given by (2). A local computation shows that $D\alpha$ is globally well-defined (See [1], Proposition 7.11).

Finally, to check (3), we differentiate twice for $\alpha \in \mathcal{A}^q(M)$ and $\sigma \in \mathcal{A}^0(E)$, and (2) gives

$$\begin{aligned} D^2(\alpha \wedge \sigma) &= D(d\alpha \wedge \sigma + (-1)^q \alpha \wedge D\sigma) \\ &= (-1)^{q+1} d\alpha \wedge D\sigma + (-1)^q d\alpha \wedge D\sigma + \alpha \wedge D^2\sigma \\ &= \alpha \wedge D^2\sigma. \end{aligned}$$

This shows that the action of D^2 on $\mathcal{A}^q(E)$ is determined by that on $\mathcal{A}^0(E)$. Moreover, by taking α to be a 0-form it shows that D^2 is linear over $C^\infty(M)$. Thus, it suffices to check (3) on elements of a local frame for E . Then it follows from (2) that

$$D^2 s_\nu = D(\theta_\nu^\mu \otimes s_\mu) = d\theta_\nu^\mu \otimes s_\mu - \theta_\nu^\mu \wedge \theta_\mu^\rho \otimes s_\rho = (d\theta_\nu^\mu - \theta_\nu^\rho \wedge \theta_\rho^\mu) s_\mu = \Theta_\nu^\mu \otimes s_\mu,$$

which is (3) in this case. $\square$

Remark 4.6. Using Theorem 4.5, it is common in complex algebraic geometry to abuse notation and denote the extended operators D by ∇ , and denote the 2-form Θ again by F (or F_∇ if we want to stress it comes from the connection ∇). Then part 3 of Theorem 4.5 using this identification becomes $F_\nabla = \nabla^2 : \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E)$. It takes some getting use to but it is extremely convinient since we can compute curvature in a coordinate free way. We will use this convention to prove Theorem 4.8.

We recall that given two complex vector bundles E_1, E_2 , we can form their direct sum, tensor product, Hom and dual bundles. The next example shows how to define connections on these bundles.

Example 4.7. Let $E_1, E_2 \rightarrow M$ be complex vector bundles, with connections ∇_1, ∇_2 .

1. We can give $E_1 \oplus E_2$ a connection by setting $\nabla(s_1 \oplus s_2) := \nabla_1 s_1 \oplus \nabla_2 s_2$. Conversely, a connection ∇ on $E_1 \oplus E_2$ gives connections ∇_i on E_i for $i = 1, 2$ in a natural way: we can define $\nabla_i s_i := \text{pr}_i(\nabla s_i)$.
2. We can give $E_1 \otimes E_2$ a connection by setting $\nabla(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$.
3. A connection ∇ on $\text{Hom}(E_1, E_2)$ can also be defined. We note that a section f on $\text{Hom}(E_1, E_2)$ is a fiberwise morphism $f : E_1 \rightarrow E_2$ over M . ∇f should be a another section of $\text{Hom}(E_1, E_2)$, which sends a section s_1 of E_1 to a section of E_2 . We can define

$$\nabla f(s_1) = \nabla_2(f(s_1)) - f(\nabla_1 s_1).$$

We note that in the second term, $\nabla_1 s_1 \in \mathcal{A}^1(E_1)$ is a E_1 -valued 1-form. By $f(\nabla_1 s_1)$ we mean applying f to its $\mathcal{A}^0(E_1)$ part: That is, if locally we have $\nabla_1 s_1 = \alpha^\mu s_\mu$ where $\alpha^\mu \in \mathcal{A}^1(M)$ and $s_\mu \in \mathcal{A}^0(E_1)$ are local frames of E_1 , then we define $f(\nabla_1 s_1) := \alpha^\mu f(s_\mu)$.

4. A connection on the trivial bundle $\underline{\mathbb{C}} = M \times \mathbb{C}$ is $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$, which is simply $\nabla : \mathcal{A}^0(M) \rightarrow \mathcal{A}^1(M)$. We can simply take ∇ to be the exterior derivative operator $d : \mathcal{A}^0(M) \rightarrow \mathcal{A}^1(M)$. Note that since $d^2 = 0$, by Theorem 4.5 this connection has vanishing curvature, hence the trivial bundle is flat, as expected.
5. Given a connection ∇ on E , we can define a connection ∇^* on the dual bundle $E^* = \text{Hom}(E, \underline{\mathbb{C}})$ using d as the connection on $\mathbb{C}$ as discussed in the previous two examples. Then this becomes

$$\nabla^* f(s) = d(f(s)) - f(\nabla s).$$

Or using the notation of a pairing from $E^* \otimes E$ to $\underline{\mathbb{C}}$, this is

$$\langle \nabla^* f, s \rangle = d\langle f, s \rangle - \langle f, \nabla s \rangle. \quad (4.5)$$

6. Let $f : M \rightarrow N$ be a smooth map and ∇ a connection on a vector bundle $E \rightarrow N$. Then there is a **pullback connection** $f^*\nabla$ on $f^*E \rightarrow M$. Recall from (4.1) and (4.2) we have a local expression $\nabla = d + \theta$. Then we can define locally $f^*\nabla = d + f^*\theta$. It is easy to check that the locally given connection glue to a global one on f^*E .

Using the definition of connections on various bundles in Example 4.7, we can also obtain information of curvature 2-forms on these bundles.

Proposition 4.8. Let E_1, E_2 be vector bundles on M with connections ∇_1, ∇_2 , with curvatures $F_{\nabla_1}, F_{\nabla_2}$, respectively. Then:

- (a) The curvature of the induced connection on the direct sum $E_1 \oplus E_2$ is given by $F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}$.
- (b) On the tensor product $E_1 \otimes E_2$ the curvature is given by $F_\nabla = F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}$.
- (c) For the induced connection ∇^* on the dual bundle E^* the curvature is given by

$$F_{\nabla^*} = -F_\nabla^t.$$

- (d) Let $f : M \rightarrow N$ be a smooth map and ∇ the connection on $E \rightarrow N$ with curvature Θ . Then $f^*\nabla$ has curvature $f^*\Theta$.

Proof. (a) is straightforward so we omit it. (b) is a straightforward computation. Let $s_1 \in \Gamma(E_1)$ and $s_2 \in \Gamma(E_2)$ be local sections. The connection ∇ on $E_1 \otimes E_2$ is defined by its action on decomposable sections:

$$\nabla(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$$

To find the curvature $F_\nabla = \nabla^2$, we apply ∇ a second time. We have to be careful here because $\nabla_1 s_1$ and $\nabla_2 s_2$ are bundle-valued 1-forms, not 0-forms (sections). When applying the connection to a tensor product of a bundle-valued k -form $\alpha \in \mathcal{A}^k(E_1)$ and a bundle-valued l -form $\beta \in \mathcal{A}^l(E_2)$, the graded Leibniz rule introduces a sign:

$$\nabla(\alpha \otimes \beta) = \nabla_1 \alpha \otimes \beta + (-1)^k \alpha \wedge \nabla_2 \beta.$$

Therefore, we have

$$\begin{aligned}\nabla^2(s_1 \otimes s_2) &= \nabla_1^2 s_1 \otimes s_2 - \nabla_1 s_1 \otimes \nabla_2 s_2 + \nabla_1 s_1 \otimes \nabla_2 s_2 + s_1 \otimes \nabla_2^2 s_2 \\ &= (F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2})(s_1 \otimes s_2).\end{aligned}$$

To prove (c), Let $s \in \Gamma(E)$ be a smooth local section of E and $f \in \Gamma(E^*)$ be a smooth local section of the dual bundle E^* . The natural pairing is defined as $\langle f, s \rangle = f(s)$. We are given the definition of the dual connection:

$$d\langle f, s \rangle = \langle \nabla^* f, s \rangle + \langle f, \nabla s \rangle.$$

To compute the curvature $F_{\nabla^*} = (\nabla^*)^2$, we must extend the Leibniz rule to bundle-valued differential forms. For $\alpha \in \mathcal{A}^k(E^*)$ and $\beta \in \mathcal{A}^l(E)$, the exterior derivative of their pairing reads:

$$d\langle \alpha, \beta \rangle = \langle \nabla^* \alpha, \beta \rangle + (-1)^k \langle \alpha, \nabla \beta \rangle.$$

Hence we have (Note that the first equation has a plus sign because $\nabla^* f$ is a 1-form so $(-1)^k = -1$ and when we move it to the other side of the equation we get a plus sign):

$$\begin{aligned}\langle \nabla^{*2} f, s \rangle &= d\langle \nabla^* f, s \rangle + \langle \nabla^* f, \nabla s \rangle \\ &= d(d\langle f, s \rangle - \langle f, \nabla s \rangle) + \langle \nabla^* f, \nabla s \rangle \\ &= -d\langle f, \nabla s \rangle + \langle \nabla^* f, \nabla s \rangle \\ &= -\langle \nabla^* f, \nabla s \rangle - \langle f, \nabla^2 s \rangle + \langle \nabla^* f, \nabla s \rangle \\ &= -\langle f, \nabla^2 s \rangle = -\langle f, F_{\nabla} s \rangle.\end{aligned}$$

Therefore, with respect to the natural pairing, we see that $F_{\nabla^*} = -F_{\nabla}^t$.

Finally, to prove (d), we recall from Example 4.7 that the pullback connection 1-form on f^*E is simply $f^*\theta$. Hence the curvature 2-form on f^*E is

$$d(f^*\theta) + f^*\theta \wedge f^*\theta = f^*(d\theta + \theta \wedge \theta) = f^*\Theta.$$

This concludes the proof. □

4.2 Chern Classes

Now let M be a complex manifold of dimension m (hence real dimension $2m$). For every $k = 0, 1, \dots, r$, and complex vector bundle $E \rightarrow M$ of rank r , we define a family of characteristic classes $c_k(E) \in H^{2k}(M; \mathbb{C})$ called the **Chern class** as follows: Recall that the k -th **elementary symmetric polynomial** is defined by

$$P_k(\lambda_1, \dots, \lambda_r) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

We define $c_0(E) = 1$ for any E . For $k > 0$, the k -th **Chern class** of E is defined to be

$$c_k(E) = P_k\left(\frac{i\Theta}{2\pi}\right),$$

where by P_k evaluated on an operator (in this case a 2-form valued matrix) we mean the P_k evaluated on eigenvalues of the matrix. Note that for a 2-form valued matrix, its eigenvalues are also 2-forms, as evident from the eigenvalue equation $\Theta_\nu^\mu \alpha^\nu = \lambda \alpha^\mu$ where $\Theta_\nu^\mu \in \mathcal{A}^2(M)$. Since eigenvalues of a linear operator is independent of its matrix

representation, $c_k(E)$ is well-defined in $\mathcal{A}^{2k}(M)$. The next key technical result is that $c_k(E)$ are all closed forms, i.e., they define cohomology classes in $H^{2k}(M; \mathbb{C})$.

One tool we shall use in the proofs is the **supercommutator** for operators in $\mathcal{A}^k(\text{End } E)$, defined by

$$[\omega \otimes A, \eta \otimes B] := (\omega \wedge \eta) \otimes [A, B],$$

where ω, η are locally defined forms and A, B are local sections of $\text{End } E$. It is easy to see that

$$[\omega \otimes A, \eta \otimes B] = \omega A \wedge \eta B - (-1)^{|\omega| \cdot |\eta|} \eta B \wedge \omega A.$$

The appearance of the extra factor $(-1)^{|\omega| \cdot |\eta|}$ is the reason why sometimes it is called a “super”-commutator. We sometimes extend the definition. We define for the connection D and $\omega \otimes A$:

$$[D, \omega \otimes A] := D(\omega \otimes A) - (-1)^{|\omega|} \omega \otimes A \wedge D.$$

We state four useful lemmas. The key results to prove that Chern classes are closed (which is an easy corollary of the Chern-Weil theorem, Theorem 4.13), is the Bianchi identity (Lemma 4.11), and Lemma 4.12.

Lemma 4.9. *If D' is another connection on E , then $D' - D \in \mathcal{A}^1(M, \text{End } E)$.*

Proof. Any connection is of the form $D = d + \theta$. Hence $D' - D = \theta' - \theta$ which is in $\mathcal{A}^1(M, \text{End } E)$. $\square$

Lemma 4.10. *If P, Q are both $\text{End } E$ -valued differential forms, then $\text{Tr}[P, Q] = 0$.*

Proof. Since the super commutator is bilinear, it suffices to prove the claim for the simple case where $P = \omega \otimes A$ and $Q = \eta \otimes B$. Then $\text{Tr}[P, Q] = (\omega \wedge \eta) \cdot \text{Tr}[A, B]$. But $\text{Tr}[A, B] = \text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA) = 0$. $\square$

Lemma 4.11 (Bianchi identity). *For any $k \geq 0$ we have $[D, \Theta^k] = 0$.*

Proof. Simply note that $\Theta = D^2$, so $[D, \Theta^k] = [D, D^{2k}] = 0$. $\square$

Lemma 4.12. *For $A \in \mathcal{A}^k(M, \text{End } E)$, we have $d \text{Tr}(A) = \text{Tr}[D, A]$.*

Proof. First note that the left-hand side is obviously independent of the connection. For the right-hand side, if we use another connection D' , by Lemma 4.9 and 4.10, we have

$$\text{Tr}[D', A] = \text{Tr}[D' - D, A] + \text{Tr}[D, A] = \text{Tr}[D, A].$$

So the right-hand side is also independent of the connection. Therefore, we can in fact choose a trivial connection locally to carry out the computation: Let $D_0 = d$ be a trivial connection on $E|_U \rightarrow U$, then

$$\begin{aligned} [D_0, A] s &= D_0(As) - (-1)^{|A|} A \wedge D_0 s \\ &= d(A_\alpha^\beta f^\alpha) e_\beta - (-1)^{|A|} A_\alpha^\beta \wedge d f^\alpha e_\beta \\ &= d A_\alpha^\beta f^\alpha e_\beta. \end{aligned}$$

Hence $\text{Tr}[D_0, A] = d \text{Tr}(A)$. $\square$

For any formal power series in one variable $f(t) = a_0 + a_1 t + a_2 t^2 + \dots$, we define $f(\Theta) := a_0 + a_1 \Theta + \dots + a_n \Theta^n \in \mathcal{A}^*(M)$.

Theorem 4.13 (Chern–Weil). *Let $E \rightarrow M$ be a complex vector bundle with connection D . For f defined as above, we have:*

1. $d \operatorname{Tr} f(\Theta) = 0$;
2. *Let D, D' be connections on E with curvature 2-forms Θ, Θ' respectively. Then there is a differential form $\eta \in \mathcal{A}^*(M)$ such that*

$$\operatorname{Tr} f(\Theta') - \operatorname{Tr} f(\Theta) = d\eta.$$

So the cohomology class of $\operatorname{Tr} f(\Theta)$ is independent of the connection. We call it the **characteristic class** of E associated to f , and $\operatorname{Tr} f(\Theta)$ the corresponding **characteristic form** of E associated to f and D .

Proof. For the first conclusion, by Lemma 4.3.4, we have

$$d \operatorname{Tr} f(\Theta) = \operatorname{Tr}[D, f(\Theta)] = \sum_k a_k \operatorname{Tr}[D, \Theta^k] = 0,$$

where we used the Bianchi identity (Lemma 4.11) in the last step.

For the second claim, we first note that if D, D' are two connections, then so are $tD' + (1-t)D$ for any $t \in [0, 1]$. Thus we can choose a family of connections

$$D_t := tD' + (1-t)D.$$

Then

$$\dot{D}_t := \frac{dD_t}{dt} = D' - D \in \mathcal{A}^1(M, \operatorname{End} E),$$

and

$$\dot{\Theta}_t := \frac{d\Theta_t}{dt} = \frac{d}{dt}(D_t D_t) = D_t \dot{D}_t + \dot{D}_t D_t = [D_t, \dot{D}_t].$$

Therefore, we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Tr} f(\Theta_t) &= \operatorname{Tr}(\dot{\Theta}_t f'(\Theta_t)) \\ &= \operatorname{Tr}([D_t, \dot{D}_t] f'(\Theta_t)) \\ &= \operatorname{Tr}([D_t, \dot{D}_t f'(\Theta_t)]) \\ &= d \operatorname{Tr}(\dot{D}_t f'(\Theta_t)), \end{aligned}$$

where the third equality is by the Bianchi identity (Lemma 4.11), and the last equality is by Lemma 4.12. So we conclude that

$$\operatorname{Tr} f(\Theta') - \operatorname{Tr} f(\Theta) = d \int_0^1 \operatorname{Tr}(\dot{D}_t f'(\Theta_t)) dt.$$

This concludes the proof. □

Now we can finally prove that

Theorem 4.14. *The Chern classes $c_k(E) := P_k(\Theta)$ defined by the elementary symmetric polynomials are closed forms, hence they descend to cohomology classes $c_k(E) \in H^{2k}(M; \mathbb{C})$.*

Proof. Notice that for any matrix B , we have $\det(I + tB) = \sum_{k=0}^r P_k(B)t^k$. Hence we see that

$$c_0(E) + c_1(E)t + \cdots + c_r(E)t^r = \det\left(I + \frac{it\Theta}{2\pi}\right). \quad (4.6)$$

Evaluating at $t = 1$ gives

$$c_0(E) + c_1(E) + \cdots + c_r(E) = \det\left(I + \frac{i\Theta}{2\pi}\right)$$

in $\mathcal{A}^*(M)$. Now notice that

$$\det\left(I + \frac{i\Theta}{2\pi}\right) = \exp\left(\text{Tr} \log\left(I + \frac{i\Theta}{2\pi}\right)\right).$$

Expanding the exponential using power series, taking exterior derivatives term by term, and using the first part of Chern-Weil theorem (Theorem 4.13), we see that all the $c_k(E)$'s are closed. $\square$

Moreover, the second part of Theorem 4.13 implies that

Theorem 4.15. $c_k(E)$ does not depend on the connection chosen. In other words, the Chern classes $c_k(E)$ are topological invariants.

Motivated by (4.6), we define the **total Chern class** to be

$$c(E) := c_0(E) + \cdots + c_r(E) = \det\left(I + \frac{i\Theta}{2\pi}\right).$$

Using Proposition 4.8, we can easily obtain

Theorem 4.16. Chern classes are natural in the sense that:

- (a) $c(E_1 \oplus E_2) = c(E_1)c(E_2)$. In particular, for every short exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ of vector bundles one has $c(E) = c(E_1)c(E_2)$.
- (b) $c_k(E^*) = (-1)^k c_k(E)$. In particular $c_1(E^*) = -c_1(E)$.
- (c) Let $f : M \rightarrow N$ be a smooth map and $E \rightarrow N$ a vector bundle. Then $c_k(f^*E) = f^*c_k(E)$.

Proof. (a) Since for direct sum, we have $\Theta = \Theta_1 \oplus \Theta_2$ by Proposition 4.8, and note that

$$c(E_1 \oplus E_2) = \det\left(I + \frac{i\Theta}{2\pi}\right) = \det\left(I_{E_1} + \frac{i\Theta_1}{2\pi}\right) \det\left(I_{E_2} + \frac{i\Theta_2}{2\pi}\right) = c(E_1)c(E_2).$$

Moreover, since every short exact sequence of vector bundle splits in the C^∞ -category, and the total Chern class c is a topological invariant, the last statement follows.

(b) Since $\Theta^* = -\Theta^T$, we have

$$c(E^*) = \det\left(I + \frac{i\Theta^*}{2\pi}\right) = \det\left(I + \frac{i(-\Theta^T)}{2\pi}\right) = \det\left(I + \frac{i(-\Theta)}{2\pi}\right),$$

it follows that $c_k(E^*) = (-1)^k c_k(E)$.

(c) This follows immediately from Proposition 4.8 which says f^*E has curvature 2-form given by $f^*\Theta$. $\square$

Also, we can deduce that the first Chern class

$$c_1(E) = \text{Tr} \left(\frac{i\Theta}{2\pi} \right)$$

is additive for line bundles under tensor product:

Theorem 4.17. $c_1(E \otimes E') = c_1(E) \text{rk } E' + c_1(E') \text{rk } E$. In particular for line bundles L, L' we have $c_1(L \otimes L') = c_1(L) + c_1(L')$.

Proof. By Proposition 4.8, we have $\Theta_{E \otimes E'} = \Theta_E \otimes \text{id}_{E'} + \text{id}_E \otimes \Theta_{E'}$. The result follows from taking trace. $\square$

4.3 Chern Connection

Recall that a **Hermitian metric** on a complex vector space V is a map $\langle -, - \rangle : V \times V \rightarrow \mathbb{C}$ that satisfies

- Conjugate symmetry: $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
- Sesquilinearity: $\langle av, bw \rangle = a\bar{b}\langle v, w \rangle$.
- Positive definiteness: $\langle v, v \rangle > 0$ unless $v = 0$.

Suppose M is a smooth manifold and $E \rightarrow M$ a smooth complex vector bundle of rank m . A **Hermitian fiber metric** on E is a choice h_p of Hermitian inner product on each fiber E_p that is smoothly varying in the sense that for any pair of smooth sections σ, τ of E over an open set $U \subset M$, the value $h_p(\sigma(p), \tau(p))$ depends smoothly on $p \in U$. For convenience, we will simply write $\langle \sigma, \tau \rangle$ instead of $h(\sigma, \tau)$ or even $h_p(\sigma(p), \tau(p))$ when there is no danger of confusion. A smooth complex vector bundle endowed with a Hermitian fiber metric is called a **Hermitian vector bundle**. A simple partition of unity argument shows that every smooth complex vector bundle admits a Hermitian fiber metric.

Recall that we defined an operator $\bar{\partial}_E : \mathcal{A}^0(E) \rightarrow \mathcal{A}^{0,1}(E)$. We can compose the connection $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$ with the projection map $\mathcal{A}^1(E) \rightarrow \mathcal{A}^{0,1}(E)$ and get an operator

$$\nabla^{0,1} : \mathcal{A}^0(E) \rightarrow \mathcal{A}^{0,1}(E).$$

One can ask what is the relation between $\nabla^{0,1}$ and $\bar{\partial}_E$. In fact there is a case when the two coincide, and this connection is even compatible with the given Hermitian metric in the following sense:

Definition 4.18. We say a connection ∇ on E is **compatible** with the Hermitian metric $\langle -, - \rangle$ if for two sections σ, τ , we have

$$d\langle \sigma, \tau \rangle = \langle \nabla \sigma, \tau \rangle + \langle \sigma, \nabla \tau \rangle. \quad (4.7)$$

The right hand side involves inner product with E -valued 1-forms, hence by inner product we of course mean just contracting the component in $\mathcal{A}^0(E)$. Note that one should be careful since the Hermitian metric is sesquilinear, so $\langle s, \omega \otimes s' \rangle = \bar{\omega} \langle s, s' \rangle$ for $s, s' \in \mathcal{A}^0(E)$ and $\omega \in \mathcal{A}^1(E)$.

Proposition 4.19. Let $E \rightarrow M$ be a complex Hermitian vector bundle. There exists a unique connection ∇ on E that is compatible with the given Hermitian metric, and such that $\nabla^{0,1} = \bar{\partial}_E$. This connection is called the **Chern connection**.

Proof. To construct ∇ , of course we need to define $\nabla^{0,1} = \bar{\partial}_E$. Hence it remains to prove the existence and uniqueness of the $\nabla^{1,0}$ part. Taking the $(1,0)$ part of the compatibility equation (4.7), we see that

$$\partial\langle\sigma, \tau\rangle = \langle\nabla^{1,0}\sigma, \tau\rangle + \langle\sigma, \bar{\partial}_E\tau\rangle.$$

To specify $\nabla^{1,0}$, we can pick a local holomorphic basis $\sigma_1, \dots, \sigma_m$ of E , and we only need to know what $\nabla^{1,0}\sigma_i$ are to specify the matrix of $\nabla^{1,0}$. Since these are holomorphic basis, $\bar{\partial}_E\sigma_i = 0$. Hence the above equation becomes

$$\langle\nabla^{1,0}\sigma_i, \sigma_j\rangle = \partial\langle\sigma_i, \sigma_j\rangle. \quad (4.8)$$

These equations completely characterizes $\nabla^{1,0}$, which proves existence and uniqueness. $\square$

Lemma 4.20. *Let ∇ be a connection on a complex vector bundle E . Then $\nabla^{0,1} = \bar{\partial}_E$ iff for each holomorphic local frame (s_ν) , we have $\nabla s_\nu = \theta_\nu^\mu \otimes s_\mu$ where the 1-forms θ_ν^μ are all of type $(1,0)$.*

Proof. Choose a holomorphic local frame (s_ν) . Take the projection of (4.1) onto its $(0,1)$ component gives

$$\nabla^{0,1}(\sigma^\mu s_\mu) = \bar{\partial}\sigma^\mu \otimes s_\mu + \sigma^\mu(\theta_\mu^\rho)^{0,1} \otimes s_\rho.$$

On the other hand, the definition of $\bar{\partial}_E$ operator gives

$$\bar{\partial}_E(\sigma^\mu s_\mu) = \bar{\partial}\sigma^\mu \otimes s_\mu.$$

Comparing the two equations, we see that $\nabla^{0,1} = \bar{\partial}_E$ iff $(\theta_\mu^\rho)^{0,1} = 0$. $\square$

Proposition 4.21. Suppose $E \rightarrow M$ is a Hermitian holomorphic vector bundle over a complex manifold. The curvature Θ associated with the Chern connection on E is of type $(1,1)$. That is, $\Theta \in \mathcal{A}^{1,1}(M, \text{End } E)$.

Proof. In terms of a holomorphic local frame (s_ν) , we can write (4.3) as

$$\Theta_j^m = \bar{\partial}\theta_j^m + \partial\theta_j^m + \theta_k^m \wedge \theta_j^k.$$

By Lemma 4.20, the first term is of type $(1,1)$, while the last three are all of type $(2,0)$. So we must show that the last three terms adds up to zero. If we write out (4.8), we have $\nabla^{1,0}\sigma_i = \theta_i^p \sigma_p$. Hence (4.8) becomes $\theta_i^p h_{pj} = \partial h_{ij}$. Thus we have

$$\theta_j^m = h^{km} \partial h_{jk}. \quad (4.9)$$

Substituting this expression into the above equation gives

$$\begin{aligned} \partial\theta_j^m + \theta_k^m \wedge \theta_j^k &= \partial(h^{km} \partial h_{jk}) + (h^{pm} \partial h_{kp}) \wedge (h^{qk} \partial h_{jq}) \\ &= \partial h^{km} \wedge \partial h_{jk} + h^{pm} h^{qk} \partial h_{kp} \wedge \partial h_{jq}. \end{aligned}$$

Now differentiating $h^{kp} h_{pl} = \delta_l^k$ and multiplying by h^{lm} gives

$$\partial h^{km} = -h^{lm} h^{kp} \partial h_{pl}.$$

Substituting this above and renaming dummy indices proves the result. $\square$

Example 4.22 (Chern Connection on a Line Bundle). Let $L \rightarrow M$ be a Hermitian holomorphic line bundle and ∇ be its Chern connection. Let s be a local frame of L , i.e., a nonvanishing holomorphic local section, over an open set $U \subset M$, let θ be the corresponding connection 1-form. The fiber metric is determined completely by the strictly positive smooth function $h = |s|^2 = \langle s, s \rangle$. In this case, (4.9) reduces to

$$\theta = h^{-1} \partial h = \partial(\log h).$$

Its curvature is given by 2-form

$$\Theta = d\theta = \bar{\partial}\partial(\log h) = \bar{\partial}\partial(\log |s|^2).$$

Note that Θ is globally defined, for if we go to a different local frame then they will differ by a transition function which is holomorphic, taking the log then applying $\bar{\partial}$ kills the additional term.

Thus the Chern form for this connection has the local expression

$$c_1(L, \nabla) = \frac{i}{2\pi} \bar{\partial}\partial(\log h) = \frac{i}{2\pi} \bar{\partial}\partial(\log |s|^2).$$

Theorem 4.23. *Let X be a compact complex manifold and $D \subset X$ be an irreducible hypersurface. Then $c_1(\mathcal{O}_X(D)) = \text{PD}(D)$.*

Proof. Let $L = \mathcal{O}_X(D)$ for convenience. Let Θ be the curvature 2-form of the Chern connection on L . Then $c_1(\mathcal{O}_X(D)) = i\Theta/2\pi$. To show that this is equal to the Poincaré dual $\text{PD}(D)$, one needs to show that for any closed real form α of degree $2n - 2$ (where $n = \dim X$), we have

$$\frac{i}{2\pi} \int_X \Theta \wedge \alpha = \int_D \alpha.$$

Let's fix a trivializing open cover $X = \bigcup_i U_i$ and trivializations $\phi_i : L|_{U_i} \cong U_i \times \mathbb{C}$. We introduce a strictly positive function $h_i : U_i \rightarrow (0, \infty)$ that completely determines the Hermitian structure h , by

$$h(s(x)) = h(s(x), s(x)) = h_i(x) \cdot |\phi_i(s(x))|^2$$

for any local section s . If s is holomorphic on U_i that vanishes along D then by the previous example one has, as in the previous example,

$$\bar{\partial}\partial \log(h \circ s) = \bar{\partial}\partial \log(h_i) = \Theta|_{U_i} \text{ on } U_i \setminus D$$

because $\bar{\partial}\partial \log(\phi_i \circ s) = \bar{\partial}\partial \log(\bar{\phi}_i \circ s) = 0$ because ϕ_i is holomorphic. Let $s \in H^0(X, L)$ be the global holomorphic section defining D , given by Theorem 3.27. Define the tubular neighborhood

$$D_\epsilon = \{x \in X : |h(s(x))| < \epsilon\}.$$

Then, using the fact that $\bar{\partial}, \partial$ anticommutes (Proposition 2.60), and Stoke's theorem given $d\alpha = 0$ by assumption, we have

$$\begin{aligned} \frac{i}{2\pi} \int_X \Theta \wedge \alpha &= \lim_{\epsilon \rightarrow 0} \frac{i}{2\pi} \int_{X \setminus D_\epsilon} \Theta \wedge \alpha = \lim_{\epsilon \rightarrow 0} \frac{i}{2\pi} \int_{X \setminus D_\epsilon} \bar{\partial}\partial \log(h \circ s) \wedge \alpha \\ &= \lim_{\epsilon \rightarrow 0} \frac{i}{4\pi} \int_{X \setminus D_\epsilon} d(\partial - \bar{\partial}) \log(h \circ s) \wedge \alpha \\ &= \lim_{\epsilon \rightarrow 0} \frac{i}{4\pi} \int_{\partial D_\epsilon} (\partial - \bar{\partial}) \log(h \circ s) \wedge \alpha. \end{aligned}$$

On the open set U_i we may write

$$\begin{aligned} (\partial - \bar{\partial}) \log(h \circ s) &= (\partial - \bar{\partial}) \log(\phi_i \circ s) + (\partial - \bar{\partial}) \log(\bar{\phi}_i \circ s) + (\partial - \bar{\partial}) \log(h_i) \\ &= \partial \log(\phi_i \circ s) - \bar{\partial} \log(\phi_i \circ s) + (\partial - \bar{\partial}) \log(h_i) \\ &= (2i) \operatorname{Im}(\partial \log(\phi_i \circ s)) + (\partial - \bar{\partial}) \log(h_i). \end{aligned}$$

The second summand does not contribute to the integral as $\epsilon \rightarrow 0$, because h_i is bounded from below (since h_i is strictly positive, and we may shrink U_i such that $\bar{U}_i$ is compact on which h_i is bounded below by some $\delta > 0$) and the measure of ∂D_ϵ goes to zero as $\epsilon \rightarrow 0$. Therefore, it suffices to prove that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{\partial D_\epsilon \cap U_i} \operatorname{Im}(\partial \log(\phi_i \circ s)) \wedge \alpha = - \int_{D \cap U_i} \alpha.$$

This is a local statement. In order to prove it, we may assume D is given by $z_1 = 0$ in a polydisc B . Moreover, $|h(z_1)| = h_i \cdot |z_1|$ if h on U_i is given by h_i . Hence

$$\partial D_\epsilon = \{z : |z_1| = \epsilon/h_i\}.$$

Furthermore, $\partial \log(\phi_i \circ s) = \partial \log(z_1) = dz_1/z_1$. We can decompose the local expression of α as follows:

$$\alpha = f(dz_2 \wedge \cdots \wedge dz_n) \wedge (d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n) + dz_1 \wedge \alpha' + d\bar{z}_1 \wedge \beta'.$$

Note that $\partial \log(\phi_i \circ s) \wedge (dz_1 \wedge \alpha') = 0$ because $\partial \log(\phi_i \circ s)$ has a dz_1 in it, and moreover

$$\partial \log(\phi_i \circ s) \wedge (d\bar{z}_1 \wedge \beta') = \frac{dz_1 \wedge d\bar{z}_1}{z_1} \wedge \beta'$$

which vanishes when integrating over ∂D_ϵ , because on ∂D_ϵ we can parameterize z_1 by an angle φ since $|z_1| = \epsilon/h_i$, hence $z_1 = \epsilon e^{i\varphi}/h_i$ and $dz_1 = i\epsilon e^{i\varphi} d\varphi/h_i$. Similarly $d\bar{z}_1 = -i\epsilon e^{-i\varphi} d\varphi/h_i$, so $dz_1 \wedge d\bar{z}_1 = 0$. Thus,

$$\int_{D \cap U_i} \alpha = \int f(0, z_2, \dots, z_n) dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n.$$

On the other hand,

$$\int_{\partial D_\epsilon} \partial \log(\phi_i \circ s) \wedge \alpha = - \int_{|z_1|=\epsilon/h_i} f \frac{dz_1}{z_1} \wedge (dz_2 \wedge \cdots \wedge dz_n) \wedge (d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n).$$

The minus sign appears as we initially integrated over exterior domain when applying the Stokes' theorem. The right hand side can be evaluated by first doing the dz_1 integral. We claim that

$$\lim_{\epsilon \rightarrow 0} \int_{|z_1|=\epsilon/h_i} f(z_1, \dots, z_n) \frac{dz_1}{z_1} = 2\pi i f(0, z_2, \dots, z_n). \quad (4.10)$$

Assuming (4.10) for the moment, we proceed with the calculation. The result is that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\partial D_\epsilon} \partial \log(\phi_i \circ s) \wedge \alpha &= -2\pi i \int_{z_1=0} f(0, z_2, \dots, z_n) (dz_2 \wedge \cdots \wedge dz_n) \wedge (d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n) \\ &= -2\pi i \int_{z_1=0} \alpha \end{aligned}$$

which would conclude the proof. Hence it remains to prove (4.10). Note that f need not be holomorphic, so we cannot just apply Cauchy integral formula for a single variable function in z_1 . If f were holomorphic, then we do not even need to take the $\epsilon \rightarrow 0$ limit in (4.10) because Cauchy integral formula says the formula is true for any $\epsilon > 0$. Instead, we expand $f(z_1, \dots, z_n)$ in Taylor series around $z_1 = 0$:

$$f = f(0, z_2, \dots, z_n) + P(z_2, \dots, z_n)z_1 + Q(z_2, \dots, z_n)\bar{z}_1 + R(z_1, \dots, z_n).$$

Note that all we used is Taylor's remainder theorem, which only relies on the assumption that f is smooth. Integrating the first term $f(0, z_2, \dots, z_n)dz_1/z_1$ around ∂D_ϵ gives the desired result on the right hand side of (4.10). Hence we need to show that all the higher order term vanishes when we integrate along ∂D_ϵ . Note that the second term gives

$$P(z_2, \dots, z_n) \int_{\partial D_\epsilon} z_1 \frac{dz_1}{z_1} = P(z_2, \dots, z_n) \int_{\partial D_\epsilon} dz_1 = P(z_2, \dots, z_n) \text{diam}(\partial D_\epsilon),$$

which goes to zero as $\epsilon \rightarrow 0$. Now for the third term, we use polar coordinates $z_1 = \epsilon e^{i\varphi}/h_i$ and $\bar{z}_1 = \epsilon e^{-i\varphi}/h_i$. The third term becomes

$$Q(z_2, \dots, z_n) \int_{\partial D_\epsilon} \bar{z}_1 \frac{dz_1}{z_1} = i\epsilon Q(z_2, \dots, z_n) \int_0^{2\pi} \frac{e^{-i\varphi}}{h_i} d\varphi,$$

which goes to zero as $\epsilon \rightarrow 0$. Finally, for the remainder term, we have the estimate $|R(z_1, \dots, z_n)| \leq C(z_2, \dots, z_n)|z_1|^2 = C(z_2, \dots, z_n)\epsilon^2/|h_i|^2$ for some function C . Hence

$$\left| \int_{\partial D_\epsilon} R(z_1, \dots, z_n) \frac{dz_1}{z_1} \right| \leq i\epsilon^2 \int_0^{2\pi} C(z_2, \dots, z_n) |h_i|^{-2} d\varphi,$$

which again goes to zero as $\epsilon \rightarrow 0$. Hence we have proven (4.10), and the theorem is proven. $\square$

Since taking c_1 and taking the Poincaré dual are both linear operations, Theorem 4.23 immediately generalizes to arbitrary divisors:

$$c_1(\mathcal{O}_X(n_1 D_1 + \dots + n_k D_k)) = \text{PD}(n_1 D_1 + \dots + n_k D_k).$$

That is,

Theorem 4.24. *Let X be a compact complex manifold and D any divisor on X . Then $c_1(\mathcal{O}_X(D)) = \text{PD}(D)$.*

4.4 Other Characteristic Classes: Thom and Euler Class

For more detailed discussions of Thom and Euler classes, the readers can refer to [7] or [8]. Here, we just give a brief recollection of all the relevant results, and our goal in this section is to construct explicit expressions of Thom and Euler class of a rank 2 real vector bundle $E \rightarrow X$ as quickly as possible.

Let $\pi : E \rightarrow X$ be a real vector bundle over a smooth manifold. A form $\omega \in \Omega^k(E)$ is said to have **compact vertical support** if for every compact set $K \subset M$, the subset $\text{Supp } \omega \cap \pi^{-1}(K)$ is compact in E . Let $\Omega_{\text{cv}}^k(E)$ denote the space of all forms on E with compact vertical support. The exterior differentiation $d : \Omega_{\text{cv}}^k(E) \rightarrow \Omega_{\text{cv}}^{k+1}(E)$ makes $\Omega_{\text{cv}}^*(E)$ a chain complex, whose cohomology groups are denoted by $H_{\text{cv}}^k(E)$. Let $n = \dim X$ and $r = \text{rank } E$. We can define an operation $\pi_* : \Omega_{\text{cv}}^{k+r}(E) \rightarrow \Omega^k(X)$ called **integration**

along fiber. Locally speaking, if $x_1, \dots, x_n$ are coordinates of X and $t_1, \dots, t_r$ are fiber coordinates, then any compactly supported form on E looks like

$$\omega = (\pi^* \phi) f(x, t) dt_1 \wedge \dots \wedge dt_r + \zeta$$

where $\phi \in \Omega^k(X)$, the function f has compact support for each fixed x on X , and ζ is a form with less than n fiber differential in its wedges. We then define

$$\pi_* \omega(p) = \phi(p) \cdot \int_{E_p} f(x, t) dt_1 \wedge \dots \wedge dt_r.$$

In fact, the integration along fiber operation is an isomorphism on cohomologies if E, X are both orientable. This is the **Thom isomorphism theorem**:

$$\pi_* : H_{cv}^{k+r}(E) \xrightarrow{\cong} H^k(X).$$

The proof is given in Section 6.8 of [7]. We omit the proof here since it will divert us too far from our goals.

Definition 4.25. Let $E \rightarrow X$ be an orientable rank r vector bundle. The Thom isomorphism $\pi_* : H_{cv}^r(E) \rightarrow H^0(X)$ determines a cohomology class

$$\tau(E) := (\pi_*)^{-1}(1) \in H_{cv}^r(E)$$

called the **Thom class** of E . It is characterized by the property that it integrates to 1 along the fiber: $\pi_* \tau(E) = 1$ by definition.

Definition 4.26. Let $E \rightarrow X$ be an orientable rank r vector bundle. The **Euler class** $e(E)$ of E is the class

$$e(E) := s^* \tau(E) \in H^r(X)$$

where $\tau(E)$ is the Thom class of E , and $s : X \rightarrow E$ is a section. This is well-defined, because any section $s : X \rightarrow E$ is homotopic to the zero section via $H : X \times I \rightarrow E$ given by $H(x, t) = ts(x)$.

Now we will find explicit descriptions of $\tau(E)$ and $e(E)$ for the case where $\pi : E \rightarrow X$ is a rank 2 real vector bundle (a potential case is when we take E to be a holomorphic line bundle). We note that E has a Riemannian structure by a standard partition of unity argument. Therefore, we can pick any Riemannian metric on E . Let $X = \bigcup_{\alpha} U_{\alpha}$ be a trivializing cover. Let $x_1, \dots, x_n$ be a coordinate on U_{α} , and $r_{\alpha}, \theta_{\alpha}$ polar coordinates on the fibers of $E^0|_{U_{\alpha}}$, where E^0 is the complement of the zero section in E . We again use $\pi : E^0 \rightarrow X$ to denote the projection (a slight abuse of notation).

On the overlap $U_{\alpha} \cap U_{\beta}$, since the transition function can always be reduced to $SO(2)$, the radii r_{α} and r_{β} are equal, but the angular coordinates $\theta_{\alpha}, \theta_{\beta}$ differ by a rotation. By the orientability of E , it makes sense to speak of counterclockwise direction in each fiber. This allows us to define a (up to a constant multiple of 2π) function $\varphi_{\alpha\beta} : U_{\alpha\beta} \rightarrow \mathbb{R}$ as the angle of rotation in the counterclockwise direction from the α -coordinate system to the β coordinate system. That is,

$$\theta_{\beta} - \theta_{\alpha} = \pi^* \varphi_{\alpha\beta}. \quad (4.11)$$

Although rotating from α to β and then from β to γ is the same as rotating from α to γ , it is not true that $\varphi_{\alpha\beta} + \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$. What is true is that $\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} \in 2\pi\mathbb{Z}$. Therefore, to each triple intersection $U_{\alpha\beta\gamma}$ we can associate an integer

$$\epsilon_{\alpha\beta\gamma} = \frac{1}{2\pi} (\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma}).$$

Proposition 4.27. There exists 1-forms ξ_α on U_α such that

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha.$$

Proof. Let $\{\rho_\gamma\}$ be a partition of unity subordinate to the trivializing cover $\{U_\gamma\}$. We define

$$\xi_\alpha = -\frac{1}{2\pi} \sum_\gamma \rho_\gamma d\varphi_{\alpha\gamma}.$$

Then we have that

$$\xi_\beta - \xi_\alpha = -\frac{1}{2\pi} \sum_\gamma \rho_\gamma d(\varphi_{\beta\gamma} - \varphi_{\alpha\gamma}) = \frac{1}{2\pi} \sum_\gamma \rho_\gamma d(\varphi_{\alpha\beta} - 2\pi\epsilon_{\alpha\beta\gamma}) = \frac{1}{2\pi} d\varphi_{\alpha\beta}$$

as desired. $\square$

It follows that $d\xi_\alpha = d\xi_\beta$ on $U_\alpha \cap U_\beta$. Hence $d\xi_\alpha$ piece together to give a global 2-form e on X . Clearly e is closed, but it need not be exact because ξ_α need not piece together to a global 1-form. Moreover, it is clear that e does not depend on the choice of 1-forms ξ_α 's, for if η_α 's is another set of such forms then $\frac{1}{2\pi} d\varphi_{\alpha\beta} = \eta_\beta - \eta_\alpha = \xi_\beta - \xi_\alpha$. Therefore, $\eta_\beta - \xi_\beta = \eta_\alpha - \xi_\alpha = \sigma$ is a global form. Hence $d\xi_\alpha$ and $d\eta_\alpha$ differ by an exact form.

The punch line is that e is the Euler class $e(E)$. We will prove this shortly, but first let's construct a representative for the Thom class $\tau(E)$.

Taking exterior derivative of (4.11), and using Proposition 4.27, we have that

$$\frac{d\theta_\alpha}{2\pi} - \pi^* \xi_\alpha = \frac{d\theta_\beta}{2\pi} - \pi^* \xi_\beta.$$

Hence these forms piece together to a global 1-form ψ on E^0 . This ψ is called the **global angular form**. This is because when restricted to each fiber $E_p \cong \mathbb{R}^2$, it becomes the angular form $d\theta/2\pi$ in $\mathbb{R}^2$. Indeed, if we let $i_p : E_p \cong \mathbb{R}^2 \rightarrow E$ be the inclusion of fiber, then since $\pi \circ i_p : E_p \rightarrow \{p\}$ is constant, $i_p^* \pi^* = 0$. Hence $i_p^* \psi = d\theta/2\pi$. Now the global angular form is not closed:

$$d\psi = d\left(\frac{d\theta_\alpha}{2\pi} - \pi^* \xi_\alpha\right) = -\pi^* d\xi_\alpha = -\pi^* e. \quad (4.12)$$

Now we consider a radial function $\rho : [0, +\infty) \rightarrow \mathbb{R}$ given by Fig.4.1:

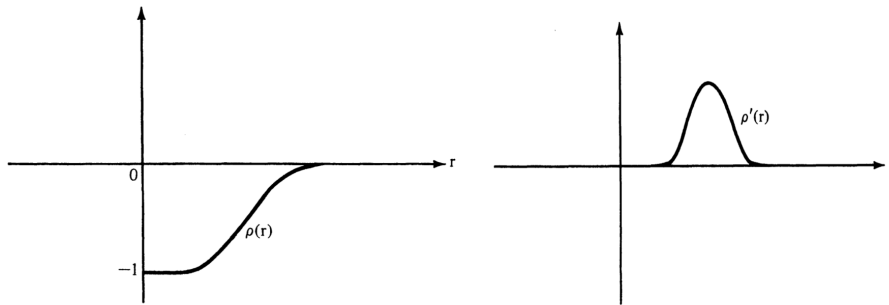


Figure 4.1: The Radial function $\rho(r)$ and its derivative $\rho'(r)$.

Proposition 4.28. In this case, the Thom class $\tau(E)$ is the cohomology class of

$$\Phi = d(\rho(r)\psi) = d\rho(r) \wedge \psi - \rho(r)\pi^*e.$$

Note that although $\rho(r)\psi$ is only defined outside of the zero section, since $d\rho \equiv 0$ near the zero section by construction (Fig. 4.1), Φ is globally defined.

Proof. Indeed, it is easy to check that Φ is compactly supported in the vertical direction. It is closed because $d\Phi = -d\rho(r) \wedge d\psi - d\rho(r)\pi^*e = 0$ by (4.12). Finally, the restriction to each fiber has integral 1 because

$$\pi_* i_p^* \Phi = \int_0^\infty \int_0^{2\pi} d\rho(r) \frac{d\theta}{2\pi} = \rho(\infty) - \rho(0) = 1.$$

Finally, the cohomology class of Φ is independent of ρ . Suppose ρ' is another such function which is -1 near 0 and 0 near infinity, which defines Φ' , then $\Phi - \Phi' = d((\rho - \rho')\psi)$ differ by an exact form. Note that $\rho(r) - \rho'(r)$ vanishes near the zero section this exact form is globally defined. This concludes the proof. $\square$

Proposition 4.29. $e = d\xi_\alpha$ defined above is a representative of the Euler class $e(E) = s^*\tau(E)$.

Proof. Indeed, we can let $s : X \rightarrow E$ be the zero section. Then by Proposition 4.28, $\tau(E) = \Phi$ in cohomology. Moreover,

$$s^*\Phi = d(\rho(0)) \wedge s^*\psi - \rho(0)s^*\pi^*e = e$$

because $\pi \circ s = \text{id}_X$ hence $s^*\pi^* = \text{id}$. This concludes the proof. $\square$

The Euler class of an oriented rank 2 vector bundle may be given in terms of transition functions, as follows. Let $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow SO(2)$ be the transition function of E . By identifying $SO(2) \cong U(1)$ via

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = e^{i\theta}.$$

Then $g_{\alpha\beta}$ may be thought of as complex valued functions. In this case the angle from the β -coordinate to the α -coordinate is $-i \log g_{\alpha\beta}$. Thus

$$\theta_\alpha - \theta_\beta = -i\pi^* \log g_{\alpha\beta}.$$

Thus using (4.11) we have

$$\pi^* \varphi_{\alpha\beta} = i\pi^* \log g_{\alpha\beta}.$$

The projection $\pi : E \rightarrow X$ has maximal rank, i.e., the differential π_* is onto. Dualizing the relative tangent sequence (Lemma 3.14) we see that the pullback π^* is injective. Hence

$$\varphi_{\alpha\beta} = i \log g_{\alpha\beta}.$$

Then using the expression in Proposition 4.27, we see that

$$\xi_\alpha = -\frac{1}{2\pi} \sum_\gamma \rho_\gamma d\varphi_{\alpha\gamma} = \frac{1}{2\pi i} \sum_\gamma \rho_\gamma d \log g_{\alpha\gamma}.$$

Therefore, taking exterior derivative gives

Theorem 4.30. *Let $E \rightarrow X$ be a real rank 2 vector bundle with trivializing cover $X = \bigcup_{\alpha} U_{\alpha}$ and transition functions $\{g_{\alpha\beta}\}$. Let $\{\rho_{\gamma}\}$ be a partition of unity subordinate to this trivializing cover $\{U_{\gamma}\}$. Then the Euler class $e(E)$ of E can be expressed explicitly on U_{α} by*

$$e(E) = \frac{1}{2\pi i} \sum_{\gamma} d(\rho_{\gamma} d \log g_{\alpha\gamma}). \quad (4.13)$$

Proposition 4.31. The Euler class is functorial: If $f : X \rightarrow Y$ is a smooth map and E a rank 2 oriented vector bundle over Y , then $e(f^*E) = f^*e(E)$.

Proof. The transition function of f^*E is $f^*g_{\alpha\beta}$. Then the statement follows from (4.30). $\square$

Proposition 4.32. If L, L' are two holomorphic line bundles over a complex manifold X , viewed as real rank 2 vector bundles on X , then $e(L \otimes L') = e(L) + e(L')$.

Proof. The transition function of $L \otimes L'$ is $g_{\alpha\beta}g'_{\alpha\beta}$, then the statement follows from (4.13). $\square$

4.5 First Chern Class and Intersection Theory

Theorem 4.23 says that $c_1(\mathcal{O}_X(D))$ is the Poincaré dual $\text{PD}(D)$ to the divisor D . One big advantage of Poincaré dual is that intersection numbers of two transversal submanifolds defined in differential topology can be computed easily by integrating over wedges of Poincaré duals:

Theorem 4.33 ([7], Theorem 7.18). *Suppose that S^s, L^l are closed, oriented submanifolds of an oriented manifold N^n of finite type (i.e., has a finite good cover), such that S is compact, $s + l = n$ (complementary dimension), and S intersects L transversally, then*

$$S \cdot L = \int_N \text{PD}(S) \wedge \text{PD}(L).$$

Now, let $E \rightarrow X$ be a smooth vector bundle of rank r . We have the zero section $\sigma_0 : M \rightarrow E$ which is a smooth embedding. Moreover, we can identify M with the submanifold $\sigma_0(E)$ of zero section.

Theorem 4.34 (Thom class). *Let $E \rightarrow X$ be a vector bundle of rank r over a compact smooth manifold of dimension m . Then the Thom class $\tau(E) \in H^r(E)$ can be chosen to be the Poincaré dual $\text{PD}(\sigma_0(M)) = \text{PD}(M)$ of the zero section M in E .*

Proof. See Corollary 6.95 of [7]. $\square$

Our goal is to show that if $s : M \rightarrow E$ is any section that intersects the zero section of E transversely, then the Euler class $e(E)$ is the Poincaré dual $\text{PD}(Z(s))$ of the zeros of the section s .

Theorem 4.35. *Let $\pi : E \rightarrow M$ be a vector bundle of rank r over a smooth manifold M of dimension n . Let $s : M \rightarrow E$ be a section that intersects $M = \sigma_0(M)$ transversely. Then*

$$e(E) = \text{PD}(Z(s)) \in H^r(M).$$

Proof. We will identify M with the image S_0 of the zero section. By Lemma 3.35, we know $Z = Z(s)$ is a closed oriented submanifold and $N_{Z/M} = E|_Z$. Moreover, the exact sequence in the proof of Proposition 3.35 shows that we also have $N_{Z/S} = E|_Z$, which is not surprising since S is diffeomorphic to M . The normal bundles $N_{Z/M}$ and $N_{Z/S}$ will be identified with the tubular neighborhoods of Z in M and in S respectively, as Figure 7.2 suggests.

Let us choose the Thom class $\tau(E)$ to have support so close to the zero section (we can do this by Corollary 6.95 of [7]), so that $\tau(E)$ restricted to the tubular neighborhood $N_{Z/S}$ in S has compact support in the vertical direction. We now show that $e(E) = s^*\tau(E)$ is the Thom class of the tubular neighborhood $N_{Z/M}$ in M .

Let E_p, S_p, M_p be the fibers of $E|_Z \cong N_{Z/S} \cong N_{Z/M}$ respectively above the point $p \in Z$. Because $\tau(E)$ has compact support in S_p , in fact $s^*\tau(E)$ has compact support in M_p . It suffices to check that fiber integration along M_p of $e(E)$ is 1 by Remark ???. First, we note that $s : M_p \rightarrow S_p$ is an orientation preserving diffeomorphism. Therefore, we have

$$\int_{M_p} e(E) = \int_{M_p} s^*\tau(E) = \int_{S_p} \tau(E).$$

Next, we note that E_p is homotopic to S_p modulo the region in E where $\tau(E)$ is zero (see Figure 7.2). Therefore,

$$\int_{S_p} \tau(E) = \int_{E_p} \tau(E) = \pi_*\tau(E)|_p = 1.$$

Therefore, we have shown that $e(E) = s^*\tau(E)$ is the Thom class of $N_{Z/M}$. By Theorem 6.94 of [7], this is precisely the Poincaré dual $\text{PD}(Z(s))$ to Z in E . $\square$

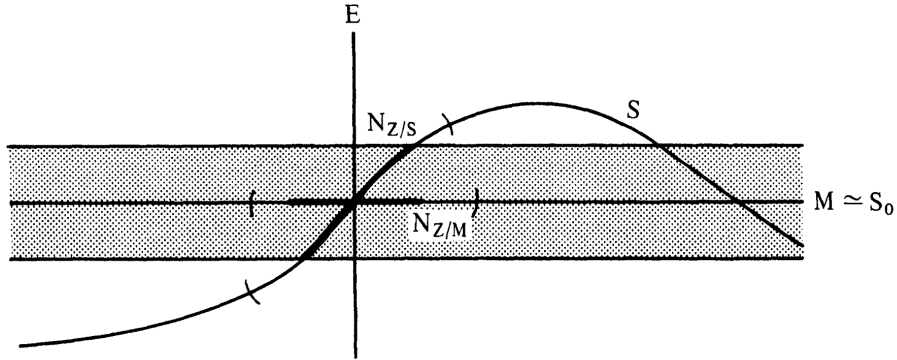


Figure 4.2: An illustration. Here, Z is a single point, and $N_{Z/M} \cong N_{Z/S} \cong E$. The support of the Thom class is contained in a thin strip near $M \cong S_0$, which is shaded in gray in the figure.

With the previous theorem and Theorem 4.23, we may give the following equivalent definition of Chern class for a line bundle on a complex manifold:

Definition 4.36. Let $L \rightarrow X$ be a holomorphic line bundle on X . Then its **first Chern class** $c_1(L) \in H^2(X; \mathbb{Z})$ is defined to be its Euler class $e(L)$, where we view L as a real rank 2 vector bundle over X .

Note that as a sanity check, for line bundles we indeed have $e(L \otimes L') = e(L) + e(L')$ by Proposition 4.32. Hence the resulting c_1 satisfies $c_1(L \otimes L') = c_1(L) + c_1(L')$. This

is necessary as we will see later. It also makes sense because in our original definition $c_1 : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X; \mathbb{Z})$ is a group homomorphism.

Next, we show that the degree of a line bundle is compatible with degree of a divisor:

Theorem 4.37. *Let X be a Riemann surface and D a divisor on X . Then $\deg \mathcal{O}_X(D) = \deg D$.*

Proof. From Theorem 4.23, we already know that $c_1(\mathcal{O}_X(D)) = \text{PD}(D)$. Hence

$$\deg \mathcal{O}_X(D) = \int_X c_1(\mathcal{O}_X(D)) = \int_X 1 \wedge c_1(\mathcal{O}_X(D)) = \int_D 1 = \deg D,$$

as desired. $\square$

Corollary 4.38. *On a curve, if $\deg L < 0$, then $H^0(X, L) = 0$.*

Proof. We argue by contradiction. Suppose $\dim H^0(X, L) > 0$, and we take a section $s \in H^0(X, L)$ nonzero. Then $\mathcal{O}_X(Z(s)) \cong L$ by Theorem 3.26. Taking degree gives

$$\deg L = \deg \mathcal{O}_X(Z(s)) = \deg Z(s) \geq 0$$

because $Z(s)$ is effective: Indeed, $Z(s)$ is given by the zeros of holomorphic functions f_α as we showed in the proof of Theorem 3.26. Therefore, we get the desired contradiction. $\square$

Now using Chern class, we can do intersection theory on a surface. Let X be a complex surface, and C, D be two divisors on X (so they are real submanifolds of dimension 2). Then the intersection number is given by

$$C \cdot D = \int_X \text{PD}(C) \wedge \text{PD}(D).$$

Corollary 4.39. *Let C, D be two divisors on a complex surface X . Then the intersection number can be calculated via*

$$C \cdot D = \deg(\mathcal{O}_X(C)|_D) = \deg(\mathcal{O}_X(D)|_C) = D \cdot C.$$

In particular, the intersection number of two divisors is symmetric.

Proof. This immediately follows from Theorem 4.23. Note that since $\text{PD}(C), \text{PD}(D)$ are both 2-forms, we have $\text{PD}(C) \wedge \text{PD}(D) = \text{PD}(D) \wedge \text{PD}(C)$, which proves symmetry. $\square$

We conclude this section by calculating the degree of the Veronese and Segre embedding.

Definition 4.40. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension k . Then its **degree** is defined to be the intersection number $\deg X = X \cdot L$ where L is a linear subspace $L \cong \mathbb{P}^{N-k}$ of complementary dimension.

Now we compute the degree of the veronese variety, which is the degree of the image of

$$v_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$$

given by the complete linear series $\mathcal{O}_{\mathbb{P}^n}(d)$, in other words, $v_d(z) = (z^I)_I$, where $I = (i_0, \dots, i_n)$ runs through all multi-index with $|I| = \sum_k i_k = d$, and $z^I = \prod_k z_k^{i_k}$.

Proposition 4.41. The degree of the Veronese variety $v_d(\mathbb{P}^n)$ is d^n .

Proof. A general linear subspace $\mathbb{P}^{N-n}$ is the intersection of n hyperplanes. In $\mathbb{P}^N$, a hyperplane H is given by the equation $z_0 = 0$, which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ by Theorem 3.26. Hence the Poincaré dual $\text{PD}(H)$ of H corresponds to $c_1(\mathcal{O}_{\mathbb{P}^N}(1)) \in H^2(\mathbb{P}^N; \mathbb{Z})$. Therefore, the Poincaré dual of a general linear subspace $\mathbb{P}^{N-n}$ which corresponds to intersection of n hyperplanes is given by

$$\text{PD}(H \cap \cdots \cap H) = \text{PD}(H) \wedge \cdots \wedge \text{PD}(H) = \text{PD}(H)^n.$$

Therefore, using Theorem 3.39, we have

$$\begin{aligned} \deg X &= X \cdot \mathbb{P}^{N-n} = \int_{v(\mathbb{P}^n)} \text{PD}(H)^n \\ &= \int_{\mathbb{P}^n} (v^* \text{PD}(H))^n = \int_{\mathbb{P}^n} (v^* c_1(\mathcal{O}_{\mathbb{P}^N}(1)))^n = \int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(d))^n. \end{aligned}$$

As discussed above, the Poincaré dual $\text{PD}(h)$ of a hyperplane $h \subset \mathbb{P}^n$ is given by $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence $c_1(\mathcal{O}_{\mathbb{P}^n}(d)) = d \text{PD}(h)$. Therefore

$$\deg X = \int_{\mathbb{P}^n} (d \text{PD}(h))^n = d^n \int_{\mathbb{P}^n} \text{PD}(h)^n = d^n$$

because n hyperplanes in $\mathbb{P}^n$ intersect at a single point: for example, take $h_i = Z(z_i - 1)$. Therefore, $\int_{\mathbb{P}^n} \text{PD}(h)^n = 1$. $\square$

Next, we consider the Segre embedding

$$\sigma : \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^N \quad \sigma(z, w) = (z_i w_j)_{0 \leq i \leq m, 0 \leq j \leq n}.$$

In this case $N = (m+1)(n+1) - 1$. In fact, we can define a line bundle $\mathcal{O}(1, 1)$ over $\mathbb{P}^m \times \mathbb{P}^n$ given by

$$\mathcal{O}(1, 1) = p_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^n}(1).$$

Then recall that $\sigma : \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^N$ is precisely the canonical map determined by the standard basis of $\mathcal{O}(1, 1)$.

Proposition 4.42. The degree of the Segre variety $\sigma(\mathbb{P}^m \times \mathbb{P}^n)$ is $\binom{m+n}{m}$.

Proof. Let $a = c_1(\mathcal{O}_{\mathbb{P}^m}(1, 0))$ and $b = c_1(\mathcal{O}_{\mathbb{P}^n}(0, 1))$. In $\mathbb{P}^N$, a hyperplane H is given by the equation $z_0 = 0$, which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ by Theorem 3.26. Hence the Poincaré dual $\text{PD}(H)$ of H corresponds to $c_1(\mathcal{O}_{\mathbb{P}^N}(1)) \in H^2(\mathbb{P}^N; \mathbb{Z})$. Therefore, by the previous lemma again,

$$\sigma^* \text{PD}(H) = \sigma^*(c_1(\mathcal{O}_{\mathbb{P}^N}(1))) = c_1(\mathcal{O}(1, 1)) = c_1(\mathcal{O}(1, 0)) + c_1(\mathcal{O}(0, 1)) = a + b.$$

Therefore,

$$\begin{aligned} \deg X &= \int_{\sigma(\mathbb{P}^m \times \mathbb{P}^n)} \text{PD}(H)^{m+n} \\ &= \int_{\mathbb{P}^m \times \mathbb{P}^n} (a + b)^{m+n} = \binom{m+n}{m} \int_{\mathbb{P}^m \times \mathbb{P}^n} a^m b^n = \binom{m+n}{m}, \end{aligned}$$

where in the second to last step, we used the observation that only $a^m b^n$ survives the integration among all the terms in the expansion $(a + b)^{m+n}$. This is because $a = p_1^* c_1(\mathcal{O}_{\mathbb{P}^m}(1)) =: p_1^* \alpha$ and $b = p_2^* c_1(\mathcal{O}_{\mathbb{P}^n}(1)) =: p_2^* \beta$, and $\alpha^{m+1} = 0, \beta^{n+1} = 0$. The last equality follows from Fubini's theorem:

$$\int_{\mathbb{P}^m \times \mathbb{P}^n} a^m b^n = \int_{\mathbb{P}^m} \alpha^m \cdot \int_{\mathbb{P}^n} \beta^n = 1.$$

This completes the proof. $\square$



Chapter 5

Kähler Manifolds

Introduction

In this chapter, we study a very special type of complex manifold: Kähler manifold. These are complex manifolds with the additional structure of a Kähler form: a closed $(1,1)$ -form ω that is positive in the sense that $\omega(X, JX) > 0$ for all nonzero vectors X . At first, this seems like a strange condition, but as we will discover, many of our most familiar examples of complex manifolds (complex plane, torus, smooth curves, projective spaces, and more) are all Kähler. Indeed, we will spend a considerable amount of time studying examples and nonexamples of Kähler manifolds.

One might wonder why we spend so much effort on a very specialized class of complex manifolds. Probably the most reasonable answer is that Kähler manifolds have many nice and special properties, which means we have a lot of tools at hand when Kähler manifold is our object of study. For example, their volume form is given by the Kähler metric: $d\text{vol} = \omega^n/n!$ where n is the dimension of the manifold. Moreover, there are topological obstructions (or rather requirements) for a manifold to be Kähler. In fact, the topological requirement is even stronger than what we will encounter in this chapter: after we start learning Hodge theory, we will discover that the cohomology of Kähler manifolds can be nicely decomposed into pieces of Dolbeault cohomology groups of (p, q) -forms.

5.1 Hermitian and Kähler Metrics

Let M be a complex manifold of complex dimension n and J the induced almost complex structure. This turns the tangent bundle TM into a complex vector bundle $T_J M$ of rank n , which is just TM but now viewed as a complex vector bundle via $(a + bi)X := aX + bJX$.

Definition 5.1. A *Hermitian fiber metrics* h on $T_J M$ is a map $h : \Gamma(T_J M) \times \Gamma(T_J M) \rightarrow C^\infty(M; \mathbb{C})$ such that:

1. h is bilinear over $C^\infty(M; \mathbb{R})$.

2. $h(JX, Y) = ih(X, Y)$ and $h(X, JY) = -ih(X, Y)$.
3. $h(Y, X) = \overline{h(X, Y)}$.
4. $h(X, X) > 0$ at points where $X \neq 0$.

Lemma 5.2. *Suppose M is a complex manifold and h a Hermitian fiber metric on $T_J M$. Then $g = \operatorname{Re} h$ is a Riemannian metric on M . Moreover, $\omega = -\operatorname{Im} h$ is a 2-form of type $(1,1)$.*

Proof. It follows directly from definition that g is smooth and positive definite, and bilinear over $C^\infty(M; \mathbb{R})$. It suffices to show that $g(X, Y)$ is symmetric, which is obvious because

$$h(X, Y) = \frac{1}{2}(h(X, Y) + h(Y, X)).$$

Next, ω is clearly smooth and bilinear over $C^\infty(M; \mathbb{R})$ as well. To show that it is antisymmetric, we note that

$$\omega(X, Y) = -\frac{1}{2i}(h(X, Y) - h(Y, X))$$

which is clearly antisymmetric. To see that ω is of type $(1,1)$, note that $h(JX, JY) = i(-i)h(X, Y) = h(X, Y)$. Hence it follows that

$$\omega(JX, JY) = -\operatorname{Im} h(JX, JY) = -\operatorname{Im} h(X, Y) = \omega(X, Y).$$

We claim that this shows that $\omega \in \mathcal{A}^{1,1}$. Indeed, we consider the decomposition

$$\omega = \omega_{\alpha\beta} dz^\alpha \wedge dz^\beta + \omega_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \omega_{\bar{\alpha}\beta} d\bar{z}^\alpha \wedge dz^\beta.$$

Recall that J acts as multiplication by i on $T^{1,0}$ and multiplication by $-i$ on $T^{0,1}$. Then

$$\omega_{\alpha\beta} = \omega\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) = \omega\left(J\frac{\partial}{\partial z^\alpha}, J\frac{\partial}{\partial z^\beta}\right) = -\omega\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) = -\omega_{\alpha\beta}.$$

Thus $\omega_{\alpha\beta} = 0$, similarly $\omega_{\bar{\alpha}\bar{\beta}} = 0$. This concludes the proof. $\square$

Lemma 5.3. *Conversely, let $\omega \in \mathcal{A}^{1,1}$ that is positive in the sense that $\omega(X, JX) > 0$ whenever X is nonzero. This uniquely determines a Hermitian structure h such that $\omega = -\operatorname{Im} h$.*

Proof. Set $g(X, Y) = \omega(X, JY)$ and $h(X, Y) = g(X, Y) - i\omega(X, Y)$. We check that

$$h(X, JY) = g(X, JY) - i\omega(X, JY) = -\omega(X, Y) - ig(X, Y) = -ih(X, Y).$$

Since $\omega \in \mathcal{A}^{1,1}$, we have $\omega(JX, JY) = \omega(X, Y)$. So we also have

$$\begin{aligned} h(JX, Y) &= g(JX, Y) - i\omega(JX, Y) \\ &= \omega(JX, JY) + i\omega(X, JY) \\ &= \omega(X, Y) + ig(X, Y) = ih(X, Y). \end{aligned}$$

As ω is alternating, we have $\operatorname{Im} h(X, Y) = \operatorname{Im} h(Y, X)$. Moreover, as $\omega(X, JY) = -\omega(JX, Y)$ since $\omega \in \mathcal{A}^{1,1}$, we have $g(X, Y) = \omega(X, JY) = -\omega(JX, Y) = \omega(Y, JX) = g(Y, X)$. Therefore, $h(X, Y) = \overline{h(Y, X)}$. Clearly h is bilinear over $C^\infty(M; \mathbb{R})$ because ω is and J is linear. Finally, positivity $h(X, X) > 0$ whenever X is nonzero follows from positivity of ω : Indeed, $h(X, X) = g(X, X) = \omega(X, JX) > 0$ whenever X is nonzero. $\square$

Corollary 5.4. *Let M be a complex manifold with the induced complex structure J . There is a bijective correspondence between Hermitian fiber metric on $T_J M$ and positive $(1,1)$ -forms, where the bijections are given by $h \mapsto -\operatorname{Im} h$, and $\omega \mapsto \omega(-, J-) - i\omega(-, -)$.*

Proof. This follows from Lemma 5.2 and 5.3. $\square$

Lemma 5.5. *The set of Hermitian fiber metrics on M is also in bijection with Riemannian metrics g that is compatible with complex structures, i.e., $g(JX, JY) = g(X, Y)$.*

Proof. By Lemma 5.2, the assignment $h \mapsto \operatorname{Re} h$ gives the map in one direction. Since $h(JX, JY) = h(X, Y)$, taking the real part gives $g(JX, JY) = g(X, Y)$.

Conversely, suppose we are given Riemannian metrics g such that $g(JX, JY) = g(X, Y)$. Now define $\omega(X, Y) = g(JX, Y)$, and define $h(X, Y) = g(X, Y) - i\omega(X, Y)$. Clearly this h is linear over $C^\infty(M; \mathbb{R})$ since g is and J is linear. Moreover,

$$h(JX, Y) = g(JX, Y) - i\omega(JX, Y) = \omega(X, Y) + ig(X, Y) = ih(X, Y).$$

Similarly,

$$\begin{aligned} h(X, JY) &= g(X, JY) - i\omega(X, JY) \\ &= g(JX, J^2 Y) - ig(JX, JY) \\ &= -\omega(X, Y) - ig(X, Y) = -ih(X, Y). \end{aligned}$$

Next, we note that since

$$\omega(X, Y) = g(JX, Y) = g(Y, JX) = g(JY, J^2 X) = -g(JY, X) = -\omega(Y, X),$$

it follows that $h(X, Y) = \overline{h(Y, X)}$. Finally, note that $\omega(X, X) = g(JX, X) = g(J^2 X, JX) = -g(X, JX) = -g(JX, X) = -\omega(X, X)$, hence $\omega(X, X) = 0$. Therefore, $h(X, X) = g(X, X) > 0$ whenever X is nonzero because g is positive definite. This concludes the proof. $\square$

To summarize, we have the following result:

Theorem 5.6. *On a complex manifold with induced almost complex structure J , the following three sets are in bijection with each other:*

1. *Hermitian metrics on M ;*
2. *Riemannian metrics g on M compatible with J in the sense that $g(JX, JY) = g(X, Y)$;*
3. *$(1,1)$ -forms on M that is positive in the sense that $\omega(X, JX) > 0$ whenever X is nonzero.*

In fact, given a Hermitian structure h on M , it is always the case that $h = g - i\omega$ where g is a Riemannian metric, and ω is a positive $(1,1)$ -form on M , that satisfies $g(X, Y) = \omega(X, JY)$, $\omega(X, Y) = g(JX, Y)$.

Remark 5.7. Because of the above bijections, quite often we will also use the term that a Riemannian metric g is Hermitian, which means that $g(JX, JY) = g(X, Y)$.

Proposition 5.8. Every complex manifold admits a Hermitian metric.

Proof. Every smooth manifold admits a Riemannian metric g_0 . We can define another Riemannian metric g by $g(X, Y) = g_0(X, Y) + g_0(JX, JY)$. Then $g(X, Y) = g(JX, JY)$, and we can construct h following Lemma 5.5. $\square$

Definition 5.9. Given a Hermitian metric h , its **fundamental 2-form** is $\omega = -\text{Im } h$.

Definition 5.10. We say that a complex manifold M is **Kähler** if it admits a Hermitian metric h whose fundamental 2-form $\omega \in \mathcal{A}^{1,1}(M)$ is closed. We also say that h is Kähler, or ω is Kähler (or a **Kähler** form).

Now, we can give local descriptions of the Hermitian metric h and its fundamental 2-form ω . It is clear from the definition that h restricts to a Hermitian metric on each fiber of $T_J M$. Therefore, pointwise speaking it suffices to study expressions of a Hermitian metric on a complex vector space V .

To this end, let V be a complex vector space, and h a Hermitian metric on it. By the sesquilinearity of h , we see that if we pick a basis $e_1, \dots, e_n$ of V and set $h_{\alpha\beta} = h(e_\alpha, e_\beta)$, then for any two vectors $u, w \in V$, we must have

$$h(u, w) = h_{\alpha\beta} u^\alpha \overline{w^\beta}.$$

Hence the fundamental 2-form ω of h on V is given by

$$\omega(u, w) = -\frac{1}{2i}(h(u, w) - h(w, u)) = \frac{i}{2} h_{\alpha\beta} (u^\alpha \overline{w^\beta} - \overline{w^\alpha} u^\beta). \quad (5.1)$$

Note that since $h(e_\alpha, e_\beta) = \overline{h(e_\beta, e_\alpha)}$, we must have $h_{\alpha\beta} = \overline{h_{\beta\alpha}}$. Now we study the case of a Hermitian metric h on $T_J M$. Recall that if M has coordinates (as a real manifold) $x^1, y^1, \dots, x^n, y^n$, and holomorphic coordinates $z^1, \dots, z^n$ with $z^k = x^k + iy^k$, then a basis (over $\mathbb{C}$) of $T_J M$ will be $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$. Note that $\frac{\partial}{\partial y^k}$ is no longer linearly independent with $\frac{\partial}{\partial x^k}$ because $i \frac{\partial}{\partial x^k} := J \frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}$ on $T_J M$ by definition. Therefore, if we have a vector field $\chi \in \Gamma(TM)$ that is given by

$$\chi = a^\mu \frac{\partial}{\partial x^\mu} + b^\mu \frac{\partial}{\partial y^\mu},$$

in terms of the standard basis $\chi = \chi^\mu \frac{\partial}{\partial x^\mu}$ of $T_J M$, this vector will have coordinate representation

$$\chi = (a^1 + ib^1, \dots, a^n + ib^n) \in \Gamma(T_J M).$$

Now it is easy to verify that $dz^\mu = dx^\mu + idy^\mu$ and $d\bar{z}^\mu = dx^\mu - idy^\mu$ acts on χ by

$$dz^\mu(\chi) = a^\mu + ib^\mu = \chi^\mu \quad d\bar{z}^\mu(\chi) = a^\mu - ib^\mu = \overline{\chi^\mu}.$$

Therefore, generalizing the fiberwise expression (5.1) locally, the fundamental 2-form of h on $T_J M$ will have local expression

$$\omega = \frac{i}{2} h_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta. \quad (5.2)$$

Theorem 5.11. Let M be a complex n -manifold with a Hermitian metric. Then the volume form of M is given by

$$\frac{\omega^n}{n!} = \frac{1}{n!} (\omega \wedge \dots \wedge \omega).$$

Proof. Let $e_1, \dots, e_n$ be a basis of $T_p M$ over $\mathbb{C}$ such that $h(e_i, e_j) = \delta_{ij}$ at a point $p \in M$. Then it follows that $e_1, J e_1, \dots, e_n, J e_n$ is a real basis of $T_p M$, and they are orthonormal with respect to g , which can be checked by taking the real part of the $h(e_i, e_j) = \delta_{ij}$. The volume form of (M, g) at p is then the unique form which has value 1 on $(e_1, J e_1, \dots, e_n, J e_n)$. Now let us choose coordinates $x^1, y^1, \dots, x^n, y^n$ on M locally around p such that $e_k = \frac{\partial}{\partial x^k}$. Then since $h_{ij} = \delta_{ij}$, (5.2) becomes

$$\omega = \frac{i}{2} \sum_{k=1}^n dz^k \wedge d\bar{z}^k.$$

Using $\frac{i}{2} dz^k \wedge d\bar{z}^k = dx^k \wedge dy^k$, we have that

$$\frac{\omega^n}{n!} = \left(\frac{i}{2}\right)^n \prod_{k=1}^n dz^k \wedge d\bar{z}^k = dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n.$$

Therefore, we see that $\frac{\omega^n}{n!}(e_1, J e_1, \dots, e_n, J e_n) = 1$, which concludes the proof. $\square$

5.2 Examples and Nonexamples of Kähler Manifolds

In this section, our focus will be on examples and nonexamples of Kähler manifolds.

Example 5.12 ($\mathbb{C}^n$). On $\mathbb{C}^n$ with the standard holomorphic coordinates $z^j = x^j + iy^j$, we have the Euclidean metric

$$g_E = \delta_{\alpha\beta} dz^\alpha d\bar{z}^\beta = \sum_{j=1}^n (dx^j)^2 + (dy^j)^2.$$

Since the almost complex structure J satisfies $J \frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}$ and $J \frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}$, we see that $g(X, Y) = g(JX, JY)$ for any vector fields $X, Y \in \Gamma(T\mathbb{C}^n)$, because we can simply check on the basis elements of $\Gamma(T\mathbb{C}^n)$. By Lemma 5.5 and its proof, we can define $\omega(X, Y) = g(JX, Y)$. Simple computation gives

$$\omega = \sum_{j=1}^n dx^j \wedge dy^j = \frac{i}{2} \sum_{j=1}^n dz^j \wedge d\bar{z}^j.$$

Clearly ω is closed, hence $\mathbb{C}^n$ is Kähler.

In fact, we can even show that every Kähler form must locally be given by the Kähler expression on $\mathbb{C}^n$ in some coordinate (we call this is **Kähler normal coordinate**).

Theorem 5.13. *Let M be a Kähler manifold with Kähler form ω . Then around every point, we can always choose holomorphic coordinates such that*

$$\omega = \frac{i}{2} \sum_{j=1}^n dz^j \wedge d\bar{z}^j + O(|z|^2).$$

Proof. We use the local expression (5.2). Let $p \in M$ be a point such that in a holomorphic chart it corresponds to the point $p = 0$. Then locally $h_{\alpha\beta}(0)$ can certainly be diagonalized, so we might as well assume that $h_{\alpha\beta}(0) = \delta_{\alpha\beta}$. Now we Taylor expand $h_{\alpha\beta}$ around $z = 0$ and get

$$h_{\alpha\beta} = \delta_{\alpha\beta} + \epsilon_{\alpha\beta} + O(|z|^2)$$

where $\epsilon_{\alpha\beta}$ is the linear term, so it depends linearly on z and $\bar{z}$'s. Now since $h_{\alpha\beta}$ is Hermitian, i.e., $h_{\alpha\beta} = \bar{h}_{\beta\alpha}$. Hence we must also have $\epsilon_{\alpha\beta} = \bar{\epsilon}_{\beta\alpha}$. Now if we decompose $\epsilon_{\alpha\beta}$ into the holomorphic part $\lambda_{\alpha\beta}$ and antiholomorphic part $\theta_{\alpha\beta}$, this the symmetry of $\epsilon_{\alpha\beta}$ implies

$$\theta_{\alpha\beta} = \bar{\lambda}_{\beta\alpha}.$$

Since ω is d-closed, this means that the form (i.e., the holomorphic linear part of its Taylor expansion)

$$\lambda_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

must be ∂ -closed. By the $\bar{\partial}$ -Poincare lemma (Theorem 2.63) and conjugation, this means there exists holomorphic functions $\phi^\beta(z_1, \dots, z_n)$ such that

$$\frac{\partial \phi_\beta}{\partial z^\alpha} = \lambda_{\alpha\beta}.$$

In fact, since $\lambda_{\alpha\beta}$ is holomorphic and depends only linearly on z , we can directly integrate it to find ϕ_β . Now let us consider a holomorphic change of coordinates

$$w^\beta = z^\beta + \phi_\beta(z).$$

Then

$$dw^\beta = dz^\beta + \frac{\partial \phi_\beta}{\partial z^\alpha} dz^\alpha = dz^\beta + \lambda_{\alpha\beta} dz^\alpha.$$

Therefore, we obtain

$$\begin{aligned} \sum_j dw^j \wedge d\bar{w}^j &= \sum_j dz^j \wedge d\bar{z}^j + \sum_{i,k} \lambda_{ki} dz^k \wedge d\bar{z}^i + \bar{\lambda}_{ki} dz^i \wedge d\bar{z}^k + O(|z|^2) \\ &= \sum_j dz^j \wedge d\bar{z}^j + \sum_{i,k} \lambda_{ki} dz^k \wedge d\bar{z}^i + \theta_{ik} dz^i \wedge d\bar{z}^k + O(|z|^2) \\ &= \sum_j dz^j \wedge d\bar{z}^j + \sum_{i,k} \epsilon_{ki} dz^k \wedge d\bar{z}^i + O(|z|^2) \\ &= -2i\omega + O(|z|^2). \end{aligned}$$

Therefore, we see that

$$\omega = \frac{i}{2} \sum_j dw^j \wedge d\bar{w}^j + O(|w|^2).$$

Note that $O(|w|^2) = O(|z|^2)$ because ϕ_β only depends quadratically on z , hence to first order $w \simeq z$. $\square$

Remark 5.14. Since the Kähler metric ω uniquely determines the Hermitian metric h and the Riemannian metric g , it follows that the Kähler normal coordinates also makes h and g into the corresponding expression for $\mathbb{C}^n$, and the error term is given by $O(|z|^2)$.

We now shift our focus back to examples of Kähler manifolds.

Example 5.15 (complex torus). A complex torus is the quotient $Q = \mathbb{C}^n / \Lambda$ of $\mathbb{C}^n$ by a lattice. It is easy to check that g_E on $\mathbb{C}^n$ descends to a Kähler metric on Q . since the quotient map $\mathbb{C}^n \rightarrow Q$ is locally a biholomorphism. Therefore, any complex torus is a Kähler manifold.

Example 5.16 (Riemann surface). Every Riemann surface Σ_g has a Hermitian metric by Proposition 5.8. By dimension reason, every 2-form on Σ_g is closed. Hence Σ_g is Kähler.

Example 5.17 ($\mathbb{P}^n$). We will show that the complex projective space $\mathbb{P}^n$ is a Kähler manifold. Recall the definition of tautological bundle $\mathcal{O}(-1)$ (Definition 3.29) on $\mathbb{P}^n$. Let $U_\alpha \subset \mathbb{P}^n$ be the open set defined by $w^\alpha \neq 0$, where $[w^0 : \dots : w^n]$ are homogeneous coordinates. Then it is easy to check that $s_\alpha : U_\alpha \rightarrow \mathcal{O}(-1)|_{U_\alpha}$ given by

$$s_\alpha([w]) = ([w], \frac{1}{w^\alpha}(w^0, \dots, w^n))$$

is a local section that does not vanish on U_α . Since each fiber of $\mathcal{O}(-1)$ is canonically identified with a line in $\mathbb{C}^{n+1}$, it has a Hermitian metric inherited from the standard Hermitian metric on $\mathbb{C}^{n+1}$. With respect to this metric, s_α has norm given by

$$|s_\alpha([w])|^2 = \frac{|w|^2}{|w^\alpha|^2}.$$

We may take the Chern connection of $\mathcal{O}(-1)$ for this metric (Example 4.22), which is given by

$$\partial \log |s_\alpha|^2 = \partial \log \frac{|w|^2}{|w^\alpha|^2}$$

and the curvature 2-form is

$$\bar{\partial} \partial \log |s_\alpha|^2 = \bar{\partial} \partial \log \frac{|w|^2}{|w^\alpha|^2}.$$

Now for the hyperplane bundle $\mathcal{O}(1) \rightarrow \mathbb{P}^1$, Proposition 4.8 shows that the curvature is the negative of what we have above, so

$$\Theta_\alpha = \bar{\partial} \partial \log \frac{|w^\alpha|^2}{|w|^2}.$$

On affine coordinates of U_α , points $[z^1 : \dots : 1 : \dots : z^n]$ are identified with $(z^1, \dots, z^n) \in \mathbb{C}^n$. Hence the local expression of Θ_α is given by

$$\Theta_\alpha = \bar{\partial} \partial \log \frac{1}{1 + |z|^2} = \partial \bar{\partial} \log(1 + |z|^2) = \frac{\sum_j dz^j \wedge d\bar{z}^j}{1 + |z|^2} - \frac{\sum_{j,k} \bar{z}^j z^k dz^j \wedge d\bar{z}^k}{(1 + |z|^2)^2}.$$

Hence this is a closed (by Chern-Weil, Theorem 4.13) real (1,1)-form. We define the **Fubini-Study metric** ω_{FS} on $\mathbb{P}^n$ by

$$\omega_{\text{FS}} = \frac{i}{2} \Theta.$$

We claim that ω is a Kähler metric on $\mathbb{P}^n$. By Lemma 5.3, we only need to show that ω is positive. To see this, let X be a (locally) nonvanishing real smooth vector field. We can always write

$$X = \chi + \bar{\chi}$$

for some $\chi \in T_{\mathbb{P}^n}^{1,0}$ (which are just smooth sections, whereas by $T_{\mathbb{P}^n}$ we mean holomorphic sections of holomorphic tangent bundles). Indeed, this is given by $\chi = (X - iJX)/2$. Then we compute:

$$\omega_{\text{FS}}(X, JX) = \omega_{\text{FS}}(\chi + \bar{\chi}, i\chi - i\bar{\chi}) = -2i\omega_{\text{FS}}(\chi, \bar{\chi}) = \Theta(\chi, \bar{\chi}).$$

And note that

$$\begin{aligned}\Theta(\chi, \bar{\chi}) &= \frac{(1 + |z|^2) \sum_j \chi^j \bar{\chi}^j - \sum_{j,k} \bar{z}^j z^k \chi^j \bar{\chi}^k}{(1 + |z|^2)^2} \\ &= \frac{(1 + |z|^2) |\chi|^2 - |\chi \cdot \bar{z}|^2}{(1 + |z|^2)^2} \\ &\geq \frac{(1 + |z|^2) |\chi|^2 - |\chi|^2 |z|^2}{(1 + |z|^2)^2} = \frac{|\chi|^2}{(1 + |z|^2)^2} > 0\end{aligned}$$

whenever X is nonzero, where we used Cauchy-Schwarz inequality. This shows that ω_{FS} is positive. Hence $\mathbb{P}^n$ is a Kähler manifold.

Next, we will show that submanifolds of Kähler manifold is Kähler.

Proposition 5.18. Let M be a Kähler manifold, then any submanifold $N \subset M$ is also Kähler.

Proof. Let ω be the Kähler form on M , and $i : N \rightarrow M$ be the inclusion map. Then $i^*\omega$ is clearly a Kähler form on N . $\square$

Corollary 5.19. Any projective manifold (i.e., submanifolds of $\mathbb{P}^n$ for some n) is Kähler.

Proof. This follows from Example 5.17 and Proposition 5.18. $\square$

Our next two examples will have slightly more technical proofs, but they are very useful to know. We will show that if X is a compact Kähler manifold and E is a holomorphic vector bundle on X , then $\mathbb{P}(E)$ is Kähler (recall Theorem 3.15). We will also show that if X is Kähler and $Y \subset X$ is a compact submanifold, then the blow up (recall section 3.4) $\hat{X} = \text{Bl}_Y X$ is also Kähler.

Theorem 5.20. If X is compact Kähler and $E \rightarrow X$ is a holomorphic vector bundle, then the manifold $\mathbb{P}(E)$ is Kähler.

In the case that X is a single point, then $E \cong \mathbb{C}^{n+1}$ where $n + 1 = \text{rk } E$, hence $\mathbb{P}(E) \cong \mathbb{P}^n$. Recall that to show that $\mathbb{P}^n = \mathbb{P}(E)$ is Kähler, we considered the curvature on the tautological bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, and showed that this gives a Kähler form which is the Fubini-Study metric ω_{FS} .

Let $\pi : \mathbb{P}(E) \rightarrow X$ be the projectivized bundle map. We can generalize the pointwise picture as follows: A point in the complex manifold $\mathbb{P}(E)$ is a pair (x, L_x) where $x \in X$ is a point and L_x is a line in the fiber E_x . We can define a tautological line bundle

$$\mathcal{O}_{\mathbb{P}(E)}(-1) = \{((x, L_x), v) : v \in L_x, x \in X\} \subset \pi^*E \subset \mathbb{P}(E) \times E.$$

This is a line bundle on $\mathbb{P}(E)$ with the projection $\mathcal{O}_{\mathbb{P}(E)}(-1) \rightarrow \mathbb{P}(E)$ being the obvious one. At every point $(x, L_x) \in \mathbb{P}(E)$, the fiber is precisely L_x . Clearly, the restriction of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ to each fiber, which is isomorphic to $\mathbb{P}^1$, is naturally isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. We now define

$$\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}(E)}(-1)^*.$$

Proof of Theorem 5.20. Let h be a Hermitian metric on the bundle E . Let $\pi : \mathbb{P}(E) \rightarrow X$ be the projection map. Then h induces a Hermitian metric on $\pi^*E \rightarrow \mathbb{P}(E)$. Hence by restriction, it induces a Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(-1)$ and its dual $\mathcal{O}_{\mathbb{P}(E)}(1)$.

The Chern form ω_E associated to this metric is not necessarily positive on $\mathbb{P}(E)$, but its restriction to each fiber $\pi^{-1}(x)$ is positive, since it is equal to the Fubini-Study metric

on $\mathbb{P}(E_x)$ associated to the metric h_x on E_x . Suppose now that X is Kähler, and let ω_X be a Kähler form on X . Since X is compact, we claim that we can choose some large $\lambda \gg 0$ such that the real closed form of type $(1,1)$

$$\omega = \omega_E + \lambda \pi^* \omega_X \quad (5.3)$$

is positive on $\mathbb{P}(E)$. Indeed, let χ be a vector field on $\mathbb{P}(E)$. If $\pi_* \chi = 0$ at some point x but $\chi \neq 0$ there, then $\pi_* \chi = \chi$ lives entirely in a fiber $\mathbb{P}(E_x)$, so the first component $\omega_E(\chi, J\chi) > 0$ by positivity of ω_E on each fiber, hence $\omega(\chi, J\chi) > 0$ there. Otherwise, $\pi_* \chi \neq 0$ at x , so $\pi^* \omega_X(\chi, J\chi) = \omega_X(\pi_* \chi, J' \pi_* \chi) > 0$, where J' is the almost complex structure on X . Note that here we used the fact that $\pi : \mathbb{P}(E) \rightarrow X$ is a holomorphic map, so $\pi_*(J\chi) = J' \pi_* \chi$ by Theorem 2.47. The problem is in this case $\omega_E(\chi, J\chi)$ might be negative. By compactness of X , we can choose a $\lambda \gg 0$ such that $\omega(\chi, J\chi) > 0$ for this χ . But this constant a priori depends on χ . So we need to argue there is a uniform constant λ that works for all χ .

To do this, we first note that positivity is scaling invariant: $\omega(c\chi, Jc\chi) = c^2 \omega(\chi, J\chi)$. Hence we can pick any arbitrary Riemannian metric g on $\mathbb{P}(E)$, and it suffices to prove positivity for any vector field χ that is a section of the unit tangent bundle

$$S = \{\chi \in \Gamma(T\mathbb{P}(E)) : \|\chi\|_g = 1\}.$$

Since $\mathbb{P}(E)$ is compact, so is S . For the sake of contradiction, suppose for every positive integer $n > 0$, there exists $\chi_n \in S$ that makes $\lambda = n$ fails to be the constant that makes (5.3) positive. Thus $\omega_E(\chi_n, J\chi_n) + n\pi^* \omega_X(\chi_n, J\chi_n) \leq 0$. Rearranging this equality, this gives

$$\pi^* \omega_X(\chi_n, J\chi_n) \leq -\frac{1}{n} \omega_E(\chi_n, J\chi_n). \quad (5.4)$$

Since S is compact, by ArzelàAscoli theorem, the sequence (χ_n) has a convergent subsequence. Therefore, up to relabeling, we can assume that χ_n converges to a limit χ_∞ . Since ω_E is a smooth form on a compact manifold $\mathbb{P}(E)$, we know $\omega_E(\chi_\infty, J\chi_\infty)$ is bounded. Hence taking the limit of (5.4) as $n \rightarrow \infty$, we have that

$$\pi^* \omega_X(\chi_\infty, J\chi_\infty) \leq 0.$$

By positivity, $\pi^* \omega_X(\chi, J\chi) \geq 0$ for all χ . Hence the inequality above implies that $\pi^* \omega_X(\chi_\infty, J\chi_\infty) = 0$. The equality holds iff χ_∞ is vertical, i.e., $\pi_* \chi_\infty = 0$ by positivity of ω_X . For a vertical vector field we have $\omega_E(\chi_\infty, J\chi_\infty) > 0$, but our assumption suggests that $\omega_E(\chi_n, J\chi_n) \leq -n\pi^* \omega_X(\chi_n, J\chi_n)$. Since $\pi^* \omega_X$ is nonnegative, the right hand side is nonpositive, so $\omega_E(\chi_n, J\chi_n) \leq 0$ for all n . Taking the limit as $n \rightarrow \infty$ gives the desired contradiction. This shows that there is a uniform $\lambda \gg 0$ making (5.3) a positive closed $(1,1)$ form. Therefore, we conclude that $\mathbb{P}(E)$ is Kähler, as desired. $\square$

Next, we will use a similar technique to show that

Theorem 5.21. *If X is Kähler and $Y \subset X$ is a compact complex submanifold of X , then the blow up $\hat{X} = \text{Bl}_Y X$ is Kähler, and it is compact if X is.*

The proof of this result requires first a lemma:

Lemma 5.22. *Let $\hat{X} = \text{Bl}_Y X$ where Y is an embedded submanifold of X , and let $E = \tau^{-1}(Y)$ be the exceptional divisor. Then $\mathcal{O}_{\hat{X}}(-E)|_E \cong \mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$.*

Sketch of Proof. By Remark 3.36, since E is a smooth hypersurface in $\hat{X}$, we have $\mathcal{O}_{\hat{X}}(-E)|_E \cong N_{E/\hat{X}}^*$. Therefore, we are reduced to showing that $N_{E/\hat{X}}$ is isomorphic to the tautological subbundle of $\tau^*N_{Y/X}$. First, we note that we have a differential map $\tau_* : T_{\hat{X}} \rightarrow \tau^*T_X$ along E . This induces a natural map $\Phi : N_{E/\hat{X}} \rightarrow \tau^*N_{Y/X}$ induced by this differential, obtained by writting down the following diagram using the relative tangent bundle sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_E & \longrightarrow & T_{\hat{X}} & \longrightarrow & N_{E/\hat{X}} \longrightarrow 0 \\ & & \downarrow \tau_* & & \downarrow \tau_* & & \downarrow \Phi \\ 0 & \longrightarrow & \tau^*T_Y & \longrightarrow & \tau^*T_X & \longrightarrow & \tau^*N_{Y/X} \longrightarrow 0 \end{array}$$

Using the explicit local description of blowup, one can check that Φ gives an isomorphism of $N_{E/\hat{X}}$ with the tautological subbundle $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1) \subset \tau^*N_{Y/X}$. $\square$

Proof of Theorem 5.21. Let $\tau : \hat{X} \rightarrow X$ be the blow-down map. It is easy to see from the local description of τ that τ is proper. So $\hat{X}$ is compact if X is.

Let ω_X be a Kähler form on X . Then $\tau^*\omega_X$ is a real closed (1,1)-form which is positive outside $\tau^{-1}(Y)$. By positivity, $\tau^*\omega_X(\chi, J\chi) = 0$ iff $\tau_*\chi = 0$, so the kernel of $\tau^*\omega_X$ along $\tau^{-1}(Y)$ consists of the tangent space to the fibers of τ .

Suppose we have a real closed (1,1)-form α on $\hat{X}$, which is zero outside a compact neighborhood of Y and strictly positive on the fibers of τ . Then since Y is compact, repeating the compactness argument we used in the proof of Theorem 5.20, we can find $C \gg 0$ such that the form

$$\omega = C\tau^*\omega_X + \alpha \tag{5.5}$$

is positive, closed, and of type (1,1), which equips $\hat{X}$ with a Kähler structure.

By Theorem 3.45, we know that the exceptional divisor $E = \tau^{-1}(Y)$ is isomorphic to the projective bundle $\tau : \mathbb{P}(N_{Y/X}) \rightarrow Y$. On $\mathbb{P}(N_{Y/X})$ we know there is a real closed (1,1)-form which is strictly positive on the fibers of τ : namely, the form obtained as the Chern connection form of the line bundle $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$ for a hermitian metric h induced by a Hermitian metric on $N_{Y/X}$. By Lemma 5.22, we know that the holomorphic line bundle $L = \mathcal{O}_{\hat{X}}(-E)$ over $\hat{X}$ is trivial outside of $E = \tau^{-1}(Y)$ and whose restriction to E is isomorphism to $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$.

Through the isomorphism $L|_E \cong \mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$, we can transplant the hermitian metric h onto $L|_E$, and we can extend it to an open neighborhood $U \subset \hat{X}$ containing E . Let's call this hermitian metric h_1 . It has the property that the Chern curvature form Θ_1 of this metric is strictly positive on the fiber of τ_* by our construction. Outside of E , since L is trivial, we can use the flat metric h_0 on $L_{\hat{X} \setminus E}$. Now take a bump function ρ such that $\rho \equiv 1$ on E and is supported on U . Then we can define a global hermitian metric H on L by setting

$$H = \rho h_1 + (1 - \rho)h_0.$$

Then the Chern curvature form $i\Theta_H/2$ associated with this Hermitian metric H will be a closed (1,1) form by Proposition 4.21 with the desired property. So we can set $\alpha = i\Theta/2$. Then (5.5) will be a positive (1,1)-form. Hence $\hat{X}$ is Kähler, as desired. $\square$

We will conclude this section with some nonexamples of Kähler manifolds. First, there are topological obstructions for a manifold to be Kähler:

Lemma 5.23. *If N is a complex compact submanifold of a Kähler manifold M , then N is not a boundary in M .*

Proof. If $N = \partial Q$ for some Q , and ω is the Kähler form on M , then Stoke's theorem gives

$$\int_N \omega^k = \int_Q d(\omega^k) = 0$$

since ω is closed. But by Theorem 5.11, we know

$$\int_N \omega^k = (\dim N)! \operatorname{vol}(N) > 0,$$

which is the desired contradiction. $\square$

This observation allowed Hironaka to construct other examples of non-Kähler complex manifolds, which we will see later. For now, let us see another example of topological obstruction to a manifold being Kähler.

Proposition 5.24. Let M be a complex n -manifold with Kähler form ω . Then ω^k represents a nonzero element in $H^{2k}(M)$ for all $k = 1, \dots, n$. Thus, a $2n$ -dimensional manifold with $b^{2k}(M) = 0$ for some $1 \leq k \leq n$ cannot be Kähler.

Proof. By Theorem 5.11, we already know that ω^n is not exact. Now suppose $\omega^k = d\eta$ is exact for some $1 \leq k < n$. Then $\omega^n = \omega^k \wedge \omega^{n-k} = d\eta \wedge \omega^{n-k} = d(\eta \wedge \omega^{n-k})$ since ω is closed, which is a contradiction. Hence we conclude that ω^k must not be exact for any $1 \leq k \leq n$. So $H^{2k}(M)$ is nontrivial for any $1 \leq k \leq n$, and the last statement follows directly. $\square$

Example 5.25 (Hopf manifolds are not Kähler). The Hopf manifold is diffeomorphic to $S^{2n-1} \times S^1$ for some $n > 1$ by Example 2.16. By Künneth formula, we have $H^2(S^{2n-1} \times S^1) = 0$. Hence by Proposition 5.24, Hopf manifolds cannot be Kähler.

Example 5.26 (Hironaka's non-Kähler example). The following example was constructed by Hironaka in his PhD thesis at Harvard in 1960 [9].

Let X be any nonsingular projective algebraic three-fold. Take two nonsingular curves $c, d \subseteq X$ which meet transversally at two points P, Q , and nowhere else (Fig 5.1). On $X - Q$, first blow up the curve c , then blow up the strict transform of the curve d . On $X - P$, first blow up the curve d , then blow up the strict transform of the curve c . On $X - P - Q$ it doesn't matter in which order we blow up the curves c and d , so we can glue our blown-up varieties along the inverse images of $X - P - Q$. The result is a nonsingular complete algebraic variety $\tilde{X}$. We will show that $\tilde{X}$ is not Kähler.

To do this, we must examine what happens in a neighborhood of P (Fig 5.1). Let l be the inverse image in $\tilde{X}$ of a general point of c . Let m be the inverse image of a general point of d . Note that l and m are projective lines. Then the inverse image of P consists of two lines l_0 and m_0 , and we have algebraic equivalence of cycles $l \sim l_0 + m_0$ and $m \sim m_0$, i.e., they are cohomologous. Note the asymmetry resulting from the order in which we blew up the two curves. Now in the neighborhood of Q the opposite happens. So $f^{-1}(Q)$ is the union of two lines l'_0 and m'_0 , and we have algebraic equivalence $l \sim l_0$ and $m \sim l'_0 + m'_0$. Combining these equivalences, we find that $l_0 + m'_0 \sim 0$. This would be impossible if $\tilde{X}$ were Kähler, because then by Proposition 5.18 the submanifold $l_0 + m'_0$ would be Kähler. But $l_0 + m'_0 \sim 0$ contradicts Lemma 5.23.

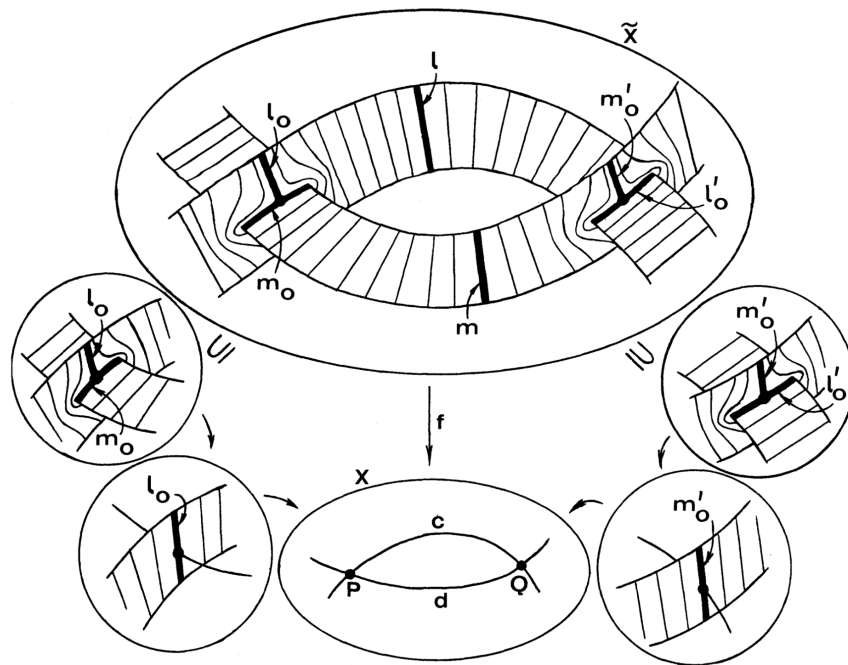
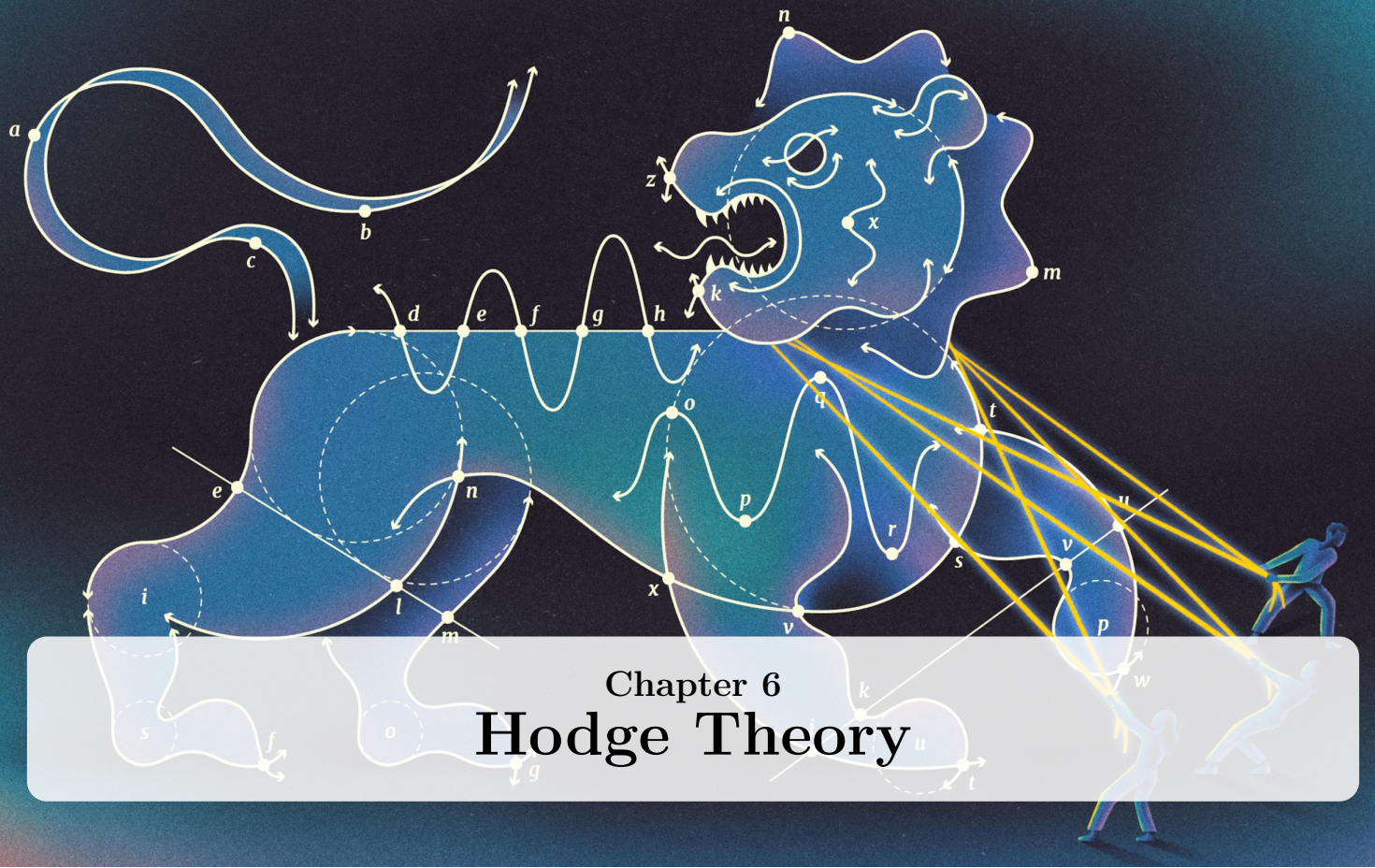


Figure 5.1: Hironaka's construction of a non-Kähler manifold $\tilde{X}$.



Introduction

The main goal of this chapter is to show that there exists special decompositions of cohomology groups for a Kähler manifold, the most important one being the Hodge decomposition $H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M)$. To prove this result, we need to use the tool of Hodge theory, which is first developed on Riemannian manifolds. Then these results are extended to complex manifolds, and eventually to Kähler manifolds. After proving the above decomposition (Theorem 6.40), we study the Lefschetz operator $L_\omega : \mathcal{A}^k \rightarrow \mathcal{A}^{k+2}$ given by wedging with the Kähler form ω on Kähler manifolds. Developing this perspective will eventually lead us to another decomposition of cohomology for Kähler manifolds, called Lefschetz decomposition (Theorem 6.48). But Hodge theory is such a rich and beautiful area that it provides more than these results. We will see many small results that can be easily obtained using the machinery of Hodge theory, including a proof of Poincaré duality, Serre duality, many topological constraints on Kähler manifolds, and classification of line bundles on $\mathbb{P}^n$.

6.1 Hodge Theory on Riemannian Manifolds

Let (M, g) be a Riemannian manifold. We have a bundle isomorphism $\Phi : TM \rightarrow T^*M$ given by $X \mapsto g(X, -)$. Let the inverse of this isomorphism be $\sharp : T^*M \rightarrow TM$, denoted by $\omega \mapsto \omega^\sharp$. This is called the **musical isomorphism**. Extending it by complex linearity we get an isomorphism $\sharp : T_{\mathbb{C}}^*M \rightarrow T_{\mathbb{C}}M$. Locally, this defines an inner product on T^*M :

Definition 6.1. For any complex valued 1-forms $\alpha, \beta \in \Gamma(T_{\mathbb{C}}^*M)$, we define the (pointwise) inner product

$$\langle \alpha, \beta \rangle := g(\alpha^\sharp, \overline{\beta^\sharp}).$$

Proposition 6.2. With respect to local coordinates, the inner product is given by the inverse metric, more specifically we have $\langle dx^i, dx^j \rangle = g^{ij}$.

Proof. To prove the statement, we first need to figure out local expressions of the musical isomorphism $\sharp : T^*M \rightarrow TM$. Assume $(dx^i)^\sharp = f^j \frac{\partial}{\partial x^j}$. Then we have that

$$f^j g_{jk} = \langle f^j \partial_j, \partial_k \rangle = \Phi \circ \sharp(dx^i) \left(\frac{\partial}{\partial x^k} \right) = dx^i \left(\frac{\partial}{\partial x^k} \right) = \delta_k^i.$$

Therefore, we see that $f^j = g^{ij}$. That is, $(dx^i)^\sharp = g^{ij} \frac{\partial}{\partial x^j}$. Hence, we have

$$\langle dx^i, dx^j \rangle = \langle g^{ik} \partial_k, g^{jl} \partial_l \rangle = g^{ik} g^{jl} g_{kl} = g^{ij}.$$

This concludes the proof. $\square$

Moreover, we can extend this inner product to the entire space of complex-valued q -forms $\mathcal{A}^q$ on M : When $q = 0$, elements in $\mathcal{A}^0(M)$ are complex valued functions u, v . We simply define $\langle u, v \rangle = u\bar{v}$. For $q = 1$, this is the inner product on $T_{\mathbb{C}}^*M$. For $q > 1$, we declare, for every orthonormal basis $\epsilon^1, \dots, \epsilon^N$ of $T_{\mathbb{C}}^*M$, the collection

$$\mathcal{B} = \{\epsilon^{j_1} \wedge \dots \wedge \epsilon^{j_q} : j_1 < \dots < j_q\}$$

is an orthonormal basis of $\mathcal{A}^q$.

Theorem 6.3. For $q \geq 1$, the inner product on $\mathcal{A}^q$ defined above satisfies

$$\langle \alpha^1 \wedge \dots \wedge \alpha^q, \beta^1 \wedge \dots \wedge \beta^q \rangle = \det \langle \alpha^j, \beta^k \rangle.$$

Proof. By linearity on both sides, it suffices to check the equality on the basis covectors $\alpha^i = \epsilon^{j_i}$ and $\beta^i = \epsilon^{k_i}$, which are orthonormal in $T_{\mathbb{C}}^*M$. Let $J = (j_1, \dots, j_q)$ and $K = (k_1, \dots, k_q)$. Let S_q be the symmetric group on q elements.

If K is not a permutation of J , then by definition the left hand side is zero. Similarly, the right hand side will have a column of zeros since for some fixed j we have $\langle \alpha^j, \beta^k \rangle = 0$ for all k . Hence the right hand side is also zero.

If $K = \sigma(J)$ for some $\sigma \in S_q$, then we have

$$\langle \alpha^1 \wedge \dots \wedge \alpha^q, \beta^1 \wedge \dots \wedge \beta^q \rangle = \text{sgn } \sigma.$$

On the other hand, the matrix $\langle \alpha^j, \beta^k \rangle$ in this case is obtained from the identity matrix by applying the permutation σ to the columns, so $\det \langle \alpha^j, \beta^k \rangle = \text{sgn } \sigma$ as well. This concludes the proof. $\square$

When M is compact, we may use the pointwise inner product to define a global inner product on $\mathcal{A}^q(M)$.

Definition 6.4. Let (M, g) be a compact Riemannian manifold. For every $\alpha, \beta \in \mathcal{A}^q(M)$, we define the global **Hodge inner product** by

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle d\text{Vol}_g.$$

We define the **norm** $\|\alpha\|$ of $\alpha \in \mathcal{A}^q(M)$ to be $\|\alpha\| := (\alpha, \alpha)^{1/2}$.

Remark 6.5. We use $|\alpha| = \langle \alpha, \alpha \rangle^{1/2}$ to denote the pointwise norm of α .

Using this inner product, we can single out a best representative for each representative of cohomology class.

Theorem 6.6. *On a compact Riemannian manifold M , a closed q -form $\alpha \in Z^q(M)$ minimizes the norm $\|\alpha\|$ within its cohomology class iff it is orthogonal to the space $\mathcal{B}^q(M)$ of exact forms. If so, it is the unique minimizer in its cohomology class.*

Proof. Suppose α is closed and is orthogonal to the space $\mathcal{B}^q(M)$ of exact forms. Then within its cohomology class, any other representative is of the form $\alpha + d\beta$. Hence by assumption

$$\|\alpha + d\beta\|^2 = (\alpha + d\beta, \alpha + d\beta) = \|\alpha\|^2 + \|d\beta\|^2 \geq \|\alpha\|^2.$$

Conversely, suppose α minimizes the norm within its cohomology class. Then for every t we have $\|\alpha + td\beta\|^2 \geq \|\alpha\|^2$ for all real t . Thus the smooth real-valued function $t \mapsto \|\alpha + td\beta\|^2$ obtains its minimum at $t = 0$. Hence we have

$$0 = \frac{d}{dt} \bigg|_{t=0} \|\alpha + td\beta\|^2 = \frac{d}{dt} \bigg|_{t=0} (\|\alpha\|^2 + 2t \operatorname{Re}(\alpha, d\beta) + t^2 \|d\beta\|^2) = 2 \operatorname{Re}(\alpha, d\beta).$$

Now replacing β by $i\beta$ and take derivative at zero again gives $\operatorname{Im}(\alpha, d\beta) = 0$. Hence $\alpha \perp \mathcal{B}^q(M)$, as desired. $\square$

To make use of the observation we obtained from Theorem 6.6, we need a practical way to identify which forms are orthogonal to the image of d . We recall from linear algebra that if we have a linear map $A : V \rightarrow W$ between finite-dimensional complex vector spaces with Hermitian inner products, then the **adjoint** A^* of A is the unique linear map $A^* : W \rightarrow V$ that satisfies $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x \in V, y \in W$.

Lemma 6.7. *In the situation described above, we have $(\operatorname{im} A)^\perp = \ker A^*$.*

Proof. Suppose $y \in (\operatorname{im} A)^\perp$. Then $\langle Ax, y \rangle = 0$ for all $x \in V$. Hence $\langle x, A^*y \rangle = 0$ for all $x \in V$. Since the inner product on V is nondegenerate, this means $A^*y = 0$, hence $y \in \ker A^*$. Conversely, if $y \in \ker A^*$, then $\langle Ax, y \rangle = \langle x, A^*y \rangle = 0$. Hence $y \in (\operatorname{im} A)^\perp$. $\square$

Turns out we can actually define adjoint operators for operator $d : \mathcal{A}^{q-1}(M) \rightarrow \mathcal{A}^q(M)$ between infinite-dimensional vector spaces as well. We are going to omit the analytical details here, and refer the readers of [1], p.259-261. The punchline is the following: we can define the formal adjoint

$$d^* : \mathcal{A}^q(M) \rightarrow \mathcal{A}^{q-1}(M)$$

of $d : \mathcal{A}^{q-1}(M) \rightarrow \mathcal{A}^q(M)$, for every $q \geq 0$. Then we have

Corollary 6.8. *On a compact Riemannian manifold M , a closed differential form α is the unique representative of its cohomology class that minimizes the Hodge norm iff $d^*\alpha = 0$.*

Proof. This follows immediately from Theorem 6.6 and the infinite dimensional generalization of Lemma 6.7. $\square$

Our next task is to study the adjoint $d^* : \mathcal{A}^{q+1}(M) \rightarrow \mathcal{A}^q(M)$ better. To do this, we need to introduce the Hodge star operator. As we will see, using the Hodge star operator we can express d^* in terms of d .

Definition 6.9. Let M be an oriented Riemannian manifold of dimension n . Let $(\epsilon^1, \dots, \epsilon^n)$ be a local *real* orthonormal basis of T^*M . Thus the collection $\{\epsilon^{j_1} \wedge \dots \wedge \epsilon^{j_q} : j_1 < \dots < j_q\}$ is an orthonormal basis of $\mathcal{A}^q(M)$. We denote $J = (j_1, \dots, j_q)$ and J' the unique increasing multi-index of length $n - q$ complementary to J . Then we define the **Hodge star operator** $*$: $\mathcal{A}^q(M) \rightarrow \mathcal{A}^{n-q}(M)$ by

$$*\epsilon^J = \text{sgn}(J, J')\epsilon^{J'} \quad (6.1)$$

where $\text{sgn}(J, J')$ is the sign of the permutation that takes $(1, 2, \dots, n)$ to $(j_1, \dots, j_q, j'_1, \dots, j'_{n-q})$.

The next theorem shows that the Hodge star operator is well-defined, independent of the choice of coframes, and satisfies several nice identities.

Theorem 6.10. *On a oriented n -dimensional Riemannian manifold (M, g) , the Hodge star operator is well-defined, and is the unique smooth complex-linear bundle isomorphism $*$: $\mathcal{A}^q(M) \rightarrow \mathcal{A}^{n-q}(M)$, such that for any $\alpha, \beta \in \mathcal{A}^q(M)$,*

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle \text{dVol}_g.$$

In terms of any local real oriented orthonormal coframe, $$ is given by (6.1). In addition, we have*

$$\overline{*\alpha} = *\bar{\alpha} \quad **\alpha = (-1)^{q(n-q)}\alpha \quad \langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle.$$

Proof. Note that to check all the equations in the statement, by (conjugate)linearity it suffices to check the case where α, β are orthonormal basis elements.

To check the first equality, we let $\alpha = \epsilon^J, \beta = \epsilon^K$, where J, K has complementary lengths. Since β is real, note that $*\bar{\beta} = *\beta = \text{sgn}(K, K')\epsilon^{K'}$. Thus $\beta \wedge *\bar{\beta} = \epsilon^1 \wedge \dots \wedge \epsilon^n = \text{dVol}_g$. Hence if $J = K$, then the first equality holds. But if $J \neq K$, then there exists a common index in J and K' , hence $\alpha \wedge *\bar{\beta} = \pm \epsilon^J \wedge \epsilon^{K'} = 0$, and $\langle \alpha, \beta \rangle = \langle \epsilon^J, \epsilon^K \rangle = 0$. Hence the first equality is proven.

Now we prove uniqueness. Suppose $\star$ is another such operator satisfying $\alpha \wedge \star\bar{\beta} = \langle \alpha, \beta \rangle \text{dVol}_g$. Define $\delta = * - \star$. Then $\alpha \wedge \delta\bar{\beta} = 0$ for all α, β . Applying this with $\bar{\beta}$ in place of β shows $\alpha \wedge \delta\beta = 0$ as well. Thus

$$|\delta\beta|^2 \text{dVol}_g = \delta\beta \wedge *\bar{\delta\beta} = \pm *\bar{\delta\beta} \wedge \delta\beta = 0.$$

This shows that $\delta = 0$. Therefore, the definition given by different local coframes agree, and $*$ is smooth because it takes smooth local frames to smooth local frames.

Now $\overline{*\alpha} = *\bar{\alpha}$ follows because we define $*$ on real forms and extend it by complex linearity.

To prove the next equality, suffices to consider $\alpha = \epsilon^J$. Then $**\alpha = \pm\alpha$, where the sign is uniquely determined by the requirement that $*\alpha \wedge **\alpha = \text{dVol}_g$. On the other hand, since $\deg \alpha = q$ and $\deg(*\alpha) = n - q$, we have

$$\text{dVol}_g = \alpha \wedge *\alpha = (-q)^{q(n-q)} *\alpha \wedge \alpha.$$

Therefore $(-1)^{q(n-q)}$ satisfies the requirement, hence is equal to $**\alpha$. This also shows that $*$ is a bundle isomorphism.

The last equality follows because $*$ takes orthonormal frame of $\mathcal{A}^q(M)$ to orthonormal frame of $\mathcal{A}^{n-q}(M)$. $\square$

Now we deduce a nice expression of d^* in terms of d and the Hodge star operator:

Theorem 6.11. *Let M be an oriented Riemannian manifold of dimension n . The formal adjoint of the exterior derivative $d : \mathcal{A}^{q-1}(M) \rightarrow \mathcal{A}^q(M)$ is the operator $d^* : \mathcal{A}^q(M) \rightarrow \mathcal{A}^{q-1}(M)$ given by*

$$d^* = (-1)^{nq+n+1} * d *.$$

Proof. Let $\beta \in \mathcal{A}^q(M)$. For brevity, let $\delta\beta := (-1)^{nq+n+1} * d * \beta$. Using Theorem 6.10, we see that the Hodge inner product is also given by

$$(\alpha, \beta) = \int_M \alpha \wedge * \bar{\beta}.$$

The formula above makes sense when M is compact. If M is not compact, then we need to restrict α, β to the space $\mathcal{A}_c^q(M)$ of compactly supported q -forms on M . Now if we can show that for every compactly supported $\alpha \in \mathcal{A}_c^{q-1}(M)$ and $\beta \in \mathcal{A}_c^q(M)$ one has

$$(\alpha, \delta\beta) = (d\alpha, \beta),$$

then the statement will be proven. We directly compute:

$$\begin{aligned} (\alpha, \delta\beta) &= \int_M \alpha \wedge * \bar{\delta\beta} \\ &= (-1)^{nq+n+1} \int_M \alpha \wedge * (* d * \bar{\beta}) \\ &= (-1)^{nq+n+1} (-1)^{(n-q+1)(q-1)} \int_M \alpha \wedge d * \bar{\beta} \\ &= (-1)^q \int_M \alpha \wedge d * \bar{\beta}, \end{aligned}$$

where we used Theorem 6.10 to deduce $*^2 = (-1)^{(n-q+1)(q-1)}$ on $d * \beta$ which has degree $n - q + 1$. Then Stokes's theorem gives

$$0 = \int_M d(\alpha \wedge * \bar{\beta}) = \int_M d\alpha \wedge * \bar{\beta} + (-1)^{q-1} \int_M \alpha \wedge d * \bar{\beta} = (d\alpha, \beta) - (\alpha, \delta\beta).$$

This concludes the proof. $\square$

Now we can finally do Hodge theory on Riemannian manifolds. Let us start with a definition:

Definition 6.12. Let M be a Riemannian manifold. The **Hodge Laplacian** is the operator $\Delta_d : \mathcal{A}^q(M) \rightarrow \mathcal{A}^q(M)$ defined by

$$\Delta_d \eta := dd^* \eta + d^* d \eta.$$

We define a **harmonic form** to be a smooth differential form η such that $\Delta_d \eta = 0$. Let $\mathcal{H}^q(M)$ denote the space of harmonic complex-valued q -forms, i.e., $\mathcal{H}^q(M) = \ker \Delta_d$.

Proposition 6.13. Let M be a compact oriented Riemannian manifold.

- (a) Δ_d is self-adjoint: $\Delta_d^* = \Delta_d$.
- (b) A differential form $\alpha \in \mathcal{A}^q(M)$ is harmonic iff $d\alpha = 0$ and $d^* \alpha = 0$.
- (c) A 0-form is harmonic iff its restriction to each connected component of M is constant.

Proof. (a) This is easy since $(PQ)^* = Q^*P^*$ and $Q^{**} = Q$ for operators. (b) Suppose $d\alpha = d^*\alpha = 0$ then it follows directly from definition that $\Delta_d\alpha = 0$. Conversely, if $\Delta_d\alpha = 0$, then

$$0 = (\Delta_d\alpha, \alpha) = (dd^*\alpha, \alpha) + (d^*d\alpha, \alpha) = \|d^*\alpha\|^2 + \|d\alpha\|^2,$$

which shows that $d\alpha = d^*\alpha = 0$. (c) At degree 0 we have $d^* : \mathcal{A}^0(M) \rightarrow \mathcal{A}^{-1}(M)$ is the zero map, so by (b) harmonic forms $u \in \mathcal{A}^0(M)$ are precisely the ones such that $du = 0$. $\square$

By (b), it follows that

Corollary 6.14. *For each q , we have $\mathcal{H}^q(M) \subset Z^q(M)$. That is, every harmonic form is closed.*

A priori, the space $\mathcal{H}^q(M)$ need not be finite-dimensional. However, it follows from some analytic computation that the Hodge Laplacian Δ_d (in fact most Laplacian-type operators) is an elliptic operator. A deep analysis result known as the Fredholm theorem shows that the kernel of an elliptic operator on a compact manifold has finite-dimensional kernel. Hence it follows from the Fredholm theorem that $\mathcal{H}^q(M)$ is finite-dimensional. We only cite this result because the proof and even the relevant definitions will take us too far:

Theorem 6.15 (Fredholm). *Let M be a compact Riemannian manifold, E, F be Hermitian vector bundles over M , and $P : \Gamma(E) \rightarrow \Gamma(F)$ an elliptic linear differential operator. Then $\ker P$ and $\ker P^*$ are finite-dimensional, and P restricts to a bijection from $(\ker P)^\perp$ to $(\ker P^*)^\perp$.*

For definitions and theory of elliptic operator we refer to [1], p.265-269. For a proof of Fredholm theorem, we refer to [3], p.80-100.

The main result of this section is:

Theorem 6.16 (Hodge Theorem for Riemannian Manifolds). *Suppose M is an oriented compact Riemannian manifold. Then $\mathcal{H}^q(M)$ is finite-dimensional for each q , and the composite map*

$$\mathcal{H}^q(M) \hookrightarrow Z^q(M) \rightarrow H_{\text{dR}}^q(M; \mathbb{C})$$

is an isomorphism. Thus every de Rham cohomology class has a unique harmonic representative.

Proof. Since Δ_d is elliptic, Fredholm theorem implies $\mathcal{H}^q(M)$ is finite-dimensional. Injectivity of $\mathcal{H}^q(M) \rightarrow H_{\text{dR}}^q(M; \mathbb{C})$ follows from Theorem 6.6. Hence it remains to show that the map is surjective.

Let $\alpha \in Z^q(M)$ be any closed q -form on M . If $q = 0$, then α is already harmonic. If $q \geq 1$, then $d^*\alpha$ is orthogonal to $\ker d$ by Lemma 6.7, and therefore orthogonal to the kernel of $\Delta_d = \Delta_d^*$. Hence it follows from the Fredholm theorem that there is a $(q-1)$ -form β such that $\Delta_d\beta = d^*\alpha$. This means $dd^*\beta + d^*d\beta = d^*\alpha$, or

$$dd^*\beta = d^*\alpha - d^*d\beta.$$

The right hand side is in the image of $d^* : \mathcal{A}^q \rightarrow \mathcal{A}^{q-1}$ and the left hand side is in the image of $d : \mathcal{A}^{q-2} \rightarrow \mathcal{A}^{q-1}$ hence is also in kernel of $d^* : \mathcal{A}^{q-1} \rightarrow \mathcal{A}^q$, which is orthogonal to the image of $d^* : \mathcal{A}^q \rightarrow \mathcal{A}^{q-1}$ by Lemma 6.7. Hence the two sides of the equation above are orthogonal to each other and thus must both be zero. This implies $d^*(\alpha - d\beta) = 0$. Since it is also the case that $d(\alpha - d\beta) = 0$, it follows that $\alpha - d\beta$ is a harmonic representative of the cohomology class of α . $\square$

Corollary 6.17 (Poincaré Duality). *Let M be a compact oriented smooth n -dimensional manifold. There is a well-defined nondegenerate bilinear map $H^k(M; \mathbb{C}) \times H^{n-k}(M; \mathbb{C}) \rightarrow \mathbb{C}$ given by*

$$(\eta, \xi) \mapsto \int_M \eta \wedge \xi.$$

Thus for each k there is a complex-linear isomorphism

$$H^k(M; \mathbb{C}) \cong H^{n-k}(M; \mathbb{C})^*.$$

Thus the Betti number of M satisfy $b^k = b^{n-k}$. Analogous results hold for cohomology with real coefficients.

Proof. The only nontrivial part of the statement is the nondegeneracy, so let's prove it. Choose any Riemannian metric on M , so that $H^k(M; \mathbb{C}) \cong \mathcal{H}^k(M)$. Suppose η is a closed k -form such that $\int_M \eta \wedge \xi = 0$ for all closed $(n-k)$ -forms ξ . Without loss of generality, we may take η to be the harmonic representative of its cohomology class. Now take $\xi = *\bar{\eta}$. To see that this is a closed form, note that $\bar{\eta}$ is also harmonic because $\Delta_d \bar{\eta} = \overline{\Delta_d \eta} = 0$, so $0 = d*\bar{\eta} = \pm *d*\bar{\eta}$. But $*$ is a bundle isomorphism, so we must have $d*\bar{\eta} = d\xi = 0$. Thus we see that

$$0 = \int_M \eta \wedge \xi = \int_M \eta \wedge (*\bar{\eta}) = \|\eta\|^2.$$

Hence $\eta = 0$. Applying the same argument to ξ shows nondegeneracy. $\square$

6.2 Hodge Theory on Complex Manifolds

Now let M be a complex manifold of complex dimension n . In particular, it is a Riemannian manifold of dimension $2n$. Some of our previous results on Riemannian manifolds simplify in this case.

Lemma 6.18. *Let M be a compact Hermitian n -manifold.*

(a) *For $\alpha \in \mathcal{A}^{p,q}(M)$, we have $**\alpha = (-1)^{p+q}\alpha$ and $d^*\alpha = -*d*\alpha$.*

(b) *$*$ maps $\mathcal{A}^{p,q}(M)$ isomorphically to $\mathcal{A}^{n-p,n-q}(M)$.*

Proof. (a) By Theorem 6.10, we have $*^2\alpha = (-1)^{(p+q)(2n-p-q)}\alpha$. In mod 2 arithmetic, we have $(p+q)(2n-p-q) \equiv -(p+q)^2 \equiv p+q$. The second claim follows from Theorem 6.11, which gives $d^*\alpha = (-1)^{2n(p+q)+2n+1} *d*\alpha = -\alpha$.

(b) Let the Hermitian metric on M be h , and we are taking the Riemannian metric $g = \text{Re } h$. Note that for $X \in \Gamma(T^{1,0}M)$ and $Y \in \Gamma(T^{0,1}M)$, by the property of h we have

$$h(X, Y) = h(JX, JY) = h(iX, -iY) = i^2 h(X, Y) = -h(X, Y).$$

Hence $h(T^{1,0}M, T^{0,1}M) = 0$. Taking the real part shows that $T^{1,0}M \perp T^{0,1}M$ with respect to g as well. Therefore, $\mathcal{A}^{1,0}(M)$ and $\mathcal{A}^{0,1}(M)$ are orthogonal. This also implies that $\mathcal{A}^{p,q}(M) \perp \mathcal{A}^{p',q'}(M)$ under the Hodge inner product unless $(p, q) = (p', q')$ by the way we constructed the Hodge inner product. Therefore, to prove the claim, we will show that for $\alpha \in \mathcal{A}^{p,q}(M)$ we have $*\alpha \in \mathcal{A}^{n-p,n-q}(M)$ by showing that this is orthogonal to $\mathcal{A}^{r,s}(M)$ unless $(r+q, s+p) = (n, n)$.

Thus let $\beta \in \mathcal{A}^{r,s}(M)$ be arbitrary, and compute

$$\langle \beta, *\alpha \rangle d\text{Vol}_g = \beta \wedge **\bar{\alpha} = (-1)^{p+q} \beta \wedge \bar{\alpha}.$$

Since $\bar{\alpha} \in \mathcal{A}^{q,p}(M)$, the last expression has bidegree $(r+q, s+p)$, which is zero unless $(r+q, s+p) = (n, n)$. $\square$

Definition 6.19. Let M be a complex manifold. We define the *Dolbeault Laplacian*

$$\Delta_{\bar{\partial}}\alpha := \bar{\partial}\bar{\partial}^*\alpha + \bar{\partial}^*\bar{\partial}\alpha$$

and its conjugate

$$\Delta_{\partial}\alpha := \partial\partial^*\alpha + \partial^*\partial\alpha$$

mapping (p, q) -forms to (p, q) -forms.

The next result is analogue to Proposition 6.13, and the proof is similar, so we omit the proof.

Proposition 6.20. Let (M, g) be a compact Hermitian manifold.

- (a) We have $\Delta_{\bar{\partial}}^* = \Delta_{\bar{\partial}}$ and $\Delta_{\partial}^* = \Delta_{\partial}$.
- (b) A form α satisfies $\Delta_{\bar{\partial}}\alpha = 0$ iff $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$, and it satisfies $\Delta_{\partial}\alpha = 0$ iff $\partial\alpha = \partial^*\alpha = 0$.
- (c) A $(p, 0)$ -form is in the kernel of $\Delta_{\bar{\partial}}$ iff it is holomorphic.

Now we prove the analogous version of Theorem 6.11.

Theorem 6.21. For $\alpha \in \mathcal{A}^{p,q}(M)$, we have $\bar{\partial}^*\alpha = - * \partial * \alpha$.

Proof. Let $\alpha \in \mathcal{A}^{p,q}(M)$ and $\beta \in \mathcal{A}^{p,q-1}(M)$. Then $\beta \wedge * \bar{\alpha} \in \mathcal{A}^{n,n-1}(M)$. Hence $d(\beta \wedge * \bar{\alpha}) = \bar{\partial}(\beta \wedge * \bar{\alpha})$. Stokes's theorem then gives

$$\begin{aligned} 0 &= \int_M \bar{\partial}(\beta \wedge * \bar{\alpha}) \\ &= \int_M \bar{\partial}\beta \wedge * \bar{\alpha} + (-1)^{p+q-1} \int_M \beta \wedge \bar{\partial} * \alpha \\ &= \int_M \bar{\partial}\beta \wedge * \bar{\alpha} + \int_M \beta \wedge * * \bar{\partial} * \alpha \\ &= (\bar{\partial}\beta, \alpha) + (\beta, * \partial * \alpha). \end{aligned}$$

Hence $\bar{\partial}^*\alpha = - * \partial * \alpha$. □

Again, by some analytical result, we have that $\Delta_{\bar{\partial}}$ is elliptic. So we can apply Fredholm theorem (Theorem 6.15).

Definition 6.22. For a compact Hermitian manifold M , we define $\mathcal{H}^{p,q}(M)$ to be the kernel of $\Delta_{\bar{\partial}} : \mathcal{A}^{p,q}(M) \rightarrow \mathcal{A}^{p,q}(M)$. Elements of $\mathcal{H}^{p,q}(M)$ are called *$\bar{\partial}$ -harmonic forms*.

By the previous result, α is harmonic iff $\bar{\partial}\alpha = \partial\alpha = 0$. The next result is proved exactly like its counterpart Theorem 6.6.

Proposition 6.23. A $\bar{\partial}$ -closed (p, q) -form α is the unique minimizer of the Hodge norm within its Dolbeault cohomology class iff it is $\bar{\partial}$ -harmonic.

The Hodge theorem for complex manifolds is proved precisely like that of Theorem 6.16:

Theorem 6.24 (Hodge-Dolbeault theorem). *Let M be a compact Hermitian manifold. Then $\mathcal{H}^{p,q}(M)$ is finite-dimensional for each (p, q) , and the composite map*

$$\mathcal{H}^{p,q}(M) \hookrightarrow Z_{\bar{\partial}}^{p,q}(M) \rightarrow H^{p,q}(M)$$

is an isomorphism. Thus every Dolbeault cohomology class on M has a unique $\bar{\partial}$ -harmonic representative.

6.3 Bundle-Valued Forms and Serre Duality

We can generalize the Hodge-Dolbeault theorem to bundle valued-forms. To do this, let M be a complex manifold and $E \rightarrow M$ be a holomorphic vector bundle. Recall (Theorem 2.67) that we have the Cauchy-Riemann operator

$$\bar{\partial}_E : \mathcal{A}^{p,q}(M; E) \rightarrow \mathcal{A}^{p,q+1}(M; E)$$

which is used to define the Dolbeault cohomology groups $H^{p,q}(M; E)$. To apply Hodge theory to these groups, we choose a Riemannian metric g on M and a Hermitian fiber metric h on E . Then we can define a Hodge inner product on $\mathcal{A}^{p,q}(M; E)$ by defining

$$\langle \alpha \otimes \sigma, \beta \otimes \tau \rangle := \langle \alpha, \beta \rangle \cdot h(\sigma, \tau)$$

for $\alpha, \beta \in \mathcal{A}^{p,q}(M)$ and $\sigma, \tau \in \Gamma(E)$, and extend this by complex linearity in the first argument and conjugate linearity in the second. Here, $\langle \alpha, \beta \rangle$ is the pointwise Hodge inner product of the ordinary (p, q) -forms on M . This defines a global Hermitian inner product on $\mathcal{A}^{p,q}(M; E)$:

$$(\eta_1, \eta_2) := \int_M \langle \eta_1, \eta_2 \rangle d\text{Vol}_g.$$

Using this inner product, we define

Definition 6.25. The *E -valued Dolbeault Laplacian* $\Delta_{\partial_E} : \mathcal{A}^{p,q}(M; E) \rightarrow \mathcal{A}^{p,q}(M; E)$ given by

$$\Delta_{\partial_E} := \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E.$$

An E -valued differential form α such that $\Delta_{\partial_E} \alpha = 0$ is said to be $\bar{\partial}_E$ -**harmonic**.

Lemma 6.26. *If E is a holomorphic vector bundle, an E -valued differential form α is $\bar{\partial}_E$ -harmonic iff $\bar{\partial}_E \alpha = \bar{\partial}_E^* \alpha = 0$.*

Proof. Similar to Proposition 6.13. □

Again, by some analytical proof we have Δ_{∂_E} is elliptic, so Fredholm theorem (Theorem 6.15) can be applied to this operator. This gives the following generalized version of the Hodge-Dolbeault theorem, whose proof we omit because it is similar to Theorem 6.16 and Theorem 6.24.

Theorem 6.27. *Suppose M is a compact Hermitian manifold and $E \rightarrow M$ is a Hermitian holomorphic vector bundle. Then the space $\mathcal{H}^{p,q}(M; E)$ of $\bar{\partial}_E$ -harmonic E -valued (p, q) -forms is finite dimensional for every (p, q) , and the composite map*

$$\mathcal{H}^{p,q}(M; E) \hookrightarrow Z^{p,q}(M; E) \rightarrow H^{p,q}(M; E)$$

is an isomorphism. Thus every E -valued Dolbeault cohomology class has a unique $\bar{\partial}_E$ -harmonic representative.

Theorem 6.27 has the following easy corollary, although the result itself is quite striking!

Corollary 6.28. *Suppose M is a compact complex manifold and $E \rightarrow M$ is a holomorphic vector bundle over M . Then the space of holomorphic sections $H^0(M, E)$ of E over M is finite-dimensional.*

Proof. By Theorem 2.68, we know that $H^0(M, E) \cong H^{0,0}(M; E)$, and we know from Theorem 6.27 that $H^{0,0}(M; E) \cong \mathcal{H}^{0,0}(M; E)$ which is finite-dimensional by the Fredholm theorem (Theorem 6.15). □

Corollary 6.29. *Suppose M is a compact complex manifold and $E \rightarrow M$ is a holomorphic vector bundle over M . Then the sheaf cohomology groups $H^q(M; \Omega^p(E)) = H^q(M; \Omega^p \otimes E)$ are all finite-dimensional, and are zero unless $0 \leq p, q \leq \dim M$.*

Proof. This follows again from Theorem 2.68, Theorem 6.27, and Theorem 6.15. $\square$

Let us now find a better expression for the adjoint $\bar{\partial}_E^*$. To do this, we note that the Hermitian metric h on E defines a conjugate-linear bundle isomorphism $\Phi : E \rightarrow E^*$ given by

$$\Phi(\sigma)(\tau) = h(\tau, \sigma)$$

for any $\tau \in \Gamma(E)$. Then, we define a conjugate-linear bundle map

$$\begin{aligned} \bar{*}_E : \mathcal{A}^{p,q}(E) &\rightarrow \mathcal{A}^{n-p, n-q}(M; E^*) \\ \alpha \otimes \sigma &\mapsto \overline{*}\alpha \otimes \Phi(\sigma). \end{aligned}$$

This map $\bar{*}_E$ is then extended bilinearly.

Proposition 6.30. The operator $\bar{*}_E : \mathcal{A}^{p,q}(M; E) \rightarrow \mathcal{A}^{n-p, n-q}(M; E^*)$ satisfies

- (a) $\bar{*}_E^2 \eta = (-1)^{p+q} \eta$ for $\eta \in \mathcal{A}^{p,q}(M; E)$.
- (b) $\bar{*}_E \bar{*}_{E^*} \xi = (-1)^{p+q} \xi$ for $\xi \in \mathcal{A}^{p,q}(M; E^*)$.
- (c) For $\omega, \zeta \in \mathcal{A}^{p,q}(M; E)$, we have $\omega \wedge \bar{*}_E \zeta = \langle \omega, \zeta \rangle \text{dVol}_g$.

Proof. (a) and (b) follows from direct computation using Lemma 6.18. To prove (c), we note that by linearity it suffices to assume that $\omega = \alpha \otimes \sigma$ and $\zeta = \beta \otimes \tau$. Then we have

$$\omega \wedge \bar{*}_E \zeta = (\alpha \otimes \sigma) \wedge (\overline{*}\beta \otimes \Phi(\tau)) = h(\sigma, \tau) \alpha \wedge \overline{*}\beta = h(\sigma, \tau) \langle \alpha, \beta \rangle \text{dVol}_g = \langle \omega, \zeta \rangle \text{dVol}_g.$$

This concludes the proof. $\square$

Therefore, we see that the Hodge-inner product on $\mathcal{A}^{p,q}(M; E)$ can also be written as

$$(\omega, \zeta) = \int_M \omega \wedge \bar{*}_E \zeta.$$

Theorem 6.31. On $\mathcal{A}^{p,q}(M; E)$, we have $\bar{\partial}_E^* = -\bar{*}_{E^*} \bar{\partial}_{E^*} \bar{*}_E$.

Proof. Let $\eta \in \mathcal{A}^{p, q-1}(M; E)$ and $\zeta \in \mathcal{A}^{p,q}(M; E)$. The form $\eta \wedge \bar{*}_E \zeta$ has bidegree $(n, n-1)$, so $d(\eta \wedge \bar{*}_E \zeta) = \bar{\partial}(\eta \wedge \bar{*}_E \zeta)$. Hence by Stokes's theorem

$$\begin{aligned} 0 &= \int_M \bar{\partial}(\eta \wedge \bar{*}_E \zeta) \\ &= \int_M \bar{\partial}_E \eta \wedge \bar{*}_E \zeta + (-1)^{p+q-1} \int_M \eta \wedge \bar{\partial}_{E^*}(\bar{*}_E \zeta) \\ &= \int_M \bar{\partial}_E \eta \wedge \bar{*}_E \zeta + \int_M \eta \wedge \bar{*}_E \bar{*}_{E^*} \bar{\partial}_{E^*}(\bar{*}_E \zeta) \\ &= (\bar{\partial}_E \eta, \zeta) + (\eta, \bar{*}_{E^*} \bar{\partial}_{E^*} \bar{*}_E \zeta). \end{aligned}$$

This shows that $\bar{\partial}_E^* = -\bar{*}_{E^*} \bar{\partial}_{E^*} \bar{*}_E$. $\square$

Now we prove a celebrated result by Serre:

Theorem 6.32 (Serre Duality). *Let M be a compact complex n -manifold and let $E \rightarrow M$ be a holomorphic vector bundle. There is a well-defined nondegenerate bilinear $H^{p,q}(M; E) \times H^{n-p, n-q}(M; E^*) \rightarrow \mathbb{C}$ given by*

$$(\eta, \zeta) \mapsto \int_M \eta \wedge \zeta.$$

Consequently, we have the following complex-linear isomorphisms for each (p, q) :

$$H^{p,q}(M; E)^* \cong H^{n-p, n-q}(M; E^*)$$

or equivalently

$$H^q(M, \Omega^p \otimes E)^* \cong H^{n-q}(M, \Omega^{n-p} \otimes E^*).$$

In addition, the Hodge numbers $h^{p,q}(M) = \dim H^{p,q}(M)$ of M satisfy

$$h^{p,q} = h^{n-p, n-q}.$$

Proof. We first show that the pairing is well-defined. Suppose $\eta \in \mathcal{A}^{p,q}(M; E)$ is $\bar{\partial}_E$ -closed and $\zeta \in \mathcal{A}^{n-p, n-q}(M; E^*)$ is $\bar{\partial}_{E^*}$ -closed. If $\eta = \bar{\partial}_E \gamma$, then it follows from Theorem 2.67 and Stokes's theorem that

$$\int_M \eta \wedge \zeta = \int_M \bar{\partial}_E \gamma \wedge \zeta = \int_M \bar{\partial}(\gamma \wedge \zeta) = \int_M d(\gamma \wedge \zeta) = 0.$$

Similarly, the integral is zero if ζ is $\bar{\partial}_{E^*}$ -exact. Hence the pairing is well-defined.

It remains to prove non-degeneracy. Suppose the cohomology class $[\eta] \in H^{p,q}(M; E)$ satisfies $\int_M \eta \wedge \zeta = 0$ for all $[\zeta] \in H^{n-p, n-q}(M; E^*)$. We can take η to be the $\Delta_{\bar{\partial}_E}$ -harmonic representative of its cohomology class, so that $\bar{\partial}_E \eta = 0$ and $\bar{\partial}_E^* \eta = 0$. Note that $\bar{*}_E \eta \in \mathcal{A}^{n-p, n-q}(M; E^*)$ satisfies

$$\bar{\partial}_{E^*}(\bar{*}_E \eta) = \pm \bar{*}_E \bar{\partial}_E^* \eta = \pm \bar{*}_E(-\bar{\partial}_E^* \eta) = 0.$$

Hence $\bar{*}_E \eta \in \mathcal{A}^{n-p, n-q}(M; E^*)$ represents a cohomology class in $H^{n-p, n-q}(M; E^*)$, and our hypothesis implies

$$0 = \int_M \eta \wedge \bar{*}_E \eta = \|\eta\|^2.$$

Hence $[\eta] = 0$. An identical argument shows non-degeneracy in the ζ -argument. Thus the duality theorem is proven. The statement about the Hodge numbers follows from Serre duality with $E = \mathcal{O}_M$. $\square$

6.4 Hodge Theory on Kähler Manifolds

Definition 6.33. Let M be a Kähler manifold with Kähler form ω . Define a complex linear bundle morphism $L_\omega : \mathcal{A}^{p,q}(M) \rightarrow \mathcal{A}^{p+1, q+1}(M)$, called the **Lefschetz operator**, by

$$L_\omega \eta = \omega \wedge \eta.$$

The key to establish Hodge theory on Kähler manifolds is a set of technical identities that holds on every Kähler manifolds.

Proposition 6.34 (Kähler Identities). On every Kähler manifold, the following identities hold:

$$[\bar{\partial}^*, L_\omega] = i\partial \quad [\partial^*, L_\omega] = -i\bar{\partial} \quad [L_\omega^*, \bar{\partial}] = -i\partial^* \quad [L_\omega^*, \partial] = i\bar{\partial}^*.$$

There are two observations that we should make: First, note that we only need to prove the first identity. Then the second one follows by taking conjugate. The last two then follows by taking adjoints, because $[A, B]^* = (AB - BA)^* = B^*A^* - A^*B^* = [B^*, A^*]$ for any operators A, B .

Next, we note that we only need to prove the first identity in the case where our Kähler manifold is $\mathbb{C}^n$. Indeed, by Theorem 5.13 and the very next remark after the theorem, we can choose Kähler normal coordinate such that up to $O(|z|^2)$ the metric is given by the standard $\mathbb{C}^n$ metric. And note that $\bar{\partial}^* = - * \partial *$ only involves one derivative of the metric coefficient, and L_ω involves no derivative at all.

To prove the first identity, we need a few more preliminary results. We first introduce some notation. For a differential form

$$\alpha = \alpha_{JK} dz^J \wedge d\bar{z}^K,$$

we define

$$\partial_j \alpha = \frac{\partial \alpha_{JK}}{\partial z^j} dz^J \wedge d\bar{z}^K \quad \partial_{\bar{j}} \alpha = \frac{\partial \alpha_{JK}}{\partial \bar{z}^j} dz^J \wedge d\bar{z}^K.$$

Then it is easy to check that

$$\bar{\partial} \alpha = d\bar{z}^j \wedge \partial_{\bar{j}} \alpha \quad \partial \alpha = dz^j \wedge \partial_j \alpha.$$

Also these derivative operators commutes with taking interior products:

$$\partial_j (\partial_{\bar{k}} \lrcorner \alpha) = \partial_{\bar{k}} \lrcorner (\partial_j \alpha).$$

Finally, assuming that we are in standard coordinate in $\mathbb{C}^n$, the coefficient of the metric are constants, one can show that

$$\langle dz^j \wedge \alpha, \beta \rangle = 2 \langle \alpha, \partial_{\bar{j}} \beta \rangle. \quad (6.2)$$

Using these identities, we will show

Lemma 6.35. *On $\mathbb{C}^n$, the formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ is given by*

$$\bar{\partial}^* \beta = - \sum_j 2 \partial_{\bar{j}} \lrcorner (\partial_j \beta).$$

Proof. First, note that for any smooth, compactly supported function f on $\mathbb{C}^n$ we have

$$\int_{\mathbb{C}^n} \partial_{\bar{j}} f dV = 0,$$

where dV is the standard volume form on $\mathbb{C}^n$. Now if $\alpha \in \mathcal{A}_c^q(\mathbb{C}^n)$ and $\beta \in \mathcal{A}_c^{q-1}(\mathbb{C}^n)$, we see that

$$\begin{aligned} 0 &= \int \partial_{\bar{j}} \langle d\bar{z}^j \wedge \alpha, \beta \rangle dV \\ &= \int \langle \partial_{\bar{j}} (d\bar{z}^j \wedge \alpha), \beta \rangle dV + \int \langle d\bar{z}^j \wedge \alpha, \partial_j \beta \rangle dV \\ &= \int \langle d\bar{z}^j \wedge \partial_{\bar{j}} \alpha, \beta \rangle dV + \int \langle \alpha, 2 \partial_{\bar{j}} \lrcorner (\partial_j \beta) \rangle dV \\ &= (\bar{\partial} \alpha, \beta) + (\alpha, \sum_j 2 \partial_{\bar{j}} \lrcorner (\partial_j \beta)). \end{aligned}$$

This proves the claim. $\square$

Using these identities together with the fact that interior multiplication is an antiderivation, i.e.,

$$X \lrcorner (\mu \wedge \nu) = (X \lrcorner \mu) \wedge \nu + (-1)^{|\mu|} \mu \wedge (X \lrcorner \nu),$$

we obtain

Proof of Proposition 6.34. As discussed, we only need to prove the first identity on $\mathbb{C}^n$. We have

$$\begin{aligned} \bar{\partial}^*(\omega \wedge \beta) &= - \sum_j 2\partial_{\bar{j}} \lrcorner \partial_j \left(\frac{i}{2} \sum_k dz^k \wedge d\bar{z}^k \wedge \beta \right) \\ &= -i \sum_{j,k} \partial_{\bar{j}}(dz^k \wedge d\bar{z}^k \wedge \partial_j \beta) \\ &= i \sum_{j,k} d\bar{z}^k (\partial_{\bar{j}}) dz^k \wedge \partial_j \beta - i \sum_{j,k} dz^k \wedge d\bar{z}^k \wedge (\partial_{\bar{j}} \lrcorner \partial_j \beta) \\ &= i \sum_j dz^j \wedge \partial_j \beta + \left(\sum_k \frac{i}{2} dz^k \wedge d\bar{z}^k \right) \wedge \left(- \sum_j 2\partial_{\bar{j}} \lrcorner \partial_j \beta \right) \\ &= i\partial\beta + \omega \wedge \bar{\partial}^*\beta. \end{aligned}$$

This concludes the proof. $\square$

The next three lemmas are the main applications of Kähler identities:

Lemma 6.36. *On every Kähler manifold, the following identities hold:*

$$\bar{\partial}^*\partial + \partial\bar{\partial}^* = 0, \quad \partial^*\bar{\partial} + \bar{\partial}\partial^* = 0.$$

Proof. Proposition 6.34 shows that $\bar{\partial}^* = -i[L_\omega^*, \partial]$. Using this and the fact that $\partial \circ \partial = 0$, we compute

$$\bar{\partial}^*\partial + \partial\bar{\partial}^* = -i[L_\omega^*, \partial]\partial - i\partial[L_\omega^*, \partial] = -iL_\omega^*\partial\partial + i\partial L_\omega^*\partial - i\partial L_\omega^*\partial + i\partial\partial L_\omega^* = 0.$$

The second identity follows from the first by conjugation. $\square$

Lemma 6.37. *On every Kähler manifold, $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}}$.*

Proof. The previous lemma yields

$$\begin{aligned} \Delta_d &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\ &= \partial\partial^* + \partial\bar{\partial}^* + \bar{\partial}\partial^* + \bar{\partial}\bar{\partial}^* + \partial^*\partial + \partial^*\bar{\partial} + \bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} \\ &= (\partial\partial^* + \partial^*\partial) + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\bar{\partial}\partial^* + \partial^*\bar{\partial}) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) \\ &= \Delta_\partial + \Delta_{\bar{\partial}}. \end{aligned}$$

This concludes the proof. $\square$

Lemma 6.38. *On every Kähler manifold, $\Delta_\partial = \Delta_{\bar{\partial}}$.*

Proof. From Proposition 6.34 again we get $\partial^* = i[L_\omega^*, \bar{\partial}]$. Thus

$$\Delta_\partial = \partial\partial^* + \partial^*\partial = i\partial[L_\omega^*, \bar{\partial}] + i[L_\omega^*, \bar{\partial}]\partial = i\partial L_\omega^* \bar{\partial} - i\partial \bar{\partial} L_\omega^* + iL_\omega^* \bar{\partial}\partial - i\bar{\partial} L_\omega^* \partial.$$

On the other hand, since $\bar{\partial}^* = -i[L_\omega^*, \partial]$,

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = -i\bar{\partial}[L_\omega^*, \partial] - i[L_\omega^*, \partial]\bar{\partial} = -i\bar{\partial} L_\omega^* \partial + i\bar{\partial}\partial L_\omega^* - iL_\omega^* \partial\bar{\partial} + i\partial L_\omega^* \bar{\partial}.$$

Because $d\bar{\partial} = -\bar{\partial}d$, these two expressions are equal. $\square$

The key theorem is the comparison of Laplacians on Kähler manifold.

Theorem 6.39. *On every Kähler manifold we have $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.*

Proof. This immediately follows from the two previous results. $\square$

The main theorem in this section (chapter even) is

Theorem 6.40 (Hodge Theorem for Kähler Manifolds). *Let M be a compact n -dimensional Kähler manifold. For each $k = 0, \dots, 2n$, the harmonic forms of degree k has the following direct sum decomposition:*

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M).$$

Therefore, the de Rham cohomology groups have a corresponding decomposition

$$H_{\text{dR}}^k(M; \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M).$$

and the Betti numbers are related to the Hodge numbers via

$$b_k = h^{k,0} + h^{k-1,1} + \dots + h^{1,k-1} + h^{0,k}.$$

Finally, for each (p, q) , conjugation gives a linear bijection from $\mathcal{H}^{p,q}(M)$ to $\mathcal{H}^{q,p}(M)$. Hence $h^{p,q} = h^{q,p}$.

Proof. Let α be a harmonic k -form. For each $p + q = k$ let $\alpha^{p,q}$ be the image of α under the projection $\mathcal{A}^k \rightarrow \mathcal{A}^{p,q}$. Because $\Delta_d = 2\Delta_{\bar{\partial}}$ maps $\mathcal{A}^{p,q}$ to itself, it follows that $\alpha^{p,q}$ are all $\bar{\partial}$ -harmonic hence lives in $\mathcal{H}^{p,q}(M)$. This shows that $\mathcal{H}^k(M)$ is a sum of $\mathcal{H}^{p,q}(M)$ for $p + q = k$. Since any two such spaces has trivial intersection, the sum is direct. Hodge decomposition follows immediately. Finally, conjugation takes $\mathcal{A}^{p,q}(M)$ to $\mathcal{A}^{q,p}(M)$ bijectively, and it maps $\bar{\partial}$ -harmonic (p, q) -forms to ∂ -harmonic (q, p) -forms. Since $\Delta_{\bar{\partial}} = \Delta_\partial$, these are also $\bar{\partial}$ -harmonic. $\square$

6.5 Lefschetz Theorems on Kähler Manifolds

In this section we study the Lefschetz operator L_ω in more details, and we conclude with another decomposition of cohomology of compact Kähler manifolds.

We first present facts about the Lefschetz operator:

Lemma 6.41. *Let M be a compact Kähler manifold with Kähler form ω . For $\alpha \in \mathcal{A}^{p,q}(M)$, we have $[L_\omega, L_\omega^*]\alpha = (p + q - n)\alpha$.*

Proof. Given $p \in M$, Theorem 5.13 shows that we can find holomorphic coordinates (z^j) on a neighborhood of p such that at p we have

$$\omega_p = \frac{i}{2} \sum_j dz^j \wedge d\bar{z}^j.$$

Equation (6.2) shows that the adjoint of the operator $\alpha \mapsto d\bar{z}^j \wedge \alpha$ is $\beta \mapsto 2\partial_{\bar{j}} \lrcorner \beta$. Using this together with its conjugate, we conclude that the following formula holds at p :

$$L_\omega^* \alpha = -\frac{i}{2} \sum_j (d\bar{z}^j \wedge)^* (dz^j \wedge)^* \alpha = -2i \sum_j \partial_{\bar{j}} \lrcorner \partial_j \lrcorner \alpha.$$

Using the fact that interior multiplication is an antiderivation, we get

$$\begin{aligned} L_\omega^* L_\omega \alpha &= -2i \sum_j \partial_{\bar{j}} \lrcorner \partial_j \lrcorner \left(\frac{i}{2} \sum_k dz^k \wedge d\bar{z}^k \wedge \alpha \right) \\ &= \sum_{j,k} \partial_{\bar{j}} \lrcorner \left(\delta_j^k d\bar{z}^k \wedge \alpha + dz^k \wedge d\bar{z}^k \wedge (\partial_j \lrcorner \alpha) \right) \\ &= \sum_{j,k} \left(\delta_j^k \delta_j^k \alpha - \delta_j^k d\bar{z}^k \wedge (\partial_{\bar{j}} \lrcorner \alpha) - \delta_j^k dz^k \wedge (\partial_j \lrcorner \alpha) + dz^k \wedge d\bar{z}^k (\partial_{\bar{j}} \lrcorner \partial_j \lrcorner \alpha) \right) \\ &= n\alpha - \sum_k dz^k \wedge (\partial_{\bar{k}} \lrcorner \alpha) - \sum_j dz^j \wedge (\partial_j \lrcorner \alpha) + L_\omega L_\omega^* \alpha. \end{aligned} \quad (6.3)$$

To interpret the two intermediate terms, note that for α of the form

$$\alpha = dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q},$$

and for a particular index j , we have $\partial_j \lrcorner \alpha = 0$ if j is not one of the indices $\{j_1, \dots, j_p\}$; while if $j = j_s$, then

$$dz^{j_s} \wedge (\partial_{j_s} \lrcorner \alpha) = (-1)^{s-1} dz^{j_s} \wedge \left(dz^{j_1} \wedge \cdots \wedge \widehat{dz^{j_s}} \wedge \cdots \wedge dz^{j_p} \right) = \alpha.$$

Since α is a (p, q) -form, there are p indices for which this expression is equal to α and the rest are zero, so $\sum_j dz^j \wedge (\partial_j \lrcorner \alpha) = p\alpha$ for α of this form; and since every (p, q) -form is a linear combination of forms of this type, this holds for all $\alpha \in \mathcal{A}^{p,q}(M)$. Similarly $\sum_k d\bar{z}^k \wedge (\partial_{\bar{k}} \lrcorner \alpha) = q\alpha$. Inserting these formulas into (6.3) finishes the proof. $\square$

Lemma 6.42. *Let M be a compact Kähler manifold with Kähler form ω . For every $0 \leq k \leq n$ where $n = \dim M$, the operator $L_\omega^k : \mathcal{A}^{n-k}(M) \rightarrow \mathcal{A}^{n+k}(M)$ is an isomorphism.*

Proof. We note that $L_\omega^k : \mathcal{A}^{n-k}(M) \rightarrow \mathcal{A}^{n+k}(M)$ is a smooth bundle homomorphism between vector bundles of the same rank. Hence it suffices to show that L_ω^k is injective. Writting $k = n - j$, we will show by induction on j that $L_\omega^{n-j} : \mathcal{A}^j(M) \rightarrow \mathcal{A}^{2n-j}(M)$ is injective.

The case $j = 0$ is immediate, for L_ω^n is just multiplication by $\omega^n = n!(d \text{Vol}_g)$. Assume now that L_ω^{n-j+1} is injective on $(j-1)$ -forms, and suppose $\alpha \in \mathcal{A}^j(M)$ such that $L_\omega^{n-j} \alpha = 0$. Then $L_\omega^{n-j+1} \alpha = 0$. Since interior multiplication is an antiderivation, direct computation shows that for every complex vector field X ,

$$0 = X \lrcorner L_\omega^{n-j+1} \alpha = X \lrcorner (\omega \wedge \cdots \wedge \omega \wedge \alpha) = (n-j+1)(X \lrcorner \omega) \wedge L_\omega^{n-j} \alpha + L_\omega^{n-j+1} (X \lrcorner \alpha).$$

The expression on the right hand side is equal to $L_\omega^{n-j+1}(X \lrcorner \alpha)$ because $L_\omega^{n-j}\alpha = 0$ by assumption. Hence we conclude that

$$0 = L_\omega^{n-j+1}(X \lrcorner \alpha).$$

Therefore, the inductive hypothesis shows that $X \lrcorner \alpha = 0$. Since this is true for every complex vector field X , we conclude that $\alpha = 0$, as desired. $\square$

We will now show that the result above is also true on cohomology groups.

Theorem 6.43 (Hard Lefschetz Theorem). *Let M be a compact n -dimensional Kähler manifold with Kähler form ω . Let L_ω be the Lefschetz operator. Then for every k , the operator L_ω^k descends to an isomorphism $L_\omega^k : H^{n-k}(M; \mathbb{C}) \rightarrow H^{n+k}(M; \mathbb{C})$ on de Rham cohomology.*

So the theorem states that the operator $L_\omega^k : H^{n-k}(M) \rightarrow H^{n+k}(M)$ maps a class from below the middle isomorphically to its symmetric reflection above the middle.

Proof. Note that ω is closed. So $d(L_\omega \alpha) = d(\omega \wedge \alpha) = \omega \wedge d\alpha = L_\omega(d\alpha)$. Hence $[L_\omega, d] = 0$. The same argument shows that $[L_\omega, \bar{\partial}] = 0$. We claim that $[L_\omega, \Delta_{\bar{\partial}}] = 0$ as well. Indeed, using the Kähler identities and $\bar{\partial}\partial + \partial\bar{\partial} = 0$, we have

$$\begin{aligned} \Delta_{\bar{\partial}} L_\omega \alpha &= \bar{\partial} \bar{\partial}^* L_\omega \alpha + \bar{\partial}^* \bar{\partial} L_\omega \alpha \\ &= \bar{\partial}(L_\omega \bar{\partial}^* \alpha + i\partial \alpha) + \bar{\partial}^* L_\omega(\bar{\partial} \alpha) \\ &= L_\omega \bar{\partial} \bar{\partial}^* \alpha + i\bar{\partial} \partial \alpha + L_\omega \bar{\partial}^* \bar{\partial} \alpha + i\partial \bar{\partial} \alpha \\ &= L_\omega \Delta_{\bar{\partial}} \alpha. \end{aligned}$$

Since $\Delta_d = 2\Delta_{\bar{\partial}}$, it follows that $[L_\omega, \Delta_d] = 0$ as well.

Since L_ω commutes with d , it descends to a well-defined map on cohomology. To show that $L_\omega^k : H^{n-k}(M) \rightarrow H^{n+k}(M)$ is injective, suppose $L_\omega^k[\alpha] = 0$ for some $[\alpha] \in H^{n-k}(M)$. By Hodge theorem, we can choose α to be the harmonic representative of its cohomology class. Then $L_\omega^k \alpha = d\beta$ for some $\beta \in \mathcal{A}^{n+k-1}$. Since L_ω commutes with Δ_d , we have $\Delta_d(d\beta) = \Delta_d(L_\omega^k \alpha) = L_\omega^k(\Delta_d \alpha) = 0$. Therefore, $d\beta$ is harmonic. But all harmonic forms are in the kernel of d^* , which is orthogonal to the image of d , thus $d\beta = 0$. This implies $L_\omega^k \alpha = 0$. By Lemma 6.42, we conclude that $\alpha = 0$. Hence $L_\omega^k : H^{n-k}(M) \rightarrow H^{n+k}(M)$ is injective.

To show that $L_\omega^k : H^{n-k}(M) \rightarrow H^{n+k}(M)$ is surjective, let $[\beta] \in H^{n+k}(M)$ be arbitrary. Again we can assume β is the harmonic representative of its cohomology class. Lemma 6.42 shows that there exists some $\alpha \in \mathcal{A}^{n-k}$ such that $L_\omega^k \alpha = \beta$. Then $L_\omega^k(\Delta_d \alpha) = \Delta_d(L_\omega^k \alpha) = \Delta_d \beta = 0$, and injectivity of L_ω^k on forms shows that $\Delta_d \alpha = 0$. Hence we must have $d\alpha = 0$. This shows that α represents a cohomology class such that $L_\omega^k[\alpha] = [\beta]$. $\square$

For our next result of the Lefschetz operator, recall that the sheaf theoretic Chern map is $c_1 : \text{Pic}(M) \rightarrow H^2(M, \mathbb{Z})$. Composing this map with the following sequence of maps gives another map which we will still denote by c_1 :

$$c_1 : \text{Pic}(M) \rightarrow H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{C}) \xrightarrow{\cong} H_{\text{dR}}^2(M; \mathbb{C}).$$

Under this map, we say that a cohomology class in $H_{\text{dR}}^2(M; \mathbb{C})$ is of **type** (p, q) if it lies in the direct summand $H^{p,q}(M)$ under the Hodge decomposition. We say a cohomology class is **integral** if it lies in the image of the coefficient morphism $H^*(M; \mathbb{Z}) \rightarrow H^*(M; \mathbb{C})$.

Theorem 6.44 (Lefschetz Theorem on (1,1)-Classes). *Let M be a compact Kähler manifold. Every integral cohomology class of type (1,1) is the first real Chern class of a holomorphic line bundle. That is, the restricted Chern map $c_1 : \text{Pic}(M) \rightarrow H^{1,1}(M; \mathbb{Z})$ is surjective.*

Proof. Let $[\eta] \in H_{\text{dR}}^2(M; \mathbb{C})$ be both integral and of type (1,1). The hypothesis that $[\eta]$ is integral means there exists a class $\gamma \in H^2(M, \mathbb{Z})$ that maps to the image of $[\eta]$ in $H^2(M, \mathbb{C})$ under the coefficient isomorphism. From the long exact sequence of the exponential sheaf sequence, we have

$$\cdots \rightarrow \text{Pic}(M) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \xrightarrow{i_*} H^2(M, \mathcal{O}) \rightarrow \cdots$$

where $i : \mathbb{Z} \rightarrow \mathcal{O}$ is inclusion. We will show that $i_*\gamma = 0$. By exactness, this shows that for some holomorphic line bundle L we have $c_1(L) = \gamma$, then under the above map $c_1 : \text{Pic}(M) \rightarrow H_{\text{dR}}^2(M; \mathbb{C})$ will map L to $[\eta]$, concluding the proof.

To this end, consider the following commutative diagram of sheaf inclusions:

$$\begin{array}{ccc} & & \mathbb{C} \\ & \nearrow j & \downarrow k \\ \mathbb{Z} & \xrightarrow{i} & \mathcal{O} \end{array}$$

which gives the following diagram of sheaf cohomology, where ρ, ρ' are the isomorphisms given by the de Rham-Weil theorem.

$$\begin{array}{ccccc} & & H^2(M, \mathbb{C}) & \xleftarrow{\rho} & H_{\text{dR}}^2(M; \mathbb{C}) \\ & \nearrow j_* & \downarrow k_* & & \downarrow \pi^{0,2} \\ H^2(M, \mathbb{Z}) & \xrightarrow{i_*} & H^2(M, \mathcal{O}) & \xleftarrow{\rho'} & H^{0,2}(M) \end{array}$$

The left triangle of the diagram above commutes by the functoriality of sheaf cohomology, and the right hand side square commutes because of the following commutative diagram of sheaf resolutions and the naturality statement of the de Rham-Weil theorem:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{A}^0 & \xrightarrow{d} & \mathcal{A}^1 & \xrightarrow{d} & \mathcal{A}^2 & \longrightarrow & \cdots \\ & & \downarrow i & & \parallel & & \downarrow \pi^{0,1} & & \downarrow \pi^{0,2} & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{A}^{0,0} & \xrightarrow{\bar{\partial}} & \mathcal{A}^{0,1} & \xrightarrow{\bar{\partial}} & \mathcal{A}^{0,2} & \longrightarrow & \cdots \end{array}$$

Now, we have that $j_*\gamma = \rho[\eta]$ and η is of type (1,1), so $\pi^{0,2}[\eta] = 0$. Therefore,

$$i_*\gamma = k_*j_*\gamma = k_*\rho[\eta] = \rho'\pi^{0,2}[\eta] = 0.$$

This concludes the proof. $\square$

Finally, we will give another decomposition of cohomology groups for compact Kähler manifolds. For this we need some preparation. We introduce the following notation for brevity: Let

$$V = \bigoplus_{k \geq 0} H_{\text{dR}}^k(M; \mathbb{C}).$$

We define three operators on V : The Lefschetz operator $L := L_\omega$, its adjoint $\Lambda := L_\omega^*$, and the counting operator H , which is defined by $H\alpha = (k - n)\alpha$ for $\alpha \in H^k(M)$. Then one can check the following commutation relations directly and using Lemma 6.41:

$$[H, L] = 2L \quad [H, \Lambda] = -2\Lambda \quad [L, \Lambda] = H.$$

These are commutation relations for the Lie algebra $\mathfrak{sl}(2)$. Hence we conclude that V is a representation of $\mathfrak{sl}(2)$.

Lemma 6.45. *For all $\alpha \in H^k(M)$, we have*

$$[L^i, \Lambda]\alpha = i(k - n + i - 1)L^{i-1}\alpha.$$

Proof. This is easily seen by induction on i as follows:

$$\begin{aligned} [L^i, \Lambda]\alpha &= L^i\Lambda - \Lambda L^i\alpha \\ &= L(L^{i-1}\Lambda\alpha - \Lambda L^{i-1}\alpha) + L\Lambda L^{i-1}\alpha - \Lambda L L^{i-1}\alpha \\ &= L[L^{i-1}, \Lambda]\alpha + [L, \Lambda]L^{i-1}\alpha \\ &= (i-1)(k - n + (i-1) - 1)L^{i-1}\alpha + (2i - 2 + k - n)L^{i-1}\alpha \\ &= i(k - n + i - 1)L^{i-1}\alpha. \end{aligned}$$

This concludes the inductive step. The base case is just the relation $[L, \Lambda] = H$. $\square$

Definition 6.46. Let M be a compact Kähler manifold of dimension n with L and Λ as defined above. A cohomology class $\alpha \in H_{\text{dR}}^k(M; \mathbb{C})$ is said to be **primitive** if $\Lambda\alpha = 0$. Let $P^k(M) \subset H_{\text{dR}}^k(M; \mathbb{C})$ denote the space of primitive degree k cohomology classes, and $P^{p,q}(M) \subset H^{p,q}(M)$ denote the space of primitive (p, q) -classes.

Note that for $\gamma \in H^k(M; \mathbb{C})$, the map $L^{n-k} : H^k(M; \mathbb{C}) \rightarrow H^{2n-k}(M)$ is an isomorphism by the Hard Lefschetz theorem (Theorem 6.43): Indeed, the Hard Lefschetz theorem says $L_\omega^j : H^{n-j}(M) \rightarrow H^{n+j}(M)$ is an isomorphism. In our case $n - j = k$, so $j = n - k$ and $n + j = 2n - k$. Hence we look at the Lefschetz operator composed one step further, i.e., L^{n-k+1} . This needs not be an isomorphism. The next result says when $k \leq n$, the subset $P^k(M) \subset H^k(M)$ is precisely the kernel of this map L^{n-k+1} .

Proposition 6.47. If $k > n$, then $P^k(M) = 0$. If $k \leq n$, then

$$P^k(M) = \ker(L^{n-k+1} : H^k(M) \rightarrow H^{2n-k+2}(M)).$$

Proof. Assume $k > n$ first. If $\alpha \in P^k(M)$, we let i be the minimal integer such that $L^i\alpha = 0$ (such i always exist by finite dimensionality of V). Then by Lemma 6.45 one has

$$0 = [L^i, \Lambda]\alpha = i(k - n + i - 1)L^{i-1}\alpha.$$

Note that since $k > n$ and $L^{i-1}\alpha \neq 0$, we must have $i = 0$. Hence $\alpha = 0$.

Now assume $k \leq n$. Let $\alpha \in P^k(M)$. Again by Lemma 6.45 in the case where $i = n - k + 1$, we see that $[L^{n-k+1}, \Lambda]\alpha = 0$. Thus $L^{n-k+1}\Lambda\alpha = \Lambda L^{n-k+1}\alpha$. If $\Lambda\alpha = 0$, then $\Lambda L^{n-k+1}\alpha = 0$ as well. Then $\Lambda^{n-k+1}L^{n-k+1}\alpha = 0$. Therefore,

$$0 = \langle \Lambda^{n-k+1}L^{n-k+1}\alpha, \alpha \rangle = \langle L^{n-k+1}\alpha, L^{n-k+1}\alpha \rangle = \|L^{n-k+1}\alpha\|^2.$$

Hence $L^{n-k+1}\alpha = 0$. This shows that $P^k(M) \subset \ker(L^{n-k+1})$. Conversely, let $\alpha \in H^k(M)$ such that $L^{n-k+1}\alpha = 0$. Then

$$L^{n-k+2}\Lambda\alpha = L^{n-k+2}\Lambda\alpha - \Lambda L^{n-k+2}\alpha = (n - k + 2)L^{n-k+1}\alpha = 0.$$

Since $\deg(\Lambda\alpha) = k - 2 \leq n$, by Lemma 6.42, we have that $\Lambda\alpha = 0$. $\square$

The key result of this section is another decomposition theorem on cohomology groups of Kähler manifolds:

Theorem 6.48 (Lefschetz Decomposition Theorem). *For any compact n -dimensional Kähler manifold, there are direct sum decompositions*

$$H_{\text{dR}}^k(M; \mathbb{C}) = \bigoplus_{0 \leq r \leq k/2} L_{\omega}^r P^{k-2r}(M)$$

and

$$H^{p,q}(M) = \bigoplus_{0 \leq r \leq (p+q)/2} L_{\omega}^r P^{p-r, q-r}(M).$$

Proof. The second decomposition follows directly from the first decomposition and the Hodge decomposition theorem for Kähler manifolds. Thus it suffices to prove the first decomposition.

To prove the first decomposition, we note that as we already discussed V form a representation of $\mathfrak{sl}(2)$. Hence it is a direct sum of irreducible representations. Any finite dimensional representation of $\mathfrak{sl}(2)$ admits a primitive vectors, i.e., a vector α such that $\Lambda\alpha = 0$. Such vectors exist because $\Lambda^k\alpha = 0$ for k sufficiently large by dimension reason. Using Lemma 6.45 one finds that for any primitive α the subspace generated by $\alpha, L\alpha, L^2\alpha, \dots$ defines an irreducible sub-representation. Moreover, all irreducible $\mathfrak{sl}(2)$ -representations are of this form. Together this proves that

$$V = \bigoplus_j W_j$$

where each W_j is an irreducible representation generated by $\alpha_j, L\alpha_j, L^2\alpha_j, \dots$ for some $\alpha_j \in P^{s_j}(M) \subset H^{s_j}(M)$. Now we look at each graded pieces. We have

$$H_{\text{dR}}^k(M; \mathbb{C}) = \bigoplus_j W_j \cap H_{\text{dR}}^k(M; \mathbb{C}).$$

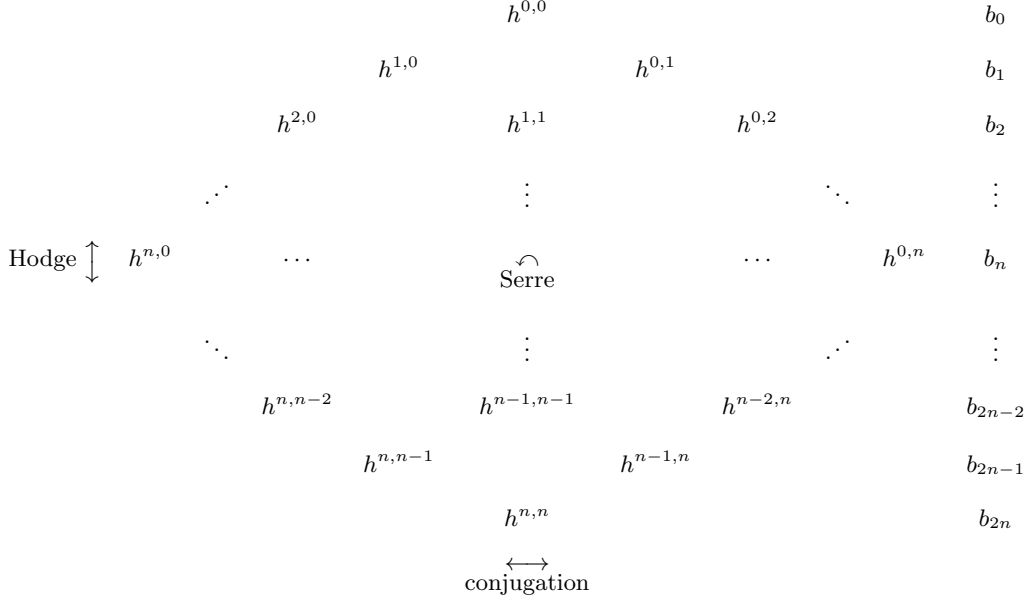
Now $W_j \cap H_{\text{dR}}^k(M; \mathbb{C}) \neq \emptyset$ iff $H_{\text{dR}}^k(M; \mathbb{C})$ contains one element in the string $\alpha_j, L\alpha_j, L^2\alpha_j, \dots$. That is, there exists an integer $r \geq 0$ such that

$$k = \deg(L^r\alpha_j) = s_j + 2r.$$

Thus $s_j = k - 2r$. Moreover, since $s_j \geq 0$ is a requirement, we need $k - 2r \geq 0$, hence $r \leq k/2$. This proves the existence of Lefschetz decomposition as written in the statement. $\square$

6.6 Applications of Hodge Theory

In this last section of this chapter, we discuss some applications of Hodge theory. Given a compact complex manifold of dimension n , its **Hodge numbers** are defined to be $h^{p,q} = \dim H^{p,q}(M)$. We can arrange these Hodge numbers into a diamond called the **Hodge diamond**, as shown in the picture below.



As we can see, there are certain symmetries of this diamond. Serre duality (Theorem 6.32) shows that $h^{p,q} = h^{n-p,n-q}$. That is, the diamond is symmetric under 180 degree rotation. Furthermore, conjugation gives bijection $H^{p,q}(M) \rightarrow H^{q,p}(M)$, hence $h^{p,q} = h^{q,p}$, so the diamond is symmetric about its vertical axis. Combining these two symmetries, we get a Hodge symmetry, $h^{p,q} = h^{n-q,n-p}$. That is, the diamond is also symmetric about its horizontal axis. Moreover, if M is Kähler, then Hodge decomposition theorem (Theorem 6.40) shows that the sum of Hodge numbers in each horizontal row is equal to the corresponding Betti number.

Using the above observations, we can obtain several results on the topology of compact Kähler manifolds.

Corollary 6.49. *On a compact Kähler manifold, the odd Betti numbers are even and the even Betti numbers are positive.*

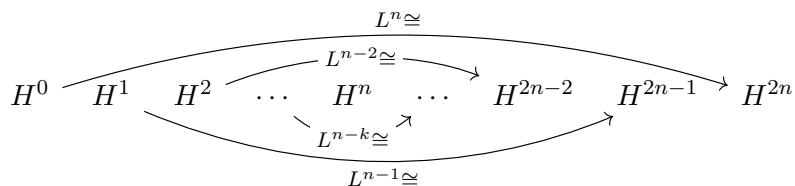
Proof. $b_{2m} > 0$ follows from Proposition 5.24. By the Hodge decomposition theorem (Theorem 6.40) and the symmetry of the Hodge diamond, we have

$$b^{2m+1} = (h^{2m+1,0} + h^{2m,1} + \dots + h^{m+1,m}) + (h^{m,m+1} + \dots + h^{1,2m} + h^{0,2m+1})$$

and the two terms in the parentheses are equal. Hence b^{2m+1} is even. $\square$

Corollary 6.50. *Let M^n be compact Kähler. The Lefschetz operator $L_\omega : H^k(M) \rightarrow H^{k+2}(M)$ is injective for $k \leq n-1$ and surjective for $k \geq n-1$. Thus the Betti numbers of M satisfy $b_k \leq b_{k-2}$ for $k \leq n-1$, and $b_k \geq b_{k+2}$ for $k \geq n-1$.*

Proof. This is a direct application of the hard Lefschetz theorem (Theorem 6.43). The following diagram might help with the understanding. Note that the Lefschetz isomorphisms always goes from the a group to left of H^n to a group to the right of H^n , that are equidistant from H^n :



By the hard Lefschetz theorem, $L_\omega^{n-k} : H^k(M) \rightarrow H^{2n-k}(M)$ is an isomorphism for $k \leq n-1$. Hence $L_\omega : H^k(M) \rightarrow H^{k+2}(M)$ must be an injection, because note that $k+2 \leq n+1$, so $L_\omega : H^k(M) \rightarrow H^{k+2}(M)$ must be at the start of some Lefschetz isomorphism. When $k \geq n-1$, we have $k+2 \geq n+1$, then $L_\omega : H^k(M) \rightarrow H^{k+2}(M)$ must be at the end of some Lefschetz isomorphism, hence must be surjective. $\square$

Since $\omega \in \mathcal{A}^{1,1}(M)$, we have in fact $L_\omega : H^{p,q}(M) \rightarrow H^{p+1,q+1}(M)$. Hence the following corollary is also immediate:

Corollary 6.51. *Let M be compact Kähler. Then for each (p, q) , the operator $L_\omega^{n-p-q} : H^{p,q}(M) \rightarrow H^{n-q, n-p}(M)$ is an isomorphism. Thus $L_\omega : H^{p,q}(M) \rightarrow H^{p+1,q+1}(M)$ is injective for $p+q \leq n-1$ and surjective for $p+q \geq n-1$; and the Hodge numbers satisfy $h^{p,q} \leq h^{p+1,q+1}$ for $p+q \leq n-1$ and $h^{p,q} \geq h^{p+1,q+1}$ for $p+q \geq n-1$.*

Note that this corollary gives another proof that the Hodge diamond is symmetric about its horizontal axis, which we previously deduced from Serre duality and conjugation symmetry.

Next, we compute the Hodge diamond of a Riemann surface and $\mathbb{P}^n$.

Proposition 6.52. Let $M = \Sigma_g$ be a connected compact Riemann surface of genus g . Then its Hodge diamond is given by $h^{0,0} = 1, h^{1,0} = h^{0,1} = g, h^{1,1} = 1$.

Proof. By Example 5.16, we know that its Hodge diamond sums to Betti numbers in each row. Now by standard algebraic topology we know $b_0 = 1, b_1 = 2g, b_2 = 1$. Hence the shape of the diamond follows from Hodge symmetries. $\square$

Proposition 6.53. The Hodge numbers of $\mathbb{P}^n$ is given by $h^{p,q} = 0$ for $p \neq q$ and $h^{p,p} = 1$ for $0 \leq p \leq n$.

Proof. By Example 5.17, we know that $\mathbb{P}^n$ is Kähler. From algebraic topology $H_k(\mathbb{P}^n) \cong \mathbb{Z}$ when k is even and $0 \leq k \leq 2n$ and is zero otherwise. If one of the Hodge numbers $h^{p,q}$ were nonzero for $p \neq q$, then $b_{p+q} \geq h^{p,q} + h^{q,p} \geq 2$. So only $h^{p,p}$ can be nonzero. The hodge decomposition then shows that $h^{p,p} = b_{2p} = 1$ for $0 \leq p \leq n$. $\square$

Now we can classify line bundles on $\mathbb{P}^n$:

Theorem 6.54. *For each n , the sheaf theoretic Chern map $c_1 : \text{Pic}(\mathbb{P}^n) \rightarrow H^2(\mathbb{P}^n; \mathbb{Z})$ is an isomorphism. Therefore, $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. Moreover, all the holomorphic line bundles on $\mathbb{P}^n$ are isomorphic to $\mathcal{O}_{\mathbb{P}^n}(d)$ for some integer d .*

Proof. In the long exact exponential sequence, we have

$$\cdots \rightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \rightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^*) \xrightarrow{c_1} H^2(\mathbb{P}^n; \mathbb{Z}) \rightarrow H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \rightarrow \cdots$$

Now $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = H^{0,1}(\mathbb{P}^n)$, which is zero because $h^{0,1}(\mathbb{P}^n) = 0$. Similarly $H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$. Thus c_1 is indeed an isomorphism. Hence we have

$$\text{Pic}(\mathbb{P}^n) = H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^*) \cong H^2(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z}.$$

Finally, From Example 5.17, we saw that $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is given by the Fubini-Study metric ω_{FS} . On $\mathbb{P}^1$, the curvature 2-form on a chart U_α is given by

$$\Theta = \frac{(1 + |z|^2)dz \wedge d\bar{z} - |z|^2 dz \wedge d\bar{z}}{(1 + |z|^2)^2} = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

We can integrate this over the subset U_0 which is biholomorphic to $\mathbb{C}$, since $\mathbb{P}^1 \setminus U_0$ has measure zero. Making change of variables $z = re^{i\theta}$, we have $dz \wedge d\bar{z} = -2ir dr \wedge d\theta$. Hence we see that

$$\deg c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} = \frac{i}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{-2ir}{(1 + r^2)^2} dr d\theta = \int_0^\infty \frac{ds}{s^2} = 1,$$

where in the last step we change variable $s = 1 + r^2$. Since $c_1(\mathcal{O}_{\mathbb{P}^1}(d)) = d\omega_{\text{FS}}$, we have $\deg c_1(\mathcal{O}_{\mathbb{P}^1}(d)) = d$. The isomorphism $H^2(\mathbb{P}^1; \mathbb{Z}) \cong \mathbb{Z}$ is obtained by Poincaré duality, i.e., integration on $\mathbb{P}^1$, hence we have $\text{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ is classified by $\mathcal{O}_{\mathbb{P}^1}(d)$ for integers d .

For $\mathbb{P}^n$ with $n > 1$, we already know that $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. So there is a holomorphic line bundle $L \rightarrow \mathbb{P}^n$ that represents a generator of $\text{Pic}(\mathbb{P}^n)$. The hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is thus isomorphic to L^k for some integer k . Under the embedding $i : \mathbb{P}^1 \rightarrow \mathbb{P}^n$, we have $i^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore, $k \deg(i^*L) = 1$. Hence we must have $k = \pm 1$. This shows that $\mathcal{O}_{\mathbb{P}^n}(1)$ generates $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, concluding the proof. $\square$

We next provide the proof of Chow's theorem in the case of hypersurfaces.

Theorem 6.55. *Let $S \subset \mathbb{P}^n$ be a hypersurface (closed codimension 1 complex submanifold), then S is the variety defined by a single homogeneous polynomial. Hence S is algebraic.*

Proof. By Theorem 3.26 and Theorem 3.27, the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^n}(S)$ has a section $\sigma : \mathbb{P}^n \rightarrow \mathcal{O}_{\mathbb{P}^n}(S)$ that vanishes on S and nowhere else. Then by Theorem 6.54, $\mathcal{O}_{\mathbb{P}^n}(S) \cong \mathcal{O}_{\mathbb{P}^n}(d)$ for some d , and furthermore $d > 0$ because $\deg \mathcal{O}_{\mathbb{P}^n}(S) = d$ and $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(S)) > 0$ since there exists a section σ (Corollary 3.32). By Theorem 3.31, σ is given by a homogenous polynomial of degree d . Hence $S = Z(\sigma)$ is algebraic. $\square$

Finally, we prove a stronger version of Corollary 2.65 on a compact Kähler manifold, which says the local $\partial\bar{\partial}$ -lemma also holds globally provided the form is exact.

Proposition 6.56. *Let M be a compact Kähler manifold and θ a d-exact (p, q) -form on M with $p, q > 0$. Then there exists a smooth $(p-1, q-1)$ -form α defined on all of M , such that $\theta = i\partial\bar{\partial}\alpha$. If θ is a real (p, p) -form, then α can be chosen to be real.*

Proof. Since θ is exact, suppose $\theta = d\eta$. We can decompose η into bidegree forms and just as in the proof of Corollary 2.65, we can assume $\eta = \eta^{p, q-1} + \eta^{p-1, q}$. Then

$$\theta = \bar{\partial}\eta^{p-1, q} + (\partial\eta^{p-1, q} + \bar{\partial}\eta^{p, q-1}) + \partial\eta^{p, q-1}.$$

Since $\theta \in \mathcal{A}^{p, q}$, we see that the first and the last term above in the summand is zero. Therefore, $\eta^{p-1, q}$ and $\overline{\eta^{p, q-1}}$ are $\bar{\partial}$ -closed. By the Hodge Dolbeault theorem, there are $\bar{\partial}$ -harmonic representatives of these, call them σ, ρ , that are in the same Dolbeault cohomology class. That is,

$$\eta^{p-1, q} = \sigma + \bar{\partial}\beta \quad \overline{\eta^{p, q-1}} = \rho + \bar{\partial}\gamma$$

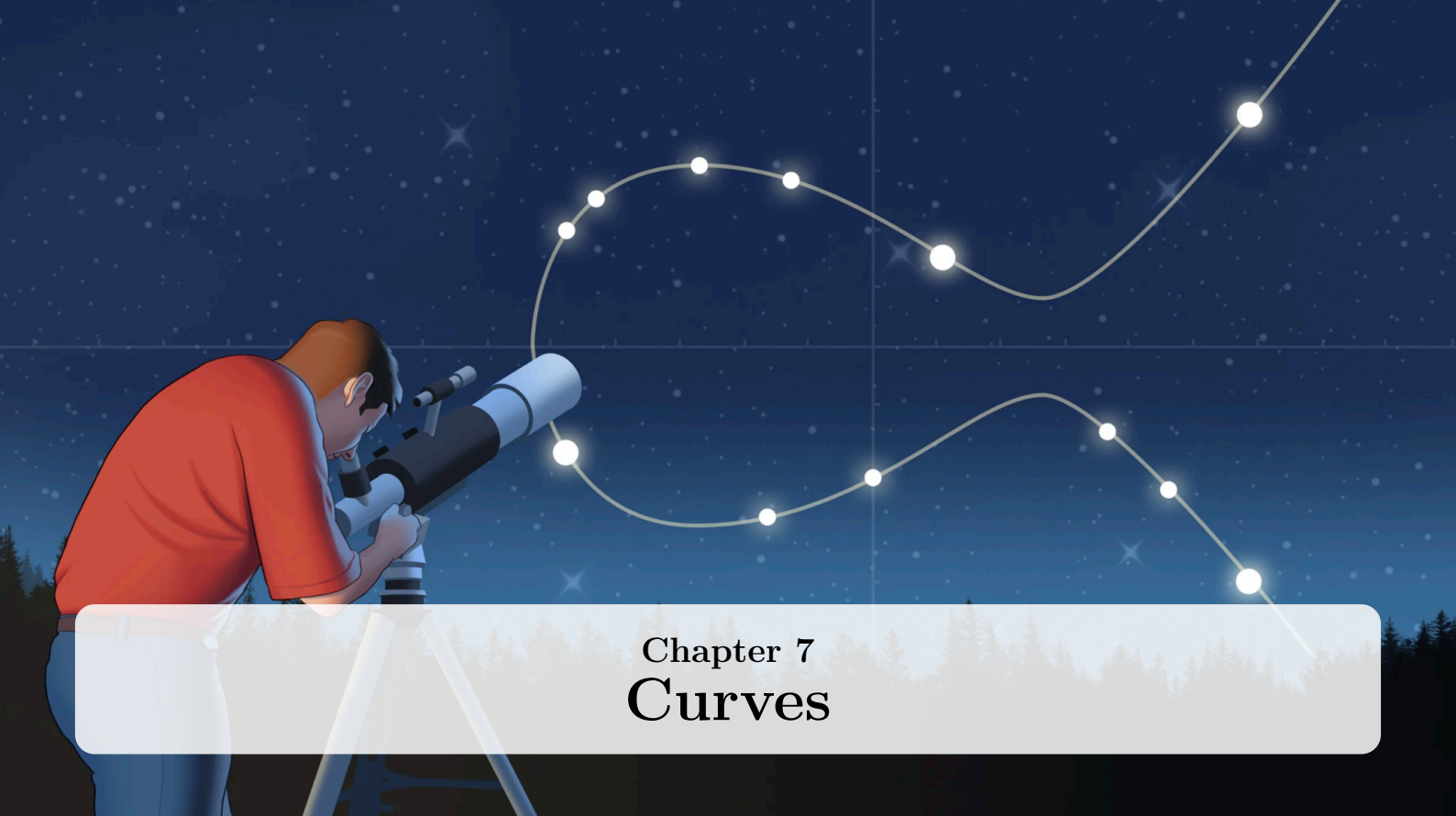
for some $\beta \in \mathcal{A}^{p-1, q-1}$ and $\gamma \in \mathcal{A}^{q-1, p-1}$. Since $\bar{\partial}$ -harmonic forms on a Kähler manifold are also $\bar{\partial}$ -harmonic, it follows that σ, ρ are both ∂ -closed and $\bar{\partial}$ -closed. Therefore, we set

$$\alpha = i\bar{\gamma} - i\beta.$$

Then we see that

$$i\partial\bar{\partial}\alpha = i\bar{\partial}\partial(i\bar{\gamma}) - i\partial\bar{\partial}(i\beta) = \bar{\partial}\partial\bar{\gamma} + \partial\bar{\partial}\gamma = \bar{\partial}(\eta^{p, q-1} - \bar{\rho}) + \partial(\eta^{p-1, q} - \sigma) = \theta.$$

If θ is a real (p, p) -form, we can choose η to be real, which means $\overline{\eta^{p, p-1}} = \eta^{p-1, p}$. Then we can choose $\gamma = \beta$, so that $\alpha = i\bar{\beta} - i\beta$ is real. $\square$



Chapter 7

Curves

Introduction

Curves are just complex manifolds of complex dimension one. In this chapter we will develop some basic theory for curves, including the famous Riemann-Roch theorem, degree genus formula, and the Riemann-Hurwitz theorem.

7.1 Basic Computational Tools

In the first section of this chapter, we collect three of the most foundational results in the theory of curves: degree genus formula, Riemann-Roch theorem, and Riemann-Hurwitz formula. Together with the adjunction formula, these are the most important computational tools that one has when studying algebraic curves and Riemann surfaces. We will see how these results are used in studying linear systems, and classification of curves of low genus in the later sections of this chapter.

7.1.1 Degree Genus Formula

Definition 7.1. A *plane curve* C is a complex submanifold of $\mathbb{P}^2$ of the form $C = Z(F)$, where F is a homogeneous polynomial of some degree d .

Since $\mathbb{P}^2$ is compact, and C is defined by a polynomial equation, C is closed and thus compact. Hence C is a compact Riemann surface. However, not all complex Riemann surface can be embedded in $\mathbb{P}^2$ as plane curves. The next result gives a restriction on the genus of plane curves.

Theorem 7.2 (degree genus formula). *Let $C \subset \mathbb{P}^2$ be a smooth algebraic curve of degree d . Then its genus g is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof. This is a simple application of adjunction formula (Theorem 3.37). We have

$$K_C = (K_{\mathbb{P}^2} + C)|_C = \mathcal{O}_{\mathbb{P}^2}(d-3)|_C.$$

by Theorem 3.27, Theorem 3.26, and Corollary 3.34.

Now we take the degree. Since C has degree d , its intersection number with a plane $\mathbb{P}^1$ is d , thus $\deg(\mathcal{O}_{\mathbb{P}^2}(1)|_C) = d$. Therefore, $\deg(\mathcal{O}_{\mathbb{P}^2}(d-3)|_C) = d(d-3)$. Thus, we conclude that $\deg K_C = d(d-3)$.

But Poincaré-Hopf theorem also tells us

$$\deg K_C = \deg T_C^* = -\deg T_C = -\int_C c_1(T_C) = \int_C e(T_C) = -\chi(C) = 2g - 2.$$

Hence $2g - 2 = d(d-3)$. Solving g in terms of d gives the desired formula. $\square$

This shows that the topology (i.e., genus g) of compact Riemann surfaces that can be realized as embedded submanifolds of $\mathbb{P}^2$ is restricted by the degree. For $d = 1, 2, \dots, 10$, the possible genera are $g = 0, 0, 1, 3, 6, 10, 15, 21, 28, 36$.

7.1.2 Riemann-Roch Theorem

In this section, let M be a compact Riemann surface. We will provide the promised proof in earlier chapter that the map $\rho : \text{Cl } M \rightarrow \text{Div } M$ is actually an isomorphism, and provide a proof of the famous Riemann-Roch theorem.

We first recall that given a sheaf $\mathcal{F}$ on a topological space X , then its *sheaf theoretic Euler characteristic* is defined to be

$$\chi(\mathcal{F}) = \sum_{k=0}^{\infty} (-1)^k \dim H^k(X, \mathcal{F}).$$

Proposition 7.3. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be a short exact sequence of sheaves (of vector spaces) on a locally compact Hausdorff space X . If their Euler characteristics are all defined, then

$$\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H}).$$

Proof. Look at the long exact sequence

$$\cdots \rightarrow H^k(X, \mathcal{F}) \xrightarrow{\alpha_k} H^k(X, \mathcal{G}) \xrightarrow{\beta_k} H^k(X, \mathcal{H}) \rightarrow H^{k+1}(X, \mathcal{F}) \rightarrow \cdots$$

Let $Z^k(X, \mathcal{G}) := \ker \beta_k = \text{im } \alpha_k$, and define $Z^k(X, \mathcal{F})$ and $Z^k(X, \mathcal{H})$ similarly. Then by rank-nullity theorem, we have

$$\begin{aligned} \dim H^k(X, \mathcal{F}) &= \dim Z^k(X, \mathcal{F}) + \dim Z^k(X, \mathcal{G}) \\ \dim H^k(X, \mathcal{G}) &= \dim Z^k(X, \mathcal{G}) + \dim Z^k(X, \mathcal{H}) \\ \dim H^k(X, \mathcal{H}) &= \dim Z^k(X, \mathcal{H}) + \dim Z^{k+1}(X, \mathcal{F}). \end{aligned}$$

Now taking the alternating sum of Euler characteristics on the left hand side, we find that all the terms on the right hand side cancel, i.e., $\chi(\mathcal{F}) - \chi(\mathcal{G}) + \chi(\mathcal{H}) = 0$. $\square$

Definition 7.4. Let M be a Riemann surface and $p \in M$ be a point. We introduce the *ideal sheaf* $\mathcal{I}_p := \mathcal{O}_M(-p)$ and its dual $\mathcal{I}_p^* := \mathcal{O}_M(p)$.

We first need a technical lemma:

Lemma 7.5. *Let M be a compact Riemann surface and $L \rightarrow M$ be a holomorphic line bundle, identified with its sheaf of holomorphic sections. Then for any $p \in M$, we have*

$$\chi(L \otimes \mathcal{I}_p^*) = \chi(L) + 1 \quad \chi(L \otimes \mathcal{I}_p) = \chi(L) - 1.$$

Proof. We consider the exact sequence of sheaves

$$0 \rightarrow L(-p) \rightarrow L \xrightarrow{\text{ev}} (L_p)_p \rightarrow 0,$$

where $L(-p)$ is the sheaf of holomorphic sections of L that vanishes at p , and $(L_p)_p$ is the skyscraper sheaf whose stalk at p is the vector space L_p (fiber of L at p), and all other stalks are zero. The map $\text{ev} : L \rightarrow (L_p)_p$ is the evaluation map. Note that by definition $L \otimes \mathcal{I}_p = L(-p)$. Moreover, $(L_p)_p$ is a fine sheaf whose higher degree cohomology all vanishes and $H^0(M, (L_p)_p) = L_p$. Hence $\chi((L_p)_p) = \dim L_p = 1$. By Proposition 7.3, we have

$$\chi(L) = \chi(L(-p)) + 1 = \chi(L \otimes \mathcal{I}_p) + 1.$$

This proves the second equation. The first equation then follows by applying the second equation to the line bundle $L \otimes \mathcal{I}_p^*$. $\square$

Next, we show that the map $\rho : \text{Cl } M \rightarrow \text{Pic } M$ is an isomorphism. We already know this map is injective, because we quotiented the kernel of the map $\rho : \text{Div } M \rightarrow \text{Pic } M$ to obtain $\rho : \text{Cl } M \rightarrow \text{Pic } M$ (recall Proposition 3.23).

Theorem 7.6. *The map $\rho : \text{Cl } M \rightarrow \text{Pic } M$ is surjective if M is a compact Riemann surface.*

Proof. Let $L \rightarrow M$ be an arbitrary holomorphic line bundle. By Dolbeault theorem $H^q(M, L) = H^q(M, \mathcal{O} \otimes L) = H^{0,q}(M; L)$. Since there are no nontrivial $(0, q)$ -forms on a Riemann surface for $q \geq 2$, this implies $H^q(M, L) = 0$ for $q \geq 2$. Thus

$$\chi(L) = \dim H^0(M, L) - H^1(M, L).$$

Let $p \in M$ be arbitrary. An easy induction based on the previous result shows

$$\chi(L \otimes \mathcal{I}_p^{*k}) = \chi(L) + k$$

for any positive integer k . Thus

$$\begin{aligned} \dim H^0(M, L \otimes \mathcal{I}_p^{*k}) &= \chi(L \otimes \mathcal{I}_p^{*k}) + \dim H^1(M, L \otimes \mathcal{I}_p^{*k}) \\ &\geq \chi(L \otimes \mathcal{I}_p^{*k}) = \chi(L) + k, \end{aligned}$$

which is strictly positive for k large enough. This implies that $L \otimes \mathcal{I}_p^{*k}$ has a nontrivial holomorphic section σ . Then by Theorem 3.26, if we let $D_1 = Z(\sigma)$ then we have $L \otimes \mathcal{I}_p^{*k} \cong \mathcal{O}(D_1)$. If we let $D_2 = -kp$, then $\mathcal{I}_p^k = \mathcal{O}(D_2)$ by definition. Therefore, we have

$$L \cong (L \otimes \mathcal{I}_p^{*k}) \otimes \mathcal{I}_p^k \cong \mathcal{O}(D_1 + D_2).$$

This concludes the proof. $\square$

Next, we prove the famous theorem of Riemann and Roch. Without any dispute, this is the most frequently used result in algebraic geometry.

Theorem 7.7 (Riemann-Roch Theorem). *Let M be a compact Riemann surface of genus g and canonical bundle K . Denote $h^n(E) := \dim H^n(M, E)$ for any holomorphic vector bundle E . Let $L \rightarrow M$ be any holomorphic line bundle. Then*

$$h^0(L) = \deg L + 1 - g + h^0(K \otimes L^*).$$

Proof. As we observed in the proof of Theorem 7.6, on a Riemann surface we have $\chi(L) = h^0(L) - h^1(L)$. Serre duality shows that $h^1(L) = h^0(K \otimes L^*)$. Thus

$$\chi(L) = h^0(L) - h^0(K \otimes L^*),$$

and the Riemann-Roch theorem is equivalent to

$$\chi(L) = \deg L + 1 - g \tag{7.1}$$

for every holomorphic line bundle $L \rightarrow M$. Note that we already have a formula for $h^0(K)$. Since $H^0(M, K) = H^{1,0}(M)$, on Riemann surface we have $h^0(K) = g$.

By Theorem 7.6, $L \cong \mathcal{O}(D)$ for some D . We first begin with the case where D is effective, i.e., $D = \sum_j n_j p_j$ where $n_j > 0$ for all j . We will prove (7.1) by induction on $\deg D = \sum_j n_j$.

Suppose $\deg D = 0$, then since D is effective we must have $D = 0$. Thus $L \cong \mathcal{O}$ is trivial. Then $h^0(L) = 1$ by Liouville's theorem (Theorem 2.5), and $h^0(K \otimes L^*) = h^0(K) = g$, so (7.1) holds.

Now let $k \geq 1$, and assume (7.1) holds for every line bundle $L = \mathcal{O}(D)$ with $\deg D < k$. Let D be an effective divisor of degree k . Write $D = D' + p$ where D' has degree $k - 1$ and $p \in M$ is some point. Then $L := \mathcal{O}(D) = \mathcal{O}(D') \otimes \mathcal{I}_p^*$. And $\deg L = \deg \mathcal{O}(D') + 1$. By inductive hypothesis

$$\chi(\mathcal{O}(D')) = \deg \mathcal{O}(D') + 1 - g.$$

Then Lemma 7.5 implies

$$\chi(L) = \chi(\mathcal{O}(D')) + 1 = \deg L + 1 - g.$$

This concludes the proof in the case where D is effective.

For the general case, we write $L = \mathcal{O}(D)$ and $D = D_+ - D_-$, where $D_{\pm}$ are effective. Then $L \cong \mathcal{O}(D_+) \otimes \mathcal{O}(D_-)^*$. Then by Lemma 7.5, induction, and the Riemann-Roch theorem for effective divisors, we have

$$\begin{aligned} \chi(L) &= \chi(\mathcal{O}(D_+ - D_-)) \\ &= \chi(\mathcal{O}(D_+)) - \deg \mathcal{O}(D_-) \\ &= \deg D_+ + 1 - g - \deg D_- \\ &= \deg D + 1 - g = \deg L + 1 - g. \end{aligned}$$

This concludes the proof. □

We will see how useful Riemann-Roch Theorem is later in this chapter.

7.1.3 Riemann-Hurwitz Formula

In this section, we consider the scenario where X, Y are two compact Riemann surfaces, and $f : X \rightarrow Y$ is a non-constant holomorphic map between them. Our first result is that locally f takes very simple form.

Proposition 7.8. Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then at every point $p \in X$, there exists local coordinate z, w on X, Y , such that f is given locally by $w = z^m$ for some unique integer m , which possibly depends on p .

Proof. Let s be local holomorphic coordinate on X such that $s(p) = 0$. Then in the corresponding local chart, we may expand $f(s)$ around $s = 0$:

$$f(s) = c_0 + c_1s + c_2s^2 + c_3s^3 + \cdots.$$

Let m be the smallest positive integer such that $c_m \neq 0$. Then $f(s) = s^m H(s)$ where H is some holomorphic function of s such that $H(0) \neq 0$. Thus $H(z) \neq 0$ in a small disk around the origin. Let $h(z) = H^{1/m}(s)$. Then h is well-defined and holomorphic, and $f(s) = (sh(s))^m$. Now consider a holomorphic change of coordinate $z = sh(s)$ on X . With respect to this coordinate, we have $f(z) = z^m$, concluding the proof. $\square$

This motivates the following definitions:

Definition 7.9. Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. For each $p \in X$, let m_p be the integer defined as in Proposition 7.8, known as the **multiplicity** of f at p .

Definition 7.10. Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. Let $q \in Y$, viewed as a divisor on Y . We define the **inverse image divisor** $f^*(q) := \sum_{p \in f^{-1}(q)} m_p p$. Extending this definition linearly, let $D = \sum_{q \in Y} n_q q$ be a divisor on Y . We define the **pullback divisor** $f^*(D) = \sum_{q \in Y} n_q f^*(q)$.

Definition 7.11. Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. We define the **ramification divisor** of f to be the divisor on X given by

$$R := \sum_{p \in X} (m_p - 1)p.$$

The ramification divisor R will be central in the statement of the Riemann-Hurwitz theorem. Now R is a divisor on X . We will first show that its sheaf of holomorphic functions is isomorphic to the relative cotangent bundle. Throughout this section, for notational convenience we will use Ω_M (instead of Ω_M^1) to denote the sheaf of holomorphic 1-forms on M .

Lemma 7.12. Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces and let R be the ramification divisor of f defined as above. Let $\Omega_{X/Y}$ be the relative cotangent sheaf given by the exact cotangent sequence induced by f . Then we have an isomorphism

$$\mathcal{O}_R \cong \Omega_{X/Y}.$$

Proof. We first need to understand $\Omega_{X/Y}$ better. This comes from the relative cotangent sequence

$$0 \rightarrow f^*\Omega_Y \rightarrow \Omega_X \rightarrow \Omega_{X/Y},$$

which is obtained from the dual of the relative tangent sequence (Theorem 3.14). It suffices to understand the local picture. Let z, w be holomorphic coordinates on X, Y around p , such that f takes the local expression $w = f(z) = z^m$, where $m = m_p$. Then

$\Omega_Y = \mathcal{O}_Y(dw)$ and $\Omega_X = \mathcal{O}_X(dz)$. Since $f^*dw = d(f^*w) = dz^m = mz^{m-1}dz$, we see that $f^*\Omega_Y = \mathcal{O}_X(mz^{m-1}dz) = \mathcal{O}_X(z^{m-1}dz)$. Therefore, we see that locally

$$\Omega_{X/Y} \cong \Omega_X/f^*\Omega_Y = \mathcal{O}_X(dz)/\mathcal{O}_X(z^{m-1}dz) \cong \mathcal{O}_X/(z^{m-1}).$$

On the other hand, $\mathcal{O}_R$ fits into the exact sequence

$$0 \rightarrow \mathcal{O}_X(-R) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_R \rightarrow 0.$$

We proved this for an irreducible hypersurface R (Theorem 3.28), and in general this is how we define $\mathcal{O}_D$ for any effective divisor D (Note that R is effective since f is non-constant). Now in the chosen local chart around p , we have $\mathcal{O}_X(-R) = \mathcal{O}_X(z^{m-1})$ by definition. Then $\mathcal{O}_R \cong \mathcal{O}_X/\mathcal{O}_X(-R) \cong \mathcal{O}_X/(z^{m-1})$, as desired. $\square$

Now we prove a general version of Riemann-Hurwitz theorem (the actual Riemann-Hurwitz formula follows from this by taking degrees).

Theorem 7.13 (Riemann-Hurwitz Theorem). *Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. Let K_X, K_Y be canonical bundles of X and Y , respectively. Then*

$$K_X \cong f^*K_Y \otimes \mathcal{O}_X(R).$$

*In the divisor language this is $K_X = f^*K_Y + R$.*

Proof. Tensoring the relative cotangent sequence $0 \rightarrow f^*\Omega_Y \rightarrow \Omega_X \rightarrow \Omega_{X/Y} \rightarrow 0$ with Ω_X^* , we get an exact sequence

$$0 \rightarrow f^*\Omega_Y \otimes \Omega_X^* \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_R \rightarrow 0, \quad (7.2)$$

where we used Lemma 7.12. Note that the map $\mathcal{O}_X \rightarrow \mathcal{O}_R$ in (7.2) is the same as in

$$0 \rightarrow \mathcal{O}_X(-R) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_R \rightarrow 0,$$

since the map $\mathcal{O}_X \rightarrow \mathcal{O}_R$ is obtained from dualizing the map $T_Y \rightarrow T_X$ in the relative tangent sequence, hence is the restriction onto R . Thus we have an isomorphism

$$f^*\Omega_Y \otimes \Omega_X^* \cong \mathcal{O}_X(-R),$$

which is the desired result. $\square$

Now we prove the Riemann-Hurwitz formula, which says that if we have a non-constant holomorphic map $f : X \rightarrow Y$ of finite degree, then the genera of X and Y are related.

Corollary 7.14 (Riemann-Hurwitz Formula). *Let $f : X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces of degree d . Let g_X, g_Y denote the genus of X, Y , respectively, then*

$$2g_X - 2 = d(2g_Y - 2) + \deg R.$$

where $\deg R = \sum_{p \in X} (m_p - 1)$.

Proof. By Theorem 7.13, we have $K_X \cong f^*K_Y \otimes \mathcal{O}_X(R)$. Taking the degree gives

$$2g_X - 2 = \deg K_X = \int_X c_1(K_X) = \int_X f^*c_1(K_Y) + c_1(\mathcal{O}_X(R)) = d \deg K_Y + \deg R$$

which gives the desired formula because $\deg K_Y = 2g_Y - 2$. In the second to last step we used the equality

$$\int_X f^*c_1(K_Y) = d \int_Y c_1(K_Y) = d \deg K_Y$$

since f is of degree d by assumption. This concludes the proof. $\square$

7.2 Residue Theorem

In this section, we generalize the Residue theorem from the complex plane to any compact Riemann surface. On the complex plane, we integrate meromorphic functions. On a compact manifold, the natural objects to integrate are differential forms. Hence we make the following definition:

Definition 7.15. Let X be a Riemann surface, Ω_X be its sheaf of holomorphic 1-forms, and $\mathcal{K}_X$ be its sheaf of meromorphic functions. A **meromorphic 1-form** ω on an open set $U \subset X$ is an element of $H^0(U, \mathcal{K}_X \otimes \Omega_X)$.

Next, we need to define what it means to take residue of a meromorphic 1-form. Let $p \in U \subset X$ be a point contained in an open subset U of X , and $\omega \in H^0(U, \mathcal{K}_X \otimes \Omega_X)$ a meromorphic 1-form on U that is holomorphic on $U \setminus \{p\}$. Then near p one can choose a holomorphic coordinate with $z(p) = 0$ and write $\omega = f(z)dz$ where f is a meromorphic function such that $f \in \mathcal{O}_X(U \setminus \{p\})$. Then let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

be the Laurent expansion of f about p with respect to coordinate z . We define the **residue** of ω to be

$$\text{Res}_p \omega := \text{Res}_p f = c_{-1}.$$

Of course, we need to check that this is well-defined.

Lemma 7.16. *The residue $\text{Res}_p \omega$ is independent of the choice of holomorphic charts.*

Proof. Let $U \subset X$ be an open set. We first make two observations:

- (1) If g is holomorphic on $U \setminus \{p\}$, then $\text{Res}_p dg = 0$ and is thus independent of the choice of chart.

To see this, let z be any coordinate chart with $z(p) = 0$. Suppose $g = \sum_{n=-\infty}^{\infty} c_n z^n$ is the Laurent series of g about p . Then $dg = (\sum_{n=-\infty}^{\infty} n c_n z^{n-1}) dz$ and thus the coefficient of $z^{-1} dz$ is zero, and since the chart is arbitrary, the statement follows.

- (2) If φ is a holomorphic function on U which has a zero of first order at p , then $\text{Res}_p(\varphi^{-1} d\varphi) = 1$ and is thus independent of the choice of chart.

Indeed, let z be any holomorphic coordinate with $z(p) = 0$. Then by assumption $\varphi = zh$ where h is holomorphic at p and does not vanish there. Thus $d\varphi = h dz + z dh$. Therefore,

$$\varphi^{-1} d\varphi = z^{-1} dz + h^{-1} dh$$

Since $h(p) \neq 0$, the second term $h^{-1} dh$ is holomorphic at p , thus has residue zero. Therefore

$$\text{Res}_p(\varphi^{-1} d\varphi) = \text{Res}_p(z^{-1} dz) = 1.$$

Again, this is evidently independent of the choice of chart z .

Using (1) and (2) one can easily finish the proof. With respect to any holomorphic chart z such that $z(p) = 0$, let $\omega = f(z)dz$, where $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be the Laurent expansion of f about p . Now let

$$g(z) = \sum_{n=-\infty}^{-2} \frac{c_n}{n+1} z^{n+1} + \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}.$$

Then we see that

$$\omega = dg + c_{-1}z^{-1}dz.$$

But by (1) we know $\text{Res}_p dg = 0$ is independent of coordinates, and $\text{Res}_p(z^{-1}dz) = 1$ is independent of coordinates as well. Hence $\text{Res}_p \omega = c_{-1}$ is independent of coordinate. $\square$

Now we prove

Theorem 7.17 (The Residue Theorem). *Let X be a compact Riemann surface and $p_1, \dots, p_n$ be distinct points in X . Let $X' = X \setminus \{p_1, \dots, p_n\}$. Then for every holomorphic 1-form $\omega \in \Omega_{X'}$, i.e., ω is a meromorphic 1-form on X with poles at $p_1, \dots, p_n$, one has*

$$\sum_{k=1}^n \text{Res}_{p_k} \omega = 0.$$

Proof. We can choose coordinates neighborhoods (U_k, z_k) of p_k such that $U_i \cap U_j = \emptyset$ if $i \neq j$, also we may assume that $z_k(p_k) = 0$ and $z_k(U_k) \subset \mathbb{C}$ is a disk. For every $k = 1, \dots, n$, choose a function f_k with compact support in U_k such that there exists an open neighborhood $U'_k \subset U_k$ of p_k with $f_k|_{U'_k} = 1$. Set $g = 1 - (f_1 + \dots + f_n)$. Then $g|_{U'_k} = 0$. Then $g\omega$ may be extended to p_k by zero, and may thus be considered as an element of $\mathcal{A}^1(X)$. By Stoke's theorem,

$$\int_X d(g\omega) = 0.$$

Since ω is holomorphic on X' , we have $d\omega = 0$ on X' by dimensional reason. On $U'_k \cap X'$ one has $f_k\omega = \omega$ and thus $d(f_k\omega) = 0$. Hence $d(f_k\omega)$ may be considered to be an element of $\mathcal{A}^2(X)$ whose support is a compact subset of $U_k \setminus \{p_k\}$. Then notice that

$$d(g\omega) = - \sum_{k=1}^n d(f_k\omega).$$

This formula is true on X' since $d\omega = 0$ there. Indeed, we note that since $g|_{U'_k} = 0$, and by viewing $g\omega \in \mathcal{A}^1(X)$ we extend $g\omega$ by zero on U'_k , hence $d(g\omega) = 0$ on U'_k , and $d(f_k\omega) = 0$ on p_k 's as well by above. Therefore, it follows from the Stoke's theorem that

$$\sum_{k=1}^n \int_X d(f_k\omega) = 0.$$

Thus the proof of the residue theorem will be completed if we could show that

$$\int_X d(f_k\omega) = -2\pi i \text{Res}_{p_k} \omega. \quad (7.3)$$

To prove (7.3), we note that since the support of $d(f_k\omega)$ is contained in U_k , we only have to integrate over U_k . Using the coordinate z_k we may identify U_k with the unit disk. There exists $0 < \epsilon < R < 1$ such that $\text{Supp}(f_k) \subset \{|z_k| < R\}$ and $f_k|_{\{|z_k| \leq \epsilon\}} = 1$ (this means that U'_k is given by the disk $|z_k| \leq \epsilon$, hence $d(f_k\omega) = 0$ inside $|z_k| < \epsilon$). Therefore,

$$\int_X d(f_k\omega) = \int_{\epsilon \leq |z_k| \leq R} d(f_k\omega) = \int_{|z_k|=R} f_k\omega - \int_{|z_k|=\epsilon} f_k\omega = - \int_{|z_k|=\epsilon} \omega = -2\pi i \text{Res}_{p_k} \omega$$

by the residue theorem in the complex plane. This concludes the proof. $\square$

7.3 Linear Systems

Recall that if $L \rightarrow X$ is a (holomorphic) line bundle over a complex manifold, and let $V \subset H^0(X, L)$ be a vector subspace, then we can pick a basis $s_0, \dots, s_n$ of V , and define a rational map

$$\phi_V : X \dashrightarrow \mathbb{P}^n \quad p \mapsto [s_0(p) : \dots : s_n(p)].$$

This map is understood as follows: For any $p \in X$ we pick an open set $U \subset X$ containing p such that $L|_U \cong \mathcal{O}|_U$. Let $s \in H^0(U, L)$ be the local frame of this line bundle. Then we can write $s_i = f_i s$ for some $f_i \in H^0(U, \mathcal{O})$. Then $\phi_V(p) = [f_0(p) : \dots : f_n(p)]$. Clearly this does not depend on the choice of s since any other choice only differ by a transition function, which does not matter when we projectivize. The points where ϕ_V is not defined is where all $s_0, \dots, s_n$ vanishes, and is called the **base locus**, which we denote by $\text{Bs}(V)$.

Definition 7.18. A **linear system** $\mathcal{V}$ on a complex manifold X is a pair $\mathcal{V} = (L, V)$ where $L \rightarrow X$ is a holomorphic line bundle and V a subspace of $H^0(X, L)$. We define the **dimension** of the linear series $\dim \mathcal{V} := \dim_{\mathbb{C}} V - 1$.

If $\mathcal{V}$ is a linear system of degree d and dimension r we say that $\mathcal{V}$ is a g_d^r .

If $V = H^0(X, L)$, then we say this is a **complete linear system** and we denote $|L| := (L, H^0(X, L))$.

We first note that on a curve, there is a different interpretation of the complete linear series. If X is a complex curve, then every line bundle L is of the form $L = \mathcal{O}(D)$ for some divisor D . Then we have

Proposition 7.19. On a complex curve X , the complete linear series $|L|$ associated to the line bundle $L = \mathcal{O}(D)$ can be identified with the set of all effective divisors

$$|D| = \{D' \in \text{Div}(C) : D' \sim D, D' \geq 0\}$$

linearly equivalent to D .

Proof. Given any section $\sigma \in H^0(X, L)$, taking its divisor gives a divisor $(\sigma) \sim D$. Moreover, since σ is a holomorphic section it has no poles, so $(\sigma) \geq 0$. Conversely, given any effective divisor $D' \sim D$, there exists a meromorphic function f on X such that $D' = D + (f)$. Now we know there exists a section $\sigma_0 \in H^0(X, L)$ such that $(\sigma_0) = D$ since $L = \mathcal{O}(D)$. Then note that $f \cdot \sigma_0$ has divisor D' and is holomorphic because $D' \geq 0$. Therefore, $f \cdot \sigma_0$ is the desired section of L . These two constructions are inverses to each other. $\square$

The importance of linear system can be illustrated by the following theorem:

Theorem 7.20. For any complex manifold X there is a natural bijection between the set of nondegenerate morphisms $\phi : X \rightarrow \mathbb{P}^r \text{ mod } \text{PGL}_{r+1}$ and basepoint free linear systems of dimension r on X up to isomorphism.

Proof. Let $\phi : X \rightarrow \mathbb{P}^r$ be any nondegenerate morphism. If this morphism is given by a line bundle L then $L \cong \phi^* \mathcal{O}_{\mathbb{P}^r}(1)$. Therefore, we define the linear system by

$$\mathcal{V} = (\phi^* \mathcal{O}_{\mathbb{P}^r}(1), \phi^* H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))).$$

Conversely, given a linear system (L, V) of dimension r we already have a map $\phi_V : X \rightarrow \mathbb{P}^r$. $\square$

Remark 7.21. There is actually a basis independent way of defining the map $\phi_V : X \dashrightarrow \mathbb{P}^r$ associated to the linear system $\mathcal{V} = (L, V)$. In the previous discussion where we chose a basis, we have

$$\phi_V(p) = [f_0(p) : \dots : f_r(p)].$$

This actually corresponds to $\sum_{i=0}^r f_i(p)s_i = \sum_i s_i(p) \in L_p \bmod \text{scalar}$. Therefore, this is a map $X \rightarrow \mathbb{P}(\text{Hom}(V, L)) \cong \mathbb{P}(V^* \otimes L) \cong \mathbb{P}(V^*)$. Hence, the basis independent description is

$$\phi_V : X \dashrightarrow \mathbb{P}(V^*) \quad p \mapsto \text{ev}_p.$$

Definition 7.22. Let $\mathcal{D} = (L, V)$ and $\mathcal{E} = (M, W)$ be two linear systems on X . By the *sum* $\mathcal{D} + \mathcal{E}$ we mean the pair

$$\mathcal{D} + \mathcal{E} = (L \otimes M, U)$$

where $U \subset H^0(L \otimes M)$ is the subspace generated by the image of $V \otimes W$ under multiplication $H^0(L) \otimes H^0(M) \rightarrow H^0(L \otimes M)$.

In the case that X is a curve, this is the subspace of complete linear system spanned by divisors of the form $D + E$ where $D \in \mathcal{D}, E \in \mathcal{E}$ both effective.

Remark 7.23. The corresponding map associated to the $\mathcal{D} + \mathcal{E}$ (assuming basepoint free) is the map

$$\phi_{\mathcal{D}+\mathcal{E}} : X \rightarrow \mathbb{P}^r \times \mathbb{P}^s \rightarrow \mathbb{P}^{(r+1)(s+1)-1}$$

where the second map is the Segre embedding.

Definition 7.24. We say a linear system $\mathcal{D} = (L, V)$ *contains effective divisors* if $|\mathcal{D}| = \mathbb{P}(V^*)$, the set of all effective divisors equivalent to V by Proposition 7.19, is nonempty.

Proposition 7.25. If $\mathcal{D}, \mathcal{E}$ are two linear series that contain effective divisors on a curve, then

$$\dim(\mathcal{D} + \mathcal{E}) \geq \dim \mathcal{D} + \dim \mathcal{E}.$$

Proof. Saying $\dim \mathcal{D} \geq m$ is equivalent to saying we can find a divisor $D \in \mathcal{D}$ containing any given m points of C ; since $\mathcal{D} + \mathcal{E}$ contains all pairwise sums $D + E$ with $D \in \mathcal{D}, E \in \mathcal{E}$, we can certainly find a divisor $D \in \mathcal{D} + \mathcal{E}$ containing any given $\dim \mathcal{D} + \dim \mathcal{E}$ points on the curve. $\square$

For a line bundle $L \rightarrow C$ on a curve, we give a useful criterion that it is basepoint free and (or) very ample:

Theorem 7.26. Let L be a line bundle on a smooth curve C . Then:

- (a) the complete linear series $|L|$ is basepoint free iff $h^0(L - p) = h^0(L) - 1$ for all $p \in C$.
- (b) $|L|$ is very ample iff $h^0(L - p - q) = h^0(L) - 2$ for all $p, q \in C$ (not necessarily distinct).

Proof. (a) We consider the short exact sequence

$$0 \rightarrow L(-p) \rightarrow L \xrightarrow{\text{ev}_p} L_p \rightarrow 0.$$

The long exact sequence reads

$$0 \rightarrow H^0(L - p) \rightarrow H^0(L) \xrightarrow{\text{ev}_p} H^0(L_p) \rightarrow \dots$$

The point $p \in C$ is not a basepoint iff ev_p is surjective, by rank-nullity and exactness this is equivalent to

$$h^0(L) = h^0(L - p) + 1,$$

which is what we want.

(b) ϕ_L is injective iff for $p \neq q$ we have $\phi_L(p) \neq \phi_L(q)$. This is equivalent to saying that the evaluation $\text{ev}_p \oplus \text{ev}_q : H^0(C, L) \rightarrow L_p \oplus L_q$ is surjective. By rank-nullity theorem this is equivalent to $h^0(L - p - q) = h^0(L) - 2$.

We claim that $d\phi_L : T_p C \rightarrow T_{\phi(p)} \mathbb{P}^r$ is injective on tangent space is equivalent to the condition $h^0(L - 2p) = h^0(L) - 2$. We choose a basis $s_0, \dots, s_r$ of $H^0(L)$. Since L is basepoint free, WLOG assume $s_0(p) \neq 0$. Then in local affine coordinate

$$\phi_L = (s_1/s_0, \dots, s_r/s_0).$$

Since $h^0(L - p) = h^0(L) - 1$ because L is basepoint free, we have that $s_1(p) = \dots = s_r(p) = 0$. Then we have

$$d\left(\frac{s_i}{s_0}\right)\Big|_p = \frac{ds_i(p) \cdot s_0(p) + s_i(p) \cdot ds_0(p)}{s_0^2(p)} = \frac{ds_i(p)}{s_0(p)}.$$

Since $T_p C$ is one dimensional, we see that

$$d\phi_{L,p} = (ds_1(p)/s_0(p), \dots, ds_r(p)/s_0(p))$$

is injective iff it is nonzero, i.e., iff $ds_i(p) \neq 0$ for some $i \in \{1, \dots, r\}$. This holds iff the evaluation of derivative map $\text{ev}'_p : H^0(L(-p)) \rightarrow H^0(L_p \otimes T_p^* C)$ is surjective. Now we have a short exact sequence

$$0 \rightarrow L(-2p) \rightarrow L(-p) \xrightarrow{\text{ev}'_p} L_p \otimes T_p^* C \rightarrow 0.$$

The corresponding long exact sequence reads

$$0 \rightarrow H^0(L - 2p) \rightarrow H^0(L - p) \xrightarrow{\text{ev}'_p} H^0(L_p \otimes T_p^* C) \rightarrow \dots$$

By rank nullity theorem and exactness, the map ev'_p is surjective iff

$$h^0(L - 2p) + 1 = h^0(L - p) = h^0(L) - 1.$$

Hence we see that $d\phi_L$ is injective at p iff $h^0(L - 2p) = h^0(L) - 2$. Hence the condition $h^0(L - p - q) = h^0(L) - 2$ for any $p, q \in C$ is equivalent to ϕ_L being an injective immersion, which is an embedding since C is compact (so ϕ_L is proper). $\square$

Using Riemann-Roch theorem, we get a useful criterion to determine whether a divisor is basepoint free and very ample:

Corollary 7.27. *Let D be a divisor of degree d on a curve C of genus g .*

(a) *If $d \geq 2g$ then $\mathcal{O}(D)$ is basepoint free.*

(b) *If $d \geq 2g + 1$ then $\mathcal{O}(D)$ is very ample.*

Proof. These follows from Theorem 7.26. (a) Since $d \geq 2g$ we have $h^0(K - D + p) = 0$ for any $p \in C$. Therefore, Riemann-Roch gives

$$h^0(D) - h^0(D - p) = (h^0(K - D) + d + 1 - g) - (h^0(K - D + p) + d - 1 + 1 - g) = 1.$$

(b) If $d \geq 2g + 1$ then $h^0(K - D + p + q) = 0$. Hence

$$h^0(D) - h^0(D - p - q) = (h^0(K - D) + d + 1 - g) - (h^0(K - D + p + q) + d - 2 + 1 - g) = 2.$$

This proves the claim. $\square$

7.4 Applications of Riemann-Roch Theorem

7.4.1 The Canonical Morphism and Geometric Riemann-Roch

Theorem 7.28. *Let C be a smooth curve of genus $g \geq 2$. Then:*

- (a) $|K_C|$ is basepoint free.
- (b) $|K_C|$ is very ample iff C admits no map of degree 2 to $\mathbb{P}^1$ (i.e., C is not hyperelliptic).

Proof. (a) By Theorem 7.26, a point p is a basepoint of $|K_C|$ iff $h^0(K_C - p) = h^0(K_C) = g$. By Riemann-Roch, this is equivalent to $h^0(p) = 2$, which would imply that $C \cong \mathbb{P}^1$ because there is a meromorphic function on C with a pole at p .

(b) By Theorem 7.26, $|K_C|$ is very ample iff $p, q \in C$ we have $h^0(K_C - p - q) = h^0(K_C) - 2 = g - 2$. By Riemann-Roch, this fails iff $h^0(p + q) \geq 2$ for some $p, q \in C$.

Now assume $h^0(p + q) \geq 2$. Since $g \geq 2$, we conclude that there exists a meromorphic function on C with poles at p, q , which defines a double cover to $\mathbb{P}^1$, hence C is hyperelliptic.

Conversely, assume C is hyperelliptic. Then take $\pi : C \rightarrow \mathbb{P}^1$ be the hyperelliptic map, and pick a point $x \in \mathbb{P}^1$ that is unramified, so that $D := \pi^*x$ is given by $D = p + q$ for some $p, q \in C$. We claim that $h^0(D) = 2$, then by Riemann-Roch $h^0(K - p - q) = h^0(K) - 1$, concluding the proof.

It remains to show $h^0(D) = 2$. Since x is a degree 1 divisor, we see that $\mathcal{O}_{\mathbb{P}^1}(x) \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Then $\mathcal{O}_C(D) \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Since π is a submersion, we have

$$H^0(C, \mathcal{O}_C(D)) = H^0(C, \pi^*\mathcal{O}_{\mathbb{P}^1}(1)) = H^0(\mathbb{P}^1, \pi_*\pi^*\mathcal{O}_{\mathbb{P}^1}(1)).$$

By projection formula, we have

$$\pi_*\pi^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_*\mathcal{O}_C.$$

By standard classification of double covers, $\pi_*\mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle such that $\mathcal{L}^2 \cong \mathcal{O}_{\mathbb{P}^1}(-B)$ where B is the branch divisor. By Riemann Hurwitz, $\chi(C) = 2\chi(\mathbb{P}^1) - \deg B$, hence $\deg B = 2g - 2 + 2\chi(\mathbb{P}^1) = 2g + 2$. Hence we see that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(-g - 1)$. Therefore,

$$\pi_*\pi^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-g).$$

Hence

$$h^0(D) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-g)) = 2 + 0 = 2,$$

as desired. □

Definition 7.29. The image of the canonical morphism $\phi_{K_C} : C \rightarrow \mathbb{P}^{g-1}$ of a nonhyperelliptic curve of genus $g > 2$ is called the **canonical curve**.

Proposition 7.30. Let $\phi : C \rightarrow \mathbb{P}^{g-1}$ be the canonical morphism. If ϕ is an embedding, then the degree of the image variety $X = \phi(C)$ is $\deg K_C = 2g - 2$.

Proof. Let $\omega = c_1(\mathcal{O}_{\mathbb{P}^{g-1}}(1)) \in H^2(\mathbb{P}^{g-1}; \mathbb{Z})$ denote the Chern class of the hyperplane class in $\mathbb{P}^{g-1}$. Then the degree of the curve X is the intersection number $X \cdot H$ where H is a hyperplane. Therefore,

$$\deg X = \int_X \omega = \int_{\phi(C)} \omega = \langle \omega, [\phi(C)] \rangle.$$

Here, the bracket is the natural pairing between homology and cohomology. Now the definition of degree of ϕ is that $\phi_*[C] = \deg \phi \cdot [\phi(C)]$. Hence we get that

$$\langle \omega, \phi_*[C] \rangle = \deg \phi \cdot \deg X.$$

Using the fact that $K_C \cong \phi^* \mathcal{O}_{\mathbb{P}^{g-1}}(1)$, we see that after taking Chern classes we have $c_1(K_C) = \phi^* c_1(\mathcal{O}_{\mathbb{P}^{g-1}}(1)) = \phi^* \omega$. Using the property that pullback and pushforward are dual to each other, we have

$$\deg \phi \cdot \deg X = \langle \omega, \phi_*[C] \rangle = \langle \phi^* \omega, [C] \rangle = \int_C \phi^* \omega = \int_C c_1(K_C) = \deg K_C = 2g - 2.$$

Now $\phi : C \rightarrow \mathbb{P}^{g-1}$ is an embedding by assumption. Hence we see that $\deg X = \deg K_C = 2g - 2$, as claimed. $\square$

Now we study a geometric version of Riemann-Roch theorem. Suppose $\phi : C \rightarrow \mathbb{P}^{g-1}$ is the canonical embedding. Let D be an effective divisor on C . We define

$$\overline{\phi(D)} := \bigcap_{D \subset \phi^{-1}(H)} H,$$

where H runs over all hyperplane in $\mathbb{P}^{g-1}$ containing D . Recall that we have $K_C \cong \phi^* \mathcal{O}_{\mathbb{P}^{g-1}}(1)$. Hence, $D \subset \phi^{-1}(H)$ is equivalent to saying that the section σ_H of $\mathcal{O}_{\mathbb{P}^{g-1}}(1)$ corresponding to the hyperplane H satisfies the condition that $\phi^* \sigma_H$ vanishes on D . Therefore, there is a bijective correspondence

$$H^0(K_C - D) \xleftrightarrow{1:1} \{H \in \mathcal{O}_{\mathbb{P}^{g-1}}(1) : D \subset \phi^{-1}(H)\}.$$

Now if we pick a basis $H_1, \dots, H_\ell$ on the right hand side, we see that

$$\overline{\phi(D)} = H_1 \cap \dots \cap H_\ell.$$

Therefore, we conclude that

$$h^0(K_C - D) = \text{codim } \overline{\phi(D)} = (g - 1) - \dim \overline{\phi(D)}.$$

Applying the Riemann-Roch theorem we get the geometric Riemann-Roch theorem:

Theorem 7.31 (Geometric Riemann-Roch). *If D is an effective divisor on a smooth curve C of genus $g \geq 2$, then the dimension $r(D) = h^0(D) - 1$ of the linear system $|D|$ is given by*

$$r(D) = d - 1 - \dim \overline{\phi(D)}.$$

Now let us interpret the expression on the right hand side in a more geometric manner. Suppose $D = \sum_{i=1}^d p_i$. Then these points on C under the canonical map become points $\phi(p_1), \dots, \phi(p_d)$ on the canonical curve, which lives inside $\mathbb{P}^{g-1} \cong \mathbb{P} H^0(K_C)^*$. Hence, each point $\phi(p_i)$ corresponds to a (projective) line in the vector space $H^0(K_C)^*$. What is the dimension that these line span in $H^0(K_C)^*$? Since there are d lines, the naive answer is d . However, there might be some linear relations among these lines, i.e., they may not be linearly independent. The number of linear relations r that this system have should therefore be the naive dimension d minus the actual dimension, i.e.,

$$r = d - \dim \text{span}\{\phi(p_1), \dots, \phi(p_d)\}.$$

But note that the span of $\{\phi(p_1), \dots, \phi(p_d)\}$ is precisely the intersection of all hyperplanes in $H^0(K_C)^*$ containing them, hence

$$\dim \text{span}\{\phi(p_1), \dots, \phi(p_d)\} = \dim \overline{\phi(D)} + 1.$$

That is, the right hand side $d - 1 - \dim \overline{\phi(D)}$ of the geometric Riemann-Roch is precisely the number of linear relations among the points of the divisor on the canonical curve. To summarize:

Corollary 7.32 (See Figure 7.1). *For an effective divisor D on C with genus $g \geq 2$, the dimension of the linear system $|D|$ is precisely the number of linear relations among the image points of D on the canonical image of C .*

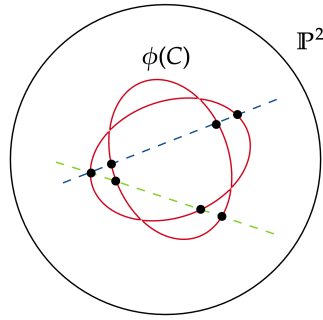


Figure 7.1: As an illustration of Corollary 7.32, this picture shows the canonical image of a genus 3 curve in $\mathbb{P}^2$, which is a plane quartic. The divisor D on C has 7 points, and so is the image divisor $\phi(D)$ on $\phi(C)$. There are two linear relations among these image points, hence the dimension of the linear system $|D|$ is 2.

Now we conclude with a study of the canonical morphism of the hyperelliptic curve. We first note that:

Lemma 7.33. *Let C be a hyperelliptic curve of genus $g \geq 2$, with hyperelliptic map $\pi : C \rightarrow \mathbb{P}^1$. Then*

$$K_C \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(g - 1).$$

Proof. By Riemann-Hurwitz, we have $K_C \cong \pi^* K_{\mathbb{P}^1} \otimes \mathcal{O}_C(R)$. Taking degree gives $\deg R = 2g + 2$. Since every branch point $p \in B$ is simple for a double cover, the branch divisor B has degree $\deg B = 2g + 2$ and satisfies $\pi^* B = 2R$. Thus $\pi^* \mathcal{O}_{\mathbb{P}^1}(B) = \pi^* \mathcal{O}_{\mathbb{P}^1}(2g + 2)$, which implies $\mathcal{O}_C(R) = \pi^* \mathcal{O}_{\mathbb{P}^1}(g + 1)$. Hence Riemann-Hurwitz gives

$$K_C \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(g + 1) = \pi^* \mathcal{O}_{\mathbb{P}^1}(g - 1)$$

as desired. □

Proposition 7.34. *Let C be a hyperelliptic curve of genus $g \geq 2$, with hyperelliptic map $\pi : C \rightarrow \mathbb{P}^1$. The canonical morphism $\phi_{K_C} : C \rightarrow \mathbb{P}^{g-1}$ is given by the composition*

$$\phi_{K_C} : C \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\nu} \mathbb{P}^{g-1},$$

where $\nu : [x : y] \mapsto [x^{g-1} : x^{g-2}y : \dots : y^{g-1}]$ is the Veronese embedding.

Proof. We already know from the previous lemma that $K_C \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(g-1)$. Thus a basis of $H^0(K_C)$ is given by pulling back degree $(g-1)$ monomials on $\mathbb{P}^1$ under the map π . Thus if x, y are homogeneous coordinates on $\mathbb{P}^1$, we know that the canonical map is given by

$$\phi(p) = [x(p)^{g-1} : x(p)^{g-2}y(p) : \cdots : y(p)^{g-1}]$$

where $x(p), y(p)$ are coordinates of $\pi(p)$ in $\mathbb{P}^1$. $\square$

7.4.2 Clifford's Theorem

The Riemann-Roch theorem reads $h^0(L) = h^0(K - L) + d + 1 - g$. Hence gives an upper bound $r(L) \geq d - g$. We would like a lower bound for $r(L)$ as well.

The first thing to note is that if $d = \deg L > 2g - 2$, then $h^0(K - L) = 0$. Hence the Riemann-Roch inequality becomes an equality. Hence it is enough to study the case where $d \leq 2g - 2$. This is

Theorem 7.35 (Clifford's Theorem). *Let C be a curve of genus g and L a line bundle of degree $0 \leq d \leq 2g - 2$. Then*

$$r(L) \leq \frac{d}{2}.$$

Moreover, when the equality is achieved, one of the following must be true:

- $d = 0$ and $L = \mathcal{O}_C$;
- $d = 2g - 2$ and $L = K_C$;
- C is hyperelliptic.

Proof. We will omit the proof of the claim when the equality is achieved.

First consider the case where L is nonspecial, i.e., when $h^1(L) = 0$. Then since $g \geq \frac{d}{2} + 1$, Riemann-Roch gives

$$r(L) = d - g \leq d - \frac{d}{2} - 1 = \frac{d}{2} - 1 < \frac{d}{2}.$$

Hence let us assume that L is special, i.e., $h^1(L) > 0$. Then by Riemann-Roch we have

$$r(K_C - L) = r(L) + g - d - 1.$$

If L does not contain any effective divisor, then $h^0(L) = 0$, then $r(L) = -1$. Since $d = \deg L \geq 0$, the Clifford equality holds trivially. Hence we may assume L contains effective divisors. If $K_C - L$ does not contain effective divisors, by Serre duality $h^0(K_C - L) = h^1(L) = 0$, hence L is nonspecial, which we have already dealt with. Hence we may assume $K_C - L$ contains effective divisors as well. Now applying Proposition 7.25 to L and $K_C - L$ gives

$$g = r(K_C) + 1 \geq r(L) + r(K_C - L) + 1 = 2r(L) + g - d$$

by Riemann-Roch. Hence we get $r(L) \leq d/2$, concluding the proof. $\square$

Remark 7.36. The assumption $d \geq 0$ is necessary. A counterexample can be given otherwise: Let C be a curve of genus 2 and let $p \in C$ be a point. Consider the line bundle $L = \mathcal{O}_C(-4p)$. Then $d = -4 \leq 2g - 2 = 2$. Since $d < 0$ we have $h^0(L) = 0$, so $r(L) = -1$, but $d/2 = -2$, so we get $-1 \leq -2$, which is a contradiction.

7.5 Abel-Jacobi Theory

Let C be a complex curve of genus g . We will associate, for every C , a complex torus $\text{Jac } C = \mathbb{C}^g / \Lambda$ of dimension g , called the Jacobian of C . We will see that this definition will lead to a necessary and sufficient condition for a degree zero divisor on C to be principal.

There are two definitions of the Jacobian, one algebraic and one analytic. We first start with the analytic definition.

Definition 7.37. Let C be a complex curve and $\Omega_C = K_C$ be the sheaf of differentials of C . Integration defines a map $I : H_1(C; \mathbb{Z}) \rightarrow H^0(C, K_C)^*$ given by

$$I(\gamma) : \omega \mapsto \int_{\gamma} \omega.$$

The **Jacobian** of C is defined to be

$$\text{Jac } C := H^0(C, K_C)^* / I(H_1(C; \mathbb{Z})).$$

We will first show that

Proposition 7.38. If C has genus g , then $\text{Jac } C$ is a complex torus of dimension g .

Proof. Since $H^0(C, K_C) \cong H^{1,0}(C) \cong \mathbb{C}^g$ by Hodge theory, and $I(H_1(C; \mathbb{Z}))$ is a subspace inside $H^0(C, K_C)^*$, it suffices to show that $I(H_1(C; \mathbb{Z}))$ is a lattice in $H^0(C, K_C)^* \cong \mathbb{C}^g$.

Let $\gamma_1, \dots, \gamma_{2g}$ be a basis of $H_1(C; \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Then $I(H_1(C; \mathbb{Z}))$ is generated by $I(\gamma_1), \dots, I(\gamma_{2g})$. Hence we just need to show that these are linearly independent over $\mathbb{R}$. Indeed, let $\lambda_1, \dots, \lambda_{2g} \in \mathbb{R}$ be such that

$$\lambda_1 I(\gamma_1) + \dots + \lambda_{2g} I(\gamma_{2g}) = 0.$$

Now let $\omega_1, \dots, \omega_g$ be a basis of $H^0(C, \Omega_C)$. Then this implies

$$\lambda_1 \int_{\gamma_1} \omega_j + \dots + \lambda_{2g} \int_{\gamma_{2g}} \omega_j = 0$$

for $j = 1, \dots, g$. Taking complex conjugate gives

$$\lambda_1 \int_{\gamma_1} \bar{\omega}_j + \dots + \lambda_{2g} \int_{\gamma_{2g}} \bar{\omega}_j = 0$$

for $j = 1, \dots, g$. But $\omega_1, \dots, \omega_g, \bar{\omega}_1, \dots, \bar{\omega}_g$ is a basis of $H_{\text{dR}}^1(C; \mathbb{C})$. Hence taking Poincaré duals gives the equality

$$\lambda_1 [\gamma_1] + \dots + \lambda_{2g} [\gamma_{2g}] = 0$$

in $H_1(C; \mathbb{C})$. Therefore $\lambda_1 = \dots = \lambda_{2g} = 0$. This concludes the proof. $\square$

Hence we see that $\text{Jac } C$ has a natural structure as a complex torus. Our next result states that over the complex numbers we have isomorphisms $\text{Jac } C \cong \text{Pic}^0 C$ on the level of vector spaces. Therefore, $\text{Pic}^0 C$ has a natural structure of a complex torus under this isomorphism.

Theorem 7.39 (Abel-Jacobi). *We have an isomorphism $\text{Jac } C \cong \text{Pic}^0 C$, where $\text{Pic}^0 C$ is the elements in $\text{Pic } C$ of degree zero.*

Sketch of Proof. The exponential sequence give rise to a long exact sequence

$$\cdots \rightarrow H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}^*) \rightarrow H^1(C; \mathbb{Z}) \rightarrow H^1(\mathcal{O}) \rightarrow H^1(\mathcal{O}^*) \rightarrow H^2(C; \mathbb{Z}) \rightarrow H^2(\mathcal{O}) \rightarrow \cdots$$

We have $H^0(\mathcal{O}) = \mathbb{C}$, $H^0(\mathcal{O}^*) = \mathbb{C}^*$, $H^1(C; \mathbb{Z}) = \mathbb{Z}^{2g}$, and $H^2(\mathcal{O}) = 0$. The map $H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}^*)$ is given by the exponential map $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$, which is surjective. Hence exactness implies $H^0(\mathcal{O}^*) \rightarrow H^1(C; \mathbb{Z})$ is the zero map, hence $H^1(C; \mathbb{Z}) \rightarrow H^1(\mathcal{O})$ is injective. Exactness also implies $H^1(\mathcal{O}^*) \rightarrow H^2(C; \mathbb{Z})$ is surjective. By definition,

$$\begin{aligned} \text{Pic}^0 C &= \ker(H^1(\mathcal{O}^*) \rightarrow H^2(C; \mathbb{Z})) \\ &= \text{im}(H^1(\mathcal{O}) \rightarrow H^1(\mathcal{O}^*)) \\ &\cong H^1(\mathcal{O}) / \ker(H^1(\mathcal{O}) \rightarrow H^1(\mathcal{O}^*)) \\ &= H^1(\mathcal{O}) / \text{im}(H^1(C; \mathbb{Z}) \rightarrow H^1(\mathcal{O})) \\ &\cong H^1(\mathcal{O}) / H^1(C; \mathbb{Z}). \end{aligned}$$

By Serre duality we have $H^1(\mathcal{O}) \cong H^0(K_C)^*$. Hence we can identify the Jacobian as

$$J(C) = H^0(K_C)^* / H_1(C; \mathbb{Z}),$$

We may view $H_1(C; \mathbb{Z})$ naturally as a subspace of $H^0(K_C)^*$ as we described before. This is the desired result. $\square$

Therefore, we see that $\text{Pic}^0 C$ also has a natural structure of a complex torus. In fact it also has a group structure given by tensor of line bundles. We can thus make the following equivalent definition of a Jacobian:

Definition 7.40. Let C be a complex curve, the **Jacobian** $\text{Jac } C$ is defined to be $\text{Pic}^0 C$.

We can think of $q - p$ as a divisor on C of degree 0. Then the map $q - p \mapsto \int_p^q$ extends to a well-defined map μ_0 from divisors of degree 0 to $J(C)$, because if p', q' are two more points then

$$\int_p^q + \int_{p'}^{q'} - \left(\int_p^{q'} + \int_{p'}^q \right)$$

is the integral around a closed loop $p \rightarrow q \rightarrow p' \rightarrow q' \rightarrow p$.

The point of identifying $J(C)$ as a complex torus is that now we can define a holomorphic map, called the **Abel-Jacobi map**, by

$$\mu : C \rightarrow J(C) \quad q \mapsto \int_p^q$$

where $p \in C$ is a fixed basepoint. Note that this is well-defined, i.e., independent of the choice of path from p to q , since the difference of integrals along any two such path is in the image of $H^1(C; \mathbb{Z})$. More generally,

Definition 7.41. Let $C^{(d)} = C^d / S_d$ be the d -fold symmetric product of C , i.e., points in $C^{(d)}$ are equivalent classes of points in C^d under the action of symmetric group S_d which permutes the points in C^d . Then note that $C^{(d)}$ is equivalent to the set of effective divisors on C of degree d .

We then have maps

$$\mu_d : C^{(d)} \rightarrow J(C) \quad (q_1, \dots, q_d) \mapsto \sum_{i=1}^d \int_p^{q_i}.$$

If we instead want to view $J(C)$ as the space of degree zero line bundles on C , then the maps are given by

$$\begin{aligned} \mu : q &\mapsto \mathcal{O}(q - p) \\ \mu_d : (q_1, \dots, q_d) &\mapsto \mathcal{O}(q_1 + \dots + q_d - dp). \end{aligned}$$

Recall that a divisor D on C is called **principal** if there exists some meromorphic function f on C such that $D = \operatorname{div} f$ which equals the zeros of f minus the poles of f (with multiplicity). Note that a principal divisor must always have degree zero since the number of zeros and poles of a meromorphic function is always equal on a compact Riemann surface.

Now we prove the famous theorem of Abel.

Theorem 7.42 (Abel). *A divisor D of degree zero on C is principal iff its image under the Abel map $D \mapsto \mathcal{O}(D)$ is zero. That is, $D \sim 0$ iff $\mathcal{O}(D) \cong \mathcal{O}$.*

Proof. If $D \sim 0$, then say $D = \operatorname{div} f$ for some meromorphic function f . Then multiplication (resp. division) by f gives an isomorphism $\mathcal{O} \cong \mathcal{O}(D)$.

Conversely, if $\mathcal{O}(D) \cong \mathcal{O}$, then $h^0(C, \mathcal{O}(D)) = h^0(C, \mathcal{O}) = 1$ where the last equality follows from Theorem 2.5. Therefore, we have a nonzero section $s \in H^0(C, \mathcal{O}(D))$. By the definition of $\mathcal{O}(D)$, namely (3.1), this section s corresponds to a meromorphic function f such that $\operatorname{div} f + D \geq 0$. Taking the degree of this effective divisor we find

$$\deg(\operatorname{div} f + D) = \deg(\operatorname{div} f) + \deg(D) = 0 + 0 = 0.$$

Hence it follows that $\operatorname{div} f + D = 0$. That is, $D = -\operatorname{div} f = \operatorname{div}(-f)$. Hence $D \sim 0$. This concludes the proof. $\square$

Theorem 7.43. *If $g \geq 1$, then the Abel Jacobi map $\mu : C \rightarrow J(C)$ given by $\mu : q \mapsto \mathcal{O}(q - p)$, where $p \in C$ is a fixed basepoint, is an embedding.*

Proof. Since C is compact it suffices to show that μ is an injective immersion. Injectivity is easy: Suppose $\mu(q) = \mu(q')$ for two distinct points q, q' on C . Then $\mathcal{O}(q - p) \cong \mathcal{O}(q' - p)$, which implies $\mathcal{O}(q - q') \cong \mathcal{O}$. Thus $q \sim q'$, which means there exists a meromorphic function f on C such that $(f) = q - q'$. But then f defines an isomorphism $C \rightarrow \mathbb{P}^1$, which is impossible since $g \geq 1$ by assumption.

Next we need to show that the differential $\mu_* : T_q C \rightarrow T_{\mu(q)} J(C)$ is injective at an arbitrary point $q \in C$. This time, it is more convenient for us to go back to the integral definition, namely we identify $\mu(q) = \int_p^q$, and $J(C) = H^0(K_C)^* / H^1(C; \mathbb{Z})$. Since $H^0(K_C)^*$ is a vector space and $J(C)$ is this vector space quotient out by the lattice $H^1(C; \mathbb{Z})$, we see that $T_{\mu(q)} J(C) \cong T_{\mu(q)} H^0(K_C)^* \cong H^0(K_C)^*$. Now the map μ_* , schematically, is given by

$$\mu_* : T_q C \rightarrow H^0(K_C)^* \quad \frac{\partial}{\partial q} \mapsto \frac{\partial}{\partial q} \int_p^q.$$

Dually, the induced map on the cotangent space is given by

$$\mu^* : H^0(K_C)^{**} \cong H^0(K_C) \rightarrow T_q^* C \quad \omega \mapsto \omega \left(\frac{\partial}{\partial q} \int_p^q \right) := \frac{\partial}{\partial q} \int_p^q \omega = \operatorname{ev}_q \omega,$$

where the last equality is given by the fundamental theorem of calculus, and the first equality is by the very construction of how we identify $H^0(K_C)^{**}$ with $H^0(K_C)$.

Now we note that K_C is basepoint free. Indeed, for any $q \in C$, we must have $h^0(q) = 1$, otherwise we have a meromorphic function with a pole at q , giving an isomorphism $C \cong \mathbb{P}^1$. Therefore, Riemann-Roch gives

$$h^0(K_C - q) = h^0(q) + (2g - 3) + 1 - g = g - 1 = h^0(K_C) - 1.$$

Hence K_C is basepoint free by Theorem 7.26. Therefore, the evaluation map $\text{ev}_q : H^0(K_C) \rightarrow T_q^*C$ is surjective. Hence $\mu_* : T_qC \rightarrow T_{\mu(q)}J(C)$ is injective, concluding the proof. $\square$

7.6 Classification of Curves of Low Genus

In this section we classify curves with low genus, more specifically $0 \leq g \leq 5$. We mostly focus on smooth curves. The cases $g = 0, 1$ are very easy:

7.6.1 $g = 0, 1$: Projective Line and Complex Torus

Theorem 7.44. *Every smooth curve of genus 0 is isomorphic to $\mathbb{P}^1$.*

Proof. By Riemann-Roch, we have $h^0(p) = h^0(K_C - p) + 1 + 1 - 0 = 2$, since $\deg(K_C - p) < 0$, i.e., $h^0(K_C - p) = 0$. Thus there exists a meromorphic function f on C with exactly one pole, giving an isomorphism $f : C \rightarrow \mathbb{P}^1$. $\square$

Theorem 7.45. *Every smooth curve of genus 1 is a complex torus. More precisely, the Abel-Jacobi map $\mu : C \rightarrow J(C)$ is an isomorphism.*

Proof. By Theorem 7.43, the Abel-Jacobi map $\mu : C \rightarrow J(C)$ is an embedding. Hence it is an isomorphism because both the domain and codomain are connected one dimensional complex manifolds and μ is holomorphic. $\square$

Remark 7.46. In fact, every genus 1 curve is hyperelliptic. To see this, let $p, q \in C$ be any points. Riemann-Roch immediately gives $h^0(p + q) = 2$. Therefore, we have a meromorphic function with poles along p and q (note that it cannot have only one pole otherwise $C \cong \mathbb{P}^1$). Thus we conclude that C admits a 2:1 cover to $\mathbb{P}^1$.

7.6.2 $g = 2$: Hyperelliptic Curves

Theorem 7.47. *Every genus 2 curve is hyperelliptic and the canonical morphism $\phi_K : C \rightarrow \mathbb{P}^1$ is a 2:1 cover, i.e., the canonical morphism is a hyperelliptic map.*

Proof. By Riemann-Roch, we have $h^0(p + q) = h^0(K - p - q) + 2 + 1 - 2 = h^0(K - p - q) + 1 = 2$, since $\deg(K - p - q) = 0$ so $h^0(K - p - q) = h^0(\mathcal{O}) = 1$. Since $h^0(p + q) = 2$, there exists some meromorphic function with poles at p or q , or both p and q . But again we cannot have a meromorphic function with just one pole, otherwise $C \cong \mathbb{P}^1$. Thus we conclude that there exists a meromorphic function f on C with two poles, which gives a 2:1 map $f : C \rightarrow \mathbb{P}^1$, showing that C is hyperelliptic.

It remains to compute the degree of the canonical map $\phi : C \rightarrow \mathbb{P}^1$. Take a volume form ω of $\mathbb{P}^1$, and WLOG we may assume ω is a representative of $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Since

$K \cong \phi^* \mathcal{O}_{\mathbb{P}^1}(1)$, taking Chern class on both sides gives $c_1(K) = \phi^* c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = \phi^* \omega$. Therefore,

$$\deg \phi = \int_C \phi^* \omega = \int_C c_1(K) = \deg K = 2g - 2 = 2.$$

This concludes the proof. $\square$

Hence, every curve of genus 2 is hyperelliptic. Although the goal of this section primarily concentrates on genus 2 curve, but let us divert a little bit and study hyperelliptic curve of any genus $g \geq 2$. The next result shows that we have a very simple description on the vector space $H^0(C, K_C)$.

Proposition 7.48. We can always write the affine part of a hyperelliptic curve C as the equation $y^2 = \prod_{i=1}^{2g+2} (x - x_i)$ in $\mathbb{C}^2$. Moreover, the differential form $\omega_0 = dx/y$ is nonvanishing on the affine part of C . Then a basis of $H^0(C, K_C)$ can be given explicitly by

$$H^0(C, K_C) = \langle \omega_0, x\omega_0, \dots, x^{g-1}\omega_0 \rangle.$$

Proof. A hyperelliptic curve, by definition, is the one that admits a 2:1 branched cover $\pi : C \rightarrow \mathbb{P}^1$. Using the Riemann-Hurwitz formula $\chi(C) = 2\chi(\mathbb{P}^1) - \sum_p (e_p - 1)$, and using that $e_p - 1 = 1$ for all ramified point since the cover is degree 2, we see that the number of ramification points is $2g + 2$. Call these points $x_1, \dots, x_{2g+2}$.

Let B denote the divisor of branched points in $\mathbb{P}^1$. If $\tilde{\pi} : \tilde{C} \rightarrow \mathbb{P}^1$ is another 2:1 branched cover with the same branched locus B , then we have an isomorphism $\pi^{-1}(\mathbb{P}^1 \setminus B) \cong \tilde{\pi}^{-1}(\mathbb{P}^1 \setminus B)$. And this isomorphism can be extended by an isomorphism $\tilde{C} \cong C$. Hence we conclude that any two degree 2 branched cover of $\mathbb{P}^1$ with the same branched divisor are isomorphic.

Now here is one such cover: Let x, y be coordinates in $\mathbb{C}^2$ and consider the curve S

$$y^2 = \prod_{i=1}^{2g+2} (x - x_i).$$

with projection $\pi : S \rightarrow \mathbb{C}$ to the x coordinate. Clearly this is a degree 2 branched cover, branched exactly at $x_1, \dots, x_{2g+2}$. Now by compactifying them, we can construct a hyperelliptic curve $\pi : C \rightarrow \mathbb{P}^1$, and we showed that this is the unique hyperelliptic curve branched at $x_1, \dots, x_{2g+2}$.

Now we find a basis for $H^0(C, K_C)$. Let $j : (x, y) \mapsto (x, -y)$ be the automorphism on C given by flipping two sheets of the cover. Then j^* acts on forms so we have a map $j^* : H^0(C, K_C) \rightarrow H^0(C, K_C)$. Since $j^2 = 1$, we know $H^0(C, K_C)$ decomposes into the (± 1) -eigenspace of j^* . In fact the $(+1)$ -eigenspace is trivial, because any form $\omega \in H^0(C, K_C)$ with $j^* \omega = \omega$ descends to a $(1, 0)$ -form on $\mathbb{P}^1$, and $H^{1,0}(\mathbb{P}^1) = 0$. Hence we conclude that $j^* \omega = -\omega$ for all holomorphic 1-forms on C .

Now consider the form

$$\omega_0 = \frac{dx}{y}$$

on C . We claim that ω_0 is nowhere vanishing on C and holomorphic. Indeed, away from branched points $y \neq 0$, here $\pi : C \rightarrow \mathbb{P}^1$ is unramified, so the projection x is a valid local coordinate, and $\omega_0 = dx/y$ is clearly nonvanishing and holomorphic. Near each branched point x_i , we note that since $f(x) := \prod_{i=1}^{2g+2} (x - x_i)$ satisfies $f'(x) = \sum_{i=1}^{2g+2} \prod_{j \neq i} (x - x_j)$,

we have $f'(x_i) \neq 0$ since $x_1, \dots, x_{2g+2}$ are distinct. Now the equation $y^2 = f(x)$ defining C has differential $2ydy = f'(x)dx$, so $dx = \frac{2y}{f'(x)}dy$. Thus near x_i we have

$$\omega_0 = \frac{dx}{y} = \frac{\frac{2y}{f'(x)}dy}{y} = \frac{2}{f'(x)}dy$$

is holomorphic and nonzero. Hence we conclude that ω_0 is holomorphic and has no zeros anywhere on finite part of C .

Now the space $\mathcal{K}^1(C)$ of meromorphic 1-forms on C is 1-dimensional over the field $\mathcal{K}(C)$ of meromorphic functions on C , which can be checked locally since $\mathcal{K}^1(C)|_U \cong \mathcal{K}(C)|_U \otimes \Omega^1(C)|_U$. Hence any holomorphic 1-form $\omega \in H^0(C, K_C)$ is given by

$$\omega = h\omega_0$$

where h is a meromorphic function on C . Moreover, $h = \omega/\omega_0$ cannot have any poles on finite part of C , since ω is holomorphic and ω_0 is nonvanishing on finite parts of C .

We argue that h must then be a polynomial. Recall that for $\omega \in H^0(C, K_C)$ we must have $j^*\omega = -\omega$. But note that $j^*\omega_0 = -\omega_0$. Hence $j^*h = h$. This implies that h must be a function of x alone, not y . Hence we see that $h(x)$ is a meromorphic function on $\mathbb{P}^1$, and is holomorphic everywhere on finite part of C , i.e., $h(x)$ is a holomorphic function on $\mathbb{C} \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. By standard complex analysis, an entire function with at most a pole at ∞ must be a polynomial.

So let d be the degree of h . We argue that $\deg h \leq g-1$. We note that if h is a polynomial of degree d , then h has $2d$ zeros on finite parts of C , so by residue theorem it must have a pole of order d at each of the points over ∞ .

Now we look at $\omega_0 = dx/y$. We claim that ω_0 has zeros of order $g-1$ at ∞ . Indeed, let $t = 1/x$ be local coordinate on $\mathbb{P}^1$ near ∞ . Then $x = 1/t$, so $dx = -\frac{1}{t^2}dt$. Next, we see how y behaves at ∞ : We have

$$y^2 = \prod_{i=1}^{2g+2} \left(\frac{1}{t} - z_i \right) = \frac{1}{t^{2g+2}} \prod_{i=1}^{2g+2} (1 - z_i t) \sim \frac{1}{t^{2g+2}}$$

as $t \rightarrow 0$. Hence $y \sim \frac{1}{t^{g+1}}$ as $t \rightarrow 0$. Thus

$$\omega_0 = \frac{dx}{y} \sim \frac{-\frac{1}{t^2}dt}{\frac{1}{t^{g+1}}} = -t^{g-1}dt.$$

Thus

$$\omega = h\omega_0 \sim \left(\frac{1}{t^d} \right) (t^{g-1}dt) = t^{(g-1)-d}dt.$$

Hence for ω to be holomorphic, we must have $d \leq g-1$, as claimed. This shows that any element in $H^0(C, K_C)$ is spanned by $\omega_0, x\omega_0, \dots, x^{g-1}\omega_0$, as desired. $\square$

7.6.3 $g = 3$: Generically Quartic

By Theorem 7.28, assuming that C is not hyperelliptic, then the canonical morphism $\phi : C \rightarrow \mathbb{P}^2$ embeds C as a smooth plane curve of degree 4 (Proposition 7.30). In fact, a generic genus 3 curve is not hyperelliptic, as hyperelliptic curves form a codimension 1 subspace in the moduli space M_3 of genus 3 curves.

7.6.4 $g = 4$: Intersection of Quadric and Cubic

Let C be a nonhyperelliptic curve of genus 4. The canonical map $\phi : C \rightarrow \mathbb{P}^3$ is an embedding with image being a degree 6 curve in $\mathbb{P}^3$, and we identify C with its image.

Theorem 7.49. *The canonical model of a nonhyperelliptic curve of genus 4 is a complete intersection of a quadric Q and a cubic S in $\mathbb{P}^3$. Conversely, any smooth curve that is the complete intersection of a quadric and cubic in $\mathbb{P}^3$ is the canonical model of a nonhyperelliptic curve of genus 4.*

Proof. We consider the restriction maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \rightarrow H^0(\mathcal{O}_C(m)) = H^0(mK_C).$$

Here, we used the fact that $K_C \cong \phi^*\mathcal{O}_{\mathbb{P}^3}(1)$, hence $K_C(m) = \phi^*\mathcal{O}_{\mathbb{P}^3}(m)$. Since we are identifying C with $\phi(C)$, we can identify $\phi^*\mathcal{O}_{\mathbb{P}^3} \cong \mathcal{O}_C$. In particular, let us consider the case where $m = 2$:

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow H^0(\mathcal{O}_C(2)).$$

We have $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$ and $h^0(\mathcal{O}_C(2)) = h^0(2K_C) = 9$ by Riemann-Roch. Hence the restriction map of quadrics in $\mathbb{P}^3$ to C has at least 1-dimensional kernel. That is, the curve $C \subset \mathbb{P}^3$ must lie on at least one quadric surface $Q \subset \mathbb{P}^3$. The quadric must be irreducible, since any reducible quadric must be a union of planes, which cannot contain the irreducible degenerate curve C . If $Q' \subset \mathbb{P}^3$ is another quadric then, by Bézout's theorem, $Q \cap Q'$ is a curve of degree 4 which cannot contain C . We conclude that Q is unique, and ρ_2 is thus surjective.

Now consider the restriction map

$$\rho_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_C(3)) = H^0(3K_C).$$

We have $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = \binom{6}{3} = 20$, and by Riemann-Roch $h^0(\mathcal{O}_C(3)) = h^0(3K_C) = 15$. This shows that $\dim \ker \rho_3 \geq 5$. The space of cubic surface containing the above quadric Q can be constructed by multiplying the equation of Q with a linear polynomial, hence is 4-dimensional. This shows that there exists a cubic S containing C that does not contain Q . By Bézout's theorem, $Q \cap S$ has degree 6, hence must be equal to C .

Conversely, let $C = Q \cap S$ be the intersection of a quadric Q and a cubic S . By adjunction formula

$$K_Q = K_{\mathbb{P}^3} + Q|_Q = \mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(2)|_Q = \mathcal{O}_Q(-2).$$

The curve C is the intersection of Q with the cubic surface S . Because S has degree 3, the divisor C on Q is cut out by a section of $\mathcal{O}_Q(3)$. We apply the adjunction formula a second time, now for $C \subset Q$:

$$K_C = K_Q + C|_Q = \mathcal{O}_Q(-2) \otimes \mathcal{O}_Q(3)|_C = \mathcal{O}_C(1).$$

This shows that the embedding $\phi : C \rightarrow \mathbb{P}^3$ satisfies $K_C = \phi^*\mathcal{O}_{\mathbb{P}^3}(1)$. Therefore C is a canonical curve. $\square$

Now that we have classified nonhyperelliptic curve of genus 4, an interesting question to ask is whether a nonhyperelliptic curve of genus 4 is **trigonal**, that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$? We introduce the following terminology:

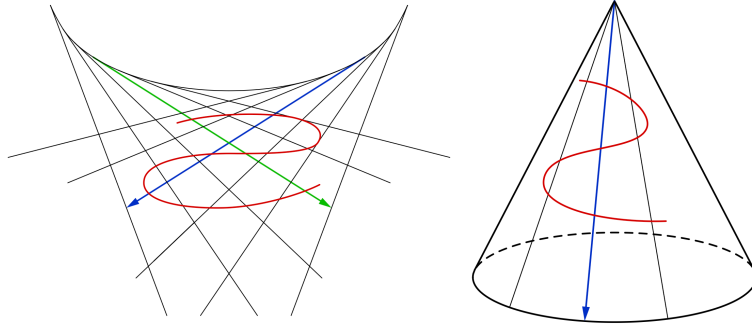


Figure 7.2: Left: canonical curve of genus 4 on a smooth quadric, showing two g_3^1 's. Right: canonical curve of genus 4 on a quadric cone, with just one g_3^1 .

Definition 7.50. Let C be a curve. A divisor D with $h^0(D) - 1 = r$ and $\deg D = d$ on C is said to be a g_d^r . Equivalently, this is a degree d map $C \rightarrow \mathbb{P}^r$. When such a divisor (map) exists, we say that C has a g_d^r .

Hence the question we are asking is: whether a nonhyperelliptic curve of genus 4 has a g_3^1 ? How many g_3^1 's does it have? This is answered by the following

Proposition 7.51 (See Fig 7.2). A nonhyperelliptic curve $C = Q \cap S$ of genus 4 may be expressed as a 3-sheeted cover of $\mathbb{P}^1$ in either one or two ways, depending on whether the unique quadric Q containing the canonical model of the curve is singular or smooth. If Q is smooth, then C has two g_3^1 's, otherwise it has a unique g_3^1 .

Proof. Finding g_3^1 's on C is equivalent to finding divisors D on C such that $r(D) = 1$ and $\deg D = 3$. Our first claim is that there is a correspondence between g_3^1 's on C with $\mathbb{P}^1$ -rulings on the quadric Q containing C .

First, suppose $\pi : C \rightarrow \mathbb{P}^1$ is a g_3^1 . Then we can take D to be a general fiber of the map π , which has the form $D = p + q + r$ for three distinct points. By the geometric Riemann-Roch theorem (Theorem 7.31), we have $1 = r(D) = 3 - 1 - \dim \text{span}\{p, q, r\}$. Hence $p, q, r \in C$ are collinear. If three points $p, q, r \in C \subset Q$ lies on a line $L \subset \mathbb{P}^3$, then Q and L meet in at least three points, hence restricting the degree 2 equation of Q on L shows that L must be contained in Q . Hence we have showed that the g_3^1 map is cut out by lines, and those lines must lie inside the quadric Q .

Conversely, if a line L is contained in Q , then the divisor $D = C \cap L = (Q \cap S \cap L) = (Q \cap L) \cap S = S \cap L$ on C has degree 3 by Bézout's theorem. Thus, if we can find a family lines completely contained in Q , it will sweep out a valid degree 3 divisor on C , thus giving us our g_3^1 . This proves our claim.

Now any smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and contains two families of lines, or rulings. On the other hand, any quadric of rank 3 is a cone over a smooth plane conic (which is isomorphic to $\mathbb{P}^1$ by degree genus formula), thus has only one ruling. Applying the claim immediately concludes the proof. $\square$

7.6.5 $g = 5$: Generically Triple Intersection of Quadrics

For nonhyperelliptic genus 5 curve, the canonical map $\phi : C \rightarrow \mathbb{P}^4$ has degree 8 image. Hence C can be identified with a octic curve in $\mathbb{P}^4$. As in the genus 4 case, it is useful to consider the restriction map

$$\rho_2 : H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \rightarrow H^0(\mathcal{O}_C(2)).$$

We have $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = \binom{6}{4} = 15$. By Riemann-Roch we have $h^0(\mathcal{O}_C(2)) = 12$. So $\dim \ker \rho_2 \geq 3$. We deduce that C lies on at least 3 independent quadrics. If the intersection of the three quadrics is 1-dimensional, then by Bézout's theorem we have

$$C = Q_1 \cap Q_2 \cap Q_3$$

is a complete intersection of quadrics. This is the generic case. The discussion for detailed classification of genus 5 curve is a bit long, and in the generic case similar argument as in the genus 4 case can be used. We will merely present the result:

Theorem 7.52. *Let $C \subset \mathbb{P}^4$ be a canonical curve of genus 5. Then C lies on exactly three quadrics. Either:*

- (a) *C is the intersection of these quadrics, in which case C is not trigonal (i.e., has no g_3^1), but has a one-dimensional family of g_4^1 's.*
- (b) *The quadrics containing C intersect in a cubic surface that is a nonsingular rational normal scroll. In this case, C has a unique g_3^1 .*

Chapter 8

Kodaira Vanishing Theorem

Introduction

This chapter is devoted to the proof of the Kodaira vanishing theorem and its applications. The statement of the theorem is that if L is a positive line bundle on a compact Kähler manifold, then $H^q(X, \Omega_X^p \otimes L) = 0$ for $p + q > n$, i.e., cohomology groups of L vanishes at high degree.

The Kodaira vanishing theorem provides powerful structural control over cohomological data. By asserting the vanishing of higher cohomology groups for sufficiently positive line bundles, it forces long exact sequences to degenerate, imposing geometric rigidity. We will see several applications of this principle in the second section of this chapter.

8.1 Proof of the Kodaira Vanishing Theorem

The goal of this section is to prove Kodaira vanishing theorem. This requires a bit of preparation. We first prove a result of vanishing of connections at a point, which will be useful in our later calculations.

Lemma 8.1. *Suppose $E \rightarrow X$ is a smooth Hermitian vector bundle and ∇ its metric connection on E . In a neighborhood of each $p \in M$, there is a smooth orthonormal frame (s_μ) of E such that $\nabla s_\mu = 0$ at p .*

Proof. Start with any orthonormal frame (t_μ) for E . We first note that ∇ is a metric connection (compatible with metric), hence the coefficient 1-forms must be skew-Hermitian: $\theta_\beta^\alpha = -\bar{\theta}_\alpha^\beta$. Indeed, for every complex vector field χ we have

$$\begin{aligned} 0 &= \chi(\delta_{\mu\nu}) = \chi\langle t_\mu, t_\nu \rangle = \langle \nabla_\chi t_\mu, t_\nu \rangle + \langle t_\mu, \nabla_\chi t_\nu \rangle \\ &= \langle \theta_\mu^\alpha(\chi) t_\alpha, t_\nu \rangle + \langle t_\mu, \theta_\nu^\beta(\bar{\chi}) t_\beta \rangle = \theta_\mu^\alpha(\chi) \delta_{\alpha\nu} + \overline{\theta_\nu^\beta(\bar{\chi})} \delta_{\mu\beta} = \theta_\mu^\nu(\chi) + \bar{\theta}_\nu^\mu(\chi). \end{aligned}$$

Now we choose any smooth coordinates x^α centered at $p \in M$ defined on some open set $U \subset M$ and write

$$\theta_\nu^\mu(p) = A_{\nu\alpha}^\mu dx^\alpha(p)$$

for some constants $A_{\nu\alpha}^\mu$. Let $F : U \rightarrow M(r, \mathbb{C})$ be the matrix-valued function

$$F_\nu^\mu(x) = A_{\nu\alpha}^\mu x^\alpha.$$

Since θ_ν^μ is skew-Hermitian, so is F_ν^μ . Define another matrix valued function

$$B(x) = \exp(-F(x)).$$

It satisfies $B_\nu^\mu(p) = \delta_\nu^\mu$ and

$$\partial_\alpha B_\nu^\mu(p) = -\partial_\alpha F_\nu^\mu(p) = -A_{\nu\alpha}^\mu(p).$$

Since the space of skew-Hermitian matrices is the Lie algebra of the unitary group, it follows that $B(x)$ is unitary for each x . Thus define the local frame

$$s_\mu(x) = B_\nu^\mu(x)t_\nu(x).$$

Then (s_μ) are orthonormal. Moreover, at p we have $s_\mu = t_\mu$, and

$$\nabla_{\partial_\alpha} s_\nu(p) = \partial_\alpha B_\nu^\mu(p)t_\mu + \delta_\nu^\mu A_{\mu\alpha}^\rho t_\rho = 0,$$

as desired. $\square$

Kodaira vanishing theorem is a theorem about positive line bundles. So we need to give definition here:

Definition 8.2. A holomorphic line bundle $L \rightarrow X$ is said to be **positive** if its first *real* Chern class $c_1(L)$ is represented by positive (1,1)-form, that is, a real (1,1)-form ω such that $\omega(\chi, J\chi) > 0$ for all nonzero χ .

Example 8.3. The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is positive, since $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is a positive multiple of ω_{FS} , which is positive (Example 5.17).

Now we will establish some properties of positive line bundles. We first need a lemma:

Lemma 8.4. *Let M be compact Kähler and $L \rightarrow M$ a holomorphic line bundle. If η is any closed (1,1)-form representing the real Chern class $c_1(L)$, there is a Hermitian fiber metric on L whose Chern curvature Θ satisfying $\eta = \frac{i}{2\pi}\Theta$.*

Proof. We choose an arbitrary Hermitian metric h_0 on L , and let Θ_0 be the curvature form of its Chern connection. Both η and $\frac{i}{2\pi}\Theta_0$ are real (1,1)-forms representing $c_1(L)$. Hence they differ by an exact form. By the global $\partial\bar{\partial}$ -lemma (Proposition 6.56), we can find a smooth real function u such that

$$i\partial\bar{\partial}\left(\frac{u}{2\pi}\right) = \frac{i}{2\pi}\Theta_0 - \eta.$$

Now we define a new fiber metric $h = e^u h_0$ and let Θ be the curvature form of its Chern connection. Then we see that in terms of any holomorphic local frame we can write

$$\Theta = \bar{\partial}\partial \log |s|_h^2 = \bar{\partial}\partial \log(e^u |s|_{h_0}^2) = \partial\bar{\partial}u + \Theta_0 = -2\pi i\eta.$$

This concludes the proof. $\square$

Now we prove two basic properties of positive line bundle:

Proposition 8.5. Let M be a compact complex manifold and $L \rightarrow M$ is a positive holomorphic line bundle. Then:

- (a) There exists Kähler metric on M and a Hermitian fiber metric on L such that $\frac{i}{2\pi}\Theta_L$ is equal to the Kähler form and Θ_L is the curvature of the Chern connection on L .
- (b) If M' is another complex manifold and $F : M' \rightarrow M$ is a holomorphic immersion, then $F^*L \rightarrow M'$ is also positive.

Proof. (a) Let ω be a closed, positive (1,1)-form representing $c_1(L)$. Then ω is a Kähler form and we can endow M with Kähler metric $g = \omega(-, J-)$. By Lemma 8.4, there is a hermitian metric on L whose Chern connection has a curvature form Θ_L such that $\frac{i}{2\pi}\Theta_L = \omega$.

(b) Give L the Hermitian fiber metric as in (a), and give F^*L the pullback metric. Then $\Theta_{F^*L} = F^*\Theta_L$. Because F is a holomorphic immersion, it follows that

$$\frac{i}{2\pi}\Theta_{F^*L}(\chi, J\chi) = \frac{i}{2\pi}F^*\Theta_L(\chi, J\chi) = \frac{i}{2\pi}\Theta_L(F_*\chi, JF_*\chi) > 0$$

for all nonzero complex vector fields χ , which shows that F^*L is positive. $\square$

Now we work towards proving the Kodaira vanishing theorem. We first need to derive Kähler identities for bundle-valued forms. The setting is the following: Let X be a complex manifold and $E \rightarrow X$ a Hermitian vector bundle. Let ∇ be a connection on E , and $D : \mathcal{A}^k(X; E) \rightarrow \mathcal{A}^{k+1}(X; E)$ be the exterior covariant derivative determined by ∇ (Theorem 4.5). On (p, q) -forms, we can decompose $D = D' + D''$, where

$$D' : \mathcal{A}^{p,q}(X; E) \rightarrow \mathcal{A}^{p+1,q}(X; E) \quad D'' : \mathcal{A}^{p,q}(X; E) \rightarrow \mathcal{A}^{p,q+1}(X; E).$$

We define two Laplacians $\Delta', \Delta'' : \mathcal{A}^{p,q}(X; E) \rightarrow \mathcal{A}^{p,q}(X; E)$:

$$\Delta'\alpha = D'D'^*\alpha + D'^*D'\alpha \quad \Delta''\alpha = D''D''^*\alpha + D''^*D''\alpha.$$

Theorem 8.6 (Kähler Identities). *Let X be a Kähler manifold with Kähler form ω , let $E \rightarrow X$ a Hermitian vector bundle, ∇ a metric connection on E . Let D, D'' be defined as above, and $L_\omega : \mathcal{A}^{p,q}(X; E) \rightarrow \mathcal{A}^{p+1,q+1}(X; E)$ be the Lefschetz operator. Then the following identities hold:*

$$[D''^*, L_\omega] = iD' \quad [D'^*, L_\omega] = -iD'' \quad [L_\omega^*, D''] = -iD'^* \quad [L_\omega^*, D'] = iD''^*.$$

Proof. We will only prove the first identity. The proof of the second identity is exactly the same, with single-primed and double-primed expressions interchanged. The last two identity follows from the first two by taking adjoints.

We now prove the first identity. Suppose (s_j) is a local frame for E on an open set $U \subseteq M$, and $\alpha = \alpha^j \otimes s_j$ is an E -valued $(p, q-1)$ -form. If θ_j^k denotes the matrix of connection 1-forms with respect to this frame, then Theorem 4.5 gives the following local expression for $D\alpha$:

$$D\alpha = d\alpha^j \otimes s_j + (-1)^{p+q-1}(\alpha^j \wedge \theta_j^k) \otimes s_k,$$

and therefore

$$D'\alpha = \partial\alpha^j \otimes s_j + (-1)^{p+q-1}(\alpha^j \wedge (\theta')_j^k) \otimes s_k,$$

$$D''\alpha = \bar{\partial}\alpha^j \otimes s_j + (-1)^{p+q-1}(\alpha^j \wedge (\theta'')_j^k) \otimes s_k,$$

where θ' and θ'' represent the $(1,0)$ - and $(0,1)$ -parts of θ , respectively. Let us write the second terms on the right-hand sides as $\theta'\alpha$ and $\theta''\alpha$, so

$$D'\alpha = \partial\alpha^j \otimes s_j + \theta'\alpha,$$

$$D''\alpha = \bar{\partial}\alpha^j \otimes s_j + \theta''\alpha.$$

Now suppose α is as above and $\beta = \beta^k \otimes s_k$ is an E -valued (p,q) -form, and both are compactly supported in U . Then by the way we have defined the inner product of E -valued forms and the fact that the frame (s_j) is orthonormal, we have

$$\begin{aligned} (D''\alpha, \beta) &= \int_X \langle \bar{\partial}\alpha^j \otimes s_j, \beta^k \otimes s_k \rangle dV_g + (\theta''\alpha, \beta) \\ &= \int_X \sum_j \langle \bar{\partial}\alpha^j, \beta^j \rangle dV_g + (\theta''\alpha, \beta) \\ &= \int_X \sum_j \langle \alpha^j, \bar{\partial}^* \beta^j \rangle dV_g + (\alpha, \theta''^* \beta), \end{aligned}$$

which shows that

$$D''^* \beta = \bar{\partial}^* \beta^k \otimes s_k + \theta''^* \beta$$

for such forms. Since every form agrees on a neighborhood of each point with a form compactly supported in U , this expression holds everywhere in U .

Since $L_\omega \beta = (L_\omega \beta^k) \otimes s_k$, we have

$$D''^* L_\omega \beta = D''^* ((L_\omega \beta^k) \otimes s_k) = (\bar{\partial}^* L_\omega \beta^k) \otimes s_k + \theta''^* L_\omega \beta,$$

and

$$L_\omega D''^* \beta = L_\omega (\bar{\partial}^* \beta^k) \otimes s_k + L_\omega \theta''^* \beta.$$

Therefore, using the corresponding Kähler identity for scalar-valued forms (Proposition 6.34), we find

$$\begin{aligned} [D''^*, L_\omega] \beta &= ([\bar{\partial}^*, L_\omega] \beta^k) \otimes s_k + [\theta''^*, L_\omega] \beta \\ &= (i\partial \beta^k) \otimes s_k + [\theta''^*, L_\omega] \beta \\ &= iD'\beta - i\theta'\beta + [\theta''^*, L_\omega] \beta. \end{aligned}$$

This formula holds with respect to every local orthonormal frame. On the other hand, since both sides the first Kähler identity are defined independently of any choice of frame, for each $p \in M$ we can use any convenient frame to verify the identity at p . Lemma 8.1 shows that we can choose an orthonormal frame with the property that $\theta_j^k = 0$ at p . Because θ' and θ'' do not involve differentiation, it follows that θ' and θ'' both vanish at p , and then (10.4) shows that the first identity holds at p . This proves the first identity. $\square$

Now we prove the final technical lemma which is required in the proof of the Kodaira vanishing theorem.

Lemma 8.7. *Let $E \rightarrow X$ be a Hermitian vector bundle over a compact Kähler manifold. Let ∇ be the Chern connection on E , and $F_\nabla : \mathcal{A}^{p,q}(X; E) \rightarrow \mathcal{A}^{p+1,q+1}(X; E)$ be the curvature operator $F_\nabla \alpha = \Theta \wedge \alpha$ where Θ is the curvature of ∇ . Then for any arbitrary harmonic form $\alpha \in \mathcal{H}^{p,q}(X; E)$, we have*

$$\frac{i}{2\pi} \langle [L_\omega^*, F_\nabla] \alpha, \alpha \rangle \geq 0.$$

Proof. Let $\nabla = D' + D''$, where $D' = \nabla^{1,0}$ and $D'' = \nabla^{0,1}$. Since the Chern connection is Hermitian with $D'' = \bar{\partial}_E$, one has

$$F_\nabla = (D' + \bar{\partial}_E)(D' + \bar{\partial}_E) = D'\bar{\partial}_E + \bar{\partial}_E D'$$

since $\Theta \in \mathcal{A}^{1,1}$. The assumption that α is Harmonic means that $\bar{\partial}_E \alpha = \bar{\partial}_E^* \alpha = 0$. Now using harmonicity and Theorem 8.6, one computes

$$\begin{aligned} \langle iF_\nabla L_\omega^* \alpha, \alpha \rangle &= i\langle D'\bar{\partial}_E L_\omega^* \alpha, \alpha \rangle + i\langle \bar{\partial}_E D' L_\omega^* \alpha, \alpha \rangle \\ &= i\langle \bar{\partial}_E L_\omega^* \alpha, D'^* \alpha \rangle + i\langle D' L_\omega^*, \bar{\partial}_E^* \alpha \rangle \\ &= \langle \bar{\partial}_E L_\omega^* \alpha, -iD'^* \alpha \rangle + 0 \\ &= \langle \bar{\partial}_E L_\omega^* \alpha, [L_\omega^*, \bar{\partial}_E] \alpha \rangle \\ &= -\langle \bar{\partial}_E L_\omega^* \alpha, \bar{\partial}_E L_\omega^* \alpha \rangle = -\|\bar{\partial}_E L_\omega^* \alpha\|^2 \leq 0. \end{aligned}$$

Similarly, one has

$$\begin{aligned} \langle iL_\omega^* F_\nabla \alpha, \alpha \rangle &= i\langle L_\omega^* \bar{\partial}_E D' \alpha, \alpha \rangle \\ &= i\langle [L_\omega^*, \bar{\partial}_E] D' \alpha, \alpha \rangle + i\langle \bar{\partial}_E L_\omega^* D' \alpha, \alpha \rangle \\ &= i\langle -iD'^* D' \alpha, \alpha \rangle + i\langle L_\omega^* D \alpha, \bar{\partial}_E^* \alpha \rangle = \|D' \alpha\|^2 + 0 \geq 0. \end{aligned}$$

Subtracting the first expression by the second gives the desired result. $\square$

Finally, we come to the main theorem of this section:

Theorem 8.8 (Kodaira Vanishing Theorem). *Let $L \rightarrow X$ be a positive line bundle on a compact Kähler manifold. Then $H^q(X, \Omega_X^p \otimes L) = 0$ for $p + q > n$.*

Proof. Choose a hermitian structure on the positive line bundle L such that the curvature of the Chern connection is positive, i.e., $\frac{i}{2\pi} \Theta_L = c_1(L) = \omega$ is the Kähler form on X (Proposition 8.5). With respect to this Kähler structure, we have $\frac{i}{2\pi} F_\nabla = L_\omega$. Using Lemma 8.7 and Lemma 6.41, we obtain

$$0 \leq \frac{i}{2\pi} \langle [L_\omega^*, F_\nabla] \alpha, \alpha \rangle = \langle [L_\omega^*, L_\omega] \alpha, \alpha \rangle = (n - p - q) \|\alpha\|^2.$$

The inequality above holds for any $\alpha \in \mathcal{H}^{p,q}(X, L) = H^q(X, \Omega_X^p \otimes L)$. Note that when $p + q > n$, we must have $\|\alpha\|^2 = 0$. This completes the proof. $\square$

8.2 Applications of Kodaira Vanishing Theorem

In this section, we study three applications of Kodaira vanishing theorem: the Weak Lefschetz theorem, Serre vanishing theorem, and Grothendieck's vector bundle classification on $\mathbb{P}^1$. The core theme of this section is that positivity controls geometry via vanishing.

To illustrate the power of Kodaira vanishing theorem, we first show that we can compute most cohomology of $\mathcal{O}(m)$ on $\mathbb{P}^n$.

Theorem 8.9. *Let n be given and the cohomology of $\mathcal{O}(m) \rightarrow \mathbb{P}^n$ is given by*

$$H^q(\mathbb{P}^n, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, \dots, z_n]_m & q = 0, m \geq 0 \\ H^0(\mathbb{P}^n, \mathcal{O}(-n - m - 1))^* & q = n, m \leq -n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. There are several cases we already know: When $q = 0$, we know $H^0(\mathbb{P}^n, \mathcal{O}(m)) = \mathbb{C}[z_0, \dots, z_n]_m$ when $m \geq 0$ and is zero when $m < 0$ by Corollary 3.32. By Serre duality $H^q(X, E) \cong H^{n-q}(X, K_X \otimes E^*)^*$, we know that $H^n(\mathbb{P}^n, \mathcal{O}(m)) \cong H^0(\mathbb{P}^n, \mathcal{O}(-n-m-1))^*$.

Now onto the cases where $q > 0$. By Kodaira vanishing theorem, since $\mathcal{O}(m)$ is positive for $m > 0$, we have that $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for all $q > 0, m > 0$. When $m < -n-1$, we have $-m-n-1 > 0$, so $H^q(\mathbb{P}^n, \mathcal{O}(m)) \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}(-m-n-1))^* = 0$ by Kodaira. Finally, for the remaining range $-n-1 \leq m \leq 0$, we claim that $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for any n given, any $-n-1 \leq m \leq 0$, and for any q . To see this, we induct on n . The base case $n = 0$ is trivial as $\mathbb{P}^0$ is just a point. Inductively, we use the hyperplane restriction sequence on $\mathbb{P}^n$:

$$0 \rightarrow \mathcal{O}(m-1) \rightarrow \mathcal{O}(m) \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(m) \rightarrow 0.$$

The long exact sequence we get reads

$$\dots \rightarrow H^{q-1}(\mathbb{P}^{n-1}, \mathcal{O}(m)) \rightarrow H^q(\mathbb{P}^n, \mathcal{O}(m-1)) \rightarrow H^q(\mathbb{P}^n, \mathcal{O}(m)) \rightarrow H^q(\mathbb{P}^{n-1}, \mathcal{O}(m)) \rightarrow \dots$$

For $0 < q < n-1$, we have by inductive hypothesis and exactness that

$$H^q(\mathbb{P}^n, \mathcal{O}(m-1)) \cong H^q(\mathbb{P}^n, \mathcal{O}(m-1)).$$

Therefore we can keep subtracting from m until we get into the range where Kodaira vanishing theorem applies, which would give $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for $0 < q < n-1$ and for all m . The last remaining case is $H^{n-1}(\mathbb{P}^n, \mathcal{O}(m))$, which by Serre duality is isomorphic to $H^1(\mathbb{P}^n, \mathcal{O}(-n-m-1))^* = 0$ by our previous result. Therefore, we have shown that

$$H^q(\mathbb{P}^n, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, \dots, z_n]_m & q = 0, m \geq 0 \\ H^0(\mathbb{P}^n, \mathcal{O}(-n-m-1))^* & q = n \\ 0 & 0 < q < n. \end{cases}$$

The last case that $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for $q > n$ is because by Dolbeault theorem we have $H^q(\mathbb{P}^n, \mathcal{O}(m)) = H^{0,q}(\mathbb{P}^n; \mathcal{O}(m))$, and there are no $(0, q)$ -forms on $\mathbb{P}^n$ for $q > n$. $\square$

Our next application is

Theorem 8.10 (Weak Lefschetz Theorem). *Let X be a compact Kähler manifold of dimension n , and $Y \subset X$ a hypersurface such that $\mathcal{O}_X(Y)$ is positive. Then the restriction map*

$$H^k(X; \mathbb{C}) \rightarrow H^k(Y; \mathbb{C})$$

is bijective for $k \leq n-2$ and injective for $k \leq n-1$.

Proof. Since both $H^k(X)$ and $H^k(Y)$ admits decomposition $H^k = \bigoplus_{k=p+q} H^{p,q}$, and restriction respects Hodge structure, it suffices to show that the restriction map

$$H^q(X, \Omega_X^p) \rightarrow H^q(Y, \Omega_Y^p)$$

is bijective for $p+q \leq n-2$ and injective for $p+q \leq n-1$.

To prove this, we use two exact sequences. The first one is the hyperplane sequence

$$0 \rightarrow \mathcal{O}_X(-Y) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0. \quad (*)$$

The second one is the dual to the sequence $0 \rightarrow T_Y \rightarrow T_X|_Y \rightarrow N_{Y/X} \rightarrow 0$, and using $N_{Y/X}^* = \mathcal{O}_X(-Y)$ by Remark 3.36, we have

$$0 \rightarrow \mathcal{O}_Y(-Y) \rightarrow \Omega_X|_Y \rightarrow \Omega_Y \rightarrow 0. \quad (**)$$

Now we tensor (*) by Ω_X^p and get

$$0 \rightarrow \Omega_X^p(-Y) \rightarrow \Omega_X^p \rightarrow \Omega_X^p|_Y \rightarrow 0. \quad (**)$$

We take the p -th exterior power of (**) and get

$$0 \rightarrow \Omega_Y^{p-1}(-Y) \rightarrow \Omega_X^p|_Y \rightarrow \Omega_Y^p \rightarrow 0. \quad (**')$$

To see (**'), we can choose coordinates $x_1, \dots, x_n$ on X such that Y is locally given by $x_n = 0$. Then the kernel of the restriction map $\Omega_X^p|_Y \rightarrow \Omega_Y^p$ must be given by $dx_n \wedge \alpha$ where $\alpha \in \Omega_Y^{p-1}$ that does not contain dx_n . Hence the kernel is isomorphic to $\Omega_Y^{p-1}(-Y)$.

Now, using Serre duality and Kodaira vanishing theorem, we have

$$H^q(\Omega_X^p(-Y)) = H^{n-q}(\Omega_X^{p*} \otimes \mathcal{O}(Y) \otimes K_X)^* = H^{n-q}(\Omega_X^{n-p} \otimes \mathcal{O}(Y))^* = 0$$

for $p+q < n$. Now the LES of (*) reads

$$\dots \rightarrow H^q(\Omega_X^p(-Y)) \rightarrow H^q(\Omega_X^p) \xrightarrow{\alpha} H^q(\Omega_X^p|_Y) \rightarrow H^{q+1}(\Omega_X^p(-Y)) \rightarrow \dots$$

So we see that α is bijective for $p+q+1 < n$ ($p+q \leq n-2$) and injective for $p+q < n$ ($p+q \leq n-1$) by exactness. But to prove our claim, we must compose α with the map β :

$$H^q(X, \Omega_X^p) \xrightarrow{\alpha} H^q(Y, \Omega_X^p|_Y) \xrightarrow{\beta} H^q(Y, \Omega_Y^p).$$

Now we look at the LES of (**):

$$\dots \rightarrow H^q(\Omega_Y^{p-1}(-Y)) \rightarrow H^q(\Omega_X^p|_Y) \xrightarrow{\beta} H^q(\Omega_Y^p) \xrightarrow{\gamma} H^{q+1}(\Omega_Y^{p-1}(-Y)) \rightarrow \dots$$

Now we apply the same argument to $\Omega_Y^{p-1}(-Y)$. We see that

$$H^{q+1}(\Omega_Y^{p-1}(-Y)) \cong H^{n-q-1}(\Omega_Y^{p-1*} \otimes \mathcal{O}(Y) \otimes K_Y)^* = H^{n-q-1}(\Omega_Y^{n-p} \otimes \mathcal{O}(Y))^* = 0$$

if $n-q-1+n-p > n$ by Kodaira vanishing theorem since restricting $\mathcal{O}(Y)$ to Y . That is, $p+q \leq n-2$. Hence we see that $\beta : H^q(Y, \Omega_X^p|_Y) \rightarrow H^q(Y, \Omega_Y^p)$ is bijective for $p+q \leq n-2$ and injective for $p+q \leq n-1$, concluding the proof. $\square$

The proof of next theorem is a slight modification of Kodaira's vanishing theorem:

Theorem 8.11 (Serre Vanishing Theorem). *Let $L \rightarrow X$ be a positive line bundle on a compact Kähler manifold. For any holomorphic vector bundle $E \rightarrow X$ there exists a constant m_0 such that for all $m > m_0$ and $q > 0$ one has*

$$H^q(X, E \otimes L^m) = 0.$$

Proof. Choose hermitian structures on E and L and denote their Chern connections by ∇_E, ∇_L . By Proposition 8.5 we may assume that $\frac{i}{2\pi}F_{\nabla_L}$ is the Kähler form ω on X . Let ∇_{L^m} be the connection on L^m induced by ∇_L , and define the connection $\nabla = \nabla_E \otimes 1 + 1 \otimes \nabla_{L^m}$ on $E \otimes L^m$. By Lemma 8.7 we have

$$\frac{i}{2\pi} \langle [L_\omega^*, F_\nabla] \alpha, \alpha \rangle \geq 0$$

for any $\alpha \in \mathcal{H}^{p,q}(X, E \otimes L^m)$. In particular, $\frac{i}{2\pi}F_{\nabla_{L^m}} = m\omega$. Hence

$$\frac{i}{2\pi}F_\nabla = \frac{i}{2\pi}F_{\nabla_E} \otimes 1 + m(1 \otimes \omega).$$

Therefore,

$$\begin{aligned}
\frac{i}{2\pi} \langle [L_\omega^*, F_\nabla] \alpha, \alpha \rangle &= \frac{i}{2\pi} \langle [L_\omega^*, F_{\nabla_E}] \alpha, \alpha \rangle + m \langle [L_\omega^*, L_\omega] \alpha, \alpha \rangle \\
&= \frac{i}{2\pi} \langle [L_\omega^*, F_{\nabla_E}] \alpha, \alpha \rangle + m(n - p - q) \|\alpha\|^2 \\
&\leq \left(\frac{i}{2\pi} \| [L_\omega^*, F_{\nabla_E}] \| + m(n - p - q) \right) \|\alpha\|^2
\end{aligned}$$

by Lemma 6.41 and the fiberwise Cauchy-Schwarz inequality. Let $C = \frac{i}{2\pi} \| [L_\omega^*, F_{\nabla_E}] \|$. This does not depend on m . Hence if $C + m(n - p - q) < 0$ one necessarily have $\alpha = 0$.

Now if $p = n$, then we can choose $m_0 \geq 0$ such that for all $m \geq m_0$ we have $C - mq < 0$. Thus $H^q(X, E \otimes K_X \otimes L^m) = 0$ for all $q > 0$ and $m > m_0$. To finish the proof, we replace E by $E \otimes K_X^*$. This might change the constant m_0 (since it depends on the vector bundle E), but the desired result follows. $\square$

Using Serre vanishing theorem, Grothendieck showed that there are no interesting vector bundles other than line bundles on $\mathbb{P}^1$. We will prove Grothendieck's lemma. But first, we need a version of Riemann-Roch theorem for vector bundles (which generalizes 7.7). The proof is based on Hirzebruch-Riemann-Roch theorem, which is beyond this book, so we will not prove it.

Lemma 8.12 (Riemann-Roch for Vector Bundles). *Let $E \rightarrow C$ be a holomorphic vector bundle of rank r over a compact curve C of genus g . Then*

$$\chi(C, E) = \deg E + r(1 - g).$$

Now we conclude this section with a proof of

Theorem 8.13 (Grothendieck's Lemma). *Every holomorphic vector bundle $E \rightarrow \mathbb{P}^1$ is a direct sum of line bundles: $E \cong \bigoplus_i \mathcal{O}(a_i)$. The ordered sequence $a_1 \geq a_2 \geq \dots \geq a_r$ is uniquely determined.*

Proof. We induct on the rank r of E . When $r = 1$, we know $\text{Pic}(\mathbb{P}^1) = \mathbb{Z}$, and we are done. Now let E be a vector bundle of rank r . Let

$$a_1 := \max\{a : \text{Hom}(\mathcal{O}(a), E) = H^0(\mathbb{P}^1, E(-a)) \neq 0\}.$$

First, there exists a such that $H^0(\mathbb{P}^1, E(-a)) \neq 0$. To see this, note that Serre vanishing (Theorem 8.11) shows that $H^1(\mathbb{P}^1, E(-a)) = 0$ for $a \ll 0$. Thus by Lemma 8.12, we have

$$h^0(\mathbb{P}^1, E(-a)) = \chi(\mathbb{P}^1, E(-a)) = \deg E - (1 - a)r > 0$$

for $a \ll 0$. We also note that a maximal a exists because by Serre vanishing, for $a \gg 0$ we have $H^0(\mathbb{P}^1, E(-a)) = H^1(\mathbb{P}^1, E^*(a - 2))^* = 0$.

Thus, we have a short exact sequence

$$0 \rightarrow \mathcal{O}(a_1) \rightarrow E \rightarrow E_1 \rightarrow 0. \quad (!)$$

We note that the map $\mathcal{O}(a_1) \rightarrow E$ in (!) is injective by the fiberwise rank-nullity theorem, since the map is assumed to be nonzero. Now E_1 is a vector bundle of lower rank, so by inductive hypothesis we may assume $E_1 = \bigoplus_{i>1} \mathcal{O}(a_i)$. Hence it remains to show that (1) splits.

We recall some homological algebra result: Over X the obstruction to splitting of the short exact sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$ of vector bundles is parameterized by the abelian group $\text{Ext}^1(\mathcal{G}, \mathcal{F}) = H^1(X, \text{Hom}(\mathcal{G}, \mathcal{F})) = H^1(X, \mathcal{F} \otimes \mathcal{G}^*)$. Hence, in our case the obstruction to splitting of (!) is parameterized by the group

$$H^1(\mathbb{P}^1, \bigoplus_{i>1} \mathcal{O}(a_1 - a_i)) = \bigoplus_{i>1} H^1(\mathbb{P}^1, \mathcal{O}(a_1 - a_i)).$$

Using Theorem 8.9, we see that the vanishing of the group above requires $a_1 - a_i > -2$ for all i . We will show that in fact a stronger result holds: $a_i \leq a_1$ for all i .

To this end, suppose $a_i > a_1$ for at least one i . Then we have $E_1(-a_1 - 1) = \bigoplus_{i>1} \mathcal{O}(a_i - a_1 - 1)$, and since $a_i - a_1 - 1 \geq 0$ for at least one i , we have

$$H^0(\mathbb{P}^1, E_1(-a_1 - 1)) = \bigoplus_{i>1} H^0(\mathbb{P}^1, \mathcal{O}(a_i - a_1 - 1)) \neq 0.$$

Now we twist (!) by $\mathcal{O}(-a_1 - 1)$ to get a SES

$$0 \rightarrow \mathcal{O}(-1) \rightarrow E(-a_1 - 1) \rightarrow E_1(-a_1 - 1) \rightarrow 0. \quad (!!)$$

We look at the LES of (!!):

$$0 \rightarrow H^0(\mathcal{O}(-1)) \rightarrow H^0(E(-a_1 - 1)) \xrightarrow{\alpha} H^0(E_1(-a_1 - 1)) \xrightarrow{\beta} H^1(\mathcal{O}(-1)) \rightarrow \dots$$

Since $H^1(\mathcal{O}(-1)) = 0$ by Theorem 8.9, we see that $\ker \beta = H^0(E_1(-a_1 - 1)) \neq 0$ by our computation above. Now if $H^0(E(-a_1 - 1)) = 0$, then

$$\text{im } \alpha = \ker \beta = H^0(E_1(-a_1 - 1)) = 0,$$

which is a contradiction. Hence we conclude that $H^0(E_1(-a_1 - 1)) = 0$. But this contradicts maximality of a_1 . We have arrived at the desired contradiction, and the proof is now complete. $\square$

Index

- E -valued Dolbeault Laplacian, 88
- $\bar{\partial}$ -harmonic forms, 87
- $\bar{\partial}_E$ -harmonic, 88
- k -slice, 14

- Abel-Jacobi map, 118
- adjoint, 82
- affine algebraic variety, 14
- almost complex manifold, 19
- almost complex structure, 19
- almost complex vector space, 19
- antiholomorphic tangent bundle, 21

- base locus, 42, 110
- base point, 42
- biholomorphic, 11
- blow up, 44
- blow-down, 44
- bundle-valued forms, 27

- canonical bundle, 31
- canonical curve, 113
- characteristic class, 47, 54
- characteristic form, 54
- Chern class, 52
- Chern connection, 56
- Chern map, 30
- compact vertical support, 60
- compatible, 56
- complete linear system, 110
- complex Grassmannian, 12
- complex hypersurface, 13
- complex manifold, 11
- complex projective space, 12
- complex structure, 11
- complex submanifold, 13
- complex torus, 13
- complex vector field, 16
- complexification, 16
- connection, 47
- connection form, 48
- contains effective divisors, 111
- covariant derivative, 48
- curvature, 49
- curvature form, 49

- degree, 30, 35, 66
- determinant bundle, 31
- dimension, 11, 110
- direct sum, 31
- discrete Lie group, 13
- divisor, 35
- divisor associated to f , 36
- divisor class group, 37
- divisor group, 35
- Dolbeault cohomology group, 25
- Dolbeault cohomology group of vector bundle, 28
- Dolbeault Laplacian, 87
- Dolbeault operator, 24
- dominant, 37
- dual bundle, 31

- effective, 35
- elementary symmetric polynomial, 52
- equivalent, 10
- Euler class, 61
- Euler sequence, 40
- exceptional divisor, 45
- exponential sequence, 30

- first Chern class, 30, 65
- free, 13
- Fubini-Study metric, 74
- function field, 11
- fundamental 2-form, 71

- global angular form, 62

- harmonic form, 84
- Hermitian fiber metric, 56
- Hermitian fiber metrics, 68

Hermitian metric, 56
 Hermitian vector bundle, 56
 Hodge diamond, 98
 Hodge inner product, 81
 Hodge Laplacian, 84
 Hodge numbers, 98
 Hodge star operator, 83
 holomorphic, 5, 11, 13
 holomorphic p -forms, 23
 holomorphic atlas, 10
 holomorphic Jacobian, 18
 holomorphic tangent bundle, 21
 holomorphic vector bundle, 29
 hom bundle, 31
 homogeneous, 15
 Hopf manifold, 13
 hypersurface, 35

 ideal sheaf, 103
 integrable, 22
 integral, 95
 integration along fiber, 61
 inverse image divisor, 106
 isomorphic, 11

 Jacobian, 117, 118

 Kähler, 71
 Kähler normal coordinate, 72

 Lefschetz operator, 90
 linear system, 110

 meromorphic 1-form, 108
 meromorphic functions, 11
 multiplicity, 106
 musical isomorphism, 80

 nonsingular, 15

 norm, 81

 order, 35

 Picard group, 30
 plane curve, 102
 positive, 127
 primitive, 97
 principal, 36, 119
 projective algebraic variety, 15
 projective manifold, 15
 projectivization, 12
 proper, 13
 pullback bundle, 31
 pullback connection, 51
 pullback divisor, 37, 106

 ramification divisor, 106
 residue, 108
 Riemann surface, 23

 Segre map, 42
 sheaf theoretic Euler characteristic, 103
 smooth, 15
 sum, 111
 supercommutator, 53

 tautological bundle, 38
 tensor product, 31
 Thom class, 61
 Thom isomorphism theorem, 61
 total Chern class, 55
 transition functions, 29
 trigonal, 123
 type (p, q) , 95

 veronese map, 42

 wedge bundle, 31

Bibliography

- [1] John Lee, *Introduction to Complex Manifolds*, American Mathematical Society, 2024.
- [2] Daniel Huybrechts, *Complex Geometry: An Introduction*, Springer, 2005.
- [3] Phillip Griffiths, Joseph Harris, *Principles of Algebraic Geometry*, Wiley, 1978.
- [4] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry (I)*, Cambridge University Press, 2002.
- [5] Otto Forster, *Lectures on Riemann Surfaces*, Springer, 1981.
- [6] Friedrich Hartogs, *Einige Folgerungen aus der Cauchyschen Integralformel bei Funktionen mehrerer Veränderlichen*, Sitzungsberichte der Königlich Bayerischen Akademie der Wissenschaften zu München, Mathematisch-Physikalische Klasse **36** (1906), 223-242.
- [7] Tianyi Wang, Lecture Notes on Smooth Manifolds. ([Link](#))
- [8] Raoul Bott, Loring W. Tu, *Differential Forms in Algebraic Topology*, Springer, GTM 82.
- [9] Heisuke Hironaka, *On the Theory of Birational Blowing-up*, Thesis, Harvard (1960).
- [10] David Eisenbud, Joseph Harris, *The Practice of Algebraic Curves*, AMS 2024.