Complex Algebraic Geometry Oral Exam Practice Problems

Tianyi Wang

Problems Given by Tony Pantev

Contents

1 Problems

1. Let X be a complex manifold. Let Z be a analytic subvariety (something cut out by analytic equations, but not necessarily smooth) of codimension ≥ 2. Suppose that on $X - Z$ we have a holomorphic line bundle $\mathcal{L}$. Show that $\mathcal{L}$ extends uniquely to a holomorphic line bundle on X. Is the extension to X still possible if instead $\mathcal{L}$ is defined on Z, not on $X - Z$?

2. Suppose that we are in $\mathbb{P}^4$. Let H be a smooth hypersurface of degree $d \geq 3$. Can it contain a plane? That is, could there be a linearly embedded $\mathbb{P}^2$ (by linearly embedded $\mathbb{P}^2$ we mean induced by linear maps $V \to W$ of vector spaces on $\mathbb{P}(V) \to \mathbb{P}(W)$, or equivalently zero locus cut out by linear equations) such that H contains that $\mathbb{P}^2$?

3. Let $\mathbb{P}^k \subset \mathbb{P}^n$ be a linearly embedded projective k-plane. Blow up $\mathbb{P}^n$ along $\mathbb{P}^k$ gives a new manifold X. What is the Picard group of X? That is, describe all the line bundles on X up to isomorphism.

4. Describe all possible morphisms $\mathbb{P}^k \to \mathbb{P}^n$.

5. Let F_1, F_2 be two smooth curves in $\mathbb{P}^2$ of degree d that intersects transversely. In $\mathbb{P}^1 \times \mathbb{P}^2$, with coordinates $(u_1 : u_2)$ of $\mathbb{P}^1$ and $(x : y : z)$ of $\mathbb{P}^2$. Consider the bihomogeneous polynomial $F = u_1 F_1(x, y, z) + u_2 F_2(x, y, z)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Denote $X = Z(F)$.

(a) Compute the Picard group of $\mathbb{P}^1 \times \mathbb{P}^2$, identify the line bundle of which F is a section.

(b) Check that X is smooth.

(c) We have the second projection map $p_2 : X \to \mathbb{P}^2$. Show that p_2 is a blow up of $\mathbb{P}^2$ at some number of points. Find that number.

(d) Look at the first projection $p_1 : X \to \mathbb{P}^1$. That is a map from X (a surface) to a curve $\mathbb{P}^1$. Show that the fibers of p_1 is connected.

(e) What is the genus of the general fiber which is smooth?

(f) Suppose F_1, F_2 are generic, and assume that all the singular fiber of p_1 has a single ordinary double point. How many singular fibers are there?

6. (a) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^2$ as a curve of degree 2?

   (b) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^2$ as a curve of degree 3?

   (c) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve of by degree (a, b) with $a, b \geq 3$?

   (d) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^d$ as a curve of degree d?

7. Consider the surface S given by the equation $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ in $\mathbb{P}^3$. Show that S is smooth. What is $\mathrm{Pic}(S)$?

8. Prove the Birkhoff-Grothendieck theorem: If E is a vector bundle of rank r over $\mathbb{P}^1$, then it decomposes into a direct sum of line bundles

$$E \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

   for some integers $a_1, ..., a_r$.

9. Let $n \geq 0$ be an integer. The Hirzebruch surface Y_n is a ruled surface (in fact a $\mathbb{P}^1$-bundle) over $\mathbb{P}^1$, defined by

$$Y_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)).$$

   Given Y_n for any n, compute $H^{\cdot}(Y_n)$ and $\mathrm{Pic}(Y_n)$.

10. Let S be a smooth surface of degree d in $\mathbb{P}^3$. Compute its Hodge diamond. In particular, what is the Hodge diamond of a quadric ($d = 2$) and cubic ($d = 3$) surface in $\mathbb{P}^3$?

11. Let S be a Hopf surface (2-dim Hopf manifold). Recall that if q is a nonzero complex number, then $\mathbb{Z}$ acts on $\mathbb{C}^2 - 0$ by $n \cdot (z, w) = (q^n z, q^n w)$. And $S = (\mathbb{C}^2 - 0)/\mathbb{Z}$. Check that S is compact, and it is not Kähler.

12. Let S be a smooth quadric (degree 2) surface in $\mathbb{P}^3$. Compute the dimension of its Lefschetz decomposition (take the Kähler form to be the Fubini-Study restricted to the quadric S). Do the same for a smooth cubic surface.

13. Let X be a smooth d dimensional variety, and p a point in X. Let $\hat{X}$ be the blow up of X at p. Then the exceptional divisor is $\mathbb{P}(N_{p/X}) = \mathbb{P}(T_p X) \cong \mathbb{P}^{d-1}$. Then $\mathcal{O}_{\hat{X}}(E)|_E \cong \mathcal{O}_{\mathbb{P}^{d-1}}(n)$ for some n. What is n?

14. Let X be a smooth variety of dimension d, and $A \subset X$ a submanifold of dimension a. Denote $\hat{X} = \operatorname{Bl}_A X$. The exceptional divisor is $E = \mathbb{P}(N_{A/X})$, which is a $\mathbb{P}^{d-a-1}$-bundle over A. Then restricting to the fibers of $E \to A$, which is a $\mathbb{P}^{d-a-1}$, we have $\mathcal{O}_{\hat{X}}(E)|_{\mathbb{P}^{d-a-1}} \cong \mathcal{O}_{\mathbb{P}^{d-a-1}}(n)$ for some n. What is n?

15. Compute the cohomology groups $H^q(\mathbb{P}^n, \mathcal{O}(m))$ for all q, n, m.

16. Consider the surface X given by $x_0^2 + x_1^2 + x_2^2 = 0$ in $\mathbb{P}^3$.

    (a) Show that X is singular at the point $p = (0 : 0 : 0 : 1)$, and that $X - p$ is smooth.

    (b) Blow up $\mathbb{P}^3$ at p and let $Y \subset \operatorname{Bl}_p \mathbb{P}^3$ be the strict transform of $X \subset \mathbb{P}^3$, i.e. Y is the closure of $X - p$ inside $\operatorname{Bl}_p \mathbb{P}^3$. Show that Y is smooth.

    (c) Let $q \in X$ be a point in X with $q \neq p$. Let L be a line passing through p, q. Check that $L \subset X$. In other words, X is a projective cone in $\mathbb{P}^3$ with vertex p.

    (d) Show that Y is a Hirzebruch surface, so that $Y \cong Y_n$ for some n (you don't need to show that is the value of n in this part).

    (e) What is $\operatorname{Pic}(Y)$?

    (f) By (d) we know $Y \cong Y_n$ for some n. What is n?

17. Let C be a smooth curve in $\mathbb{P}^2$. What are the possible genera of C? Is every genus allowed?

18. Let C be a smooth curve in $\mathbb{P}^2$ which has genus 3 (assume hyperelliptic require $g > 1$ instead of $g \geq 1$ if you want). Can C be hyperelliptic?

19. Suppose C is a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, what are the possible genera?

20. Describe a way to compute the Hodge numbers of a hypersurface X in the weighted projective space $\mathbb{P}(Q) = \mathbb{P}(q_0, ..., q_n)$ defined by the equation $f(x_0, ..., x_n) = 0$ of degree d.

21. Take a smooth conic in $\mathbb{P}^2$. Show that there is a unique double cover of $\mathbb{P}^2$ branched along the conic. That is, a surface $S \to \mathbb{P}^2$ smooth and 2:1 and branched along the conic. Show that S is smooth, and find its Hodge numbers.

22. Take a smooth quartic curve in $\mathbb{P}^2$. Show that there exists a unique double cover X of $\mathbb{P}^2$ branched along the quartic. Show that X is smooth and find its Hodge numbers.

23. Now let X be a degree 2 cover of $\mathbb{P}^2$ branched along a degree 6 curve in $\mathbb{P}^2$. What is the Hodge diamond of X?

24. Find an example of a complex manifold X such that $\operatorname{Pic}(X) \ncong H^2(X; \mathbb{Z})$.

25. Let $\mathcal{L}$ be a line bundle over a curve X of genus g. How many square root does it have?

2 Solutions to Problems

Problem 1. Let X be a complex manifold. Let Z be a analytic subvariety (something cut out by analytic equations, but not necessarily smooth) of codimension ≥ 2. Suppose that on $X - Z$ we have a holomorphic line bundle $\mathcal{L}$. Show that $\mathcal{L}$ extends uniquely to a holomorphic line bundle on X. Is the extension to X still possible if instead $\mathcal{L}$ is defined on Z, not on $X - Z$?

Proof. We cover X by small open sets $X = \bigcup_i U_i$ such that on each $U_i - Z$ the line bundle $\mathcal{L}$ is trivial. Then the line bundle $\mathcal{L}$ is completely characterized by the transition functions $g_{ij} \in \mathcal{O}_X^*(U_{ij} - Z)$ where $U_{ij} = U_i \cap U_j$. Since $\operatorname{codim}_X Z \geq 2$, we have that Z is a codimension ≥ 2 subset in U_{ij}. Hence Hartog's theorem gives unique extensions $\psi_{ij} \in \mathcal{O}_X(U_{ij})$ of g_{ij}.

This form of Hartog's theorem we are using is stronger than the usual Hartog's theorem. It states that if Z has codimension at least 2 then $\mathcal{O}_X(U - Z) \cong \mathcal{O}_X(U)$ for all open set U in X, so we can always extend if the missing region has large enough codimension. [4] Theorem 1.25 gives a proof for the simple case, but it is not hard to generalize this proof.

We need to check that ψ_{ij} does not vanish at anywhere. Otherwise, $Z(\psi_{ij})$ would be an analytic subset of codimension 1. Since on $U_{ij} - Z$ the function ψ_{ij} never vanishes, we conclude that $Z(\psi_{ij}) \subset Z$. But then $1 = \operatorname{codim}_X(Z(\psi_{ij})) \geq \operatorname{codim}_X(Z) = 2$ which is a contradiction. Hence we conclude that ψ_{ij} are nowhere vanishing hence $\psi_{ij} \in \mathcal{O}_X^*(U_{ij})$.

We still need to show that ψ_{ij}'s are honest transition functions. That is, we need to check the cocycle condition $\psi_{ij}\psi_{jk}\psi_{ki} = 1$ on $U_{ijk} = U_i \cap U_j \cap U_k$. This is true on $U_{ijk} - Z$. Hence by identity theorem of holomorphic functions, this must also be true on the entire U_{ijk}.

Therefore, the transition functions $\psi_{ij} \in \mathcal{O}_X^*(U_{ij})$ defines a holomorphic line bundle on X that uniquely extends $\mathcal{L}$. $\square$

Problem 2. Suppose that we are in $\mathbb{P}^4$. Let H be a smooth hypersurface of degree ≥ 3. Can it contain a plane? That is, could there be a linearly embedded $\mathbb{P}^2$ (by linearly embedded $\mathbb{P}^2$ we mean induced by linear maps $V \to W$ of vector spaces on $\mathbb{P}(V) \to \mathbb{P}(W)$, or equivalently zero locus cut out by linear equations) such that H contains that $\mathbb{P}^2$?

Proof. No, a smooth hypersurface H can never contain a projective plane $\mathbb{P}^2$, otherwise it would be singular. Here is the proof.

Up to a projective change of coordinates, we can always assume that our plane $\Pi \cong \mathbb{P}^2$ is given by $\Pi = Z(z_0, z_1)$. Suppose $H = Z(Q)$ for some homogeneous polynomial Q of degree $d \geq 3$. Now for the sake of contradiction, assume $\Pi \subset H$. Then by projective Nullstellensatz, we see that $Q \in (z_0, z_1)$. Hence we can write

$$Q = z_0 S + z_1 T$$

for some homogeneous polynomial S, T of degree $d - 1$. Now we take derivatives. Note that for $i \geq 2$,

$$\partial_i Q = z_0 \partial_i S + z_1 \partial_i T$$

which vanishes identically on Π because $z_0 = z_1 = 0$ there. Now note that

$$\partial_0 Q = S + z_0 \partial_0 S + z_1 \partial_0 T \qquad \partial_1 Q = z_0 \partial_1 S + T + z_1 \partial_1 T.$$

Hence we see that when restricted to to Π we have $\partial_0 Q = S$ and $\partial_1 Q = T$ again because $z_0 = z_1 = 0$ on Π. If $S = T = 0$ on some point of Π, then that point would be a singularity of H; otherwise, $S|_\Pi, T|_\Pi$ are some homogenous polynomials with positive degrees that do not vanish entirely on Π. But since $\Pi \cong \mathbb{P}^2$, by Bézout's theorem, $Z(S|_\Pi) \cap Z(T|_\Pi) \neq \emptyset$. This is the desired singularity of H, and we are done. $\square$

Problem 3. Let $\mathbb{P}^k \subset \mathbb{P}^n$ be a linearly embedded projective k-plane. Blow up $\mathbb{P}^n$ along $\mathbb{P}^k$ gives a new manifold X. What is the Picard group of X? That is, describe all the line bundles on X up to isomorphism.

Proof. Let $X = \mathrm{Bl}_{\mathbb{P}^k} \mathbb{P}^n$. By Proposition 3.24 of [4] we see that X is Kähler, hence we can use Hodge theory. Consider the long exact sequence of sheaf cohomology induced by the exponential sequence:

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots.$$

We will show that $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathcal{O}_X) = 0$, so $c_1 : H^1(X, \mathcal{O}_X^*) \cong H^2(X, \mathbb{Z})$, which reduces the problem to calculating the singular cohomology of X in degree 2 by de Rham theorem. We note that X is the blow up of a Kähler manifold along a compact submanifold, hence is again Kähler by Proposition 3.24 of [4]. Hence we may use Hodge theory.

To show that $H^1(X, \mathcal{O}_X) = 0$, we show that $h^{0,1} = 0$. Note that $b_1 = h^{10} + h^{01} = 2h^{0,1}$ by Hodge decomposition and conjugation symmetry, it suffices to show that $b_1 = 0$. We do so by showing that X is simply connected. Let $N \to \mathbb{P}^k$ be the normal bundle of $\mathbb{P}^k$ in $\mathbb{P}^n$. Then we have

$$X = \mathbb{P}^n \cup_{\mathbb{P}^k} \mathbb{P}(N).$$

Thus by van Kampen theorem,

$$\pi_1(X) = \pi_1(\mathbb{P}^n) *_{\pi_1(\mathbb{P}^k)} \pi_1(\mathbb{P}(N)).$$

Now $\pi_1(\mathbb{P}^n) = \pi_1(\mathbb{P}^k) = 0$. To compute $\pi_1(\mathbb{P}(N))$, we note that $\mathbb{P}(N) \to \mathbb{P}^k$ has fiber $\mathbb{P}^{n-k-1}$. Hence the long exact sequence of homotopy group in part reads

$$\cdots \to \pi_1(\mathbb{P}^{n-k-1}) \to \pi_1(\mathbb{P}(N)) \to \pi_1(\mathbb{P}^k) \to \cdots$$

Since $\pi_1(\mathbb{P}^{n-k-1}) = \pi_1(\mathbb{P}^k) = 0$, by exactness $\pi_1(\mathbb{P}(N)) = 0$. Hence we see that $\pi_1(X) = 0$. Thus X is indeed simply connected. This shows that $h^{0,1} = 0$, hence $H^1(X, \mathcal{O}_X) = 0$.

Next we show that $H^2(X, \mathcal{O}_X) = 0$. By conjugation symmetry, it suffices to show that $H^0(X, \Omega_X^2) = 0$. Suppose otherwise, we take a nonzero holomorphic 2-form ω on X. Let E be the exceptional divisor of X. Then $X \setminus E \cong \mathbb{P}^n \setminus \mathbb{P}^k$. Therefore this gives a nonzero holomorphic 2-form on $\mathbb{P}^n \setminus \mathbb{P}^k$. By Hartog's theorem, this extends to a nonzero 2-form on $\mathbb{P}^n$ for $k \leq n-2$. But

$h^{2,0}(\mathbb{P}^n) = 0$, which gives the desired contradiction. In fact, we have $H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = 0$ if $p \neq q$ and $H^p(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = \mathbb{C}$ for $0 \leq p \leq n$. Thus $H^2(X, \mathcal{O}_X) = 0$.

We conclude that the Chern map

$$c_1 : H^1(X, \mathcal{O}_X^*) \xrightarrow{\cong} H^2(X, \mathbb{Z})$$

is an isomorphism. Hence the question reduces to calculate the singular cohomology $H^2(X, \mathbb{Z})$ with integer coefficients. Very generally, we have the following result from Theorem 7.31 of [4]:

$$H^k(\mathrm{Bl}_Y\, S; \mathbb{Z}) \cong H^k(S, \mathbb{Z}) \oplus \bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y, \mathbb{Z}), \tag{1}$$

where $r = \operatorname{codim}_S Y$.

In our case, $S = \mathbb{P}^n$ and $Y = \mathbb{P}^k$, and $\mathrm{Bl}_Y\, S = X$. Hence the formula gives

$$H^2(X, \mathbb{Z}) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \oplus H^0(\mathbb{P}^k, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

Hence we conclude that

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The proof of (1) in [4] used Hodge theory. But in our case this can be directly obtained using a Mayer-Vietoris argument. Let $p : X \to \mathbb{P}^n$ be the blow down map, and U a tublar neighborhood of $\mathbb{P}^k$ in $\mathbb{P}^n$. Since U retracts to $\mathbb{P}^k$, we see that $p^{-1}(U)$ retracts to $\mathbb{P}(N)$. We also take another open set $V = \mathbb{P}^n \setminus \mathbb{P}^k$. Then $X = p^{-1}(U) \cup p^{-1}(V)$. A Mayer-Vietoris argument should yield the result. □

Problem 4 (Side question)**.** Describe all possible morphisms $\mathbb{P}^k \to \mathbb{P}^n$.

Proof. Any morphism $\varphi : \mathbb{P}^k \to \mathbb{P}^n$ must be of the form

$$\varphi(z_0 : \cdots : z_k) = (f_0(z) : \cdots f_n(z)),$$

where $f_0, ..., f_n$ are homogenous polynomials of degree d on $\mathbb{P}^k$. Hence, in [2]'s notation, we choose a linear system $L < H^0(\mathbb{P}^k, \mathcal{O}(d))$ of dimension $n+1$, then $\varphi = \varphi_L$. Hence morphisms $\mathbb{P}^k \to \mathbb{P}^n$ are in bijective correspondence with $(n+1)$-dimensional linear systems in $H^0(\mathbb{P}^k, \mathcal{O}(d))$, where d runs over all nonnegative integers. □

Problem 5. Let F_1, F_2 be two smooth curves in $\mathbb{P}^2$ of degree d that intersects transversely. In $\mathbb{P}^1 \times \mathbb{P}^2$, with coordinates $(u_1 : u_2)$ of $\mathbb{P}^1$ and $(x : y : z)$ of $\mathbb{P}^2$. Consider the bihomogeneous polynomial $F = u_1F_1(x, y, z) + u_2F_2(x, y, z)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Denote $X = Z(F)$.

(a) Compute the Picard group of $\mathbb{P}^1 \times \mathbb{P}^2$, identify the line bundle of which F is a section.

(b) Check that X is smooth.

(c) We have the second projection map $p_2 : X \to \mathbb{P}^2$. Show that p_2 is a blow up of $\mathbb{P}^2$ at some number of points. Find that number.

(d) Look at the first projection $p_1 : X \to \mathbb{P}^1$. That is a map from X (a surface) to a curve $\mathbb{P}^1$. Show that the fibers of p_1 is connected.

(e) What is the genus of the general fiber which is smooth?

(f) Suppose F_1, F_2 are generic, and assume that all the singular fiber of p_1 has a single ordinary double point. How many singular fibers are there?

Proof.

(a) Denote $Y = \mathbb{P}^1 \times \mathbb{P}^2$. We use the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$. The long exact sequence in cohomology reads

$$\cdots \to H^1(Y, \mathbb{Z}) \to H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y^*) \to H^2(Y, \mathbb{Z}) \to H^2(Y, \mathcal{O}_Y) \to \cdots .$$

Since products of Kähler manifold is Kähler, we see that Y is Kähler so we can use Hodge theory. By Künneth formula we have $H^1(Y, \mathbb{Z}) = 0$. So that $0 = b_1 = h^{1,0} + h^{0,1}$ which means $h^{0,1} = 0$. Therefore, $H^1(Y, \mathcal{O}_Y) = 0$. Next, we show that $H^2(Y, \mathcal{O}_Y) = 0$ so that $c_1 : H^1(Y, \mathcal{O}_Y^*) \cong H^2(Y, \mathbb{Z})$. To see this, note that by conjugation symmetry $h^{0,2} = h^{2,0}$, so it suffices to show $H^0(Y, \Omega_Y^2) = 0$. Now suppose ω is a nonzero holomorphic 2-form on Y. Then consider the projection $p_2 : Y \to \mathbb{P}^2$ (taking p_1 will also do). The pullback $p_2^*\omega$ will be a nonzero holomorphic 2-form on $\mathbb{P}^2$. But $h^{2,0}(\mathbb{P}^2) = 0$, so this is the desired contradiction. Thus we have shown that $H^1(Y, \mathcal{O}_Y^*) \cong H^2(Y, \mathbb{Z}) = \mathbb{Z}^2$, where the last equality was by Künneth formula. Therefore, we conclude that

$$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^2) = \mathbb{Z}^2.$$

Recall that the isomorphism $H^2(Y) \cong H^2(\mathbb{P}^1) \oplus H^2(\mathbb{P}^2)$ in Künneth formula is explicitly given by pullback: $\omega \mapsto (p_1^*\omega, p_2^*\omega)$. Now we also have $\mathrm{Pic}(\mathbb{P}^1) \cong H^2(\mathbb{P}^1) = \mathbb{Z}$ and $\mathrm{Pic}(\mathbb{P}^2) \cong H^2(\mathbb{P}^2) = \mathbb{Z}$ via the Chern map, with generators $\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^2}(1)$. The generators of $H^2(\mathbb{P}^1) \oplus H^2(\mathbb{P}^2)$ is thus identified with $p_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $p_2^*\mathcal{O}_{\mathbb{P}^2}(1)$ in $\mathrm{Pic}(Y)$. Therefore, we see that the line bundles on Y, i.e., $\mathrm{Pic}(Y)$, is completely classified by the isomorphism classes of bundles

$$\mathcal{O}_Y(a, b) = p_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(b).$$

This is because

$$c_1(\mathcal{O}_Y(a,b)) = c_1(p_1^*\mathcal{O}_{\mathbb{P}^1}(a)) + c_1(p_2^*\mathcal{O}_{\mathbb{P}^2}(b)) = p_1^*c_1(\mathcal{O}_{\mathbb{P}^1}(a)) + p_2^*\mathcal{O}_{\mathbb{P}^2}(b)),$$

where we used $c_1(L\otimes L') = c_1(L)+c_1(L')$ and $c_1(f^*L) = f^*c_1(L)$. Now F is a bi-homogenous polynomial of degree $(1,d)$ in $Y = \mathbb{P}^1\times\mathbb{P}^2$. Hence it is a section of the line bundle $\mathcal{O}_Y(1,d)$.

(b) To check smoothness, we check that the partial derivatives with respect to u_1, u_2, x, y, z do not vanish simultaneously at any point. Note that

$$\partial_{u_1}F = F_1 \qquad \partial_{u_2}F = F_2 \qquad \partial_\mu F = u_1\partial_\mu F_1 + u_2\partial_\mu F_2.$$

for $\mu = x, y, z$. Suppose that such a point $p = ((u_1:u_2)(x:y:z)) \in \mathbb{P}^1\times\mathbb{P}^2$ exits. Then the first two equations being zero implies $(x:y:z)\in Z(F_1)\cap Z(F_2)$. If F_1, F_2 are generic smooth curves of degree d, then they meet at d^2 distinct points by Bézout theorem. Thus the last three equation being equal zero would mean that $\nabla F_1, \nabla F_2$ are linearly dependent. But ∇F_i is normal to the tangent space of $Z(F_i)$ so this would mean that $Z(F_1)$ and $Z(F_2)$ intersect non-transversally, which is a contradiction. Hence $X = Z(F)$ is smooth.

(c) Recall that if $Y\subset\mathbb{P}^2$ is a submanifold of codimension k, and is defined by the equations $f_1 = f_2 = \cdots = f_k = 0$, then by definition

$$\mathrm{Bl}_Y\,\mathbb{P}^2 = \{(Z,z)\in\mathbb{P}^{k-1}\times\mathbb{P}^2 : Z_if_j(z) = Z_jf_i(z), \forall i,j\le k\}.$$

Also, we see that

$$Z(F) = \{(Z,z)\in\mathbb{P}^1\times\mathbb{P}^2 : Z_1F_1(z) = Z_2F_2(z) = 0\}.$$

Hence, setting $f_1 = -F_2$ and $f_2 = F_1$, and $k = 2$, we immediately see that $X = Z(F)$ is the blow up of $\mathbb{P}^2$ at the intersection points $Y = Z(F_1)\cap Z(F_2)$. Then it follows from Bézout's theorem that X is the blow up of $\mathbb{P}^2$ at d^2 distinct points.

(d) For a fixed $(u_1:u_2)\in\mathbb{P}^1$, the fiber is given by

$$X_{(u_1:u_2)} \cong \{(x:y:z)\in\mathbb{P}^2 : u_1F_1(x,y,z) + u_2F_2(x,y,z) = 0\}.$$

Hence it is the zero locus of the degree d homogeneous polynomial $u_1F_1(x,y,z)+u_2F_2(x,y,z)$ in $\mathbb{P}^2$, hence a plane curve, which we call C. Now we use the fact that C is connected in Euclidean topology iff it is connected in Zariski topology. We argue by contradiction. Suppose $C = C_1\sqcup C_2$ is the disjoint union of two plane curves $C_1 = Z(g_1)$ and $C_2 = Z(g_2)$, where $\deg g_1 = d_1$ and $\deg g_2 = d_2$, for $d_1, d_2 > 0$. By then by Bézout's theorem C_1 and C_2 always intersect in $\mathbb{P}^2$, hence cannot be disjoint. Thus we conclude that $C = X_{(u_1:u_2)}$ is connected.

(e) The general fiber X_u of some fixed points $u = (u_1:u_2)$ is given by the smooth curve $C =$

$Z(u_1F_1 + u_2F_2)$ of degree d. By the degree genus formula,

$$g(X_u) = g(C) = \frac{(d-1)(d-2)}{2}.$$

(f) We can find the number of singular fibers N by calculating the Euler characteristic $\chi(X)$ of X in two different ways. From (c), we already know that $p_2 : X \to \mathbb{P}^2$ is a blow up of $\mathbb{P}^2$ at d^2 distinct points. So for each point we are blowing up, we attach a copy of $\mathbb{P}^1 \cong S^2$ to the base $\mathbb{P}^2$. Using $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, we see that adding a $\mathbb{P}^1$ changes the Euler characteristics of $\chi(\mathbb{P}^2) = 3$ by $\chi(\mathbb{P}^1) - \chi(\mathrm{pt}) = 1$. Thus

$$\chi(X) = 3 + d^2.$$

We can calculate $\chi(X)$ in a different way using the first projection $p_1 : X \to \mathbb{P}^1$. The useful result we will use here is that if we have a fibration $F \to E \to B$, the Euler characteristic is multiplicative: $\chi(E) = \chi(F)\chi(B)$. Unfortunately, the map $p_1 : X \to \mathbb{P}^1$ is not a fibration because we have singular fibers and smooth fibers, so the fibers are not homotopy equivalent. However, if we remove the singular points from $\mathbb{P}^1$, which we denote by $\mathbb{P}^{1*}$, and we remove the singular fibers in X, which we denote by X^*, then the restriction $p_1 : X^* \to \mathbb{P}^{1*}$ will be a fibration. In this case, the fiber $F \cong \Sigma_g$, and $\mathbb{P}^{1*}$ is just $\mathbb{P}^1$ with n singular points removed. So we have that

$$\chi(X^*) = \chi(F)\chi(\mathbb{P}^{1*}) = (2-2g) \cdot (\chi(\mathbb{P}^1) - n\chi(\mathrm{pt})) = (2-2g)(2-n).$$

Now, X is just a disjoint union of X^* and the singular fibers F'. We recall that for disjoint union $\chi(A \sqcup B) = \chi(A) + \chi(B)$. In our case, A singular fiber F' with one node is topologically equivalent to a smooth curve with two points identified, so $\chi(F') = \chi(F) + 1 = (2-2g) + 1 = 3 - 2g$. Therefore,

$$\chi(X) = \chi(X^*) + n\chi(F') = (2-2g)(2-n) + n(3-2g) = 4 - 4g + n.$$

Now equating the two expression for $\chi(X)$ above gives

$$3 + d^2 = 4 - 4g + n.$$

Using the degree genus formula for g in part (e) and solve for n, we see that the final answer is

$$\boxed{n = 3(d-1)^2.}$$

Tony thought that this might be wrong at first, because he was a little bit worried that it is always divisible by 3. So we did a concrete example to check this. Let $d = 2$. So $F = u_1F_1 + u_2F_2$ is a quadratic curve for a fixed $(u_1 : u_2)$. We can always write such a quadratic form as

$$F(\mathbf{x}) = \mathbf{x}^T M(u_1, u_2)\mathbf{x},$$

where $M(u_1, u_2)$ is a symmetric 3×3 matrix, depending on u_1, u_2. Now, F is singular iff $\det M = 0$, which is a cubic equation in $(u_1 : u_2)$, which has three solutions, consistent with for formula $n = 3(d-1)^2 = 3$ for $d = 2$.

□

Problem 6.

(a) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^2$ as a curve of degree 2?

(b) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^2$ as a curve of degree 3?

(c) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve of bidegree (a, b) with $a, b \geq 3$?

(d) Can you embed $\mathbb{P}^1$ in $\mathbb{P}^d$ as a curve of degree d?

Proof.

(a) Yes, we have the Veronese embedding $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ via $\nu : [x : y] \mapsto [x^2 : xy : y^2]$ Note that this is a curve of degree 2 because it is given by $z_0 z_2 - z_1^2 = 0$ in $\mathbb{P}^2$.

(b) For a smooth embedding no, by degree genus formula.

(c) No. The claim is that for a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a, b), the genus is given by $g(C) = (a-1)(b-1)$. Hence for $a, b \geq 3$ we have $g \geq 4$ which is impossible for a smooth embedding. So it remains to prove the assertion. We first need the adjunction formula as a lemma:

Lemma (Adjunction Formula)**.** *Let H be a smooth hypersurface in a variety X, then we have*

$$K_X|_H = K_H \otimes \mathcal{O}_X(-H)|_H.$$

Proof of Lemma. The normal bundle sequence of H in X reads

$$0 \to T_H \to T_X|_H \to N_{H/X} \to 0.$$

We note that $N_{H/X} \cong \mathcal{O}_X(H)$. We note that $N_{H/X} \cong \mathcal{O}_X(H)$. Indeed, we can choose open cover $X = \cup_\alpha U_\alpha$ such that $H \cap U_\alpha = Z(f_\alpha)$ for some $f_\alpha \in \mathcal{O}_X(U_\alpha)$. On the overlap $U_\alpha \cap U_\beta$ these equations are related by $f_\alpha = g_{\alpha\beta} f_\beta$ for some $g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$. Moreover, these $g_{\alpha\beta}$ satisfies the cocycle condition, hence they define a line bundle L. The functions f_α are then local expressions of a global holomorphic section $s \in H^0(X, L)$: locally $s|_{U_\alpha} = f_\alpha e_\alpha$. The zero locus of this section s is exactly H, so we have $L = \mathcal{O}_X(-H)$

On the other hand, $\mathrm{d}f_\alpha$ is a holomorphic 1-form on U_α. Restricted to points of H this 1-form annihilates tangent vectors tangent to H, hence $\mathrm{d}f_\alpha|_{H \cap U_\alpha} \in \Gamma(U_\alpha, N^*_{H/X})$. On the overlap $U_\alpha \cap U_\beta$, we have

$$\mathrm{d}f_\alpha|_H = \mathrm{d}(g_{\alpha\beta} f_\beta)\,|_H = g_{\alpha\beta} \mathrm{d}f_\beta|_H,$$

since $f_\beta|_H = 0$. Hence we see that $g_{\alpha\beta}$ is the transition function of $N^*_{H/X}$. Therefore, we conclude that $N^*_{H/X} \cong \mathcal{O}_X(-H)$. Dualizing gives $N_{H/X} \cong \mathcal{O}_X(H)$, as desired.

Therefore, dualizing the exact normal bundle sequence gives

$$0 \to \mathcal{O}_X(-H)|_H \to \Omega^1_X \to \Omega^1_H \to 0.$$

Taking the determinant of this exact seqence gives the desired adjunction formula. □

Proposition. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree (a, b). Then the genus g of C is given by*

$$g = (a-1)(b-1).$$

Proof of Proposition. Let $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to the i-th coordinate, where $i = 1, 2$. Let $p_i : C \to C_i$ be the restriction of π_i to the curve C. The goal is to compute $\deg K_C = 2g - 2$, which will in turn give the genus of C.

Using the adjunction formula, we have that

$$K_{\mathbb{P}^1 \times \mathbb{P}^1}|_C = K_C \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-C)|_C.$$

It is a standard calculation that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$, and every line bundle up to isomorphism is represented by

$$\mathcal{O}(a, b) = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(b).$$

Therefore, we see that $K_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}(-2, -2)$ and $\mathcal{O}(-C) = \mathcal{O}(-a, -b)$ because C has bidegree (a, b). Therefore we conclude that

$$K_C = \mathcal{O}(a-2, b-2)|_C = p_1^* \mathcal{O}_{\mathbb{P}^1}(a-2) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b-2)|_C.$$

Taking Chern class gives

$$c_1(K_C) = (p_1|_C)^* c_1(\mathcal{O}_{\mathbb{P}^1}(a-2)) + (p_2|_C)^* c_1(\mathcal{O}_{\mathbb{P}^1}(b-2)).$$

In the above, $p_j|_C = i_j \circ p_j$ for $j = 1, 2$, where $i_j : C_j \to \mathbb{P}^1$ is the inclusion map.

Recall that for a line bundle $L \to X$ where X is a curve, $\deg L = \int_X c_1(L)$. On $\mathbb{P}^1$, we find a 2-form $\omega = c_1(\mathcal{O}_{\mathbb{P}^1}(1))$, which integrates to 1 because $\deg \mathcal{O}_{\mathbb{P}^1}(1) = 1$. Then ω is a volume form on $\mathbb{P}^1 \cong S^2$. Then it follows that $c_1(\mathcal{O}_{\mathbb{P}^1}(n)) = n\omega$ for any integer n. Therefore,

$$(p_1|_C)^* c_1(\mathcal{O}_{\mathbb{P}^1}(a-2)) = (a-2)(p_1|_C)^* \omega$$

where $p_1|_C = i_1 \circ p_1 : C \to \mathbb{P}^1$ is a map between orientable 2-dimensional manifolds. Integrating gives

$$\deg(p_1^* \mathcal{O}_{\mathbb{P}^1}(a-2)|_C) = (a-2) \int_C (p_1|_C)^* \omega = (a-2) \cdot \deg(p_1|_C).$$

since ω is the volume form on $\mathbb{P}^1$. But $p_1|_C = i_1 \circ p_1 : C \to \mathbb{P}^1$ is just the projection map. To calculate its degree, we fix any regular value $z \in \mathbb{P}^1$. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be given by the bihomogeneous polynomial $F(x, y)$ of degree (a, b). Then $\deg(p_1|_C)$ is the number of solutions to the equation $F(z, y) = 0$ in $\mathbb{P}^1$, and for a generic point this number is b because $F(z, y)$ is a degree b polynomial in y. Hence we conclude

$$\deg(p_1^*\mathcal{O}_{\mathbb{P}^1}(a-2)|_C) = b(a-2).$$

Now the computation for $p_2^*\mathcal{O}_{\mathbb{P}^1}(b-2)|_C$ is entirely analogous. The result is

$$\deg(p_2^*\mathcal{O}_{\mathbb{P}^1}(b-2)|_C) = (b-2)\int_C (p_2|_C)^*\omega = (b-2)\deg(p_2|_C) = a(b-2).$$

Therefore, we conclude that

$$\deg K_C = b(a-2) + a(b-2) = 2ab - 2a - 2b.$$

Now using $\deg K_C = 2g - 2$ and solving for g gives

$$g = (a-1)(b-1),$$

as desired. This concludes the proof of the proposition. □

(d) Yes, we have the Veronese embedding $\nu : \mathbb{P}^1 \to \mathbb{P}^d$ by

$$\nu : [x : y] \mapsto [x^d : x^{d-1}y : \cdots : xy^{d-1} : y].$$

Why is it degree d? Hint: Degree of a curve C is defined by its intersection number $C \cdot H$ where H is a hyperplane.

□

Problem 7. Consider the surface S given by the equation $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ in $\mathbb{P}^3$. Show that S is smooth. What is $\mathrm{Pic}(S)$?

Proof. The fact that S is smooth follows from taking partial derivatives, and the singular points is $x_0 = x_1 = x_2 = x_3 = 0$ which is not allowed in $\mathbb{P}^3$.

To compute the picard group of S, we again use the LES of exponential sequence:

$$\cdots \to H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbb{Z}) \to H^2(S, \mathcal{O}_S) \to \cdots$$

Again our goal is to show $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$ and conclude $H^1(S, \mathcal{O}_S^*) \cong H^2(S, \mathbb{Z})$. We first show $H^2(S, \mathcal{O}_S) = 0$. We have a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-S) \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_S \to 0$$

where $i : S \to \mathbb{P}^3$ is inclusion. We note that (i) $H^\cdot(\mathbb{P}^3, i_*\mathcal{O}_S) = H^\cdot(S, \mathcal{O}_S)$; and (ii) S is a divisor hence $\mathcal{O}_{\mathbb{P}^3}(-S)$ is a line bundle on $\mathbb{P}^3$, which is completely characterized by $\mathcal{O}_{\mathbb{P}^3}(m)$ for integers m. The integer m should corresponds to negative of degree of the divisor S, which is of course 2. Hence we conclude that $\mathcal{O}_{\mathbb{P}^3}(-S) = \mathcal{O}_{\mathbb{P}^3}(-2)$. More detailed discussion: Prop 2.3.18 on p.83 of [2]. Therefore, the LES reads

$$\cdots \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^2(S, \mathcal{O}_S) \to H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)) \to H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to \cdots$$

Because $h^{0,2}(\mathbb{P}^3) = h^{0,3}(\mathbb{P}^3) = 0$, we have

$$H^2(S, \mathcal{O}_S) \cong H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)) \cong H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2))^* = 0$$

where in the last isomorphism we used Serre duality

$$H^p(X, E) = H^{n-p}(X, E^* \otimes K_X)^*$$

where X is Kähler of complex dimension n, and $K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. By similar argument, the LES shows

$$H^1(S, \mathcal{O}_S) \cong H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$$

where the last equality follows from Kodaira vanishing theorem ([2], p.241, Example 5.2.5). Hence we have proven our claim and we conclude that

$$\mathrm{Pic}(S) = H^1(S, \mathcal{O}_S^*) \cong H^2(S, \mathbb{Z}).$$

It remains to compute the singular cohomology $H^2(S; \mathbb{Z})$. We first claim that

Lemma. *Any two smooth quadratic surface $\mathcal{Q}, \mathcal{Q}'$ of $\mathbb{P}^3$ is equivalent up to a projective transformation.*

Proof of Lemma. Any such quadratic surface is defined by a symmetric matrix A via $\mathcal{Q}(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. For $\mathcal{Q}$ smooth this means $\det A \neq 0$, so A is nonsingular. Any such A defines an inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T A\mathbf{y} = A_{ij}x^i y^j$. Recall that $\mathcal{Q}, \mathcal{Q}'$ are equivalent iff $A' = PAP^T$ for some change of basis matrix P. But note that since any bilinear form over $\mathbb{C}$ can be diagonalized, $A \sim \mathrm{id}$. □

Thus we have concluded that all smooth quadric surfaces in $\mathbb{P}^3$ are diffeomorphic. One particular quadric surface is obtained via the image of the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \qquad ([s_0 : s_1], [t_0 : t_1]) = [s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1].$$

Indeed, the image is a quadric $x_0x_3 - x_1x_2 = 0$. Thus

$$H^2(S; \mathbb{Z}) = H^2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

by the Künneth formula. □

Problem 8. Prove the Birkhoff-Grothendieck theorem: If E is a holomorphic vector bundle of rank r over $\mathbb{P}^1$, then it decomposes into a direct sum of line bundles

$$E \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

for some integers $a_1, ..., a_r$.

Proof. We induct on the rank r of E. When $r = 1$, we know $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$, and we are done. Now let E be a vector bundle of rank r. Let

$$a_1 := \max\{a : \mathrm{Hom}(\mathcal{O}(a), E) = H^0(\mathbb{P}^1, E(-a)) \neq 0\}.$$

First, there exists a such that $H^0(\mathbb{P}^1, E(-a)) \neq 0$. To see this, note that Serre vanishing theorem shows that $H^1(\mathbb{P}^1, E(-a)) = 0$ for $a \ll 0$. Thus by Riemann Roch for vector bundles on a curve: $\chi(E) = \deg E + r(1 - g)$, we have

$$h^0(\mathbb{P}^1, E(-a)) = \chi(\mathbb{P}^1, E(-a)) = \deg E - (1 - a)r > 0$$

for $a \ll 0$. We also note that a maximal a exists because by Serre vanishing, for $a \gg 0$ we have $H^0(\mathbb{P}^1, E(-a)) = H^1(\mathbb{P}^1, E^*(a - 2))^* = 0$.

Thus, we have a short exact sequence

$$0 \to \mathcal{O}(a_1) \to E \to E_1 \to 0. \tag{!}$$

We note that the map $\mathcal{O}(a_1) \to E$ in (!) is injective by the fiberwise rank-nullity theorem, since the map is assumed to be nonzero. Now E_1 is a vector bundle of lower rank, so by inductive hypothesis we may assume $E_1 = \bigoplus_{i>1} \mathcal{O}(a_i)$. Hence it remains to show that (1) splits.

We recall some homological algebra result: Over X the obstruction to splitting of the short exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ of vector bundles is parameterized by the abelian group $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F}) = H^1(X, \mathrm{Hom}(\mathcal{G}, \mathcal{F})) = H^1(X, F \otimes \mathcal{G}^*)$. Hence, in our case the obstruction to splitting of (!) is parameterized by the group

$$H^1(\mathbb{P}^1, \bigoplus_{i>1} \mathcal{O}(a_1 - a_i)) = \bigoplus_{i>1} H^1(\mathbb{P}^1, \mathcal{O}(a_1 - a_i)).$$

Using Problem 15, we see that the vanishing of the group above requires $a_1 - a_i > -2$ for all i. We will show that in fact a stronger result holds: $a_i \leq a_1$ for all i.

To this end, suppose $a_i > a_1$ for at least one i. Then we have $E_1(-a_1 - 1) = \bigoplus_{i>1} \mathcal{O}(a_i - a_1 - 1)$, and since $a_i - a_1 - 1 \geq 0$ for at least one i, we have

$$H^0(\mathbb{P}^1, E_1(-a_1 - 1)) = \bigoplus_{i>1} H^0(\mathbb{P}^1, \mathcal{O}(a_i - a_1 - 1)) \neq 0.$$

Now we twist (!) by $\mathcal{O}(-a_1-1)$ to get a SES

$$0 \to \mathcal{O}(-1) \to E(-a_1-1) \to E_1(-a_1-1) \to 0. \tag{!!}$$

We look at the LES of (!!):

$$0 \to H^0(\mathcal{O}(-1)) \to H^0(E(-a_1-1)) \xrightarrow{\alpha} H^0(E_1(-a_1-1)) \xrightarrow{\beta} H^1(\mathcal{O}(-1)) \to \cdots$$

Since $H^1(\mathcal{O}(-1)) = 0$ by Problem 15, we see that $\ker\beta = H^0(E_1(-a_1-1)) \neq 0$ by our computation above. Now if $H^0(E(-a_1-1)) = 0$, then

$$\operatorname{im}\alpha = \ker\beta = H^0(E_1(-a_1-1)) = 0,$$

which is a contradiction. Hence we conclude that $H^0(E_1(-a_1-1)) = 0$. But this contradicts maximality of a_1. We have arrived at the desired contradiction, and the proof is now complete. □

Problem 9. Let $n \geq 0$ be an integer. The Hirzebruch surface Y_n is a ruled surface (in fact a $\mathbb{P}^1$-bundle) over $\mathbb{P}^1$, defined by

$$Y_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)).$$

Given Y_n for any n, compute $H^\cdot(Y_n)$ and $\operatorname{Pic}(Y_n)$.

Proof. We first compute the Hodge diamond of $Y_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. For simplicity denote $\mathcal{E}_n = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. Since $\mathcal{E}_n$ is a holomorphic vector bundle over a Kähler manifold, its projectivization $Y_n = \mathbb{P}(\mathcal{E}_n)$ is again Kähler by Proposition 3.18 of [4]. We first note that in particular this implies $h^{1,1}(Y_n) \neq 0$, because Y_n is compact Kahler, there exists a Kähler form ω on Y_n, which is of type $(1,1)$. We note that $H^{1,1}(X) \subset H^2(X)$. If ω is zero in $H^{1,1}(X)$, it is zero in $H^2(X)$, but

$$\operatorname{vol}(X) = \frac{1}{n!}\int_X \omega^n > 0,$$

contradiction. Note that compactness is necessary, for noncompact Kahler manifold like $\mathbb{C}^n$, we have $h^{1,1} = 0$. c

Next, we compute the cohomology $H^\cdot(Y_n)$. We use the Leray-Hirsch theorem:

Theorem (Leray-Hirsch)**.** *Let E be a fiber bundle over M with fiber F. Suppose there are global cohomology classes $e_1, \ldots, e_r$ on E which when restricted to each fiber freely generates the cohomology of the fiber $H^*(F)$, then we have*

$$H^*(E) \cong H^*(M) \otimes \mathbb{C}\langle e_1, \ldots, e_r\rangle \cong H^*(M) \otimes H^*(F).$$

In our case, $E = Y_n = \mathbb{P}(\mathcal{E}_n)$, where $\mathcal{E}_n = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$, and $F \cong \mathbb{P}^1$. We need to justify our use of Leray-Hirsch theorem in this case. Let

$$\beta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_n)}(-1)) \in H^2(\mathbb{P}(\mathcal{E}_n); \mathbb{Z}).$$

We claim that $1 \in H^0(\mathbb{P}(\mathcal{E}_n);\mathbb{Z})$ and $\beta \in H^2(\mathbb{P}(\mathcal{E}_n);\mathbb{Z})$ restricts to a basis of each fiber of $\mathbb{P}(\mathcal{E}_n)$. Indeed, let $\pi : \mathbb{P}(\mathcal{E}_n) \to \mathbb{P}^1$ denote the bundle map, with $p \in \mathbb{P}^1$ and $x \in \pi^{-1}(p)$. Then x represent a class in the projectivized fiber $\mathbb{P}(\mathcal{E}_n|_b)$, and by definition

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_n)}(-1)|_{\pi^{-1}(b)} = \pi^{-1}(b) = \mathbb{P}(\mathcal{E}_n|_b)$$

which is exactly the fiber at b. Since Chern class is compatible with restriction maps, the statement follows. Therefore, viewing $Y_n = \mathbb{P}(\mathcal{E}_n)$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, Leray-Hirsch gives

$$H^k(Y_n) \cong \bigoplus_{p+q=k} H^p(\mathbb{P}^1) \otimes H^q(\mathbb{P}^1).$$

Hodge decomposition, combined with the fact that $h^{1,1}(Y_n) \neq 0$ and the symmetry of Hodge diamond, we see that the Hodge diamond of Y_n is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Surprisingly, we see that the Hodge diamond of Y_n is independent of n. To compute the Picard group, we use the exponential sequence. Since $h^{0,1}(Y_n) = h^{0,2}(Y_n) = 0$, exactness gives

$$\mathrm{Pic}(Y_n) = H^1(Y_n, \mathcal{O}^*_{Y_n}) \cong H^2(Y_n;\mathbb{Z}) = \mathbb{Z}^2.$$

□

Problem 10. Let S be a smooth surface of degree d in $\mathbb{P}^3$. Compute its Hodge diamond. In particular, what is the Hodge diamond of a quadric ($d = 2$) and cubic ($d = 3$) surface in $\mathbb{P}^3$?

Proof. The only nontrivial Hodge numbers we need are $h^{1,0} = h^{0,1}, h^{2,0} = h^{0,2}$, and $h^{1,1}$.

- $h^{1,0} = h^{0,1} = 0$: This is a consequence of

  Theorem (Lefschetz hyperplane theorem)**.** *Let X be an (complex) n dimensional projective variety in $\mathbb{P}^N$. Let H be a hyperplane in $\mathbb{P}^N$. Then the restriction*

  $$H^k(X;\mathbb{Z}) \to H^k(X \cap H;\mathbb{Z})$$

  is an isomorphism for $k < n - 1$ and and injective for $k = n - 1$.

  To use this theorem, we consider the degree d Veronese embedding

  $$v : \mathbb{P}^3 \to \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))^*) = \mathbb{P}^N.$$

The surface S is a global section of $\mathcal{O}_{\mathbb{P}^3}(d)$, so it corresponds to a hyperplane $H \subset \mathbb{P}^N$, such that $S = v^{-1}(H)$. Using the Lefschetz hyperplane theorem with $X = v(\mathbb{P}^3)$, and noting that $X \cap H \cong v^{-1}(X \cap H) = \mathbb{P}^3 \cap S = S$, we see that for $k < 2$, we have isomorphisms

$$H^k(\mathbb{P}^3; \mathbb{Z}) \cong H^k(S; \mathbb{Z})$$

for $k < 2$. In particular, $H^1(S; \mathbb{Z}) = H^1(\mathbb{P}^3; \mathbb{Z}) = 0$. Hence $h^{1,0} = h^{0,1} = 0$ by Hodge decomposition, as desired.

- $h^{2,0} = h^{0,2} = \binom{d-1}{3}$: To compute this, let us compute $h^{2,0}$, which is the dimension of $H^0(S, K_S)$. By adjunction formula,

$$K_S = K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(d)|_S = \mathcal{O}_{\mathbb{P}^3}(d-4)|_S = i^*\mathcal{O}_{\mathbb{P}^3}(d-4).$$

where $i : S \to \mathbb{P}^3$ is the inclusion map. We use the restriction exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_S \to 0.$$

Tensoring by $\mathcal{O}_{\mathbb{P}^3}(d-4)$, and noting that by projection formula

$$i_*\mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}^3}(d-4) = i_*(\mathcal{O}_S \otimes i^*\mathcal{O}_{\mathbb{P}^3}(d-4)) = i_*i^*\mathcal{O}_{\mathbb{P}^3}(d-4),$$

where the last equality is because $\mathcal{O}_S$ is the multiplicative unit in $\mathrm{Pic}(S)$. Therefore, the exact sequence reads

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(d-4) \to i_*i^*\mathcal{O}_{\mathbb{P}^3}(d-4) \to 0,$$

Taking the long exact sequence in cohomology, and using $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$ by degree reason and $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$ by Kodaira vanishing theorem, exactness gives

$$H^0(\mathbb{P}^3, i_*i^*\mathcal{O}_{\mathbb{P}^3}(d-4)) \cong H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4)).$$

Note that in the case where $d = 2, 3$, the left hand side has dimension zero. Then

$$H^0(S, K_S) = H^0(S, i^*\mathcal{O}_{\mathbb{P}^3}(d-4)) = H^0(\mathbb{P}^3, i_*i^*\mathcal{O}_{\mathbb{P}^3}(d-4)).$$

Hence in general

$$h^{2,0} = h^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) = \binom{d-4+3}{3} = \binom{d-1}{3}.$$

In the case where $d = 2, 3$, we have $h^{2,0} = h^{0,2} = 0$.

- $h^{1,1} = \frac{d(2d^2-6d+7)}{3}$: Now that we know all the Hodge numbers except $h^{1,1}$, so we can calculate the Euler characteristic $\chi(S)$, and then solve for $h^{1,1}$. Now $\chi(S) = \int_S c_2(T_S)$. Hence we need to find the top degree Chern class $c_2(T_S)$. We consider the normal bundle sequence:

$$0 \to T_S \to T_{\mathbb{P}^3}|_S \to N_{S/\mathbb{P}^3} \to 0.$$

Thus $c(T_S)c(N_{S/\mathbb{P}^3}) = c(T_{\mathbb{P}^3}|_S)$. Or equivalently

$$c(T_S) = c(T_{\mathbb{P}^3}|_S)/c(N_{S/\mathbb{P}^3}).$$

In the proof of the adjunction formula in Problem 6, we showed that $N_{S/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(S) = \mathcal{O}_{\mathbb{P}^3}(d)$, or technically $N_{S/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(d)|_S$.

Now, let $H = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$. Then we have that

$$c(N_{S/\mathbb{P}^3}) = c(\mathcal{O}_{\mathbb{P}^3}(d)) = c_0(\mathcal{O}_{\mathbb{P}^3}(d)) + c_1(\mathcal{O}_{\mathbb{P}^3}(d)) = 1 + dH.$$

To compute $c(T_{\mathbb{P}^3})$, we use the Euler sequence (Proposition 2.4.4 of [2]):

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4} \to T_{\mathbb{P}^3} \to 0.$$

The total chern class splits so

$$c(T_{\mathbb{P}^3}) = \frac{c(\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4})}{c(\mathcal{O}_{\mathbb{P}^3})} = (1+H)^4,$$

where we used that $c(L \oplus L') = c(L)c(L')$, and $c(\mathcal{O}_{\mathbb{P}^3}) = 1$, which is the special case of $c_1(\mathcal{O}_{\mathbb{P}^3}(d)) = 1 + dH$ when $d = 0$. Therefore,

$$c(T_S) = \frac{(1+H)^4}{1+dH} = (1 + 4H + 6H^2 + 4H^3 + H^4)(1 - dH + d^2H^2 + \cdots).$$

Collecting the term of order H^2, we find

$$c_2(T_S) = (d^2 - 4d + 6)H^2.$$

On S we have $\int_S H^2 = \deg S = d$. Hence

$$\chi(S) = \int_S c_2(T_S) = d(d^2 - 4d + 6) = d^3 - 4d^2 + 6d.$$

Now using $\chi(S) = b_0 - b_1 + b_2 - b_3 + b_4$ and $b_k = \sum_{p+q=k} h^{p,q}$ we can solve for $h^{1,1}$. The result is

$$h^{1,1} = \frac{d(2d^2 - 6d + 7)}{3}.$$

Hence, in the case where $d = 2$ the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

And in the case where $d = 3$, the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

□

Problem 11. Let S be a Hopf surface (2-dim Hopf manifold). Recall that if q is a nonzero complex number, then $\mathbb{Z}$ acts on $\mathbb{C}^2 - 0$ by $n \cdot (z, w) = (q^n z, q^n w)$. And $S = (\mathbb{C}^2 - 0)/\mathbb{Z}$. Check that S is compact, and it is not Kähler.

Proof. The Hopf surface X is diffeomorphic to $S^1 \times S^3$, by [2], p.61. Hence X is compact. However, by Künneth formula, $H^2(X) = H^2(S^1 \times S^3) = 0$. If X is compact Kähler, we know that the Kähler form will be a nontrivial element in $H^{1,1}(X)$, so $h^{1,1} \neq 0$. This is the desired contradiction. □

Problem 12. Let S be a smooth quadric (degree 2) surface in $\mathbb{P}^3$. Compute the dimension of its Lefschetz decomposition (take the Kähler form to be the Fubini-Study restricted to the quadric S). Do the same for a smooth cubic surface.

Proof. In this special case, we can apply similar method as Problem 20. Since the ordinary Hodge diamonds were already calculated in Problem 10, the only nontrivial primitive Hodge number is

$$h^{1,1}_{\text{prim}}(S) = h^{1,1}(S) - 1.$$

Hence in the case where $d = 2$ we have $h^{1,1}_{\text{prim}} = 1$ and in the case where $d = 3$ we have $h^{1,1}_{\text{prim}} = 6$, by Problem 10.

We will now outline a general method of computing dimension of Lefschetz decomposition and the primitive Hodge numbers.

Suppose now X is a compact Kähler manifold of dimension n. From now on we use shorthand $H^* = H^*(X)$ for cohomology groups of X. By hard Lefschetz theorem, for all $k \leq n$ the map $L^{n-k} : H^k \to H^{2n-k}$ is an isomorphism. In particular, this means that for $m \leq n$ the map

$$L : H^{m-2} \to H^m$$

is injective. Indeed, this is because $L^{n-m+2} : H^{m-2} \to H^{2n-m+2}$ is an isomorphism. Recall that

$$H^m_{\text{prim}} = \ker(L^{n-m+1} : H^m \to H^{2n-m+2}).$$

Now by Lefschetz decomposition, we have

$$H^m = \bigoplus_{0 \le r \le m/2} L^r H_{\text{prim}}^{m-2r} = H_{\text{prim}}^m \oplus LH_{\text{prim}}^{m-2} \oplus L^2 H_{\text{prim}}^{m-4} \oplus \cdots = H_{\text{prim}}^m \oplus LH^{m-2}$$

where in the last equality we applied the Lefschetz decomposition to H^{m-2}. Hence taking the dimension of this equation and using that $L : H^{m-2} \to H^m$ is injective for $m \le n$, we get that $b_m + h_{\text{prim}}^m + b_{m-2}$. That is,

$$h_{\text{prim}}^m = b_m - b_{m-2} \qquad m \le n.$$

This gives us a formula to compute the dimension of the Lefschetz decomposition. For $n > m$ we don't need to do additional computation by Hodge symmetry. In order to compute the primitive Hodge numbers, we note that the equation above respects direct sums. That is, we have

$$h_{\text{prim}}^{p,q} = h^{p,q} - h^{p-1,q-1} \qquad p + q \le n$$

and again we don't need to do new calculations for $p + q > n$ by symmetry. This actually comes from the equation $H^{p,q} = H_{\text{prim}}^{p,q} \oplus LH^{p-1,q-1}$ whose derivation is similar to the H^m case. In the hypersurface case as in the problem, we have $h_{\text{prim}}^{1,1} = h^{1,1} - h^{0,0} = h^{1,1} - 1$. □

Problem 13. Let X be a smooth d dimensional variety, and p a point in X. Let $\hat{X}$ be the blow up of X at p. Then the exceptional divisor is $\mathbb{P}(N_{p/X}) = \mathbb{P}(T_p X) \cong \mathbb{P}^{d-1}$. Then $\mathcal{O}_{\hat{X}}(E)|_E \cong \mathcal{O}_{\mathbb{P}^{d-1}}(n)$ for some n. What is n?

Proof. $n = -1$. Proof: Let $\pi : \hat{X} \to X$ be the blow up of X at p, and $E = \pi^{-1}(p)$ the exceptional divisor in $\hat{X}$. We have $K_{\hat{X}} = \pi^* K_X \otimes \mathcal{O}_{\hat{X}}((d-1)E)$, where $d = \dim X$. By adjunction we have

$$\mathcal{O}_{\mathbb{P}^{m-1}}(-d) = K_{\mathbb{P}^{d-1}} \cong K_E = (K_{\hat{X}} \otimes \mathcal{O}_{\hat{X}}(E))|_E = \mathcal{O}_{\hat{X}}(dE)|_E$$

because $\pi^* K_X|_E = \pi^* K_X \cdot E = 0$. Therefore we see that

$$\mathcal{O}_{\hat{X}}(dE)|_E \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-d).$$

Now since $\text{Pic}(E) \cong \text{Pic}(\mathbb{P}^{d-1}) = \mathbb{Z}$ is torsion free, we see that $\mathcal{O}_{\hat{X}}(E)|_E \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-1)$. □

Problem 14. Let X be a smooth variety of dimension d, and $A \subset X$ a submanifold of dimension a. Denote $\hat{X} = \text{Bl}_A X$. The exceptional divisor is $E = \mathbb{P}(N_{A/X})$, which is a $\mathbb{P}^{d-a-1}$-bundle over A. Then restricting to the fibers of $E \to A$, which is a $\mathbb{P}^{d-a-1}$, we have $\mathcal{O}_{\hat{X}}(E)|_{\mathbb{P}^{d-a-1}} \cong \mathcal{O}_{\mathbb{P}^{d-a-1}}(n)$ for some n. What is n?

Proof. Since this is a local question, we may assume that $X = \mathbb{C}^d$ and $A = \mathbb{C}^a$. Let $r = d - a$ be the codimension of A in X, and we write the coordinate on $\mathbb{C}^d$ as $(z_1, ..., z_a, w_1, ..., w_r)$ and $A = Z(w_1, ..., w_r)$. Then $N_{A/X} = A \times \mathbb{C}^r$ and the exceptional divisor is $E = \mathbb{P}(N_{A/X}) = A \times \mathbb{P}^{r-1}$.

Recall that the blow up is given by

$$\hat{X} = \{([u_1 : ... : u_r], (z_1, ..., z_a, w_1, ..., w_r)) \in \mathbb{P}^{r-1} \times \mathbb{C}^d : u_i w_j = u_j w_i, 1 \le i, j \le r\}.$$

and the exceptional divisor E is precisely defined by the equations $w_1 = \cdots = w_r = 0$. We look at the chart U_i defined by $u_i \neq 0$ in $\hat{X}$. Then in this chart $E \cap U_i$ is given by the equation $w_i = 0$ using the relation $u_i w_j = u_j w_i$ for all $1 \le j \le r$ and $u_j \neq 0$. Therefore, any holomorphic function on U_i that vanishes on E must be a multiple of w_i. This means that w_i is a local frame of $\mathcal{O}_{\hat{X}}(-E)$. The transition function of $\mathcal{O}_{\hat{X}}(-E)$ on $U_i \cap U_j$ is therefore

$$g_{ij} = \frac{w_j}{w_i} = \frac{u_j}{u_i}.$$

Restricting $\mathcal{O}_{\hat{X}}(-E)$ to one fiber of $\mathbb{P}^{r-1}$ in $\hat{X}$ (which means that we fix the coordinates $z_1, ..., z_a, w_1, ..., w_r$ in the $\mathbb{C}^d$ component), we get a line bundle $\mathcal{O}_{\hat{X}}(-E)|_{\mathbb{P}^{r-1}}$ whose transition functions are still given by g_{ij} above by continuity. But these are precisely the transition functions of $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Dualizing, we get that

$$\mathcal{O}_{\hat{X}}(E)|_{\mathbb{P}^{r-1}} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(-1).$$

So we still get $n = -1$. □

Problem 15. Compute the cohomology groups $H^q(\mathbb{P}^n, \mathcal{O}(m))$ for all q, n, m.

Proof. There are several cases we already know: When $q = 0$, we know $H^0(\mathbb{P}^n, \mathcal{O}(m)) = \mathbb{C}[z_0, ..., z_n]_m$ when $m \ge 0$ and is zero when $m < 0$. By Serre duality $H^q(X, E) \cong H^{n-q}(X, K_X \otimes E^*)^*$, we know that $H^n(\mathbb{P}^n, \mathcal{O}(m)) \cong H^0(\mathbb{P}^n, \mathcal{O}(-n-m-1))^*$.

Now onto the cases where $q > 0$. By Kodaira vanishing theorem, since $\mathcal{O}(m)$ is positive for $m > 0$, we have that $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for all $q > 0, m > 0$. When $m < -n-1$, we have $-m-n-1 > 0$, so $H^q(\mathbb{P}^n, \mathcal{O}(m)) \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}(-m-n-1))^* = 0$ by Kodaira. Finally, for the remaining range $-n-1 \le m \le 0$, we claim that $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for any n given, any $-n-1 \le m \le 0$, and for any q. To see this, we induct on n. The base case $n = 0$ is trivial as $\mathbb{P}^0$ is just a point. Inductively, we use the hyperplane restriction sequence on $\mathbb{P}^n$:

$$0 \to \mathcal{O}(m-1) \to \mathcal{O}(m) \to \mathcal{O}_{\mathbb{P}^{n-1}}(m) \to 0.$$

The long exact sequence we get reads

$$\cdots \to H^{q-1}(\mathbb{P}^{n-1}, \mathcal{O}(m)) \to H^q(\mathbb{P}^n, \mathcal{O}(m-1)) \to H^q(\mathbb{P}^n, \mathcal{O}(m)) \to H^q(\mathbb{P}^{n-1}, \mathcal{O}(m)) \to \cdots$$

For $0 < q < n-1$, we have by inductive hypothesis and exactness that

$$H^q(\mathbb{P}^n, \mathcal{O}(m-1)) \cong H^q(\mathbb{P}^n, \mathcal{O}(m-1)).$$

Therefore we can keep subtracting from m until we get into the range where Kodaira vanishing theorem applies, which would give $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for $0 < q < n-1$ and for all m. The last remaining case is $H^{n-1}(\mathbb{P}^n, \mathcal{O}(m))$, which by Serre duality is isomorphic to $H^1(\mathbb{P}^n, \mathcal{O}(-n-m-1))^* = 0$ by our previous result. Therefore, we conclude that

$$H^q(\mathbb{P}^n, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, ..., z_n]_m & q = 0, m \geq 0 \\ H^0(\mathbb{P}^n, \mathcal{O}(-n-m-1))^* & q = n \\ 0 & 0 < q < n \\ 0 & q > n. \end{cases}$$

The last statement that $H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for $q > n$ is because by Dolbeault theorem we have $H^q(\mathbb{P}^n, \mathcal{O}(m)) = H^{0,q}(\mathbb{P}^n; \mathcal{O}(m))$, and there are no $(0, q)$-forms on $\mathbb{P}^n$ for $q > n$. □

Problem 16. Consider the surface X given by $x_0^2 + x_1^2 + x_2^2 = 0$ in $\mathbb{P}^3$.

(a) Show that X is singular at the point $p = (0 : 0 : 0 : 1)$, and that $X - p$ is smooth.

(b) Blow up $\mathbb{P}^3$ at p and let $Y \subset \mathrm{Bl}_p \mathbb{P}^3$ be the strict transform of $X \subset \mathbb{P}^3$, i.e. Y is the closure of $X - p$ inside $\mathrm{Bl}_p \mathbb{P}^3$. Show that Y is smooth.

(c) Let $q \in X$ be a point in X with $q \neq p$. Let L be a line passing through p, q. Check that $L \subset X$. In other words, X is a projective cone in $\mathbb{P}^3$ with vertex p.

(d) Show that Y is a Hirzebruch surface, so that $Y \cong Y_n$ for some n (you don't need to show that is the value of n in this part).

(e) What is $\mathrm{Pic}(Y)$?

(f) By (d) we know $Y \cong Y_n$ for some n. What is n?

Proof.

(a) X is singular at p because all partial derivatives vanishes, and this is the only point where all partial derivative vanishes so $X - p$ is smooth.

(b) $\hat{\mathbb{P}}^3 = \mathrm{Bl}_p \mathbb{P}^3$ can be described locally on the affine chart $x_3 = 1$ of $\mathbb{P}^3$. Equivalently, set $z_i = x_i/x_3$ for $i = 0, 1, 2$. Then $\hat{\mathbb{P}}^3$ is the blow up of $\mathbb{C}^3_{z_0,z_1,z_2}$ at the origin $Z(z_0, z_1, z_2)$. Using $[a_0 : a_1 : a_2]$ for homogeneous coordinate on $\mathbb{P}^2$, this is

$$\hat{\mathbb{P}}^3 = \{([a_0 : a_1 : a_2], (z_0, z_1, z_2)) \in \mathbb{P}^2 \times \mathbb{C}^3 : a_i z_j = a_j z_i\}.$$

We can study the blow up $\hat{\mathbb{P}}^3$ on three affine charts of $\mathbb{P}^2$, namely $a_i = 1$ for $i = 0, 1, 2$.

On the affine chart $a_0 = 1$, the relation defining $\hat{\mathbb{P}}^3$ suggests $z_j = a_j z_0$ for $j = 0, 1, 2$. The equation of X is given by $z_0^2 + z_1^2 + z_2^2 = 0$ on $\mathbb{C}^3$. So its *total transform* on the chart $a_0 = 1$ is given by

$$z_0^2(1 + a_1^2 + a_2^2) = 0.$$

Note that the exceptional divisor is at $z_0 = z_1 = z_2 = 0$, using the relation $z_j = a_j z_0$, we see that the exceptional divisor in this chart is identified with $z_0 = 0$. The *strict transform* is obtained by removing the factor coming from the exceptional divisor in the above equation. Hence in this chart the strict transform Y is given by the equation

$$1 + a_1^2 + a_2^2 = 0,$$

which is smooth. The calculation in the chart $a_1 = 1$ and $a_2 = 1$ is entirely analogous. The equation of Y in the chart $a_1 = 1$ is given by

$$a_0^2 + 1 + a_2^2 = 0$$

and the equation of Y in the chart $a_2 = 1$ is given by

$$a_0^2 + a_1^2 + 1 = 0,$$

which are both smooth.

(c) Let $q = (q_0 : q_1 : q_2 : q_3) \in X \subset \mathbb{P}^3$ be another point. Then a projective line L connecting p, q (which corrsponds to a complex plane in $\mathbb{C}^4$ spanned by P, Q, where P, Q are preimages of p, q under the quotient map $\mathbb{C}^4 \to \mathbb{P}^3$) is parameterized by a $(\lambda : \mu) \in \mathbb{P}^1$ by

$$L = \lambda q + \mu p = (\lambda q_0 : \lambda q_1 : \lambda q_2 : \lambda q_3 + \mu).$$

Since $q_0^2 + q_1^2 + q_2^2 = 0$ by assumption, it is easy to see that $L \subset X$.

(d) By (c), we know that X is a projective cone. Projection from the vertex p gives a rational map

$$\pi_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \qquad (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2)$$

which is defined away from $p \in \mathbb{P}^3$. Let

$$C := X \cap Z(x_3) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : x_0^2 + x_1^2 + x_2^2 = 0\}.$$

Because X is a projective cone with vertex p, we know $\pi_p(X - p) \subset C$. Moreover, C is a smooth curve of genus 0 in $\mathbb{P}^2$. Hence $C \cong \mathbb{P}^1$. Let $\hat{\mathbb{P}}^3$ denote the blow up of $\mathbb{P}^3$ at p, with $\phi : \hat{\mathbb{P}}^3 \to \mathbb{P}^3$ the blow down map. Then the composition

$$\pi_p \circ \phi : \hat{\mathbb{P}}^3 \to \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$$

restricted to the strict transform Y of X, gives a morphism

$$\varphi : Y \to C \cong \mathbb{P}^1$$

whose fiber $\varphi^{-1}(q)$ over a point $q \in C$, the is strict transform of the projective line $L_{pq} \subset \mathbb{P}^3$ in $\hat{\mathbb{P}}^3$. Since L_{pq} is smooth and we are blowing it up at p, its strict transform is isomorphic to itself, L_{pq}, which is isomorphism to $\mathbb{P}^1$ because it is a projective line. To conclude, the map φ makes Y into a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Now we invoke a theorem of Grothendieck:

Theorem (Grothendieck-Tsen's Theorem)**.** *Let C be a curve, and let $\varphi : Y \to C$ be a smooth map from a surface Y to C with fiber isomorphic to $\mathbb{P}^1$. Then $Y \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a rank 2 vector bundle over C.*

Then it follows immediately from the above theorem and the splitting theorem of Grothendieck in Problem 8 that $Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ for some integers a, b. Now $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ for any line bundle on $\mathbb{P}^1$, hence WLOG we may tensor by $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(a)$ and assume that

$$Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

for some n.

(e) By the calculation we did in Problem 9, we know $\mathrm{Pic}(Y) \cong \mathbb{Z}^2$.

(f) Answer: $n = -2$. Proof: The Hirzebruch surface Y_n has a unique curve (divisor) C such that $C^2 = C \cdot C = n$. So it suffices to find a curve $C \subset Y$ with $C^2 = 2$.

Let $\hat{\mathbb{P}}^3$ be the blow up of $\mathbb{P}^3$ at p. Let E be the exceptional divisor in $\mathbb{P}^3$. Then there is a natural candidate C in Y given by $C = Y \cap E$. Note that since E has codimension 1 (locally cut out by 1 equation) and Y is a surface, C is indeed a curve (hence a divisor in Y). We claim that

$$\mathcal{O}_{\hat{\mathbb{P}}^3}(E)|_Y = \mathcal{O}_Y(C).$$

To see this, we cover $\hat{\mathbb{P}}^3$ by open sets $\hat{\mathbb{P}}^3 = \bigcup_i U_i$, and assume that locally $E \cap U_i$ is cut out by a single equation f_i. Then the transition function of $\mathcal{O}_{\hat{\mathbb{P}}^3}(E)$ is $\tau_{ij} = f_i/f_j$. But note that since $C = Y \cap E$, these equations f_i when restricted to Y is the defining equation for C, hence the transition function of $\mathcal{O}_Y(C)$ is the same as the transition function of $\mathcal{O}_{\hat{\mathbb{P}}^3}(E)|_Y$. This proves the claim.

Therefore, we see that $\mathcal{O}_Y(C)|_C = (\mathcal{O}_{\hat{\mathbb{P}}^3}(E)|_Y)|_C = \mathcal{O}_{\hat{\mathbb{P}}^3}(E)|_C$. Hence

$$C^2 = \deg(\mathcal{O}_Y(C)|_C) = \deg(\mathcal{O}_{\hat{\mathbb{P}}^3}(E)|_C).$$

Now since $E = \mathbb{P}(N_{p/\mathbb{P}^3}) = \mathbb{P}(T_p\mathbb{P}^3) \cong \mathbb{P}^2 \times \{p\}$, by Problem 13, we see that

$$\mathcal{O}_{\hat{\mathbb{P}}^3}(E)|_C = (\mathcal{O}_{\hat{\mathbb{P}}^3}(E)|_E)|_C \cong (\mathcal{O}_{\mathbb{P}^2}(-1))|_C.$$

Therefore,

$$C^2 = \deg(\mathcal{O}_{\mathbb{P}^2}(-1)|_C) = \int_C c_1(\mathcal{O}_{\mathbb{P}^2}(-1)) = -\int_C c_1(\mathcal{O}_{\mathbb{P}^2}(1)).$$

Recall that in (b), we already calculated what the strict transform Y look like, and we saw that if $[a_0 : a_1 : a_2]$ are homogeneous coordinates on $\mathbb{P}^2 \cong E$, the curve C is given by

$$C = Y \cap E = Z(a_0^2 + a_1^2 + a_2^2) \subset \mathbb{P}^2,$$

which is a quadratic equation. By the above equation, C^2 is equal to minus of the intersection number of C with the hyperplane class $H = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, which by definition is the degree of C as a projective variety (or we can also use Bézout's theorem). Hence we see that

$$C^2 = -2.$$

This shows that $Y = Y_{-2} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) = Y_2$ where we twisted the projectivization by the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$. Hence $n = 2$, as desired.

□

Problem 17. Let C be a smooth curve in $\mathbb{P}^2$. What are the possible genera of C? Is every genus allowed?

Proof. Not every genus is allowed. By the degree genus formula we have

$$g = \frac{(d-1)(d-2)}{2},$$

where d is the degree of the defining equation of C. For $d = 1, 2, ..., 10$, the possible genera are $g = 0, 0, 1, 3, 6, 10, 15, 21, 28, 36$. Hence we see that, for example, $g = 2$ is not allowed in $\mathbb{P}^2$.

Finally, to see that for every d there is a smooth curve of degree d in $\mathbb{P}^2$, we note that the Fermat polynomial $x^d + y^d + z^d = 0$ is smooth of degree d. □

Problem 18. Let C be a smooth curve in $\mathbb{P}^2$ which has genus 3 (assume hyperelliptic require $g > 1$ instead of $g \geq 1$ if you want). Can C be hyperelliptic?

Proof. No. Use degree genus formula $3 = (d-1)(d-2)/2$ we see that $d = 4$. The claim is that a degree 4 plane curve can never be hyperelliptic.

We note that by the adjunction formula $K_C = K_{\mathbb{P}^2} \otimes \mathcal{O}(4)|_C = \mathcal{O}(1)|_C$. Therefore, sections of K_C are sections of $\mathcal{O}(1)$ (homogeneous polynomials of degree 1) restricted to C. By Riemann-Roch, we have

$$h^0(K_C) - h^1(K_C) = \deg K_C - g + 1 = 2g - 2 - g + 1 = g - 1 = 2.$$

Since $h^1(K_C) = h^0(K_C^* \otimes K_C) = h^0(\mathcal{O}_C) = 1$, we have $h^0(K_C) = 3$. Indeed, these three linearly independent sections are given by x, y, z, linear polynomials on $\mathbb{P}^2$. Then the canonical map

$$\varphi_{K_C} : C \to \mathbb{P}^2 \qquad (x, y, z) \mapsto (x, y, z)$$

is precisely the embedding of C into $\mathbb{P}^2$. But recall that one of the equivalent definition for a curve C to be hyperelliptic is precisely that φ_{K_C} is *not* an embedding. Thus we conclude that C is not hyperelliptic. □

Problem 19. Suppose C is a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, what are the possible genera?

Proof. Such a curve C is defined by a homogenous polynomial of bidegree (a, b). Then the genus is given by

$$g = (a-1)(b-1).$$

See Problem 6 part (c) for a proof.

It remains to show that for every bidegree (a, b) there exists a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$ with that bidegree. If $a = 0$ or $b = 0$, then the Fermat polynomial will be an example. Now assume $a, b > 0$. Then the line bundle $\mathcal{O}(a, b)$ is very ample. Then the result follows from Bertini's Theorem: If a linear system $|L|$ on a smooth variety X is basepoint-free, then a general member of $|L|$ is smooth.

To check that $\mathcal{O}(a, b)$ is basepoint free, we check that for every point $p \in \mathbb{P}^1 \times \mathbb{P}^1$ there is a section $s \in H^0(\mathcal{O}(a, b))$ such that $s(p) \neq 0$ for $a, b > 0$. Note that $\mathcal{O}(a) \to \mathbb{P}^1$ is basepoint free for $a > 0$. Hence if $p = ([x : y], [u : v]) \in \mathbb{P}^1 \times \mathbb{P}^1$, there exists $f \in H^0(\mathcal{O}(a))$ such that $f(x, y) \neq 0$, and some $g \in H^0(\mathcal{O}(b))$ such that $g(u, v) \neq 0$. Then $f \otimes g(p) = f(x, y)g(u, v) \neq 0$ in $H^0(\mathcal{O}(a, b))$. □

Problem 20. Describe a way to compute the Hodge numbers of a hypersurface X in the weighted projective space $\mathbb{P}(Q) = \mathbb{P}(q_0, ..., q_n)$ defined by the equation $f(x_0, ..., x_n) = 0$ of degree d.

Proof. The proof is beyond our capabilities so we will just describe the method. Let R be the Jacobian ring which is defined by

$$R = \mathbb{C}[x_0, ..., x_n] / \left(\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_n} \right).$$

This is a graded ring (by degrees) with graded pieces $R = \bigoplus_{\lambda \geq 0} R_\lambda$. We define the Poincaré series of R

$$P(t) = \prod_{i=0}^{n} \frac{1 - t^{d-q_i}}{1 - t^{q_i}}.$$

We can expand this series as a polynomial in t. And foundation result states that

$$\sum_{a=0}^{\infty} (\dim R_a) t^a = P(t).$$

Let $D = \dim X$, then there is a theorem of Griffiths-Steenbrink-Danilov which says

$$h^{D-k,k}_{\text{prim}}(X) = \dim R_{\lambda_k}$$

where

$$\lambda_k = (k+1)d - |Q| = (k+1)d - \sum_{i=0}^{n} q_i.$$

Hence in principle all the primitve Hodge numbers can be read off from the Poincaré series $P(t)$. Finally, we compute the Hodge diamond. The Lefschetz hyperplane theorem (the restriction $H^k(\mathbb{P}(Q); \mathbb{C}) \to H^k(X; \mathbb{C})$ is an isomorphism for $k < \dim X$, and the cohomology of the weighted projective space is the same as that of the projective space $\mathbb{P}^{|Q|}$, only difference is the cup product structure) shows that the only interesting Hodge numbers are the middle ones and the one row above the middle. Here, we use that for $p+q = \dim X = D$, we have

$$h^{p,q} = h^{p,q}_{\text{prim}} + \delta_{p,q}.$$

To see this, note that for the middle cohomology $p+q = D$ we have

$$H^D_{\text{prim}}(X) = \ker(L : H^D(X) \to H^{D+2}(X)).$$

Thus

$$H^{p,q}_{\text{prim}}(X) = \ker(L : H^{p,q}(X) \to H^{p+1,q+1}(X)).$$

When $p \neq q$, then by Lefschetz hyperplane theorem we have $h^{p+1,q+1}(X) = h^{p-1,q-1}(X) = h^{p-1,q-1}(\mathbb{P}(Q)) = 0$. Hence $h^{p,q}_{\text{prim}}(X) = h^{p,q}(X)$. Again by the hyperplane theorem, when $p = q$, then $h^{p+1,q+1}(X) = 1$, and $L : H^{p,p}(X) \to H^{p+1,p+1}(X)$ is surjective because $L(\omega^p) = \omega^{p+1}$ generates $H^{p+1,p+1}(X)$. Therefore, by the rank-nullity theorem, we see that $h^{p,p}_{\text{prim}}(X) + 1 = h^{p,p}(X)$. $\square$

Problem 21. Take a smooth conic in $\mathbb{P}^2$. Show that there is a unique double cover of $\mathbb{P}^2$ branched along the conic. That is, a surface $S \to \mathbb{P}^2$ smooth and 2:1 and branched along the conic. Show that S is smooth, and find its Hodge numbers.

Proof. We first show a more general claim:

Theorem. *Let $Y \subset \mathbb{P}^2$ be a variety define the an equation $F(x, y, z) = 0$ of degree $2d$. Then there is a surface $X \to \mathbb{P}^2$ that is a double cover branched along Y. If Y is smooth then X is smooth.*

Proof of Theorem. Let X be the hypersurface in $\mathbb{P}(1, 1, 1, d)$ defined by the equation

$$w^2 = F(x, y, z),$$

where w is a variable of weight d. Note that X does not contain points such that $x = y = z = 0$, because otherwise the equation above implies $w = 0$ as well, which is not allowed in $\mathbb{P}(1, 1, 1, d)$. Therefore, there is a well-defined holomorphic map $X \to \mathbb{P}^2$ given by $(x, y, z, w) \mapsto (x, y, z)$. Moreover, note that for any $(x, y, z) \in \mathbb{P}^2$ such that $F(x, y, z) \neq 0$, there are two solutions for w, and there is only one solution if $F = 0$. Hence $X \to \mathbb{P}^2$ is precisely branched at $Z(F) = Y$. The surface X is singular at $p = (x, y, z, w)$ iff $w = 0$ and $F_x(p) = F_y(p) = F_z(p) = 0$. But since F is smooth by assumption, we conclude that X is smooth. $\square$

To prove uniqueness of X, we claim that

Proposition. *Let $\pi : X \to \mathbb{P}^2$ be any degree 2 cover branched at $Y = Z(F)$ with $\deg F = 2d$. Then we have an isomorphism*

$$\pi_* \mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-d).$$

Recall that $\pi_* \mathcal{O}_X(U) = \mathcal{O}_X(\pi^{-1}U)$ is a sheaf on $\mathbb{P}^2$.

Given this proposition, we can see how uniqueness of X follows easily: If X' is another such branched cover, then $\pi_*\mathcal{O}_X \cong \pi'_*\mathcal{O}_{X'}$. Then we can choose open cover $\mathbb{P}^2 = \cup_i U_i$, such that $\pi^{-1}U_i, (\pi')^{-1}U_i$ are charts of X, X'. Then the isomorphism $\pi_*\mathcal{O}_X \cong \pi'_*\mathcal{O}_{X'}$ allows us to identify local holomorphic coordinates on $\pi^{-1}U_i, (\pi')^{-1}U_i$, and these identifications are compatible on the intersections. Hence we can construct a global isomorphism $X \to X'$.

Proof of the Proposition. Let $\sigma : X \to X$ be the automorphism on X defined by flipping the two sheets of the cover. Then there is an induced automorphism $\sigma^* : \mathcal{O}_X \to \mathcal{O}_X$. Now consider a map

$$\tau : \pi_*\mathcal{O}_X \to \mathcal{O} \qquad \tau : s \mapsto \frac{1}{2}(s + \sigma^* s).$$

It is easy to check that if $s \in \mathcal{O}_X(\pi^{-1}U)$ then indeed $\tau(s) = \frac{1}{2}(s + \sigma^* s)$ is a well-defined function on $\mathbb{P}^2$, defined by

$$\tau(s)(p) := \frac{1}{2}(s(q) + \sigma^* s(q))$$

where q is any point in $\pi^{-1}(p)$ which contains two points if π is not ramified at p, but this definition does not depend on the choice of the preimage since whichever choice give the same value due to the average of s and $\sigma^* s$. Moreover, it is easy to check that the pullback $\pi^* : \mathcal{O} \to \pi_*\mathcal{O}_X$ gives a right splitting of τ. Therefore, we have a split exact sequence

$$0 \to \mathcal{L} \to \pi_*\mathcal{O}_X \to \mathcal{O} \to 0 \tag{*}$$

where $\mathcal{L} = \ker(\tau : \pi_*\mathcal{O}_X \to \mathcal{O})$. Thus $\pi_*\mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{L}$. Hence we need to show that $\mathcal{L} \cong \mathcal{O}(-d)$.

To see this, we note multiplication gives a bilinear map

$$\mu : \pi_*\mathcal{O}_X^{\otimes 2} \to \mathcal{O} \qquad \mu(s \otimes t) = st + \sigma^*(st).$$

This induces a map via adjunction:

$$\mu : \pi_*\mathcal{O}_X \to (\pi_*\mathcal{O}_X)^\vee.$$

The map $\pi : X \to \mathbb{P}^2$ is a covering map (unramified) at at point p iff there exists open $p \in U$ such that $\pi^{-1}U = \bigsqcup_\alpha U_\alpha$ and $\pi : U_\alpha \to U$ is an isomorphism. Then

$$\pi_*\mathcal{O}_X(U) = \mathcal{O}_X(\pi^{-1}U) = \mathcal{O}_X(\bigsqcup_\alpha U_\alpha) \cong \prod_\alpha \mathcal{O}_X(U_\alpha) \cong \prod_\alpha \mathcal{O}(U).$$

Now we choose a generator e_α of $\mathcal{O}_X(U_\alpha)$, given by a function that is identically 1 on U_α and zero on all the other U_β's. Then these e_α's form a basis of $\pi_*\mathcal{O}_X(U)$, and the map above with respect to this basis is given by

$$\mu : \Gamma(U, \pi_*\mathcal{O}_X) \to \Gamma(U, (\pi_*\mathcal{O}_X)^\vee) \qquad \mu = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

Now let us study the case where $\pi : X \to \mathbb{P}^2$ is ramified at a point $p \in \mathbb{P}^2$. In this case we can choose holomorphic coordinates (u, v) on open set U of $\mathbb{P}^2$ and (z, w) on open set $\pi^{-1}U$ of X such that π with respect to these holomorphic coordinate is given by $u = z^2$ and $v = w$. Then for any $f \in \Gamma(U, \pi_*\mathcal{O}_X)$ we can write a power series expansion

$$f(z,w) = \sum_{m,n=0}^{\infty} c_{mn} z^m w^n.$$

Writting $m = 2q + r$, for $0 \le r < 2$, we have

$$f(z,w) = \sum_{r=0}^{1} z^r \sum_{q,m=0}^{\infty} c_{2q+r,n} z^{2q} w^n = \sum_{r=0}^{1} z^r \sum_{q,m=0}^{\infty} c_{2q+r} u^q v^n.$$

Therefore, we have shown that $f(z, w) = g_0(u, v) + g_1(u, v)z$ where $g_0, g_1 \in \Gamma(U, \mathcal{O})$. Therefore, we have

$$\pi_*\mathcal{O}_X(U) = \mathcal{O}[z]/(z^2).$$

Hence in this case, the multiplication map $\mu : \pi_*\mathcal{O}_X^{\otimes 2}(U) \to \mathcal{O}(U)$ is degenerate, because we have $\mu(z, z) = 0$. Therefore the map $\mu : \Gamma(U, \pi_*\mathcal{O}_X) \to \Gamma(U, (\pi_*\mathcal{O}_X)^\vee)$ obtained by adjunction is also degenerate.

In summary, we have shown that the map $\pi : X \to \mathbb{P}^2$ is unramified at a point iff the multiplication map $\mu : \pi_*\mathcal{O}_X \to (\pi_*\mathcal{O}_X)^\vee$ is nondegenerate there. Now this becomes a linear algebra question: A map $T : V \to W$ is nondegenerate iff $\det T : \det V \to \det W$ is nondegenerate. Hence π is unramified iff the induced map

$$\det \mu : \det(\pi_*\mathcal{O}_X) \to \det(\pi_*\mathcal{O}_X)^\vee$$

is nondegenerate. This corresponds to a section

$$\xi \in H^0(\mathbb{P}^2, \det(\pi_*\mathcal{O}_X)^{\otimes 2})$$

with the property that π is ramified (branched) precisely at the place where $\xi = 0$. But we know that π is branched precisely at $Y = Z(F)$ which is a section of $\mathcal{O}(2d)$. Therefore,

$$\mathcal{O}(2d) = \mathcal{O}(Y) = \mathcal{O}(Z(\xi)) \cong \det(\pi_*\mathcal{O}_X)^{\otimes -2} \cong \mathcal{L}^{\otimes -2},$$

where the last isomorphism follows from taking the determinant of the short exact sequence (*). This shows that $\mathcal{L} \cong \mathcal{O}(-d)$, as desired. □

To compute the Hodge numbers of X, we use the result in Problem 20. In this case, we have a hypersurface of degree $2d$ (dimension $D = 2$) in the weighted projective space $\mathbb{P}(1, 1, 1, d)$. Hence

$|Q| = 3 + d$. The Poincaré series now reads

$$P(t) = \frac{(1-t^{2d-1})^3(1-t^{2d-d})}{(1-t)^3(1-t^d)} = \left(\frac{1-t^{2d-1}}{1-t}\right)^3 = (1+t+\cdots+t^{2d-2})^3.$$

Note that the factors $(1-t^d)$ in the expressions of $P(t)$ cancels out nicely. This is because our equation is defined by $w^2 - F(x,y,z) = 0$, and when we take partial with respect to w, we get $2w$ in the Jacobian ideal, so when we consider the Jacobian ring which is the quotient of $\mathbb{C}[x,y,z,w]$ by the Jacobian ideal, the variable w will be eliminated completely.

Note that when $d = 1, 2$, we have $h^{2,0} = h^{0,2} = 0$. To see this, recall that $h^{2,0} = h^{2,0}_{\text{prim}}$. Recall that from Problem 20 we have $h^{D-k,k} = \dim R_{\lambda_k}$, here $D = 2$ and $k = 0$. Therefore

$$\lambda_0 = (k+1)2d - |Q| = 2d - (3+d) = d - 3 < 0$$

since $d = 1, 2$. Hence $\dim R_{\lambda_0} = 0$. Also, by the Lefschetz hyperplane theorem (the restriction $H^k(\mathbb{P}(Q);\mathbb{C}) \to H^k(X;\mathbb{C})$ is an isomorphism for $k < \dim X$, and the cohomology of the weighted projective space is the same as that of the projective space $\mathbb{P}(1,1,1,1) = \mathbb{P}^3$, only difference is the cup product structure), we have $h^{1,0} = h^{0,1} = 0$. Hence it only remains to compute the Hodge number $h^{1,1} = h^{1,1}_{\text{prim}} + 1$. In this case $k = 1$. And we have

$$\lambda_1 = (k+1)2d - |Q| = 4d - (3+d) = 3d - 3.$$

When $d = 1$, we have $\lambda_1 = 0$. Therefore we should look for the coefficient of the constant term t^0. But when $d = 1$ the Poincaré series reads

$$P(t) = 1.$$

Thus $h^{1,1}_{\text{prim}} = 1$ and $h^{1,1} = 2$. Hence the Hodge diamond is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Alternatively, we could use the previous proposition in Problem 21, which says $\pi_*\mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-d)$. Then using

$$H^{0,p}(X) = H^p(X, \mathcal{O}_X) = H^p(\mathbb{P}^2, \pi_*\mathcal{O}_X) = H^p(\mathbb{P}^2, \mathcal{O}) \oplus H^p(\mathbb{P}^2, \mathcal{O}(-d)).$$

Then together with Lefschetz hyperplane theorem and Hodge symmetry, we can obtain all Hodge numbers except $h^{1,1}$. This can be obtained by calculating $\chi(X)$ and solving $h^{1,1}$ using $\chi(M) = \sum_k (-1)^k \sum_{p+q=k} h^{p,q}$. To calculate $\chi(X)$, we note that $\pi : X \setminus \tilde{Y} \to \mathbb{P}^2 \setminus Y$ is an honest 2:1 covering map. Hence

$$\chi(X) = \chi(X \setminus \tilde{Y}) + \chi(\tilde{Y}) = 2\chi(\mathbb{P}^2 \setminus Y) + \chi(\tilde{Y}) = 2\chi(\mathbb{P}^2) - 2\chi(Y) + \chi(\tilde{Y}).$$

Now $\pi : \tilde{Y} \to Y$ is a 1-1 map, hence is an isomorphism. Therefore $\chi(\tilde{Y}) = \chi(Y)$. Thus

$$\chi(X) = 2\chi(\mathbb{P}^2) - \chi(Y) = 6 - (2 - (2d-1)(2d-2)) = 4 + (2d-1)(2d-2).$$

In the case where $d = 1$, we have $\chi(X) = 2$. One can verify that $h^{1,1} = 2$ in this case. $\square$

Remark. Although the method of calculating $\chi(X)$ and using Lefschtez hyperplane theorem and $\pi_* \mathcal{O}_X$ to calculate the Hodge diamond seems more elementary, it is still worthwhile to develop the theory of Jacobian rings to calculate Hodge diamond. This is because the Jacobian ring method also applies to calculating the Hodge numbers of hypersurfaces in $\mathbb{P}^n$, which is just $\mathbb{P}(1, ..., 1)$.

Problem 22. Take a smooth quartic curve in $\mathbb{P}^2$. Show that there exists a unique double cover X of $\mathbb{P}^2$ branched along the quartic. Show that X is smooth and find its Hodge numbers.

Proof. The construction is the same as in the previous problem, where we did the construction for a general degree $2d$ curve. In this case $d = 2$. To compute the Hodge numbers, the Poincaré series is still given by

$$P(t) = (1 + t + \cdots + t^{2d-2})^3 = (1 + t + t^2)^3.$$

Again by the same argument as in the previous problem, the only nontrivial Hodge number to compute is $h^{1,1} = h^{1,1}_{\text{prim}} + 1$. Recall that $k = 1$ in this case, and

$$\lambda_1 = (k+1)2d - |Q| = 3d - 3 = 3.$$

Hence we need to look for coefficients in front of t^3 in the Poincaré series $P(t)$. Just expand it by brute force gives

$$P(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6.$$

Therefore, $h^{1,1} = 7 + 1 = 8$. Hence the Hodge diamond reads

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 8 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Again, as in the previous problem, one can also use the isomorphism $\pi_* \mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-d)$, the Lefschetz hyperplane theorem, and Euler characteristic $\chi(X)$ to obtain the full Hodge diamond of X. $\square$

Problem 23. Now let X be a degree 2 cover of $\mathbb{P}^2$ branched along a degree 6 curve in $\mathbb{P}^2$. What is the Hodge diamond of X?

Proof. Again by Lefschetz hyperplane theorem we only need to compute the middle row of the Hodge diamond. By Problem 21, we have $\pi_*\mathcal{O}_X = \mathcal{O} \oplus \mathcal{O}(-3)$. Therefore $h^{2,0} = h^{0,2}$ can be computed via

$$H^{0,2}(X) = H^2(X, \mathcal{O}_X) = H^2(\mathbb{P}^2, \pi_*\mathcal{O}_X) = H^2(\mathbb{P}^2, \mathcal{O}) \oplus H^2(\mathbb{P}^2, \mathcal{O}(-3)).$$

Now $H^2(\mathbb{P}^2, \mathcal{O}) = 0$ because $h^{0,2}(\mathbb{P}^2) = 0$, while $\mathcal{O}(-3) \cong K_{\mathbb{P}^2}$ so $H^2(\mathbb{P}^2, \mathcal{O}(-3)) = H^2(\mathbb{P}^2, K_{\mathbb{P}^2}) = \mathbb{C}$ because $h^{2,2}(\mathbb{P}^2) = 1$. Therefore $h^{2,0}(X) = h^{0,2}(X) = 1$. Thus the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h^{1,1} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

We also deduced in Problem 21 that

$$\chi(X) = 4 + (2d-1)(2d-2).$$

In our case $2d = 6$, so $d = 24$. Therefore $h^{1,1} = 18$, and we are done. □

Problem 24. Find an example of a complex manifold X such that $\mathrm{Pic}(X) \ncong H^2(X; \mathbb{Z})$.

Proof. The exponential sequence reads

$$\cdots \to H^1(X; \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X; \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

For $c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X; \mathbb{Z})$ to be an isomorphism, we need $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*)$ to be the zero map and $H^2(X; \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ to be an injection. So this motivates us to find an example where $H^1(X, \mathcal{O}_X) \neq 0$ and $H^2(X; \mathcal{O}_X) = 0$.

Let X be an elliptic curve (topologically a torus with $g = 1$). Let $D = p - q$ be a divisor that is the difference of two distinct points on X. Consider the line bundle $\mathcal{L} = \mathcal{O}_X(D)$. Since the map $c_1 : \mathrm{Pic}(X) \to H^2(X; \mathbb{Z})$ on a curve is given by the degree map, $\mathcal{L}$ is topologically trivial.

But we claim that $\mathcal{L}$ cannot be holomorphically trivial, i.e., $\mathcal{L} \ncong \mathcal{O}_X$. Since we have an isomorphism $\mathrm{Div}(X) \cong \mathrm{Pic}(X)$, we know $\mathcal{L} \cong \mathcal{O}_X$ iff $D \sim 0$. If this is the case, then there exists a meromorphic function f on X with $\operatorname{div} f = p - q$. Thus f has a zero and a pole on X, which induces a map

$$f : X \to \mathbb{P}^1.$$

By open mapping theorem, $f(X) \subset \mathbb{P}^1$ is open. By compactness of X, we know $f(X) \subset \mathbb{P}^1$ is compact, so $f(X) \subset \mathbb{P}^1$ is closed. Since f has a pole, it hits $\infty \in \mathbb{P}^1$. Therefore, by connectedness of

$\mathbb{P}^1$, we know $f : X \to \mathbb{P}^1$ is a surjective holomorphic map, and is in fact a branched cover. But since f has only one pole, this is a degree 1 branched cover, which means it is actually a biholomorphism. But this is the desired contradiction because genus of X is 1 while genus of $\mathbb{P}^1$ is 0. $\square$

Problem 25. Let $\mathcal{L}$ be a line bundle over a curve X of genus g of even degree. How many square root does it have?

Proof. The answer is 2^{2g}.

Let us again look at the exponential sequence:

$$\cdots \to H^1(X;\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to \cdots$$

Since $H^2(X;\mathbb{Z}) = \mathbb{Z}$ and $H^2(X,\mathcal{O}_X) = 0$, the degree map $c_1 : H^1(X,\mathcal{O}_X^*) \to \mathbb{Z}$ is surjective, and has kernel $\mathrm{Pic}^0(X)$, the degree zero line bundles. Therefore,

$$\mathrm{Pic}(X) \cong \mathrm{Pic}^0(X) \oplus \mathbb{Z}.$$

The group $\mathrm{Pic}^0(X)$ is isomorphic to the Jacobian variety $\mathrm{Jac}(X)$. If we are given a base point $p \in X$, the isomorphism above takes a degree d line bundle L to $(L(-dp), d)$. Now consider the squaring map $\mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Under the identification above it reads

$$(L(-dp), d) \mapsto (L^2(-2dp), 2d).$$

Now suppose $\mathcal{L} = (\mathcal{L}', 2d)$. Since $\mathrm{Pic}^0(X)$ is just a torus, squaring map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ is always surjective. That is, we can always find a line bundle $\mathcal{E} \in \mathrm{Pic}^0(X)$ such that $\mathcal{E}^{\otimes 2} \cong \mathcal{L}'$. Therefore, $(\mathcal{E}, d)$ squares to $\mathcal{L} = (\mathcal{L}', 2d)$, so $\mathcal{L}$ always have a square root. Moreover, it is easy to see that the number of square roots is equal to the number of preimages of the square map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$.

Since $\mathrm{Pic}^0(X) \cong \mathbb{C}^g/\Lambda$ is a torus obtained by quotient of a lattice $\Lambda \cong \mathbb{Z}^g$, we see that a point $x \in \mathbb{C}^g/\Lambda$ squares to zero iff $2x \in \Lambda$ (since $\mathrm{Pic}^0(X)$ is an abelian group under tensor product, in the abelian group $\mathbb{C}^g/\Lambda$ this corresponds to addition. More concretely, any $L \in \mathrm{Pic}^0(X)$ is of the form $\mathcal{O}_X(D - dp)$ where $d = \deg D$, under squaring this goes to $2D - 2dp$). Thus $x \in (\mathbb{Z}/2\mathbb{Z})^g$. The number of such x is

$$|(\mathbb{Z}/2\mathbb{Z})^{2g}| = 2^{2g}.$$

$\square$

3 Mock Test #1

1. Let S be a smooth hypersurface of degree 2 in $\mathbb{P}^4$. Can it contain a linear $\mathbb{P}^2$? Can it contain a linear $\mathbb{P}^1$?

2. Take a plane $\mathbb{P}^2 \subset \mathbb{P}^3$. Take a smooth cubic curve $C \subset \mathbb{P}^2$. Take a point p not on the plane and look at the cone X over the curve with vertex at that point (which is the union of all lines passing through p and a point on C). This is a surface in $\mathbb{P}^3$.

   (a) What is the degree of X?

   (b) Is X smooth?

   (c) Take $\mathbb{P}^3$ and blow it up at p. Let S be the strict transform of X in the blow up. Is S smooth? If S is smooth, what are its Hodge numbers?

Solutions to Q1. Let $S \in H^0(\mathbb{P}^4, \mathcal{O}(2))$ be a smooth section. It cannot contain a linear $\mathbb{P}^2$, otherwise it would be singular. To see this, we argue by contradiction, assuming $\Pi \subset S$ is a projective plane. We first note that up to projective transformation, we can always assume Π is given by the equation $x_3 = x_4 = 0$. Since $\Pi \subset S$, the equation F defining S must be of the form

$$F = x_3 A(x_0, x_1, x_2) + x_4 B(x_0, x_1, x_2),$$

where $A, B \in H^0(\Pi, \mathcal{O}(1))$. Taking derivatives give

$$F_i = x_3 A_i + x_4 B_i \qquad F_3 = A \qquad F_4 = B$$

for $i = 0, 1, 2$. Now on the plane $\Pi = \{x_3 = x_4 = 0\}$ we have $F_i = 0$, and when restricted to $\Pi \cong \mathbb{P}^2$, the plane curves A, B must intersect at one point by Bezóut's Theorem. Hence there is at least one point where all derivatives of F vanishes, hence F is singular there.

Next, we show that X indeed contains a line. We first consider the special quadric S defined by

$$x_0 x_1 + x_2 x_3 + x_4^2 = 0.$$

It is straightforward to verify this is smooth by the Jacobian criterion. Now, consider the line ℓ defined by $x_0 = 0$, $x_2 = 0$, and $x_4 = 0$. It is easy to see that $\ell \subset S$.

The group PGL_5 (the group of automorphisms of $\mathbb{P}^4$) acts continuously on $\mathbb{P}^4$ and the space of quadrics $\mathbb{P}(\mathrm{Sym}^2 \mathbb{C}^5) \cong \mathbb{P}^{14}$. Because the property of a line being contained in a quadric is preserved under coordinate transformations, if a quadric S contains a line, then every coordinate transformation of S is also contains a line. In the space of quadrics $\mathbb{P}^{14}$, two quadrics lie in the same PGL_5-orbit if and only if they have the same rank (as symmetric 5×5 matrices). There are exactly 5 orbits:

- Rank 5: Smooth quadrics (this is a single, dense open orbit).

- Rank 4: Quadrics with a single singular point.

- Rank 3, 2, 1: Progressively more singular quadrics.

Therefore, by transitivity and equivariance of PGL_5-action, all smooth quadrics in $\mathbb{P}^4$ contains a line. $\square$

Solution to Q2.

(a) The degree of a projective variety $X \subset \mathbb{P}^3$ is the intersection number of X with a projective linear subspace of complementary dimension, in this case it is $\#(X \cap L)$ where L is a projective line. Without loss of generality we can assume $L \subset \mathbb{P}^2$, so $\#(X \cap L) = \#(C \cap L) = 3$ since C is a cubic curve.

(b) By drawing pictures, it is clear that X is smooth everywhere except at the point p. But we will need explicit coordinate expressions in (c), so let us write down everything carefully.

Let $\mathbb{P}^3$ have homogeneous coordinates x_0, x_1, x_2, x_3, and let the plane $\mathbb{P}^2$ be given by the equation $x_3 = 0$. Let C be the cubic equation $f(x_0, x_1, x_2) = 0$, and let $p = [0, 0, 0, 1]$. Then it is clear that $X \subset \mathbb{P}^3$ is also given by the equation

$$X = \{f(x_0, x_1, x_2) = 0\}.$$

Indeed, p satisfies the equation, and since the line joining p and a point $q \in C$, where $q = [x_0, x_1, x_2] \in C \subset \mathbb{P}^2$, is given by $[tx_0, tx_1, tx_2, 1 - t]$ for $t \in \mathbb{C}$, we see that L is contained in X. Now we see that at $p = [0, 0, 0, 1]$, the gradient $\nabla f(p) = 0$. Therefore, X is singular at p.

(c) We cover $\mathbb{P}^3$ by standard open sets $U_0, ..., U_3$. Since blow up is isomorphism away from p, we just need to study the blow up $\mathrm{Bl}_p U_3$. We consider coordinates $z_i = x_i/x_3$ for $i = 0, 1, 2$. Then these coordinates identifies $U_3 \subset \mathbb{P}^3$ with $\mathbb{C}^3$, and p with the origin 0 of $\mathbb{C}^3$. Hence under such identification, we have

$$\mathrm{Bl}_p U_3 = \mathrm{Bl}_0 \mathbb{C}^3_{z_0,z_1,z_2} = \{(z, Z) \in \mathbb{C}^3 \times \mathbb{P}^2 : z_i Z_j = z_j Z_i \text{ for } i, j = 0, 1, 2\}.$$

We cover $\mathbb{P}^2$ with the charts $\{Z_i = 1\}$ for $i = 0, 1, 2$. We will only look at the chart $\{Z_0 = 1\}$, since the study of the other two charts are entirely analogous. On the chart $\{Z_0 = 1\}$, the blow up relations reads $z_i = z_0 Z_i$ for $i = 0, 1, 2$. The equation of X is given by $f(z_0, z_1, z_2) = 0$. Hence in this local chart the *total transform* of X reads

$$z_0^3 f(1, Z_1, Z_2) = 0.$$

The exceptional divisor (i.e., the origin) is given by $z_0 = z_1 = z_2 = 0$. But in this chart $z_i = z_0 Z_i$, so the exceptional divisor is just $z_0 = 0$. The *strict transform* is obtained by removing the exceptional divisor and taking the closure. In effect this just means we remove the factor of the above equation coming from the exceptional divisor. Hence the strict transform of X in this chart is given by

$$f(1, Z_1, Z_2) = 0.$$

But this is the affine chart representation of the smooth cubic C in $\mathbb{P}^2$, which is smooth by assumption. Therefore, the strict transform S of X is smooth in this chart, and similarly one can show that S is smooth in all other charts. Thus we conclude that S is smooth.

It remains to compute the Hodge numbers of S. We first claim that S is a ruled surface over C. In fact, the blow up gives a map $\pi : S \to C$ making S into a $\mathbb{P}^1$-bundle over C. (See Figure 1 while reading the explanation in the next paragraph)

To see this, we note that projection from p defines a rational map $X \dashrightarrow C$, which is not defined at p. In the blow up $\mathrm{Bl}_p \mathbb{P}^3$, the point p is replaced by the exceptional divisor $E = \mathbb{P}(T_p\mathbb{P}^3) \cong \mathbb{P}^2$, which is all the tangent directions from p. The strict transform S is obtained by removing E from the total transform $\tilde{X}$ and then taking the closure. Hence $S \cap E$ will consists of all the tangent directions in $\mathbb{P}(T_p\mathbb{P}^3)$ that comes from the cone X. Thus we can define the map $\pi : S \to C$ as follows: Away from the exceptional divisor we define π via

$$S \setminus E \xrightarrow{\cong} X \setminus \{p\} \xrightarrow{\mathrm{pr}_p} C.$$

On $S \cap E$, we define π as follows: Every element in $S \cap E$ is represented by a line $\overline{pq}$, where $q \in C$. We define

$$\pi(\overline{pq}) = q.$$

Then it is not hard to see that the fiber of a general point $q \in C$ of the map $\pi : S \to C$ is given by the projective line $\overline{pq} \cong \mathbb{P}^1$.

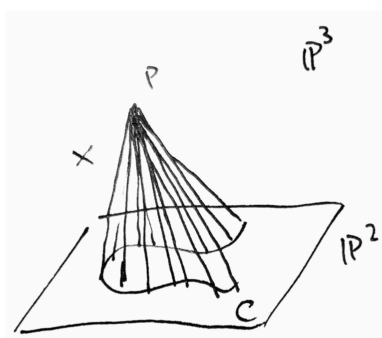

Figure 1: A drawing of the projective cone X.

Therefore, S is a $\mathbb{P}^1$-bundle over C. We first compute the cohomology of S using Leray-Hirsch theorem. We can indeed use Leray-Hirsch, because every ruled surface over a curve C is given by the projectivization of a rank 2 vector bundle $\mathcal{E} \to C$, i.e., $S \cong \mathbb{P}(\mathcal{E})$. See, from example, Hartshorne Proposition V.2.2 for a proof. Therefore, the class $c_1(\mathcal{O}_S(1))$ generates the cohomology of the fibers ($\mathbb{P}^1$). Therefore, Leray-Hirsch theorem gives

$$H^k(S) \cong \bigoplus_{p+q=k} H^p(C) \otimes H^q(\mathbb{P}^1).$$

Simple computation shows that the Betti numbers of S are given by

$$(b_0, b_1, b_2, b_3, b_4) = (1, 2, 2, 2, 1).$$

Therefore the only nontrivial part of the Hodge diamond is the middle row. We note that $h^{2,0}(S) = h^{0,2}(S) = 0$, because any nontrivial (2,0)-form on S, or similarly a (0,2)-form, restricts to zero on each fiber because $h^{2,0}(\mathbb{P}^1) = 0$, so it descends to a (2,0)-form on C, but $h^{2,0}(C) = 0$ by dimension reasons. This forces $h^{1,1}(S) = 2$. Therefore, the Hodge diamond of S is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

This concludes the solution.

$\square$

4 Mock Test #2

1. Can a smooth degree 4 curve in $\mathbb{P}^2$ be hyperelliptic?
2. Let C be a smooth genus 2 curve, and consider its Jacobian $\mathrm{Jac}(C) \cong \mathrm{Pic}^0(C)$, which is an abelian surface. Choose a point $p \in C$. Jacobi inversion theorem implies that the Abel-Jacobi map
$$u : C \to \mathrm{Pic}^0(C) \qquad x \mapsto \mathcal{O}(x-p)$$
is an embedding. The image $u(C)$ is a divisor on $\mathrm{Pic}^0(C)$. The linear system of this divisor gives a rational map $\varphi : \mathrm{Pic}^0(C) \dashrightarrow \mathbb{P}^N$.
   (a) Is φ basepoint free?
   (b) If it does have basepoint, and we resolve the basepoint, what is the image of the resolved map?
   (c) Do the same for the divisor $2u(C)$.

Solution to Q1. Answer: No. We argue by contradiction. Assuming there is a smooth curve C that is hyperelliptic, i.e., it admits a 2:1 branched cover $\pi : C \to \mathbb{P}^1$. We first claim that

Proposition. *$K_C \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(g-1)$, where g is the genus of C.*

Proof of Proposition. Let $B \subset \mathbb{P}^1$ denote the branch divisor of π and let $R \subset C$ denote the ramification divisor. By the Riemann–Hurwitz formula applied to the degree 2 map π we have as divisors
$$K_C = \pi^* K_{\mathbb{P}^1} + R.$$
Taking degrees in this equality gives
$$2g-2 = \deg(K_C) = \deg(\pi^* K_{\mathbb{P}^1}) + \deg(R) = 2 \cdot (-2) + \deg(R),$$
hence $\deg(R) = 2g+2$. Since every branch point $p \in B$ is simple for a double cover, the branch divisor B has degree $\deg(B) = 2g+2$ and satisfies $\pi^* B = 2R$.

A (standard) description of degree 2 covers shows that there exists a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ with $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_{\mathbb{P}^1}(B) \cong \mathcal{O}_{\mathbb{P}^1}(2g+2)$, and such that the ramification divisor satisfies $\mathcal{O}_C(R) \cong \pi^*\mathcal{L}$. Since $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$, so we conclude that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(g+1)$. Hence $\mathcal{O}_C(R) \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(g+1)$. Now substitute into the Riemann–Hurwitz formula:
$$K_C \cong \pi^* K_{\mathbb{P}^1} \otimes \mathcal{O}_C(R) \cong \pi^*\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(g+1) \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(g-1),$$
as required. $\square$

In our case since $d = 4$, by degree genus formula $g = 3$. Hence $K_C \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(2)$. On the other hand, we can directly apply the adjunction formula, which gives that
$$K_C = (K_{\mathbb{P}^2} + C)|_C = \mathcal{O}_{\mathbb{P}^2}(d-3)|_C = \mathcal{O}_{\mathbb{P}^2}(1)|_C.$$

Now let us look at the canonical map given by the complete linear system $H^0(C, K_C)$ which has dimension g, i.e., look at the canonical map

$$\varphi : C \to \mathbb{P}^{g-1} = \mathbb{P}^2.$$

Let x, y, z be homogeneous coordinates on $\mathbb{P}^2$. Then the adjunction formula $K_C = \mathcal{O}_{\mathbb{P}^2}(1)|_C$ tells us that a basis of $H^0(C, K_C)$ is simply given by restricting the coordinates x, y, z of $\mathbb{P}^2$ to C, hence the canonical map $\varphi : C \to \mathbb{P}^2$ is simply the inclusion map of C as a plane curve into $\mathbb{P}^2$, which is an embedding.

On the other hand, we also have $K_C \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(2)$. Let u, v be homogeneous coordinates on $\mathbb{P}^1$, then a basis for $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ is u^2, uv, v^2. So a basis of $H^0(C, K_C)$ is given by $\pi^*(u^2), \pi^*(uv), \pi^*(v^2)$, and the canonical map is therefore given by

$$\varphi : C \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\nu_2} \mathbb{P}^2$$

where $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^1$ is the degree 2 Veronese embedding $\nu_2 : [u : v] \mapsto [u^2 : uv : v^2]$. But this is not an embedding, because the projection map $\pi : C \to \mathbb{P}^1$ is 2:1. This is the desired contradiction. $\square$

Solution to Q2. Denote $J := \mathrm{Pic}^0(C)$, and $\Theta := u(C) \subset J$.

(a) No, the rational map $\varphi : J \dashrightarrow \mathbb{P}^N$ given by the linear system of Θ is not basepoint free. In fact the basepoint is precisely the curve Θ.

We claim that $h^0(J, \Theta) = 1$. We first use the Hirzebruch-Riemann-Roch to compute the Euler characteristic $\chi(J, \Theta)$. We have

$$\chi(J, \Theta) = \int_J \mathrm{ch}(\Theta)\,\mathrm{td}(J).$$

Since J is an abelian variety (i.e., a Lie group), it is parallelizable. Hence TJ is trivial. Therefore, $\mathrm{td}(J) = 1$. Since Θ is a line bundle on J, we have $\mathrm{ch}(\Theta) = \exp(c_1(\Theta))$. Its degree 4 piece is $c_1(\Theta)^2/2!$. Hence we see that

$$\chi(J, \Theta) = \int_J \frac{c_1(\Theta)^2}{2!} = \frac{\Theta \cdot \Theta}{2!}.$$

Furthermore, by the Kodaira Vanishing Theorem (since Θ is an ample divisor), all higher cohomology groups vanish: $h^i(J, \Theta) = 0$ for all $i > 0$. Hence

$$h^0(J, \Theta) = \frac{\Theta \cdot \Theta}{2!}.$$

By Riemann's formula

$$\int_J \Theta^g = g!.$$

In our case $g = 2$, so we have $\Theta \cdot \Theta = 2!$ and this gives $h^0(J, \Theta) = 1$. Therefore, let $s \in H^0(J, \Theta)$ be the generating section, then $Z(s) = \Theta$ by definition, and the canonical map

$\varphi : J \dashrightarrow \mathbb{P}(H^0(J, \Theta)) \simeq \mathbb{P}^0$ has baselocus given by $Z(s) = \Theta$, as claimed.

(b) Since the map $\varphi : J \dashrightarrow \mathbb{P}^0$ maps $J \setminus \Theta$ to a point, the resolved map $\tilde{\varphi} : J \to \mathbb{P}^0$ still maps everything to a point.

(c) By similar calculation we have

$$h^0(J, 2\Theta) = \frac{(2\Theta)^2}{2!} = 4.$$

Therefore the canonical map is $\varphi : J \dashrightarrow \mathbb{P}^3$.

A foundational theorem of abelian varieties states that for any principally polarized abelian variety, the linear system $|2\Theta|$ is always basepoint free.

Now we study the image $\varphi(J)$. Clearly $\dim \varphi(J) \leq 2$ because the domain is a surface. Also φ is not trivial because $h^0(2\Theta) = 4$. We claim that $\dim \varphi(J) = 2$, so we need to show $\dim \varphi(J) \neq 1$. Suppose for the sake of contradiction it is, then consider the intersection $\varphi(J) \cap H \cap H'$ of $\varphi(J)$ with two hyperplanes in $\mathbb{P}^3$. For generic hyperplane, $H \cap H'$ is a curve and the triple intersection will be empty. Since $\varphi^*\mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_J(2\Theta)$, we have $\varphi^*H \cap \varphi^*H' = (2\Theta)^2 = 8$ which is nonempty, contradiction. Hence $\varphi(J)$ is a surface in $\mathbb{P}^3$. We next study its degree.

First, note that C is a hyperelliptic curve. So it admits a 2:1 cover to $\mathbb{P}^1$. Let $\tau : C \to C$ be the automorphism that flips the two sheets of the cover, and by Riemann-Hurwitz C has 6 ramified points which are fixed by τ. Consider the involution on the Jacobian

$$\rho : J \to J \qquad \rho : L \mapsto L^{-1}.$$

We first note that ρ fixes the divisor Θ on J. Indeed, $\Theta = u(C)$ is the image of the Abel-Jacobi map with a chosen fixed point $p \in C$. We let p be a fixed point of τ. Then to show that $\rho(\Theta) = \Theta$ it suffices to show that ρ is surjective on Θ, since $\rho^2 = 1$. To this end, let $\mathcal{O}(x - p) \in \Theta$ and we want to find $x' \in C$ such that

$$\mathcal{O}(x - p) \cong \mathcal{O}(p - x') \Leftrightarrow x + x' - 2p \sim 0.$$

But C is of genus 2 so it has a unique g^1_2, and Riemann-Roch shows that both $h^0(2p) = 2$ and $h^0(x + \tau(x)) = 2$. Hence we conclude that the solution to the above equation is $x' = \tau(x)$. This shows that ρ fixes Θ. Hence ρ fixes Θ.

Next, since $\rho^{*2} = 1$, the line bundle $\mathcal{O}_J(2\Theta)$ decomposes into even and odd eigenspaces $V_\pm$ of ρ^*. We know $\dim V_+ + \dim V_- = h^0(2\Theta) = 4$. By computing $\operatorname{Tr} \rho^* = \dim V_+ - \dim V_-$ we can find dimensions of $V_\pm$, and it turns out that $\dim V_+ = 4$. Hence we conclude that for every $f \in \mathcal{O}_J(2\Theta)$, we have $f = \rho^* f$. That is, the map $\varphi : J \to \mathbb{P}^3$ factors through the quotient:

$$\begin{tikzcd} J \arrow[r, "\varphi"] \arrow[d, "2:1"'] & \mathbb{P}^3 \\ J/\langle \rho \rangle \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

Now we compute the degree of the variety $\varphi(J) \subset \mathbb{P}^3$. Again, this is given by the intersection number $\#(\varphi(J) \cap H \cap H')$ where $H, H' \in \mathcal{O}_{\mathbb{P}^3}(1)$ are two hyperplanes. Now consider the number $\#(\varphi^* H \cap \varphi^* H')$ on J. Note that $\varphi^* H, \varphi^* H'$ are both equivalent to the divisor 2Θ since $\varphi^* \mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_J(2\Theta)$. Hence,

$$8 = \#(\varphi^* H \cap \varphi^* H') = \deg \varphi \cdot \#(\varphi(J) \cap H \cap H') = 2 \deg \varphi(J).$$

Therefore, $\deg \varphi(J) = 4$.

Hence the image is a quartic surface in $\mathbb{P}^3$ known as the Kummer surface. The 16 fixed points of the involution $x \mapsto -x$ (which are the 16 two-torsion points of the Jacobian) map to 16 ordinary double points (nodes) on this quartic surface.

□

(Follow Up Question). Let C be a smooth curve of genus 2. Let $C \dashrightarrow \mathbb{P}^1$ be a morphism of degree 3. Is it basepoint free?

Proof. The map is basepoint free. Suppose the map is given by some linear system of the line bundle L, i.e., a subspace $V < H^0(C, L)$. Then by the information given, $d = \deg L = 3$. Then Riemann-Roch gives

$$h^0(L) = h^0(K - L) + d + 1 - g = h^0(K - L) + 2 = 2$$

because $\deg(K - L) < 0$. Hence we know $h^0(L) = 2$, and the map is given by the complete linear system $|L|$.

We know that $p \in C$ is a basepoint iff we have $h^0(L - p) = h^0(L) = 2$. By Riemann-Roch, we have

$$h^0(L - p) = h^0(K - L + p) + (d - 1) + 1 - g = h^0(K - L + p) + 1.$$

Hence p is a basepoint iff $h^0(K - L + p) = 1$. Assume this is the case. Note that

$$\deg(K - L + p) = (2g - 2) - d + 1 = 2 - 3 + 1 = 0.$$

We know that for a degree 0 line bundle $\mathcal{L}$, if $\mathcal{L} \cong \mathcal{O}_C$ then $h^0(\mathcal{L}) = 1$, otherwise $h^0(\mathcal{L}) = 0$. This is because

Lemma. *Let $\mathcal{L}$ be a degree zero line bundle. If $h^0(\mathcal{L}) \geq 1$, then $\mathcal{L} \cong \mathcal{O}_C$ and $h^0(\mathcal{L}) = 1$.*

Proof of Lemma. If $h^0(\mathcal{L}) \geq 1$, then take a section $s \in H^0(C, \mathcal{L})$. The divisor $D = (s)$ is effective and $\mathcal{O}_C(D) \cong \mathcal{L}$. Since $\deg \mathcal{L} = 0$, we must have $\deg D = 0$, but $D \geq 0$, so $D = 0$. Thus $\mathcal{L} = \mathcal{O}_C(0) = \mathcal{O}_C$. □

Hence p is a basepoint iff $K - L + p \sim 0$. This would mean $L \cong K(p)$. Therefore, we conclude that $C \dashrightarrow \mathbb{P}^1$ has basepoints iff it is of the form $L \cong K(p)$ for $p \in C$. □

5 Oral Exam Questions by Tony Pantev

1. Let X be a smooth quadric in $\mathbb{P}^3$.

   (a) Does it contain a line?

   (b) Suppose it contains a line $L \subset X$. Let B be a line disjoint from L and consider the rational map
   $$\mathrm{pr} : \mathbb{P}^3 \dashrightarrow B \qquad p \mapsto \mathrm{span}(p, L) \cap B.$$
   This map is undefined if $p \in L$. Algebraically, if we view $\mathbb{P}^3 = \mathbb{P}(V)$ for a 4-dimensional vector space V and the line $L = \mathbb{P}(W)$ where $W < V$ is a 2-dimensional subspace, then $B = \mathbb{P}(V/W)$ and the map is simply the induced quotient map $\mathrm{pr} : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/W)$, which is undefined for a point in $\mathbb{P}(W)$. By construction, its restriction
   $$\mathrm{pr}|_X : X \dashrightarrow B$$
   is not defined on L. Show that $\mathrm{pr}|_X$ extends to a morphism $\phi : X \to B$.

   (c) Describe all fibers of ϕ. How many singular fibers does it have?

2. Let $X \subset \mathbb{C}^4$ be the quadric 3-fold defined by $X = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\}$.

   (a) Where is X singular?

   (b) Show that X contains a (linear) plane Π in $\mathbb{C}^4$. Find what this plane is.

   (c) Let Y be the strict transform of X in $\mathrm{Bl}_\Pi \mathbb{C}^4$. Show that Y is smooth. Let $\phi : Y \dashrightarrow X$ be the restriction of the blow down map $\mathrm{Bl}_\Pi \mathbb{C}^4 \to \mathbb{C}^4$. Find all fibers of ϕ (generically the fiber will be a point because ϕ is an isomorphism away from the exceptional points, so the interesting fibers are the fibers of positive dimension).

Solutions

Solution to Q1. We note that $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, since all smooth quadrics in $\mathbb{P}^3$ are equivalent up to some projective transformations, since all quadratic forms over $\mathbb{C}$ can be diagonalized to to $x_0^2 + x_1^2 + x_2^2 + x_3^2$. Now $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ via the Segre embedding is quadratic, concluding the proof.

(a) Up to projective change of coordinates we may assume $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Hence of course it contains a projective line. Or if we view $X = Z(x_0^2 + x_1^2 + x_2^2 + x_3^2)$, clearly it contains the line $L = Z(x_0 + ix_1, x_2 + ix_3)$.

(b) Up to projective transformation, we can assume the line is given by $L = Z(x_0, x_1)$. Then the equation of X must be given by
$$x_0 F + x_1 G$$
for some linear polynomials F, G. The projection is a map $\mathrm{pr} : \mathbb{P}^3 \to B \cong \mathbb{P}(\Pi^\perp)$ where $\Pi = Z(x_0, x_1) \subset \mathbb{C}^4$ is the preimage of L under the projectivization. More precisely, it is given by
$$\mathrm{pr} : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1 : 0 : 0]$$

which is naturally identified with $[x_0 : x_1] \in \mathbb{P}^1$. To see this, let $p = [\mu_0 : \mu_1 : \mu_2 : \mu_3] \in \mathbb{P}^3$. Then the claim follows immediately because

$$\operatorname{span}(p, L) \cap B = \{[a\mu_0 : a\mu_1 : a\mu_2 + bx_2 : a\mu_3 + bx_3]\} \cap \{[x_0 : x_1 : 0 : 0]\} = \{[\mu_0 : \mu_1 : 0 : 0]\}.$$

Hence by the above description pr is undefined on L. When restricted to X, the equation of X tells us that $x_0 F = -x_1 G$. That is, $[-G : F] = [x_0 : x_1] \in \mathbb{P}^1$. Hence, the natural way to extend the projection map $\operatorname{pr}|_X : X \dashrightarrow \mathbb{P}^1$ is to define $\phi : X \to B$ by

$$\phi(x) = \begin{cases} [x_0 : x_1] & \text{when } x_0, x_1 \text{ do not both vanish} \\ [-G(x) : F(x)] & \text{when } F, G \text{ do not both vanish.} \end{cases}$$

We must show that the map is well-defined when $x_0 = x_1 = 0$. Note that

$$\nabla X = \nabla(x_0 F + x_1 G) = (\nabla x_0)F + x_0 \nabla F + (\nabla x_1)G + x_1 \nabla G.$$

If $x_0 = x_1 = 0$, then at that point

$$\nabla X = (\nabla x_0)F + (\nabla x_1)G = (F, G, 0, 0).$$

If $F(x) = G(x) = 0$ there, then $\nabla X = 0$ there, so X is singular at that point, which cannot happen since X is smooth. Hence we conclude that ϕ is a well-defined map that extends $\operatorname{pr}|_X : X \dashrightarrow B$.

(c) The claim is that all the fibers are isomorphic to $\mathbb{P}^1$ and there are no singular fibers. To see this, we can WLOG choose X to be the image of the Segre embedding

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \qquad \sigma : ([x : y], [x' : y']) \mapsto [xx' : xy' : yx' : yy'].$$

Then its image $X = \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ is given by the quadric equation

$$x_0 x_3 - x_1 x_2 = 0$$

in $\mathbb{P}^3$. Thus in this case $F = x_3, G = -x_2$. The map $\phi : X \to \mathbb{P}^1$ is given by

$$\phi([x_0 : x_1 : x_2 : x_3]) = \begin{cases} [x_0 : x_1] & \text{when } x_0, x_1 \text{ do not both vanish} \\ [-G : F] = [x_2 : x_3] & \text{when } x_2, x_3 \text{ do not both vanish.} \end{cases}$$

Thus the composition is

$$\phi \circ \sigma([x : y], [x' : y']) = \begin{cases} [x' : y'] & x \neq 0 \\ [x' : y'] & y \neq 0. \end{cases}$$

Therefore, we see that under the identification $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, the map $\phi : X \to \mathbb{P}^1$ is simply projection onto the second factor. Therefore, we conclude that the fiber of ϕ at a point $b \in \mathbb{P}^1$ is just $\mathbb{P}^1 \times \{b\}$ which is smooth since it is isomorphic to $\mathbb{P}^1$.

$\square$

Solution to Q2. (a) $\nabla X = 2(x_1, x_2, x_3, x_4)$. Hence X is singular at the origin.

(b) One such plane Π is given by $\Pi = Z(x_1 + ix_2, x_3 + ix_4)$.

(c) By definition we have

$$\mathrm{Bl}_\Pi \mathbb{C}^4 = \{(u, x) \in \mathbb{P}^1 \times \mathbb{C}^4 : u_1(x_3 + ix_4) = u_2(x_1 + ix_2)\}.$$

Let Y be the strict transform of X. We check that Y is smooth on two open sets $u_1 = 1$ and $u_2 = 1$ of $\mathbb{P}^1$. First on $u_1 = 1$. Then the equation gives $x_3 + ix_4 = u_2(x_1 + ix_2)$. Thus the Fermat equation becomes

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= (x_1 + ix_2)(x_1 - ix_2) + (x_3 + ix_4)(x_3 - ix_4) \\ &= (x_1 + ix_2)[(x_1 - ix_2) + u_2(x_3 - ix_4)]. \end{aligned}$$

The exceptional divisor is at $x_1 + ix_2 = x_3 + ix_4 = 0$. Since $x_3 + ix_4 = u_2(x_1 + ix_2)$, the exceptional divisor on this chart is just $x_1 + ix_2 = 0$. Hence the equation for the strict transform is given by

$$Y : x_1 - ix_2 + u_2(x_3 - ix_4) = 0.$$

This is smooth because

$$\nabla Y = \left(\frac{\partial Y}{\partial x_i}, \frac{\partial Y}{\partial u_j}\right) = (1, -i, u_2, -iu_2, 0, x_3 - ix_4)$$

which is never identically zero. Similarly one can check that Y is smooth on the other chart $u_2 = 1$. It remains to find fibers of the map $\phi : Y \dashrightarrow X$. Since $\phi : Y \setminus E \xrightarrow{\sim} X \setminus \Pi$ is an isomorphism, the fibers over $X \setminus \Pi$ will be just a single point. To determine the fiber over a point in Π, let us go back to the local coordinate expression of Y above, and look at the fiber over a point $x = (x_1, x_2, x_3, x_4)$ in Π, so that $x_1 + ix_2 = x_3 + ix_4 = 0$. Then the equation becomes

$$Y : 2(x_1 + u_2 x_3) = 0$$

for some fixed x_1, x_3. We need to discuss by cases:

- $x \neq 0 \in \Pi$: In this case x_1, x_3 cannot both be zero, otherwise by definition of Π the other two coordinates $x_2 = x_4 = 0$ as well, which means $x = 0$. Hence the equation above has one solution $[1 : u_2]$. So the fiber over any point other than zero is a single point.
- $x = 0 \in \Pi$: If this is the case then the equation above is vacuously true, so that the fiber over the origin is a copy of $\mathbb{P}^1$.

$\square$

6 Some Extra Problems (No Solutions)

The problem was given to Emmerson Hemley by Tony Pantev. I thank Emmerson for showing these problems to me two days before our oral and showed me his solution. By the way these are really good problems! Do not miss!

1. Suppose C is a smooth projective curve of genus g and let $f : C \to \mathbb{P}^1$ be a $2{:}1$ map.

   (a) Show that $f_*\mathcal{O}_C$ is a locally free sheaf of rank two on $\mathbb{P}^1$. By Grothendieck's classification theorem, $f_*\mathcal{O}_C \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ for some integers a, b. Find a and b.

   (b) Same question for f_*K_C.

2. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the degree k morphism given by $[x : y] \mapsto [x^k : y^k]$. For every integer a, show that $f_*\mathcal{O}(a)$ is locally free of rank k and determine the corresponding vector bundle.

3. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree 4 defined by $f = 0$. Let $Z \subset \mathbb{P}^2$ be the non-reduced subscheme defined by $f^2 = 0$.

   (a) The inclusion $C \subset Z$ is a closed subscheme. Does there exist a retraction $Z \to C$, i.e. a morphism whose restriction to C is the identity?

   (b) Show that there exists a line bundle $\mathcal{L}$ on Z such that $\mathcal{L}|_C$ is trivial but $\mathcal{L}$ itself is nontrivial. Compute $H^i(Z, \mathcal{L})$.

References

[1] John Lee, *Introduction to Complex Manifolds*, American Mathematical Society, 2024.

[2] Daniel Huybrechts, *Complex Geometry: An Introduction*, Springer, 2005.

[3] Phillip Griffiths, Joseph Harris, *Principles of Algebraic Geometry*, Wiley, 1978.

[4] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry (I)*, Cambridge University Press, 2002.