

lec 2.

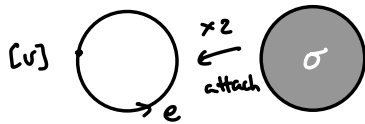
(3.2) UCT: let C be chain complex of free abelian group, then $\exists$ SES

$$0 \rightarrow \text{Ext}(H_{n-1}(C); G) \rightarrow H^n(C, G) \xrightarrow{h} \text{Hom}(H_n(C), G) \rightarrow 0.$$

h is the evaluation map (surjection). The sequence split. Thus

$$H^n(C, G) \cong \text{Hom}(H_n(C), G) \oplus \text{Ext}(H_{n-1}(C); G).$$

e.g. $\mathbb{R}P^2$.



$$\begin{array}{ccccc} C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 \\ \parallel & \times 2 & \parallel & 0 & \parallel \\ \mathbb{Z}[\sigma] & & \mathbb{Z}[e] & & \mathbb{Z}[\omega] \end{array}$$

$$H_2(\mathbb{R}P^2) = 0 \quad H_1(\mathbb{R}P^2) = \mathbb{Z}_2 \quad H_0(\mathbb{R}P^2) = \mathbb{Z}.$$

Cohomology in $\mathbb{Z}$ -coefficient: Dualize gives

$$\begin{array}{ccccc} \mathbb{Z} & \xleftarrow{\times 2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} \\ H^0(\mathbb{R}P^2) = 0 & & H^1(\mathbb{R}P^1) = 0 & & H^2(\mathbb{R}P^2) = \mathbb{Z}_2. \end{array}$$

Rmk: If G is a field, then $\text{Ext}(-, G) = 0$. Thus $H^n(-, F) \cong (H_n(-, F))^*$.

Def. Choose a **free resolution** of H (an exact seq of abelian groups):

$$\cdots \rightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

let $U^* = \text{Hom}(U, G)$ the dual group and consider

$$\cdots \leftarrow F_3^* \xleftarrow{f_3^*} F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \leftarrow 0$$

Then

$$\text{Ext}(H, G) \triangleq \text{Ker } f_2^* / \text{Im } f_1^* = H^1(F^*, G).$$

e.g. let $G = \mathbb{Z}$. $H = \mathbb{Z}_2$. A free resolution of $\mathbb{Z}_2$ is

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$$

$$\begin{array}{ccccccc} \text{(Dual)} & & \cdots & 0 & \leftarrow & \mathbb{Z}^* & \xleftarrow{\times 2} \mathbb{Z}^* \xleftarrow{f_1^*} \mathbb{Z}_2^* \xleftarrow{f_0^*} 0 \\ & & & & & \parallel & \\ & & & & & 0 & \end{array}$$

$$\begin{aligned}\text{Ext}(\mathbb{Z}_2, \mathbb{Z}) &= \text{Ker } f_2^* / \text{Im } f_1^* \\ &= \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.\end{aligned}$$

Must show Ext is independent of choices of free resolution.
(Hatcher Lemma 3.1).

pf of UCT (Assuming H Lemma 3.1):

Let (C, ∂) be a chain complex. Let $Z_n = \text{Ker } \partial_n$, $B_n = \text{Im } \partial_{n+1}$.

Then $\exists$ SES

$$0 \rightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \rightarrow 0.$$

B_{n-1} is free (subgp of C_{n-1}). Thus SES splits. Thus $\exists \sigma: B_{n-1} \rightarrow C_n$ s.t. $\sigma \partial = \text{id}$. So

$$C_n \cong Z_n \oplus \sigma(B_{n-1}) \cong Z_n \oplus B_{n-1}. \quad (*)$$

Prove evaluation h is onto. Given $[\phi] \in \text{Hom}(H_n(C), G)$, where $\phi: Z_n \rightarrow G$ and $\phi(B_n) = 0$. $(*) \Rightarrow$ can extend $\bar{\phi}: C_n \rightarrow G$ s.t. $\bar{\phi}|_{Z_n} = \phi$ by setting $\bar{\phi}(\sigma(B_{n-1})) = 0$. Then $\bar{\phi} \in C_n^*$. and $\delta \bar{\phi} = \bar{\phi} \circ \partial = 0$.

So $[\bar{\phi}] \in H^n(C; G)$. Thus h is onto.

Define $\tau[\phi] = [\bar{\phi}] \in H^n(C; G)$. Then

$$0 \rightarrow \text{Ker } h \rightarrow H^n(C; G) \xrightarrow[\tau]{h} \text{Hom}(H_n(C), G) \rightarrow 0$$

is a split SES. WTS $\text{Ker } h = \text{Ext}(H, G)$.

$$\begin{array}{ccccccc} 0 & \rightarrow & Z_{n+1} & \rightarrow & C_{n+1} & \xrightarrow{\partial} & B_n \rightarrow 0 \\ & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ 0 & \rightarrow & Z_n & \rightarrow & C_n & \xrightarrow{\partial} & B_{n-1} \rightarrow 0 \end{array}$$

is a commutative diagram with exact rows. SES splits.

Dualize:

$$\begin{array}{ccccccc} 0 & \leftarrow & Z_{n+1}^* & \leftarrow & C_{n+1}^* & \xleftarrow{\delta} & B_n^* \leftarrow 0 \\ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta \\ 0 & \leftarrow & Z_n^* & \leftarrow & C_n^* & \xleftarrow{\delta} & B_{n-1}^* \leftarrow 0 \end{array}$$

whose row is still SES (splits $\Rightarrow$ dualize still SES). This is a SES of chain complexes (C^*, δ) , $(Z^*, \delta=0)$, $(B^*, \delta=0)$. Thus get a LES of homology:

$$\cdots \leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(C, G) \leftarrow B_{n-1}^* \leftarrow Z_{n-1}^* \leftarrow \cdots$$

$\uparrow$
 (connecting homomorphism)

pf that connecting hom is i_n^* : $\delta \bar{\alpha}$ $\bar{\alpha}|_{B_n^*} = \alpha|_{B_n^*}$

$$\begin{array}{ccccccc} 0 & \leftarrow & Z_{n+1}^* & \leftarrow & C_{n+1}^* & \xleftarrow{\delta} & B_n^* \leftarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \leftarrow & Z_n^* & \xleftarrow{\alpha} & C_n^* & \xleftarrow{\delta} & B_{n-1}^* \leftarrow 0 \end{array}$$

$\uparrow$
 this map takes B_{n-1}^* to zero.
 Since SES splits: $C_n^* \cong Z_n^* \oplus B_{n-1}^*$

$\bar{\alpha}$ agrees with α on Z_n^* , hence also on the subgroup B_n^* .

get SES:

$$0 \leftarrow \ker i_n^* \leftarrow H^n(C, G) \leftarrow \operatorname{Coker} i_{n-1}^* \leftarrow 0$$

SES gives free resolution of $H_{n-1}(C)$:

$$0 \rightarrow B_{n-1} \xrightarrow{i_{n-1}^*} Z_{n-1} \rightarrow H_{n-1}(C) \rightarrow 0$$

(Dual) $0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow H_{n-1}^*(C) \leftarrow 0$

So $\operatorname{Coker} i_{n-1}^* = \operatorname{Ext}(H_{n-1}(C); G)$ by def.

□

(p 199-204) Relative Cohomology. (lec 3).

$A \subset X$. $C_n(X, A) = C_n(X) / C_n(A)$ space of relative chains.

G = ab group. let

$$C_n(X, A; G) \triangleq \text{Hom}(C_n(X, A); G) \quad \text{relative cochains.}$$

SES: (splits: FACT: $C_n(X, A)$ free abelian)

$$0 \rightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \rightarrow 0 \quad \text{free abelian}$$

(Dualize) SES by applying $\text{Hom}(-, G)$:

$$(*) \quad 0 \leftarrow C^n(A, G) \xleftarrow{i^*} C^n(X, G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0.$$

Define $\delta: C^n(X, A) \rightarrow C^{n+1}(X, A)$ restriction of $\delta: C^n(X) \rightarrow C^{n+1}(X)$.

This is because j^* is injective. So $C^n(X, A) \subseteq C^n(X)$.

So $\exists$ LES

$$\cdots \rightarrow H^n(X, A) \xrightarrow{j_n^*} H^n(X) \xrightarrow{i_n^*} H^n(A) \xrightarrow{\delta} H^{n+1}(X, A) \rightarrow \cdots$$

connecting homomorphism
↓

Induced homomorphism: $f: X \rightarrow Y$ gives $f_{\#}: C_n(X) \rightarrow C_n(Y)$. Thus the dual map $f^*: C^n(Y) \rightarrow C^n(X)$.

- f^* sends cocycle (coboundary) to cocycle (coboundary).

So induces $f^*: H^n(Y) \rightarrow H^n(X)$. Can check

- $(gf)^* = f^*g^*$.

Homotopy Invariance: let $f \simeq g: (X, A) \rightarrow (X, B)$. Then $f^* = g^*$.

pf. Use chain homotopy $P: C_n(X) \rightarrow C_{n+1}(Y)$. We have

$$\delta_{\#} - f_{\#} = \partial P + P\partial.$$

let $P^*: C^{n+1}(Y) \rightarrow C^n(X)$. Then

$$g^* - f^* = \delta P^* + P^* \delta: C^n(Y) \rightarrow C^n(X).$$

Thus $g^* - f^* = 0: H^n(Y) \rightarrow H^n(X)$. □

Excision: If $Z \subset A \subset X$ s.t. $cl(Z) \subseteq \text{int}(A)$. Then

$$i: (X-Z, A-Z) \hookrightarrow (X, A)$$

induces an isomorphism $i_*: H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X, A)$.

Therefore $i^*: H^n(X, A) \xrightarrow{\cong} H^n(X-Z, A-Z)$.

Mayer-Vietoris: Let $X = \text{int}(A) \cup \text{int}(B)$ where $A, B \subseteq X$. Then $\exists$ LES

$$\cdots \rightarrow H^n(X) \xrightarrow{i_A^* + i_B^*} H^n(A) \oplus H^n(B) \xrightarrow{j_A^* - j_B^*} H^n(A \cap B) \xrightarrow{\delta} H^{n+1}(X) \rightarrow \cdots$$

where $i_A: A \hookrightarrow X$, $i_B: B \hookrightarrow X$. $j_A: A \cap B \hookrightarrow A$, $j_B: A \cap B \hookrightarrow B$.

pf. Excision + LES of a pair.

SES: dual to singular simplicies with image in A or in B

$$0 \rightarrow C^n(A+B) \xrightarrow{i_A^* + i_B^*} C^n(A) \oplus C^n(B) \xrightarrow{j_A^* - j_B^*} C^n(A \cap B) \rightarrow 0.$$

gives a LES of homology (See Hatcher).

□

Axioms for Reduced Cohomology:

A contravariant functor $\tilde{h}^n: \text{category of CW complex} \rightarrow \text{Ab}$

& natural coboundary homomorphism

$$\delta: \tilde{h}^n(A) \rightarrow \tilde{h}^{n+1}(X/A) \quad A \text{ subcomplex of } X$$

s.t. (1) $f \simeq g \Rightarrow f^* = g^*$

(2) $\exists$ LES: $A \subset X$, $q: X \rightarrow X/A$:

$$\cdots \xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \rightarrow \cdots$$

(3) If $X = \bigvee_{\alpha} X_{\alpha}$ and $i_{\alpha}: X_{\alpha} \rightarrow X$, then

$$\prod_{\alpha} i_{\alpha}^*: \tilde{h}^n(X) \xrightarrow{\cong} \prod_{\alpha} \tilde{h}^n(X_{\alpha})$$

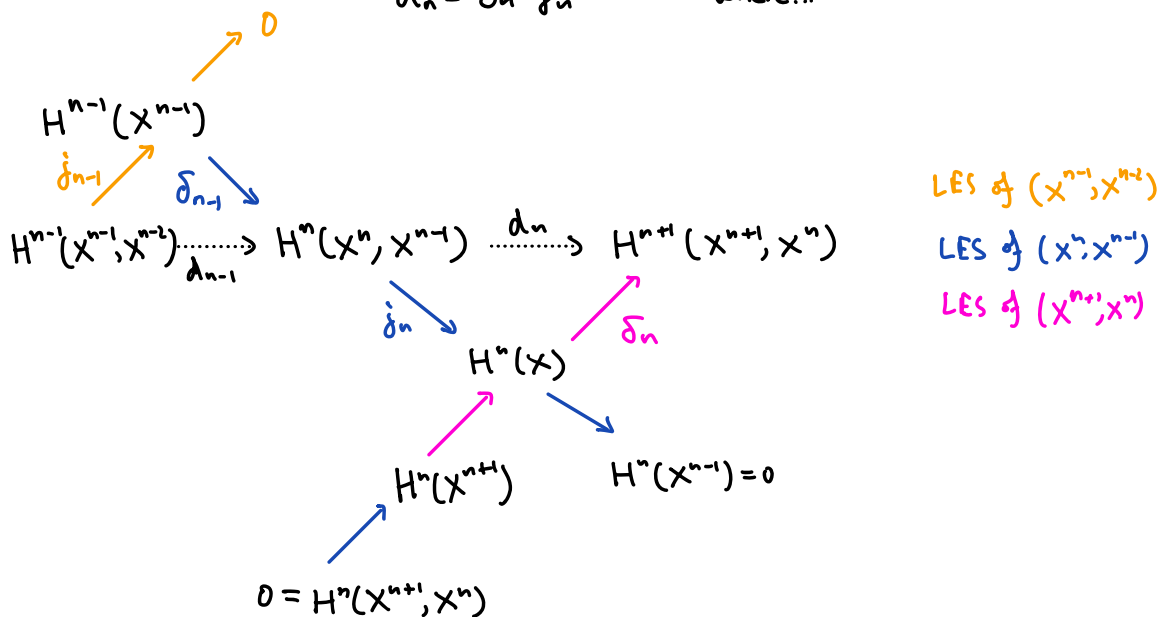
is an isomorphism.

Cellular Cohomology:

Let X be a CW complex. X^n be its n -skeleton. Then the ^(nth) cochain group is

$$H^n(X^n, X^{n-1}) \cong \prod_{\substack{(3) \\ \alpha \text{ n-cell}}} (H^n(\alpha, \partial\alpha)) \cong \prod_{\alpha} \mathbb{Z}.$$

Define $d_n: H^n(X^n, X^{n-1}) \rightarrow H^{n+1}(X^{n+1}, X^n)$ by
 $d_n = \delta_n \circ j_n$ where...



Then

$$d_n \circ d_{n-1} = \delta_n \circ \underbrace{(j_n \circ \delta_{n-1})}_0 \circ j_{n-1} = 0 \quad \text{by Exactness.}$$

Thus $d^2 = 0$. Thus $(H^n(X^n, X^{n-1}), d_n)$ gives a cohomology theory, called the cellular cohomology theory.

Cup Products. (lec 4).

Use $\mathbb{Z}$ -coefficients. But generalizes to any commutative ring R .

$$\cup: C^k(X) \times C^l(X) \rightarrow C^{k+l}(X) \quad \text{Cup Product.}$$

bilinear. Induces a well-defined map (bilinear) on $H^k(X) \times H^l(X) \rightarrow H^{k+l}(X)$.

Let $\sigma: \Delta^{k+l} \rightarrow X$ be a singular simplex. Let $\varphi \in C^k(X)$, $\psi \in C^l(X)$.

Then

$$(\varphi \cup \psi)(\sigma) \triangleq \varphi(\sigma|[\sigma_0, \dots, \sigma_k]) \psi(\sigma|[\sigma_k, \dots, \sigma_{k+l}]).$$

$$\text{Lemma. } \delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$$

pf. Hatcher Lem 3.6. □

$$\text{Claim: } \mathbb{Z}^k \cup \mathbb{Z}^l \subset \mathbb{Z}^{k+l}.$$

pf. Let $\varphi \in \mathbb{Z}^k$, $\psi \in \mathbb{Z}^l$. Then by the lemma, $\delta(\varphi \cup \psi) = 0$. □

$$\text{Claim: } \mathbb{Z}^k \cup B^l \subset B^{k+l} \quad \& \quad B^k \cup \mathbb{Z}^l \subset B^{k+l}.$$

pf. Take $\varphi \in \mathbb{Z}^k$, $\delta\psi \in B^l$. Then $\psi \in C^{l-1}$. Then

$$(-1)^k \varphi \cup \delta\psi = \delta(\varphi \cup \psi) - \underbrace{(\delta\varphi) \cup \psi}_0 = \delta(\varphi \cup \psi) \in B^{k+l}.$$

□

This induces a well-defined map $\cup: H^k(X) \times H^l(X) \rightarrow H^{k+l}(X)$

$$\text{bilinear: } [\varphi] \cup [\psi] = [\varphi \cup \psi].$$

well-defined: If $[\varphi] = [\varphi']$, $\varphi - \varphi' = \delta\eta$. Then $[\varphi] \cup [\psi] - [\varphi'] \cup [\psi] = [(\varphi - \varphi') \cup \psi]$

$$= [\delta\eta \cup \psi] = [\delta(\eta \cup \psi)] = 0. \text{ So } [\varphi] \cup [\psi] = [\varphi'] \cup [\psi]. \text{ Similarly}$$

If $[\psi] = [\psi']$ then $[\varphi] \cup [\psi] = [\varphi] \cup [\psi']$. Thus

$$[\varphi'] \cup [\psi'] = [\varphi] \cup [\psi'] = [\varphi] \cup [\psi].$$

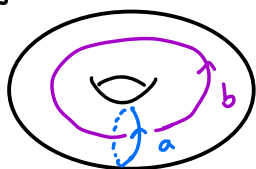
So the map is well-defined.

Thus $H^*(X)$ is a ring. $H^*(X) = \bigoplus_{k=0}^{\infty} H^k(X)$, General element: $\sum_i \alpha_i$, where $\alpha_i \in H^i(X)$. Then $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i \cup \beta_j$.

Prop. Let $f: X \rightarrow Y$ cts. Then $f^*: H^*(Y) \rightarrow H^*(X)$ is a ring homomorphism: i.e., $f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta)$.

Prop. Multiplication is graded commutative: $\alpha \in H^k, \beta \in H^l$. Then $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$.

E.g. (H. Ex 3.7). $H^*(T^2)$.



$$H_0(T^2) = \mathbb{Z}. \quad H_1(T^2) = \mathbb{Z}a \oplus \mathbb{Z}b.$$

$$H_2(T^2) = \mathbb{Z}.$$

Universal Coefficient Theorem: $H^1(T^2) \cong \text{Hom}(H_1(T^2), \mathbb{Z})$

$$H^1(T^2) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \quad \alpha(a)=1, \alpha(b)=0, \beta(a)=0, \beta(b)=1.$$

$H^2(T^2) = \mathbb{Z}\varphi$, where $\varphi(U+L) = 1$ and $\varphi = 0$ on other stuff.

$$\alpha \cup \beta = \varphi$$

$$\alpha \cup \beta = -\beta \cup \alpha.$$

$H^*(T^2)$ as a group is $\mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}$, so

$$H^*(T^2) \cong \mathbb{Z}^4$$

$$= \mathbb{Z} \oplus \langle \alpha, \beta : \alpha \cup \alpha = 0 = \beta \cup \beta, \alpha \cup \beta = -\beta \cup \alpha \rangle.$$

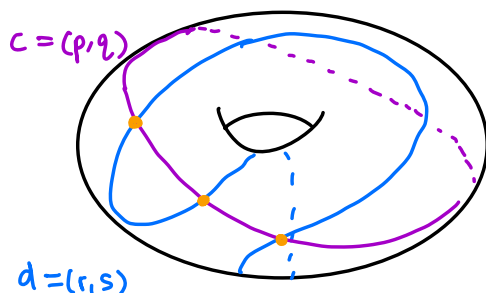
$\uparrow$
dim 0

$\uparrow$
dim 1 and dim 2.

$$\text{Hence } (p\alpha + q\beta) \cup (r\alpha + s\beta) = (ps - qr) \alpha \cup \beta$$

$$= \det \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \varphi.$$

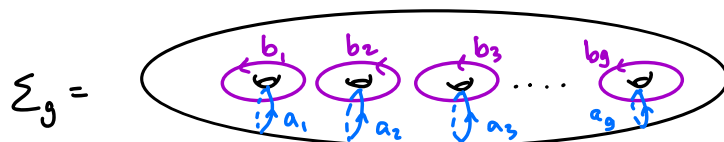
$c = p\alpha + q\beta$ is dual to $pa + qb$. Similarly $r\alpha + s\beta \xleftrightarrow{\text{dual}} ra + sb = d$



of intersections

$$= \begin{vmatrix} p & q \\ r & s \end{vmatrix}.$$

e.g.



$$H_1(\Sigma_g) = \mathbb{Z}^{2g} \text{ generated by } a_i, b_i.$$

$$\text{UCT} \Rightarrow H^1(\Sigma_g) \cong \text{Hom}(H_1(\Sigma_g), \mathbb{Z}) \cong \mathbb{Z}^{2g} \text{ generated by dual basis } \alpha_i, \beta_i.$$

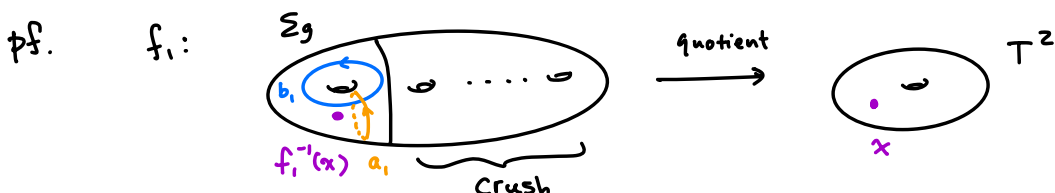
$$H^2(\Sigma_g) = \mathbb{Z} \eta.$$

$$\alpha_i \cup \beta_j = \delta_{ij} \eta. \quad \alpha_i \cup \alpha_j = 0 = \beta_i \cup \beta_j.$$

$\cup: H^1(\Sigma_g) \times H^1(\Sigma_g) \rightarrow \mathbb{Z}$ is bilinear. The matrix we get is skew symmetric:

$$\begin{matrix} & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \dots & \alpha_n & \beta_n \\ \begin{matrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_n \\ \beta_n \end{matrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}$$

Every element of $H^1 \leftrightarrow$ closed curves on Σ_g
 $\cup \leftrightarrow$ algebraic intersection number.

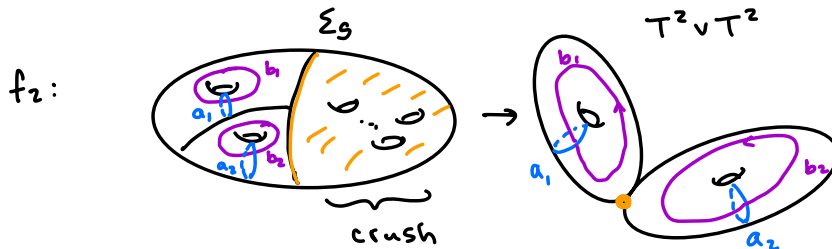


This map has deg 1.

$$f_1^*: H^*(T^2) \rightarrow H^*(\Sigma_g)$$

generators of $H^1 \mapsto \alpha_1, \beta_1 \Rightarrow \alpha_1 \cup \beta_1 = \eta.$

(in T^2 example, know $\alpha \cup \beta = \eta$.
 f_1^* homomorphism, generator $\mapsto$ generator, so $\alpha_1 \cup \beta_1 = \eta$.)



$$f_2^*: H^*(T^2 \vee T^2) \rightarrow H^*(\Sigma_g)$$

$$\Rightarrow \begin{cases} \alpha_1 \cup \alpha_2 = 0 \\ \alpha_1 \cup \beta_2 = 0 \end{cases} \quad \left(\begin{array}{l} \alpha_1, \alpha_2 \text{ does not intersect in } T^2 \vee T^2, \text{ so a} \\ \text{simplex only lives in one of the } T^2 \text{ s. So} \\ \text{When evaluate } \alpha_1 \cup \alpha_2, \text{ one of them gives 0.} \end{array} \right)$$

Conclusion: Can deduce everything in $H^*(\Sigma_g)$ from $H^*(T^2)$.

Another proof in Hatcher. □

Defⁿ. (Exterior Algebra).

Let R be a commutative ring. The **Exterior algebra**

$\Lambda_R[\alpha_1, \dots, \alpha_n] =$ Free R -Module generated by $\alpha_1, \dots, \alpha_n$ s.t.

$$\alpha_i \alpha_j = -\alpha_j \alpha_i, \quad \alpha_i^2 = 0.$$

e.g. $T^n = \prod_{i=1}^n S^1$. $\tilde{H}^*(T^n) = \Lambda[\alpha_1, \dots, \alpha_n], \quad \alpha_i \in H^1(T^n).$

pf. Deduce from $H^*(T^2)$ using induction.

$$\begin{array}{ccc} T^{n-1} \hookrightarrow T^n & \xrightarrow{\pi} & T^{n-1} \\ & \searrow \mathbb{1} & \nearrow \\ & & \end{array} \quad \pi: \text{proj onto first } n-1 \text{ factors.}$$

Induction, induced maps on H^* . □

lec 5.

Thm. $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$ for $\alpha \in H^p(X)$, $\beta \in H^q(X)$.

pf. Let $\sigma: \Delta^{p+q}(X)$. Then

$$\alpha \cup \beta (\sigma) = \alpha(\sigma|[\sigma_0, \dots, \sigma_p]) \beta(\sigma|[\sigma_p, \dots, \sigma_{p+q}])$$

$$\beta \cup \alpha (\sigma) = \beta(\sigma|[\sigma_0, \dots, \sigma_q]) \alpha(\sigma|[\sigma_q, \dots, \sigma_{p+q}]).$$

Let $R: \Delta^{p+q} \rightarrow \Delta^{p+q}$ by $R(x_0, \dots, x_n) = (x_n, x_1, \dots, x_0)$. Let

$\bar{\sigma} = \sigma \circ R$. Then

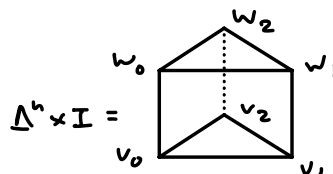
$$(\alpha \cup \beta)(\bar{\sigma}) \approx (\pm) (\beta \cup \alpha)(\sigma).$$

Let $\rho: C_{p+q}(X) \rightarrow C_{p+q}(X)$, $\sigma \mapsto \bar{\sigma}$.

Show:

- ρ is a chain map: $\partial \rho = \rho \partial$.
- ρ is chain homotopic to the identity:
 $\exists P: C_n(X) \rightarrow C_{n+1}(X)$ s.t. $\partial P + P \partial = \rho - \text{id}$.

Let $\pi: \Delta^n \times I \rightarrow \Delta^n$ be the projection.



$$\rho(\sigma) = \sum_{i=1}^n (-1)^i \epsilon_{n-i} (\sigma \circ \pi) | [\sigma_0, \dots, \sigma_i, w_n, \dots, w_i]$$

where

$$\epsilon_n = (-1)^{n(n+1)/2}.$$

Computation shows that

$$\partial P + P \partial = \rho - \text{id}.$$

Details in Hatcher pf Thm 3.11.

□

lec 6.

M^n connected orientable manifold.

if $H_n(M) \cong \mathbb{Z}$, the generator is called a **fundamental class** and determines an orientation.

Defⁿ. Let R be a commutative ring, M^n = closed n -manifold. Then M is **R -orientable** if $H_n(M; R) \cong R$.

If M^p, N^p are closed orientable manifolds. $f: M \xrightarrow{cts} N$.

Defⁿ. $\deg(f)$ = algebraic number of preimages of a generic point if f is differentiable.

Cup Product & **Cross Product**:

$$\begin{aligned} \xrightarrow{\text{cross product}} x: H^k(X) \times H^l(Y) &\rightarrow H^{k+l}(X \times Y) \\ a \times b &= (p_1^*(a)) \cup (p_2^*(b)) \end{aligned}$$

where $p_1: X \times Y \rightarrow X$, $p_2: X \times Y \rightarrow Y$ are projections.

This is well-defined and bilinear.

Thm (H 3.15) (Kunneth Formula).

Let X, Y be CW complexes and $H^k(Y; R)$ is a f.g. free R -module.

Then

$$x: H^*(X; R) \overset{\text{graded commutative}}{\otimes} H^*(Y; R) \xrightarrow{\cong} H^*(X \times Y; R)$$

is an isomorphism.

Let M, N be graded R -module. $M = \bigoplus_i M_i$, $N = \bigoplus_j N_j$. Then

$$\underset{\text{graded commutative}}{M \otimes_R N} := M \otimes_R N / \left\{ \begin{array}{l} a_i \otimes b_j = (-1)^{ij} b_j \otimes a_i \\ a_i \in M_i, b_j \in N_j \end{array} \right\}.$$

e.g. $T^n = S^1 \times \dots \times S^1$ n circles.

Thm gives

$$\begin{aligned} H^*(T^n; \mathbb{Z}) &= \wedge [\alpha_1, \dots, \alpha_n] & \dim(\alpha_i) &= 1. \\ &= \frac{\mathbb{Z}[\alpha_1, \dots, \alpha_n]}{\langle \alpha_i \alpha_j = \alpha_j \alpha_i \rangle} \end{aligned}$$

$\Rightarrow$

$H^p(T^n; \mathbb{Z})$ has basis $(\alpha_i^2 = 0)$.

$$\{\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_p} : i_1 < i_2 < \dots < i_p\}$$

So

$$\dim H^p(T^n; \mathbb{Z}) = \binom{n}{p}.$$

Thm (H 3-12). $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, $\dim \alpha = 1$.

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1}) \quad \dim \alpha = 2.$$

Thm (H 3.20). If $\mathbb{R}^n$ is a division algebra, then $n = 2^m$.

pf. Div alg $\Rightarrow$ bilinear $k: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with no zero divisors. Define

$$(S^{n-1} \subset \mathbb{R}^n) \quad g: S^{n-1} \times S^{n-1} \rightarrow S^{n-1}$$

$$g(x, y) = \frac{xy}{\|xy\|}$$

$$\text{bilinear} \Rightarrow g(-x, y) = -g(x, y) = g(x, -y).$$

get

$$h: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$$

lec 7.

$\mu: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ multiplication

$$\mu(e_i, e_j) = \sum_{k=1}^n a_k^{ij} e_k$$

- Want associative: $\mu(\mu(u, v), w) = \mu(u, \mu(v, w))$.

This restriction is a set of equations for a_k^{ij} .

- Want no zero divisors.

Lemma. $H^*(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y] / (x^{n+1}, y^{n+1})$ from Kunneth.

pf. Kunneth.

□

Thm (H 3.21). If $\mathbb{R}^n$ is a division algebra, then $n = 2^m$.

pf. D.A. $\Rightarrow k: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ multiplication is bilinear. This gives

$$h: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$$

$$h([u], [v]) = [k(u, v)]$$

is well-defined.

(use $\mathbb{Z}_2$ -coefficients).

(by the lemma)

claim:

$$h^*: H^1(\mathbb{R}P^{n-1}) \rightarrow H^1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}) \cong \mathbb{Z}_2 \alpha \oplus \mathbb{Z}_2 \beta$$

$$\cong \mathbb{Z}_2 \ni \gamma \mapsto \alpha + \beta$$

Assuming this, $\gamma^n = 0$ since $H^n(\mathbb{R}P^{n-1}) = 0$. Thus $(\alpha + \beta)^n = 0$. Thus

$$(\alpha + \beta)^n = \sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k} \in \mathbb{Z}_2[\alpha, \beta] / (\alpha^n, \beta^n)$$

Thus $\binom{n}{k} = 0 \forall k \not\equiv 0 \pmod 2 \iff n = 2^m$.
 $\uparrow$
 number theory

pf of claim: $\mathbb{R}^n \ni a \neq 0$. Define $\theta: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\theta(b) = ab$. No zero divisors:

θ is injective. Thus θ is a vector space isomorphism (since it is a linear map). Then θ covers map

$$\bar{\theta}: \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$$

$$\bar{\theta}[\sigma] \mapsto [\theta\sigma]$$

Then

$$h_*: H_1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}) \rightarrow H_1(\mathbb{R}P^{n-1})$$

$$\begin{array}{c} \downarrow \\ r \in H_1(\mathbb{R}P^{n-1}) \mapsto c \in H_1(\mathbb{R}P^n) \\ \uparrow \quad \quad \quad \nearrow \\ \text{generator} \end{array}$$

Existence of $\theta \Rightarrow h_*(r) = c$

Sends generator to generator since fixing $[a] \in \mathbb{R}P^{n-1}$ gives an isomorphism.

Similarly for the left copy: $h_*(l) = c$.

$$\Rightarrow h^*(\gamma) = \alpha + \beta.$$

///

Graded Commutative Rings: $a \cdot b = (-1)^{|a| \cdot |b|} b \cdot a$ $a \in R_i, |a| = i$.

If A, B are graded rings also R -module, then can make $A \otimes_R B$ into a ring:

$$(a \otimes b)(a' \otimes b') = (-1)^{|a'| \cdot |b|} (aa') \otimes (bb').$$

We denote this ring by

$$A \otimes_{g.c.} B.$$

$$\begin{array}{ccc} C^k(X) \times C^l(X) & \rightarrow & C^{k+l}(X) \\ \downarrow \quad \downarrow & & \downarrow \\ \phi \quad \quad \psi & \mapsto & \phi \cup \psi \end{array}$$

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[\sigma_0, \dots, \sigma_k]) \cup \psi(\sigma|[\sigma_{k+1}, \dots, \sigma_{k+l}]).$$

Relative Cup Product: $A \subset X, B \subset Y$.

$$x: H^k(X, A) \times H^l(Y, B) \rightarrow H^{k+l}(X \times Y, A \times Y \cup X \times B).$$

$$(a, b) \longmapsto a \times b$$

$$a \times b = p_1^*(a) \cup p_2^*(b)$$

where $p_1: X \times Y \rightarrow X, p_2: X \times Y \rightarrow Y$.

This formula makes sense when $A = B = \phi$.

When $A, B \neq \emptyset$, you get cochains which doesn't vanish on $(A \times Y) \cup (X \times B)$.

Modification: work with CW complex and cellular cohomology st. $A \times Y, X \times B$ are subcomplexes of $X \times Y$. Let $a = [\varphi]$, $\varphi \in C^k(X, A)$.

$$b = [\psi], \quad \psi \in C^l(Y, B).$$

Evaluate $p_1^* \varphi \cup p_2^* \psi$ on $\sigma \in C_{k+l}(X \times Y, A \times Y \cup X \times B)$ where σ a cell.

Case 1: $\sigma \in A \times Y \Rightarrow p_1 \sigma \subset A$. Then

$$\begin{aligned} p_1^*(\varphi)(\sigma|[\nu_0, \dots, \nu_k]) &= \varphi(p_{1*} \sigma|[\nu_0, \dots, \nu_k]) \\ &= 0 \quad \text{since } \varphi \in C^k(X, A). \end{aligned}$$

Case 2: Similarly if $\sigma \in X \times B$, then $(p_2^* \psi)(\sigma|[\nu_k, \dots, \nu_{k+l}]) = 0$.

Therefore:

$$(p_1^* \varphi) \cup (p_2^* \psi) \in C^{k+l}(X \times Y, A \times Y \cup B \times X).$$

Lemma. α generator of $H^1(I, \partial I) \cong \mathbb{Z}$. Then

$$\begin{array}{ccc} H^n(Y) & \xrightarrow{\cong} & H^{n+1}(I \times Y, \partial I \times Y) \\ \beta & \longmapsto & \alpha \times \beta \end{array}$$

is an isomorphism.

pf. LES for $(I \times Y, \partial I \times Y)$:

$$\dots \rightarrow H^n(I \times Y, \partial I \times Y) \xrightarrow{0} H^n(I \times Y) \rightarrow \dots$$

0 because:

$$\begin{array}{ccc} H^n(I \times Y, \partial I \times Y) & \rightarrow & H^n(I \times Y) \\ & \searrow \quad \swarrow & \\ & H^n(I \times Y, \{0\} \times Y) & \end{array}$$

Since all induced by inclusion.

$$\begin{array}{ccc} H^n(I \times Y, \{0\} \times Y) & \cong & H^n(\{0\} \times Y, \{0\} \times Y) = 0 \\ \uparrow & & \\ \text{def. rect. induces } \cong & & \end{array}$$

And commutativity proves the claim.

Thus get SES

$$0 \rightarrow H^n(I \times Y) \xrightarrow{i^*} H^n(\partial I \times Y) \xrightarrow{\delta} H^{n+1}(I \times Y, \partial I \times Y) \rightarrow 0.$$

$$\cong H^n(\{0\} \times Y) \oplus H^n(\{1\} \times Y).$$

$\ker \delta = \text{Im } i^*$ where

$$i^*: \varepsilon \mapsto i^*(\varepsilon) = (\varepsilon, \varepsilon)$$

$$H^n(\{0\} \times Y) \oplus H^n(\{1\} \times Y)$$

So

$\delta: H^n(\{0\} \times Y) \rightarrow H^{n+1}(I \times Y, \partial I \times Y)$ is an isomorphism.

Note:

$$\begin{array}{ccc} H^0(\partial I) \times H^n(Y) & \xrightarrow{(\delta, 1)} & H^1(I, \partial I) \times H^n(Y) \\ \downarrow \times & \curvearrowright & \downarrow \times \\ H^n(\partial I \times Y) & \xrightarrow{\delta} & H^{n+1}(I \times Y, \partial I \times Y) \end{array}$$

... Fuck me...

□

Consequences:

$$\textcircled{1} \exists \text{ map } H^n(Y) \xrightarrow{\cong} H^{n+1}(S' \times Y, \{s_0\} \times Y) \quad s_0 \in S'. \\ \beta \mapsto \alpha \times \beta$$

$$\textcircled{2} \exists H^k(T^n) \xrightarrow{\cong} H^{k+1}(T^{n+1}, T^n)$$

lec 8.

Recall: $H^k(X, A) \times H^l(Y, B) \rightarrow H^{k+l}(X \times Y, A \times Y \cup X \times B)$
 $\alpha \times \beta := p_1^*(\alpha) \cup p_2^*(\beta).$

Lemma. $\alpha \in H^1(I, \partial I)$. Then

$$\begin{aligned} H^n(Y) &\xrightarrow{\cong} H^{n+1}(I \times Y, \partial I \times Y) \\ \beta &\longmapsto \alpha \times \beta. \end{aligned}$$

Cor. $H^*(T^n) = \wedge[\alpha_1, \dots, \alpha_n].$

Pf. Induct on n . $n=2$ direct calculation.

$n=3$:

$$T^3 = S_1^1 \times S_2^1 \times S_3^1 \xleftarrow{\quad} T_{ij}^2 = S_i^1 \times S_j^1$$

$\searrow \pi_{ij}$

$$H^1(T^3) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle.$$

Induction shows $\alpha_i \cup \alpha_j$ generates $H^2(T_{ij}^2) \cong \mathbb{Z}$.

remains to show $\alpha_1 \cup \alpha_2 \cup \alpha_3$ generates $H^2(T^3)$.

$$\begin{array}{ccc} H^1(T^3) \times H^2(T^3) & \longrightarrow & H^3(T^3) \\ \cup & & \cup \\ \alpha_i & \xrightarrow{\quad} & \alpha_i \cup \beta \end{array}$$

deduce from lemma $H^1(I, \partial I) \xrightarrow[\uparrow]{\cong} H^1(S^1, \{s_0\})$
 excision.

□

Cor $\Rightarrow$

$$\begin{array}{ccc} H^i(T^i) \times H^j(T^j) & \longrightarrow & H^n(T^n) \\ \cup & & \cup \\ \alpha & \xrightarrow{\quad} & \alpha \times \beta \end{array} \quad \begin{array}{l} T^n = T^i \times T^j \\ \text{generator.} \end{array}$$

$$\Rightarrow \begin{array}{ccccc} & \xleftarrow{T^i - (i\text{-cell})} & & & \\ H^i(T^i, \dot{T}^i) & \times & H^j(T^j, \dot{T}^j) & \longrightarrow & H^n(T^n, \dot{T}^n) \\ \text{gen} & \downarrow \alpha & \downarrow \beta & \longmapsto & \alpha \times \beta \text{ gen.} \end{array}$$

$$\Rightarrow \begin{array}{ccccc} H^i(I^i, \partial I^i) & \times & H^j(I^j, \partial I^j) & \longrightarrow & H^n(I^n, \partial I^n) \\ \text{gen} & \downarrow \alpha & \downarrow \beta & \longmapsto & \alpha \times \beta \text{ gen} \end{array}$$

lem. $A \subset B \subset X$ & B def. ret. to A . Then $i^*: H^*(X, X-A) \xrightarrow{\cong} H^*(X, X-B)$

pf. LES of $(X, X-A)$ and $(X, X-B)$:

$$\begin{array}{ccccccc} \cdots \leftarrow H^n(A) \leftarrow H^n(X) \leftarrow H^n(X, A) \leftarrow H^{n-1}(A) \leftarrow H^{n-1}(X) \leftarrow \cdots \\ \downarrow \text{ret}^* \quad \downarrow \text{id} \quad \downarrow i^* \quad \downarrow \text{ret}^* \quad \downarrow \text{id} \\ \cdots \leftarrow H^n(B) \leftarrow H^n(X) \leftarrow H^n(X, B) \leftarrow H^{n-1}(B) \leftarrow H^{n-1}(X) \leftarrow \cdots \end{array}$$

5-Lemma
four maps isomorphism $\Rightarrow H^n(X, A) \xrightarrow{\cong} H^n(X, B)$. □

Cor. $i^*: H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \xrightarrow{\cong} H^*(I^n, \partial I^n)$.

pf. $\mathbb{R}^n - \{0\}$ def. ret. to ∂I^n , use excision to delete $\mathbb{R}^n \setminus I^n$.

projectivization: $P(V) = \frac{V - \{0\}}{v \sim \lambda v, \lambda \neq 0}$ □

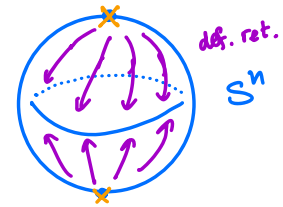
lem. Let $W = U \oplus V$ real vector space. Then $P(W) - P(U)$ def ret. to $P(V)$. In particular if $i+j = n+1$, $U = \mathbb{R}^{i+1}$, $V = \mathbb{R}^{j+1}$, $W = \mathbb{R}^{n+1}$. Let $\mathbb{R}P^n \triangleq P^n = P(\mathbb{R}^{n+1})$, then $P^n - P^i$ def. ret. to P^j . In particular, $P^n - \{pt\}$ def. ret. to P^{n-1} .

pf. $F_t: W \rightarrow W \quad t \in [0,1],$
 $F_t(u+v) = (tu) + v.$

$F_1 = \text{id}_W, F_0(u+v) = v. \quad F_0: u \oplus v \rightarrow v.$

$\Rightarrow F_t|_{W-U}: W-U \rightarrow W-U$

induces retraction $P(W-U) = P(W) - P(U)$ onto $P(V).$ $\square$

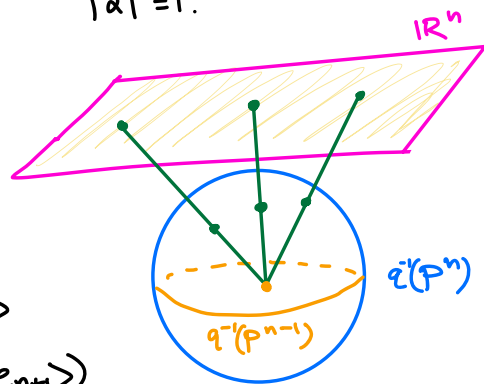


lem. $i^*: H^n(P^n, P^K) \xrightarrow{\cong} H^n(P^n)$

pf. Use LES of (P^n, P^K) . And we know what $H^n(P^n)$ is. $\square$

Thm. $H^*(P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1}) \quad |\alpha| = 1.$

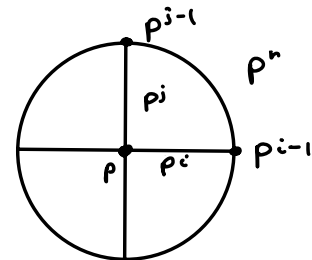
pf. $P^n = P^{n-1} \cup \mathbb{R}^n$, where the identification is shown on the right.



$P^n = P(\mathbb{R}^{n+1})$ has basis $\langle e_1, \dots, e_{n+1} \rangle$

$P^i = P(\langle e_1, \dots, e_{i+1} \rangle) \quad P^j = P(\langle e_{i+1}, \dots, e_{n+1} \rangle)$

where $i+j=n$. Thus the picture is:



We have a diagram

$$\begin{array}{ccccc}
 H^i(P^n) \times H^j(P^n) & \xrightarrow{\cup} & H^n(P^n) & & \\
 \uparrow \textcircled{3} & & \uparrow \textcircled{1} & & \\
 H^i(P^n, P^n - P^j) \times H^j(P^n, P^n - P^i) & \xrightarrow{\cup} & H^n(P^n, P^n - \{P\}) & & \\
 \downarrow \textcircled{2} & & \downarrow \textcircled{5} & & \\
 H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \xrightarrow{\cup} & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) & & \\
 \downarrow \textcircled{4} & & \downarrow \textcircled{6} & & \\
 H^i(I^i, \partial I^i) \times H^j(I^j, \partial I^j) & \xrightarrow[\cong]{\cup} & H^i(I^n, \partial I^n) & &
 \end{array}$$

vertical maps induced by inclusion & cup product natural $\Rightarrow$ diagram commutes.

Claim: All vertical maps are isomorphisms. Bottom map take generator to generator, so does the upper map. Done.

$$\textcircled{5} H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong H^n(P^n, P^n - \{p\}) \text{ excision.}$$

$$\textcircled{4} H^*(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \cong H^*(I^i, \partial I^i) \text{ by Cor.}$$

$$H^*(\mathbb{R}^j \times \mathbb{R}^i, \mathbb{R}^j \times (\mathbb{R}^i - \{0\})) \cong H^*(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j)$$

$$\textcircled{6} H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong H^i(I^n, \partial I^n) \text{ by Cor.}$$

$$\textcircled{3} H^i(P^n, P^n - p^i) \times H^j(P^n, P^n - p^i) \xrightarrow{\cong} H^i(P^n) \times H^j(P^n)$$

$$\textcircled{1}: i^*: H^n(P^n, P^n - p) \xrightarrow{\cong} H^n(P^n, P^{n-1}) \xrightarrow{\cong} H^n(P^n)$$

$\textcircled{2} P^n - p^j$ def. rel. to $P^i - p$ using F_t restricted. above lemma

$$\therefore H^i(P^n, P^n - p^j) \xrightarrow{\cong} H^i(P^n, P^i - p) \xrightarrow[\cong]{\text{cell decomp}} H^i(P^i, P^i - \{p\}) \xrightarrow[\cong]{\text{excision}} H^i(\mathbb{R}^i, \mathbb{R}^i - 0).$$

Prop (3.14). $H^*(X \sqcup Y) = H^*(X) \times H^*(Y)$ product ring (a,b)(c,d) = (ac, bd) □

pf: $i: X \rightarrow X \sqcup Y$ induces hom $i^*: H^*(X \sqcup Y) \rightarrow H^*(X)$ (a,b) \cup (c,d) = (p,q) \mapsto p = ac

$$H^*(X) \oplus H^*(Y)$$

□

lec 10.

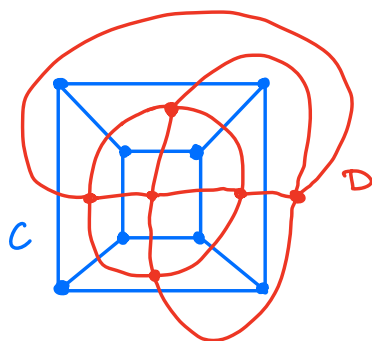
Defⁿ. Two cell decomposition C and D of X are dual if $\exists$ bijection $\theta: C \rightarrow D$ s.t. $\theta(k\text{-cell}) = (n-k)\text{-cell}$ and

$$\textcircled{1} a \in C_k, b \in D_{n-k}, a \cap b \neq \emptyset \Leftrightarrow b = \theta(a)$$

& $\exists$ nbhd $U \subset X$ of $a \cup b$, $h: U \rightarrow \mathbb{R}^n$ s.t. $h(a) \subseteq \mathbb{R}^k \times 0$, $h(b) \subseteq 0 \times \mathbb{R}^k$.

$$\textcircled{2} c, a \in C, \text{ then } c \subseteq a \Leftrightarrow \theta(c) \supseteq \theta(a). \quad (\theta \text{ inclusion reversing}).$$

eg. $\textcircled{1} X = S^2 = \partial I^3$.



Poincaré Duality (Version 1).

If space X has finite dual cell decomposition, wrt $\dim = n$, then

$$H_k(X; \mathbb{Z}_2) \cong H^{n-k}(X; \mathbb{Z}_2).$$

pf. Let C, D dual complexes. $C_k = \mathbb{Z}_2$ -chain complexes of k -cells in C ,

similarly for D_k . $D^k = \text{Hom}(D_k, \mathbb{Z}_2)$. Let $e_1, \dots, e_m$ basis of D_k .

Then $e_1^*, \dots, e_m^*$ dual basis of D^k . Let $\eta_k: D_k \rightarrow D^k$ be the isomorphism $\eta_k(e_i) = e_i^*$.

Key: $\sigma \in D_k$, then $\partial\sigma = \text{sum of } (k-1)\text{-cells in } \sigma$.

$\delta(\sigma^*) = \text{sum of duals of } (k+1)\text{-cells that contains } \sigma$.

i.e., let $f_1, \dots, f_n$ basis of D_{k+1} , then $f_1^*, \dots, f_n^*$ dual basis of D^{k+1} .

Then we have

$$\delta e_j^* = \sum_{f_i \supset e_j} f_i^*.$$

Now:

$$\begin{array}{ccccc} \partial & \rightarrow & C_k & \xrightarrow{\partial} & C_{k-1} & \xrightarrow{\partial} & \\ & & \eta \circ \theta \downarrow \cong & \curvearrowright & \cong \downarrow \eta \circ \theta & & \\ & \xrightarrow{\delta} & D^{n-k} & \xrightarrow{\delta} & D^{n-k+1} & \xrightarrow{\delta} & \end{array}$$

θ is a vector space isomorphism. So is η . Hence so is $\eta \circ \theta$.

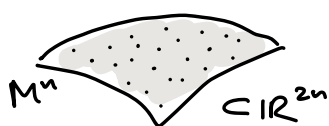
Show the diagram commutes:

$$\begin{array}{ccc} e & \xrightarrow{\partial} & \sum \text{cells in } C \text{ in } e \\ \eta \circ \theta \downarrow & \curvearrowright & \downarrow \eta \circ \theta \\ [\theta(e)]^* & \xrightarrow{\delta} & \sum (\text{cells that contain } \theta(e))^* \end{array}$$

Then it follows that $H_k(X; \mathbb{Z}_2) \cong H^{n-k}(X; \mathbb{Z}_2)$. □

Thm. Every Smooth closed manifold M^n has dual cell-decomposition.

Pf. ① [Whitney] M can be triangulated (homeomorphic to simplicial complex). Take $M \hookrightarrow \mathbb{R}^{2n}$ by Whitney embedding, take a lot of vertices, and triangulize.



(take points sufficiently st. regions bounded by points are basically flat. Thus we get a homeomorphism.)

② This triangulation has a dual. Let K be the given triangulation.

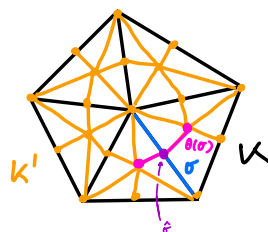
$K^{(1)}$ = first barycentric subdivision.

$\sigma \in K$ p -simplex

$\theta(\sigma) = cl(\text{union of simplices } \tau \text{ in } K^{(1)} \text{ that contain}$

$\hat{\sigma}$ & $\tau \cap \sigma = \hat{\sigma}$)

$\uparrow$
barycenter



③ $\theta(\sigma) \cong B^{n-k}$ for $\dim \sigma = k$. And θ is indeed the desired dual map. $\square$

Thus for every C^∞ -manifold, $H_k(M; \mathbb{Z}_2) \cong H^{n-k}(M; \mathbb{Z}_2)$.

Poincaré Duality (Version 2).

If M is a closed orientable n -manifold, then

$$H_k(M; \mathbb{Z}) \cong H^{n-k}(M; \mathbb{Z}).$$

Pf. Hard : $\square$

Rmk: M compact $\Rightarrow H_*$ finitely generated. So $H_k = \mathbb{Z}^{n_k} \oplus T_k$ ^{torsion.}

UCT $\Rightarrow H^k = \mathbb{Z}^{n_k} \oplus T_{k-1}$. If no torsion, then

$$H_k(M; \mathbb{Z}) \cong H_{n-k}(M; \mathbb{Z}).$$

e.g. $H_k(T^n; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{k}} \cong \mathbb{Z}^{\binom{n}{n-k}} \cong H_{n-k}(T^n; \mathbb{Z})$.

But:

$$H_k(\mathbb{R}P^3) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}_2 & k=1 \\ 0 & k=2 \\ \mathbb{Z} & k=3. \end{cases}$$

Poincaré Duality (Version 3).

Let M^n be closed, orientable, connected & $[M]$ ^{Fundamental class} generates $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$
 $\exists$ natural isomorphism

$$H^k(M; \mathbb{Z}) \longrightarrow H_{n-k}(M; \mathbb{Z})$$

$$\alpha \longmapsto [M] \cap \alpha$$

$\uparrow$ cap product

Cap Product: $k \geq l$.

$$\cap: C_k(X) \times C^l(X) \longrightarrow C_{k-l}(X)$$

$\underbrace{\sigma}_{\mathbb{R}}$

$\underbrace{\psi}_{\mathbb{R}}$

$$\sigma \cap \psi := (\psi(\sigma|[\sigma_0, \dots, \sigma_l]))(\sigma|[\sigma_{l+1}, \dots, \sigma_k]).$$

lemma. $\partial(\sigma \cap \psi) = (-1)^l ((\partial\sigma) \cap \psi - \sigma \cap \partial\psi).$

pf. H p.240

□

This induces $\cap: H_k(X) \times H^l(X) \longrightarrow H_{k-l}(X)$

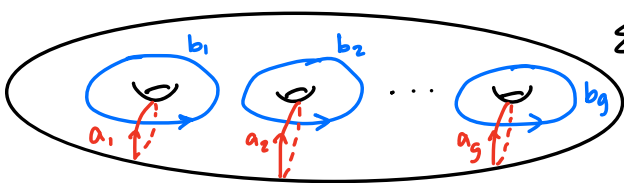
$$[z] \cap [\psi] \mapsto [z \cap \psi].$$

Special case: $k=l$:

recall $h: H^k(X) \longrightarrow \text{Hom}(H_k(X), \mathbb{R})$

$$(h[\varphi])[z] = [z] \cap [\varphi] = \varphi(z).$$

lemma. $\sigma \cap (\varphi_1 \cup \varphi_2) = (\sigma \cap \varphi_1) \cup \varphi_2.$

e.g. 

$$H_1(\Sigma_g) = \langle a_i, b_i \rangle.$$

$$UCT \Rightarrow H^1(\Sigma_g) = \langle \alpha_i, \beta_i \rangle$$

$\uparrow$
dual basis.

$$[M] \in H_2(\Sigma_g) \cong \mathbb{Z}.$$

$$([M] \cap \alpha_i) \cap \gamma \quad \gamma \in H^1(M).$$

$$= [M] \cap (\alpha_i \cup \gamma) \leftarrow \text{know } \cup$$

$$\alpha_i \cup \gamma = \begin{cases} \text{generator of } H^2(M), & \text{if } \gamma = \beta_i \\ 0 & , \text{ else.} \end{cases}$$

$$\Rightarrow [M] \cap \alpha_i = b_i$$

$$\therefore \text{PD: } H^1(M) \longrightarrow H_1(M)$$

$$\begin{matrix} \alpha_i & \mapsto & b_i \\ \beta_j & \mapsto & a_j. \end{matrix} \leftarrow \text{can work out similarly}$$

lec 11.

Poincaré Duality: Let M^n be a closed orientable manifold. Then there is an isomorphism

$$D: H^k(M; \mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M; \mathbb{Z})$$

$$D([\varphi]) = [M] \cap [\varphi].$$

Orientation (Differential Topology Version):

Defⁿ. An **orientation** on $\mathbb{R}^n$ is an equivalence class of ordered basis (i.e. frame).

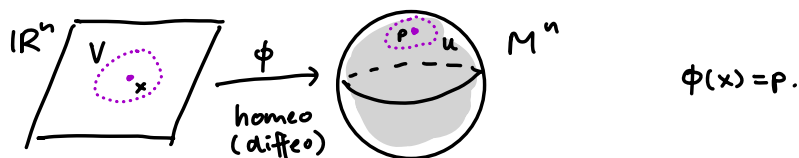
$$[v_1, \dots, v_n] \sim [w_1, \dots, w_n] \iff \det T > 0, \text{ where } T(\sum \lambda_i v_i) = \sum \lambda_i u_i.$$

Ex. There are 2 orientation on $\mathbb{R}^n$: $[\pm e_1, e_2, \dots, e_n]$.

Defⁿ. An n -dimensional **manifold** is a Hausdorff, $2^{<\aleph_0}$ -countable, locally Euclidean (locally $\cong \mathbb{R}^n$) topological space.

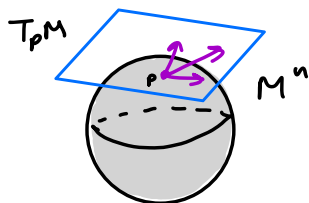
Smooth manifold: replace homeomorphism by diffeomorphism.

Thm (Whitney). Every smooth manifold M^n embeds in $\mathbb{R}^{2n+1}$.

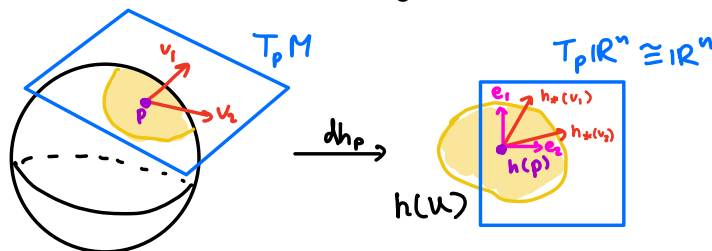


$$T_p M = \text{Im } d\phi_x \cong \mathbb{R}^n.$$

Tangent bundle $TM = \{(p, v) : p \in M, v \in T_p M\}$ a $2n$ -dim manifold.



Defⁿ. An **orientation** on M^n is an orientation on each tangent space $T_p M$, $p \in M$ that is "locally consistent": $\forall p \in M$, $\exists$ nbhd $p \in U \subset M$ and a diffeomorphism $h: U \hookrightarrow \mathbb{R}^n$ s.t. $\forall x \in U$, $dh_x: T_x M \rightarrow T_{h(x)} \mathbb{R}^n \cong \mathbb{R}^n$ is always orientation preserving.



Prop. If M^n closed (compact, $\partial M = \emptyset$) and $M \subset \mathbb{R}^{n+1}$, then M is orientable.

pf. $\exists$ continuous outward normal vector $\underline{n}(x)$. Choose $e_1, \dots, e_n \in T_x M$ s.t. $[\underline{n}(x), e_1, \dots, e_n] \sim [e_1, \dots, e_{n+1}]$. $\square$

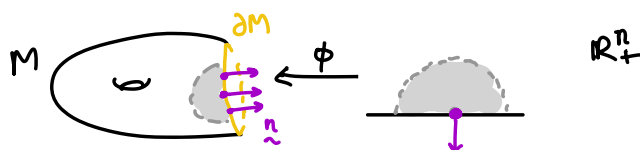
Ex. Orientable: $\mathbb{R}^n$, $\mathbb{C}P^n$, $\mathbb{R}P^{2n}$.

If M^n orientable and $W^n \subset M^n$ orientable, then W^n orientable.

If M, N orientable, $M \times N$, $M \# N$ orientable.

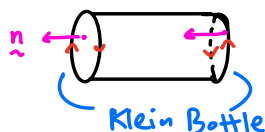
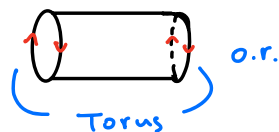


Defⁿ. M^n is a manifold with boundary if M is Hausdorff, $\mathcal{D}^{\text{nd}}$ -countable, and is locally $\mathbb{R}_+^n = \{ (x_1, \dots, x_n) : x_n \geq 0 \}$



Let $[v_1, \dots, v_{n-1}] \in T_p(\partial M)$. Then define $[v_1, \dots, v_{n-1}]$ positively oriented iff $[n, v_1, \dots, v_{n-1}]$ positively oriented in $T_p M$.

If M^n orientable with $\partial M = A^{n-1} \sqcup B^{n-1}$ and $f: A \rightarrow B$ diffeomorphism. Then $M/\sim_f = N$ orientable $\Leftrightarrow f$ orientation reversing.



Orientation Double Cover:

Defⁿ. M^n connected manifold. Define $\theta: \pi_1(M, x) \rightarrow \mathbb{Z}_2$

$\theta([\gamma]) = 0 \Leftrightarrow$ "walking around γ preserves the orientation"

$\Leftrightarrow \exists$ continuously varying frame $B_t = (v_1(t), \dots, v_n(t)) \in T_{\gamma(t)} M$
s.t. $[B_0] = [B_1]$.

Claim: θ is a well-defined homomorphism.



Prop. M orientable $\Leftrightarrow \theta \equiv 0$.

If $\theta \neq 0$, then $\text{Ker } \theta \triangleleft \pi_1(M)$. Then $\exists$ double cover $p: \tilde{M} \rightarrow M$ s.t.

$p_*(\pi_1(\tilde{M}, x)) = \text{Ker } \theta$.

Orientation (Algebraic Topology Version):

lem. $H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \mathbb{Z}$, $n \geq 2$.

pf. LES of $(\mathbb{R}^n, \mathbb{R}^n - 0)$ gives $H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong H_{n-1}(\mathbb{R}^n - 0) \cong H_{n-1}(S^{n-1})$.

□

Observation: choice of generator $\pm 1 \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$ corresponds to choice

of orientation on $\mathbb{R}^n$.

Fix $\mu \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$ generator. Let $B = [v_1, \dots, v_n]$ basis of $\mathbb{R}^n$.

Define $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ $f: (\mathbb{R}^n, \mathbb{R}^n - 0) \rightarrow (\mathbb{R}^n, \mathbb{R}^n - 0)$.
 $\sum \lambda_i e_i \mapsto \sum \lambda_i v_i$.

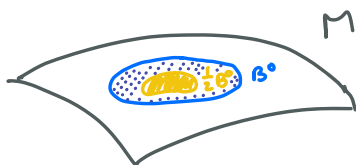
Define

$$\mu_B = f_*(\mu) \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0).$$

Check $B \sim B' \Leftrightarrow \mu_B = \mu_{B'}$.

Defⁿ. Let $x \in M^n$. An **orientation** of M at x is a choice of generator $\mu_x \in H_n(M, M-x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \mathbb{Z}$.
 $\uparrow$
 excision.

Proof of the isomorphism:



choose $B^0 \cong \mathbb{R}^n$. $\frac{1}{2}B^0 \subset B^0$.

$$A = M - \frac{1}{2}B^0 \quad B = B^0 \cong \mathbb{R}^n$$

$$(B, A \cap B) \hookrightarrow (M, A)$$

induces isomorphism

$$H_n(B, A \cap B) \cong H_n(M, A)$$

$$H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong H_n(M, M-x)$$

Local Consistency: $\forall$ open ball $B \subset M$, $x, y \in B$,

$$H_n(M, M-x) \xleftarrow[\varphi]{\cong} H_n(M, M-B) \xrightarrow[\varphi]{\cong} H_n(M, M-y)$$

$$\mu_x \xrightarrow[\varphi \circ \varphi^{-1}]{} \mu_y.$$

Isomorphism by excision.

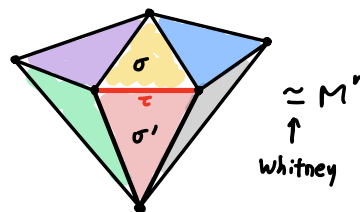
Lec 12.

Thm. M^n is connected and closed manifold. Then M is orientable iff $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$.

Pf. Use simplicial homology. The only possible n cycle is

$$\sum_{\text{all } \sigma \in C_n(M)} (\pm k) \sigma.$$

To see this, note every $(n-1)$ -simplex is contained in precisely 2 n -simplices. Thus once a $k\sigma$ ($k \in \mathbb{Z}$) is chosen, cycle condition forces us to have $k\sigma' \forall \sigma' \in C_n(M)$. □



Thm. M^n closed, connected n -manifold.

Then $H_n(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

pf. Similar to above. □

Universal Coefficient Thm for Homology (Pf later):

$$H_k(X; A) \cong (H_k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} A) \oplus \text{Tor}(H_{k-1}(X; \mathbb{Z}); A).$$

Thm. M closed connected n -manifold. Then the torsion subgroup of $H_{n-1}(M; \mathbb{Z})$ is 0 if orientable, is $\mathbb{Z}_2$ if not.

pf. M compact $\Rightarrow H_*$ is finitely generated and abelian. Thus $H_{n-1}(M) \cong \mathbb{Z}^k \oplus T$ where T torsion, with $|T| < \infty$. Choose prime $2 \neq p \mid |T|$. By UCT for homology:

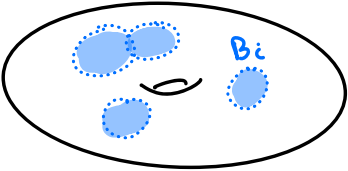
$$\underbrace{H_n(M; \mathbb{Z}_p)}_{\substack{\mathbb{Z}_p \text{ or } 0 \\ \text{orientable or not}}} \cong \underbrace{(H_n(M; \mathbb{Z}) \otimes \mathbb{Z}_p)}_{\substack{\mathbb{Z}_p \text{ or } 0 \\ \text{orientable or not}}} \oplus \text{Tor}(H_{n-1}(M; \mathbb{Z}); \mathbb{Z}_p).$$

$\therefore \text{this is } 0 \uparrow$

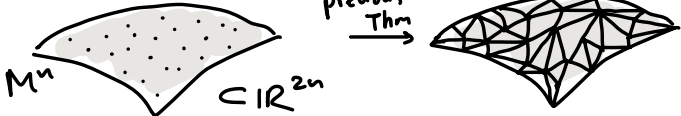
Thus $|T| = 2^k$. Otherwise if $|T|$ odd, then can choose p s.t. $\text{Tor}(T, \mathbb{Z}_p) \neq 0$.
 So only chance where $\text{Tor}(T, \mathbb{Z}_p) \neq 0$ is when $p=2$: In the orientable case, we get
 $\mathbb{Z}_2 \cong \mathbb{Z} \otimes \mathbb{Z}_2 \oplus \text{Tor}(T, \mathbb{Z}_2)$. So T is trivial. In non-orientable case,
 $\mathbb{Z}_2 \cong (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(T, \mathbb{Z}_2)$. So $T \cong \mathbb{Z}_2$. $\square$

e.g. $H_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$.

Thm. Closed n -manifold $M^p \hookrightarrow \mathbb{R}^N$.

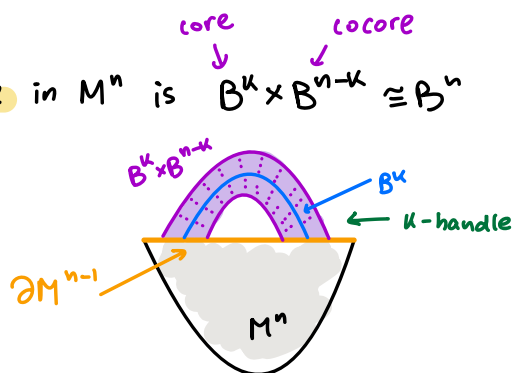
Pf.  finitely many balls $B_i \cong \mathbb{R}^p$.
 $f_i: M \rightarrow S^p$ quotient by collapsing B_i to a pt.
 $f_1 \times \dots \times f_k: M \hookrightarrow S^p \times \dots \times S^p$
 $\subset \mathbb{R}^{p+1} \times \dots \times \mathbb{R}^{p+1}$. $\square$

Thm. Smooth manifolds can be triangulated.

Pf.  previous Thm  (take points sufficiently s.t. regions bounded by points are basically flat. Thus we get a homeomorphism.) $\square$

Handle Decomposition: M^n triangulated. Thicken each n -simplex to be a n -handle.

Defⁿ. A k -handle in M^n is $B^k \times B^{n-k} \cong B^n$



Rmk. If M^n n -mfd with boundary, then $M - \partial M$ is a noncompact n -manifold.

Cohomology with Compact Support

With cellular or simplicial chain complexes. $\exists$ subcomplex of cochain complex consisting of cochains that are non-zero on finitely many cells:

$$\delta: C_c^*(X) \subset C^*(X).$$

Defⁿ. Cochain $\varphi \in C^i(X)$ has **compact support** if $\exists$ compact set $K \subset X$ s.t. $\varphi(\sigma) = 0 \quad \forall \sigma$ a chain in $X - K$, i.e., $\varphi \in C^i(X, X - K)$.

Denote $C_c^i(X)$ the cochains with compact support, and this is a subcomplex:

$$\delta: C_c^i(X) \rightarrow C_c^{i+1}(X).$$

Define

$$H_c^*(X) = H^*(C_c^*(X)).$$

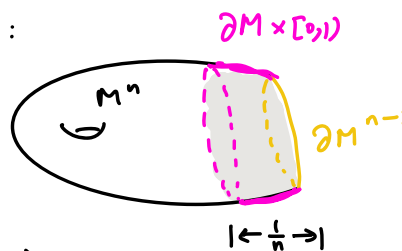
But: $H_c^*(\mathbb{R}) \cong \mathbb{Z}$ is not homotopy invariant.

Prop. M^n compact n -mfd with ∂ . Then $i: (\overset{\circ}{M}, \phi) \hookrightarrow (M, \partial M)$ induces isomorphism $i^*: H^n(M, \partial M) \cong H_c^n(\overset{\circ}{M})$.

Pf. ∂M has a collar neighborhood $\partial M \times [0, 1)$:

pf in Hatcher. Then

$$C_c^k(M) = \bigcup_{n=1}^{\infty} C^k(M, \partial M \times [0, \frac{1}{n})).$$



Note that $\forall n, H^k(M, \partial M \times [0, \frac{1}{n})) \cong H^k(M, \partial M)$.

Thm. Suppose M^n is $\mathbb{R}$ -orientable (that is the interior of a compact n -manifold), then $\exists$ natural isomorphism

$$D_M: H_c^k(M; \mathbb{R}) \xrightarrow{\cong} H_{n-k}(M; \mathbb{R}).$$

Lec 13.

By UCT, if $\mathbb{F}$ a field, then $\text{Ext}(A, \mathbb{F}) = 0$. So

$$\begin{aligned} H^n(X; \mathbb{F}) &\cong \text{Hom}(H_n(X; \mathbb{Z}); \mathbb{F}) \\ &\cong \text{Hom}_{\mathbb{F}}(H_n(X; \mathbb{F}); \mathbb{F}) \end{aligned} \quad \text{UCT for homology.}$$

Therefore

$$\begin{array}{c} \text{evaluation} \\ \downarrow \\ h: H^n(X; \mathbb{F}) \cong H_n^*(X; \mathbb{F}) \end{array} \begin{array}{c} \text{natural} \\ \uparrow \end{array} \leftarrow \text{dual vector space}$$

Poincaré Duality: M closed manifold, $\exists$ natural isomorphism

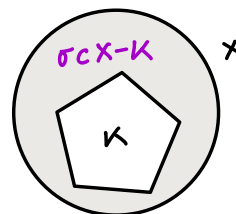
$$D_M: H^k(M; \mathbb{R}) \xrightarrow{\cong} H_{n-k}(M; \mathbb{R})$$

PD + v.s. Duality:

$$\begin{array}{ccc} H^k(M; \mathbb{F}) & \cong & H^{n-k}(M; \mathbb{F}) \\ \parallel & & \parallel \end{array}$$

$$H_k(M; \mathbb{F}) \cong H_{n-k}(M; \mathbb{F})$$

NOT NATURAL.



LAST TIME: Cohomology with compact support

$$H_c^*(X) = H^*(C_c^*(X))$$

X
 $\sigma C X - K$
 K

cochain with compact support: $\varphi \in C$
 $\exists K \subset X$ compact s.t. $\varphi(\sigma) = 0$ for $\sigma \subset X - K$.

Defⁿ. Call I a directed set if it is a set with a partial order s.t.

$$\forall \alpha, \beta \in I, \exists \gamma \in I \text{ s.t. } \alpha \leq \gamma \text{ and } \beta \leq \gamma.$$

$(G_\alpha)_{\alpha \in I}$ is called a direct system of groups if G_α 's abelian groups

- I is a directed set.
- whenever $\alpha < \beta$, $\exists$ homomorphism $f_{\alpha\beta}: G_\alpha \rightarrow G_\beta$.
- If $\alpha < \beta < \gamma$, then $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$.

Define the directed limit

$$\varinjlim G_\alpha = \coprod_{\alpha \in I} G_\alpha / \sim = \frac{\bigoplus_{\alpha \in I} G_\alpha}{\langle a-b: \begin{array}{l} a \in G_\alpha, b \in G_\beta, \\ \exists \gamma > \alpha, \beta \text{ s.t. } f_{\alpha\gamma}(a) = f_{\beta\gamma}(b) \in G_\gamma \end{array} \rangle}$$

If $a \in G_\alpha, b \in G_\beta, \exists \gamma > \alpha, \beta$
& $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$.

Rmk. Suppose γ maximal element in I . Then

$$\varinjlim G_\alpha = G_\gamma.$$

E.g. Recall: $A \subset X$, $\downarrow$ cochain on X vanishes on simplices in A .

$$0 \leftarrow C^*(A) \leftarrow C^*(X) \leftarrow C^*(X, A) \leftarrow 0$$

$$H^*(X, A) = H^*(C^*(X, A)).$$

Define $H^*(X|K) = H^*(X, X-K)$. $\leftarrow$ cohomology of X restricted to K $\leftarrow$ cocycles that vanishes on $\sigma \subset X-K$.

Similarly $H_*(X|K) = H_*(C_*(X, X-K)) = H_*(X, X-K)$.

Note: If $cl(K) \subset U$ open $\subset X$, then

$$H^*(X|K) = H^*(U|K).$$

Now take $I = \{K \subset X : K \text{ compact}\}$ with $\subseteq$ as a partial order.

Given $K \in I, L \in I$. Then $K \cup L \in I$ s.t. $K, L \subseteq K \cup L$.

Thus I directed set. Let

$$G_K = H^*(X|K).$$

Let $K \subset L$. Then

$$i: H^*(X|K) \rightarrow H^*(X|L).$$

Thus $(G_K)_{K \in I}$ directed system.

Prop (H3.33). $\exists$ natural isomorphism

$$\theta: \varinjlim H^*(X|K) \xrightarrow{\cong} H_c^*(X)$$

pf. Onto: Let $\varphi \in H_c^*(K)$ s.t. φ vanishes outside K . Then φ comes from corresponding $H^*(X|K)$.

1-1: $\theta[\varphi] = 0$. Then $\varphi = \delta\psi$ where ψ vanishes outside some K . Then $[\varphi] = 0 \in H^*(X|K)$. $\square$

Special Case: X compact. Then X is a maximal element.
Thus $H_c^*(X) \cong H^*(X)$.

lem. If M^n is a compact mnd with ∂ . Then

$$i^*: H^*(M, \partial M) \xrightarrow{\cong} H_c^*(M \cup \partial M \times [0,1]) \cong H_c^*(M^o).$$

(LAST TIME)

let $K \subset L \subset M$, K, L compact.

$$\begin{array}{ccc} H_n(M|K) \times H^k(M|K) & \searrow \cap & \\ i^* \downarrow & & \downarrow i^* \\ H_n(M|L) \times H^k(M|L) & \nearrow \cap & H_{n-k}(M) \end{array}$$

commutes. let $M =$ oriented n -mnd, define

$$\begin{aligned} D_M: H_c^k(M) &\longrightarrow H_{n-k}(M) \\ \downarrow \cup \\ [\varphi] \in H^k(M|K) &\text{ for some } K \\ D_M [\varphi] &= \mu_K \cap [\varphi] \\ &\in \bigcap_{H_n(M|K)}. \end{aligned}$$

Poincaré Duality: If M is $\mathbb{R}$ -oriented, then

$$D_M: H_c^k(M; \mathbb{R}) \xrightarrow{\cong} H_{n-k}(M; \mathbb{R}).$$

pf. Assume M is interior of some compact manifold. Then $M = \bigcup_{i=1}^p U_i$, $U_i \cong \mathbb{R}^n$

open $\cong \mathbb{B}^n$. Induct on p :

$$p=1: H_c^k(\mathbb{R}^n) \xrightarrow{\cong} H_{n-k}(\mathbb{R}^n).$$

For inductive step, use

|| lem (H 3.36). If $M = U \cup V$, where U, V open, oriented n -manifolds,
then $\exists$ commutative diagram (up to sign):

$$U_i \cap U_j = \begin{cases} \emptyset \\ \text{or } \cong \mathbb{B}^n \end{cases}$$

$$\begin{array}{ccccccccc}
\cdots & \rightarrow & H^k(U \cap V) & \rightarrow & H_c^k(U) \oplus H_c^k(V) & \rightarrow & H_c^k(M) & \rightarrow & H^{k+1}(U \cap V) & \rightarrow \cdots \\
& & \cong \downarrow \cap & & \cong \downarrow \cap & & \cong \downarrow \cap & & \downarrow \cap & & \cong \downarrow \cap \\
\cdots & \rightarrow & H_{n-k}(U \cap V) & \rightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \rightarrow & H_{n-k}(M) & \rightarrow & H_{n-(k+1)}(U \cap V) & \rightarrow \cdots
\end{array}$$

pf of lem: The diagram commutes (up to sign) and the vertical maps except the middle one are isomorphisms by inductive hypothesis. Then by 5-lemma, $H_c^k(M) \cong H_{n-k}(M)$ isomorphism as well. $\square$

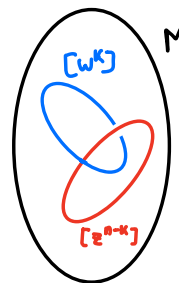
Then by induction, done. $\square$

lec 14.

3 versions of Poincaré Duality:

① M^n closed, oriented.

$$\begin{aligned} H^k(M) &\xrightarrow{\cong} H_{n-k}(M) \\ \varphi &\longmapsto [M] \cap \varphi. \end{aligned}$$



② M^n oriented.

$$D_M: H_c^k(M) \xrightarrow{\cong} H_{n-k}(M)$$

③ M^n compact, oriented manifold with ∂ .

$$H^k(M, \partial M) \xrightarrow{\cong} H_{n-k}(M)$$

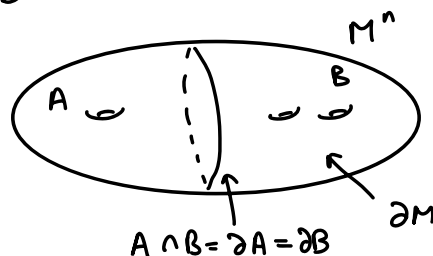
④ Lefschetz Duality:

Let M^n be a compact connected $\mathbb{R}$ -oriented n -mfd.

$A^{n-1}, B^{n-1} \subset \partial M$ compact submanifold with ∂ .

$$A \cup B = \partial M \quad A \cap B = \partial A = \partial B$$

$$\begin{aligned} D_M: H^k(M, A; \mathbb{R}) &\xrightarrow{\cong} H_{n-k}(M, B; \mathbb{R}) \\ \varphi &\longmapsto [M] \cap \varphi. \end{aligned}$$



pf.

$$\begin{array}{ccccccc} \cdots \rightarrow H^k(M, \partial M) & \rightarrow & H^k(M, A) & \rightarrow & H^k(\partial M, A) & \rightarrow & H^{k+1}(M, \partial M) \rightarrow \cdots \\ [M] \cap \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong [M] \cap \\ \cdots \rightarrow H_{n-k}(M) & \rightarrow & H_{n-k}(M, B) & \rightarrow & H_{n-k-1}(B) & \rightarrow & H_{n-k-1}(M) \rightarrow \cdots \end{array}$$

$\downarrow \cong$ excision $\cong H^k(B, \partial B)$
 $\cong \downarrow [B] \cap$

The black vertical maps are isomorphisms by Poincaré Duality. Then apply 5-lemma. $\square$

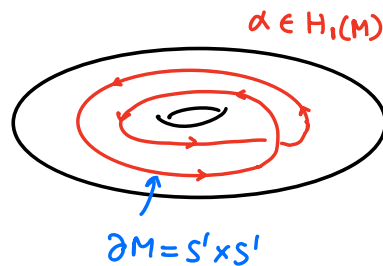
eg. $M = S^1 \times D^2$.

$$H^k(S^1 \times D^2) \cong H_{n-k}(S^1 \times D^2, S^1 \times S^1)$$

$$H_n(M, \partial M) \cong \mathbb{Z}[\hat{D}^2]$$

generator $[\varphi] \in H^1(M)$.

$$\langle \varphi, \alpha \rangle = 2 = \#(\alpha \cap D^2).$$

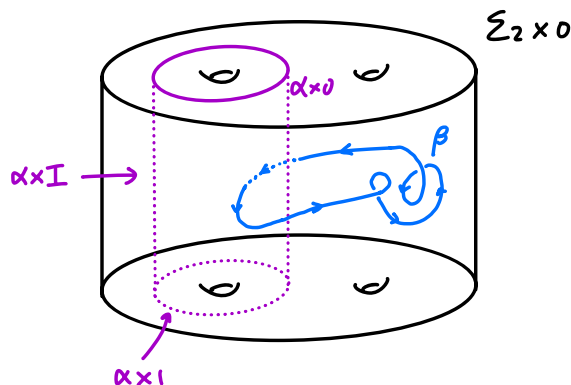


$$M = \Sigma_2 \times [0, 1] \quad \alpha \in \Sigma \times 0.$$

$$\mathbb{Z} \cong H_2(M, \partial M) \cong H^1(M)$$

$$[\alpha \times I] \mapsto [\varphi]$$

$$\varphi(\beta) = \#(\beta \cap (\alpha \times I)).$$



Summary: Correspondence of Lefschetz duality is given by

$$\alpha \longleftrightarrow \varphi_\alpha$$

↑
counts intersection # with α .

⑤ Alexander Duality:

Let K be a locally contractible, compact, non-empty, proper subspace of S^n . Then

$$\tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z}).$$

Pf. Assume $K = M^n$ = compact manifold with ∂ .

$$S^n - M = \text{int}(S^n - \bar{M})$$

↑
cpt mfd with ∂

$$H_i(S^n - M) \underset{P.O.}{\cong} H_{n-i}^{n-i}(S^n - M) \cong H^{n-i}(S^n - \bar{M}, \partial M)$$

$$\cong H^{n-i}(S^n, \partial M) \quad \text{excision.}$$

$$\cong H^{n-i-1}(M) \quad \text{LES for } (S^n, M).$$

using

$$H^*(S^n) = 0 \text{ usually.}$$

□

e.g.

$$\textcircled{1} K = S^1 \times D^2 \simeq S^1$$



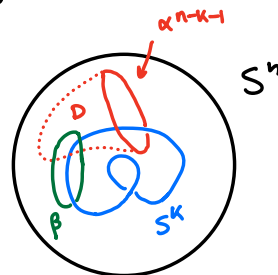
$$H_1(S^3 - K) \cong \mathbb{Z}.$$

"homology does not see knotting".

$$\textcircled{2} S^K \subset S^n.$$

$$H_{n-k-1}(S^n - S^K) \cong H^k(S^K) \cong \mathbb{Z}$$

$$[\alpha] \xleftrightarrow{AD} [\psi] \\ \alpha = \partial D^{k+1}$$



$$\psi(\beta) = \#(\beta \cap D) \in \mathbb{Z}.$$

Half Lives, Half Dies:

$M = \text{compact, orientable } (2n+1)\text{-manifold and } i_*: H_n(\partial M) \rightarrow H_n(M).$

Then $\text{rank}(\text{Ker } i_*) = \text{rank}(\text{Im } i_*).$

pf. For simplicity, assume $2n+1=3$. LES $(M, \partial M)$:

$$0 \rightarrow H_3(M) \rightarrow H_3(M, \partial M) \rightarrow H_2(\partial M) \rightarrow \dots \rightarrow H_0(M) \rightarrow H_0(M, \partial M) \rightarrow 0.$$

In the middle,

$$H_2(M, \partial M) \xrightarrow{\Delta} H_1(\partial M) \xrightarrow{i_*} H_1(M)$$

$\parallel ?$

$$K \oplus I$$

$$I \cong \text{Im } i_*$$

$$K = \text{Ker } i_* = \text{Im } \Delta.$$

Lemma 3.5. If M is a compact orientable 3-manifold then the image of the boundary map $H_2(M, \partial M) \rightarrow H_1(\partial M)$ has rank equal to one half the rank of $H_1(\partial M)$.

Here 'rank' means the number of $\mathbb{Z}$ summands in a direct sum splitting into cyclic groups. Equivalently, we can use homology with $\mathbb{Q}$ coefficients and replace 'rank' by 'dimension.'

Proof: With $\mathbb{Q}$ coefficients understood, consider the commutative diagram

$$\begin{array}{ccccc} H_2(M, \partial M) & \xrightarrow{\partial} & H_1(\partial M) & \xrightarrow{i_*} & H_1(M) \\ \parallel & & \parallel & & \parallel \\ H^1(M) & \xrightarrow{i^*} & H^1(\partial M) & \xrightarrow{\delta} & H^2(M, \partial M) \end{array}$$

where the vertical isomorphisms are given by Poincaré duality. Since we have coefficients in a field, the map i^* is the dual of i_* , obtained by applying $\text{Hom}(-, \mathbb{Q})$. Linear algebra then implies that $\dim \text{Ker } i_* = \dim \text{Coker } i^*$. Hence $\dim \text{Im } \partial = \dim \text{Ker } i_* = \dim \text{Coker } i^* = \dim \text{Coker } \partial$ which implies the result. $\square$

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Universal Coefficients in Homology.

Tensor Products :

Given two Ab Gps A, B . Want to make a group $A \otimes B$. The general element is of the form

$$a_1 \otimes b_1 + \dots + a_n \otimes b_n \quad a_i \in A, b_i \in B$$

$$\text{st. } \left. \begin{array}{l} a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 \\ (a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b. \end{array} \right\} \Rightarrow na \otimes b = a \otimes nb. \quad (*)$$

e.g. let $A = \mathbb{Z}^m = \langle e_1, \dots, e_m \rangle$, $B = \mathbb{Z}^n = \langle f_1, \dots, f_n \rangle$

$$A \otimes B = \langle e_i \otimes f_j \mid 1 \leq i \leq m, 1 \leq j \leq n \rangle.$$

Thus we may define

$$A \otimes B = \text{FAG}(A, B) / \sim \quad \text{where } \sim \text{ satisfies } (*).$$

If $f: A \rightarrow A$, $g: B \rightarrow B$ homomorphisms, then define

$$f \otimes g: A \otimes B \rightarrow A \otimes B$$

$$a \otimes b \mapsto f(a) \otimes g(b).$$

Defⁿ. A **free resolution** of a group A is an exact seq

$$0 \rightarrow R \xrightarrow{i} F \xrightarrow{\pi} A \rightarrow 0$$

st. R, F are free abelian.

Prop. Given abelian gp A , take a free resolution of A

$$0 \rightarrow R \xrightarrow{i} F \xrightarrow{\pi} A \rightarrow 0$$

then

$$R \otimes G \xrightarrow{i \otimes 1} F \otimes G \xrightarrow{\pi \otimes 1} A \otimes G \rightarrow 0$$

is exact. Now define

$$\text{Tor}(A, G) = \text{Ker}(i \otimes 1).$$

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Lem (H 3.1). Given a free resolution F of a gp H , F' of H' , and a homomorphism $\alpha_{-1}: H \rightarrow H'$. Then $\exists$ a chain map α_i s.t. the diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 \xrightarrow{f_0} H \longrightarrow 0 \\ & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 \downarrow \alpha_{-1} \\ \cdots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 \xrightarrow{f'_0} H' \longrightarrow 0 \end{array}$$

Moreover, any two such α 's are chain homotopic.

pf. Inductively, assume α_i is defined and diagram commutes.

$$\begin{array}{ccccc} x & \xrightarrow{\quad} & 0 \\ F_{n+1} & \xrightarrow{f_{n+1}} & F_n & \xrightarrow{f_n} & F_{n-1} \\ \downarrow \alpha_{n+1} & & \downarrow \alpha_n & & \downarrow \alpha_{n-1} \\ F'_{n+1} & \xrightarrow{f'_{n+1}} & F'_n & \xrightarrow{f'_n} & F'_{n-1} \end{array}$$

$\exists \gamma \mapsto \alpha_n(f_{n+1}x) \mapsto 0$

Pick free generating set for F_{n+1} , choose γ for each generator defines α_{n+1} .
Clearly diagram commutes (α is a chain map).

Suppose $\alpha, \gamma: F \rightarrow F'$ be two s.t. chain maps. They agree on $\alpha_{-1}, \gamma_{-1}: H \rightarrow H'$. Define a chain homotopy $\lambda_i: F_i \rightarrow F_{i+1}$ s.t.

$$\begin{array}{c} \partial' \lambda + \lambda \partial = \alpha - \gamma := \beta. \\ \uparrow \quad \quad \uparrow \\ f' \quad \quad f \end{array}$$

More specifically,

$$f'_{i+1} \lambda_i + \lambda_{i-1} f_i = \beta_i.$$

Define $\lambda_{-1} = 0$. Then

$$f'_{i+1} \lambda_i = \beta_i - \lambda_{i-1} f_i. \quad (*)$$

Consider the diagram below: Assume λ_{i-1} is defined. Suppose $z = \lambda_i(x)$.

Then need $f'_{i+1}(z) = \beta_i(x) - \lambda_{i-1}(f_i(x))$.

Universal Coefficient Theorem.

Let C be a free chain complex with $\mathbb{Z}$ coefficients, $G = \text{Ab Grp}$. Then $\exists$ SES that splits (not in a natural way):

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C \otimes G) \longrightarrow \text{Tor}(H_{n-1}(C), G) \longrightarrow 0$$

$$[z] \otimes g \longmapsto [z \otimes g]$$

pf. Chain complex C splits:

$$0 \longrightarrow Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0$$

Since C_n is free abelian, $C_n \cong Z_n \oplus B_{n-1}$. Since everything is free abelian, get a SES

$$0 \longrightarrow Z_n \otimes G \xrightarrow{i_n \otimes 1} C_n \otimes G \xrightarrow{\partial_n \otimes 1} B_{n-1} \otimes G \longrightarrow 0.$$

$\partial=0$ (above $Z_n \otimes G$) $\partial=0$ (above $B_{n-1} \otimes G$)

This gives a LES of homology group:

$$\cdots \longrightarrow B_n \otimes G \xrightarrow{i_n \otimes 1} Z_n \otimes G \longrightarrow H_n(C \otimes G) \longrightarrow B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes 1} Z_{n-1} \otimes G$$

$\uparrow$
 connecting homomorphism.

This gives

$$0 \longrightarrow \text{Coker}(i_n \otimes 1) \longrightarrow H_n(C \otimes G) \longrightarrow \text{Ker}(i_{n-1} \otimes 1) \longrightarrow 0$$

$$H_n(C) \otimes G = \left(\frac{Z_n}{B_n} \right) \otimes G = \frac{Z_n \otimes G}{B_n \otimes G} \qquad \qquad \text{Tor}(H_{n-1}(C), G)$$

□

Cor. Singular Homology

$$H_n^{\text{sing}}(X; G) \cong H_n(X; \mathbb{Z}) \otimes G \oplus \text{Tor}(H_{n-1}(X; \mathbb{Z}), G).$$

$$\cong H_n(C_{\text{sing}}(X) \otimes G)$$

Cor. If $H_n(X; \mathbb{Z})$ is f.g., Then $H_n(X; \mathbb{Q}) = H_n(X; \mathbb{Z}) \otimes \mathbb{Q}$.

$\forall n$.

$= \mathbb{Q}$ -vector space with
dim rank($H_n(X; \mathbb{Z})$).

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Kunneth Thm. If X, Y are CW complexes (not needed), then there is a natural SES

$$0 \rightarrow \bigoplus_{i+j=n} (H_i(X) \otimes H_j(Y)) \rightarrow H_n(X \times Y) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}(H_i(X), H_j(Y)) \rightarrow 0.$$

The sequence splits (but not naturally).

e.g. $X = Y = \mathbb{R}P^2$. $H_i(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_2 & i=1 \\ 0 & i \geq 2 \end{cases}$

$W = \mathbb{R}P^2 \times \mathbb{R}P^2$. Then

$$H_3(W) = \text{Tor}(H_1(\mathbb{R}P^2), H_1(\mathbb{R}P^2)) = \text{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Homology Cross Product

$$H_p(X) \times H_q(Y) \xrightarrow{\times} H_{p+q}(X \times Y)$$

defined using cellular homology (C_*, d) . On the chain level,

$$e^p: I^p \rightarrow X \quad e^q: I^q \rightarrow Y. \quad I^p = [0, 1]^p.$$

$e^p \in C_p(X)$, $e^q \in C_q(Y)$. Define

$$e^p \times e^q: I^p \times I^q \equiv I^{p+q} \rightarrow X \times Y$$

$$(e^p \times e^q)(x, y) = (e^p(x), e^q(y)).$$

Prop (H, 3B.1). $d_{X \times Y}(e^p \times e^q) = (de^p) \times e^q + (-1)^p e^p \times (de^q)$. □

Prop (H, 3B.2). If $f: X \rightarrow Z$, $g: Y \rightarrow W$ are cellular maps ($f(X^n) \subset Z^n \dots$).

Then $(f \times g): X \times Y \rightarrow Z \times W$ satisfies

$$(f \times g)_* = f_* \times g_*. \quad \square$$

(3B.1) is a special case of (3B.2).

Proof of Kunneth. Let X, Y be CW complexes. Define a CW structure on $X \times Y$ by letting a cell in $X \times Y$ to be $(\text{cell in } X) \times (\text{cell in } Y)$.

|| Aside: X does not depend on cell decomposition of X & Y . Every $f: X \rightarrow X'$ is homotopic to a cellular map. The result follows.

Therefore,

$$C_n(X \times Y) = \bigoplus_{i+j=n} C_i(X) \otimes C_j(Y).$$

and

$$d(e^i \otimes e^j) = (de^i) \otimes e^j + (-1)^i e^i \otimes (de^j). \quad (*)$$

Now just do algebra: Do the tensor product of chains $(C, d), (C', d')$:

$$(C \otimes C')_n := \bigoplus_{i+j=n} C_i \otimes C'_j$$

with d given by $(*)$. Need to check $d^2 = 0$ (i.e., $C' \otimes C'$ with d is a chain complex).

Special Cases: • $C'_i = \begin{cases} G & i=0 \\ 0 & i>0. \end{cases}$

Then Kunneth Thm is just UCT for homology.

- $d_{C'} \equiv 0$. So $H_*(C') = C'$. Then $d_{C'} \equiv 0$ implies $C \otimes C'_j$ is a subcomplex. So

$$(C \otimes C')_n = \bigoplus_{i+j=n} (C_i \otimes C'_j) = \bigoplus_j (C \otimes C'_j).$$

Therefore

$$H_n(C \otimes C') = \bigoplus_{i+j=n} H_i(C) \otimes C'_j = \bigoplus_{i+j=n} H_i(C) \otimes H_j(C').$$

General Case: SES of free chain complex

$$0 \rightarrow Z_i \rightarrow C_i \xrightarrow{d} B_{i-1} \rightarrow 0$$

Hence splits. Thus $\otimes C'$ gives $(C' = \bigoplus_j C'_j)$:

$$0 \rightarrow Z_i \otimes C' \rightarrow C_i \otimes C' \rightarrow B_{i-1} \otimes C' \rightarrow 0$$

is a SES. Taking $\oplus_i$ gives a SES of chain complexes

$$0 \rightarrow Z \otimes C' \rightarrow C \otimes C' \xrightarrow{\text{deg} - 1} B \otimes C' \rightarrow 0$$

gives a LES of homology:

$$\dots \xrightarrow{i_n} H_n(Z \otimes C') \rightarrow H_n(C \otimes C') \rightarrow H_{n-1}(B \otimes C') \xrightarrow{i_{n-1}} \dots$$

B & Z have $d=0$. By special case:

$$H_n(Z \otimes C') = \bigoplus_{i+j=n} Z_i \otimes H_j(C')$$

$$H_n(B \otimes C') = \bigoplus_{i+j=n-1} B_i \otimes H_j(C')$$

and $i_n: H_n(B \otimes C') \rightarrow H_n(Z \otimes C')$ comes from inclusion $B_n \hookrightarrow Z_n$.

So LES splits into SES:

$$0 \rightarrow \text{Coker } i_n \rightarrow H_n(C \otimes C') \rightarrow \text{Ker } i_{n-1} \rightarrow 0$$

where

$$\begin{aligned} \text{Coker } i_n &= \bigoplus_{i+j=n} \left(\frac{Z_i \otimes H_j(C')}{\text{Im } i_n} \right) = \bigoplus_{i+j=n} \left(\frac{Z_i}{B_i} \right) \otimes H_j(C') \\ &= \bigoplus_{i+j=n} H_i(C) \otimes H_j(C'). \end{aligned}$$

Define Tor by free resl.

$$0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i(C) \rightarrow 0.$$

Apply $\otimes H_j(C')$ gives a SES

$$0 \rightarrow \text{Tor}(H_i(C), H_j(C')) \rightarrow B_i \otimes H_j(C') \rightarrow Z_i \otimes H_j(C') \rightarrow H_i(C) \otimes H_j(C') \rightarrow 0$$

Therefore

$$\text{Ker } i_{n-1} = \bigoplus_{i+j=n-1} \text{Tor}(H_i(C), H_j(C')).$$

□