Complex Algebraic Geometry Oral Exam Practice Problems

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Problems Given by Ron Donagi

Problem Set 1

Problem 1. Show $\mathbb{P}^1 \cong S^2$ and that $\mathbb{P}^1$ is simply connected.

Problem 2. Compute the algebraic dimension of $\mathbb{P}^n$ and of the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

Problem 3. Prove that holomorphic maps $\mathbb{P}^1 \to \mathbb{T}$ are constant, where $\mathbb{T}$ is a complex torus.

Problem 4. For the Hopf manifold in $n = 1$, identify it as an elliptic curve and determine its lattice.

Problem 5. Explain how to recover the bundle E corresponding to a locally free sheaf $\mathcal{F}$. Define the sheaf $\mathcal{O}(k)$ on projective space and check that it corresponds to the correct bundle.

Problem 6. Show that the Grassmannian $\mathrm{Gr}(2,4)$ is a 4-dimensional quadric. (Use the Plücker embedding $\mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}^5$.)

Problem 7. Explain why the flag varieties are projective varieties.

Problem Set 2

Problem 8. Let $L \coloneqq \mathcal{O}_{\mathbb{P}^n}(d)$. Show that the map φ_L is a morphism iff $d \geq 0$, an isomorphism iff $d = 1$, and an embedding iff $d \geq 1$. The image is called the Veronese variety. For a hyperplane $H \subset \mathbb{P}^n$, what is $\varphi_L^{-1}(H)$?

Problem 9. In the above setting, let $p \in \mathbb{P}^n$ and let $V \subset H^0(\mathbb{P}^n, \mathcal{O}(d))$ be the linear system of sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ that vanish at p. Show that φ_V is no longer a morphism on $\mathbb{P}^n$ but only on $\mathbb{P}^n \backslash p$. Describe the closure of its image.

Problem 10. Consider $\mathbb{P}^2$ with homogeneous coordinates x, y, z. Let $p \in \mathbb{P}^2$ be the point $(0, 0, 1)$, and let $C \subset \mathbb{P}^2$ be the curve with equation $y^2z^2 = x^4$. Let Y be the blowup of $\mathbb{P}^2$ at p. Describe the strict and total transforms in Y of C. What happens if you blow up a second time?

Problem 11. Let (X, ω) be compact connected Kähler, $\dim X \geq 2$. Show: wedge with ω is injective on 1-forms. Deduce: if φ is a smooth function and φh is Kähler, then φ is constant.

Problem 12. For a holomorphic vector bundle E of rank r and a line bundle L, prove $\mathbb{P}(E^* \otimes L^*) = \mathbb{P}(E^*)$ and $\mathcal{O}_{\mathbb{P}(E^*\otimes L^*)}(1) \cong \mathcal{O}_{\mathbb{P}(E^*)}(1) \otimes \pi^* L$.

Problem 13. The space of conics in the plane $\mathbb{P}^2$ is parametrized by a $\mathbb{P}^5$. This has a stratification into three strata. If we interpret $\mathbb{P}^5$ as $\mathbb{P}(\mathrm{Sym}^2 V^*)$ where $\mathbb{P}^2 = \mathbb{P}(V)$, then the strata are distinguished by the rank of the symmetric 3×3 matrix. If we think of $\mathbb{P}^5$ as parametrizing conics, the 3 strata are characterized by the topology of the conic: double lines, line pairs, and non-singular conics, respectively.

How far can you push the analogous stratification of the space of cubics in the plane? Focusing on the topology of the cubic, you should get a finite number of strata. One of these is Zariski-open (i.e. open and dense). Try to prove that, for each non-open stratum S, all the cubics corresponding to points of S are isomorphic to each other. (You'll need to agree on what "isomorphic" means here.) Show also that this is false for the open stratum: all the corresponding cubics are topologically the same (they are tori), but they are not all isomorphic to each other.

Hint: you may want to consider the action of $\mathrm{SL}(V) = \mathrm{SL}(3, \mathbb{C})$ on everything.

Problem Set 3

Problem 14 (Residue Theorem, Voisin Vol I, p.113)**.** Let X be a compact complex curve. Let μ be a volume form on X. We can consider μ as a closed form of type $(1,1)$ on X.

(a) By considering the integral $\int_X \mu$, show that μ is not $\bar\partial$-exact. Deduce from this that $H^1(X, K_X)$ is different from $\{0\}$ and admits a surjective map

$$\mathrm{Tr} : H^1(X, K_X) \to \mathbb{C}, \quad \omega \mapsto \int_X \omega.$$

Let $D = \sum_i x_i$ be a divisor of X (all of whose multiplicities are equal to 1). We denote by $K_X(D)$ the holomorphic line bundle (or sheaf of free $\mathcal{O}_X$-modules of rank 1), whose sections are the holomorphic forms of degree 1 on any open set not containing any of the x_i's, and in the neighbourhood of each x_i, the meromorphic forms which can be written as $\phi(z_i)\mathrm{d}z_i/z_i$ where z_i is a local coordinate centred at x_i and ϕ is holomorphic.

(b) Show that we have an exact sequence

$$0 \to K_X \to K_X(D) \xrightarrow{\mathrm{Res}} \bigoplus_i \mathbb{C}_{x_i} \to 0,$$

where each sheaf $\mathbb{C}_{x_i}$ is a "skyscraper" sheaf supported at x_i, whose group of sections over any open set not containing x_i is $\{0\}$ and whose group of sections on an open set containing x_i is $\mathbb{C}$. Here the map $\mathrm{Res}_i : K_X(D) \to \mathbb{C}_{x_i}$ maps a meromorphic form ω to its residue at x_i, defined as

$$\mathrm{Res}_i(\omega) = \frac{1}{2\pi i} \int_{\partial D_i} \omega,$$

where D_i is a disk centred at x_i not containing any of the x_j's, $i \neq j$.

(c) Let $\delta : \bigoplus_i \mathbb{C} = H^0(X, \bigoplus_i \mathbb{C}_{x_i}) \to H^1(X, K_X)$ be the arrow appearing in the long exact sequence associated to the short exact sequence above. Show that $\delta(1_{x_i})$ is the class in $H^1(X, K_X)$ of the form $\bar{\partial}\mu_i$, where μ_i is a differential form of type $(1,0)$, which is C^∞ away from x_i, and equal to $\mathrm{d}z_i/z_i$ in a neighbourhood of x_i.

(d) Show that

$$\int_X \bar{\partial}\mu_i = -2i\pi.$$

Deduce from the long exact sequence associated to the short exact sequence above the following result: If ω is a meromorphic 1-form on X having poles of order at most 1 at each x_i, and holomorphic otherwise, then

$$\sum_i \mathrm{Res}_i\, \omega = 0.$$

Problem 15. For a manifold C and an integer $d \geq 2$, define the symmetric product $C^{(d)}$ to be the quotient of the Cartesian product C^d by the action of the symmetric group S_d permuting the factors. Show that $C^{(d)}$ is smooth iff $\dim C \leq 1$.

Problem 16. Let C be the projective curve in $\mathbb{P}^2$ whose affine equation is $y^2 = \Pi_{i=1}^{50}(x-i)^i$. Characterize its singularities and calculate its arithmetic genus and its geometric genus: the genus of its desingularization.

Problem 17. The dual of a plane curve $C \subset \mathbb{P}^2$ of degree d is the curve $C^\vee \subset (\mathbb{P}^2)^\vee$ consisting of all lines meeting C in fewer than d distinct points. For C non-singular, this is simply the family of tangent lines to C. Show that for most non-singular curves C of a given degree d, the only singularities of the dual curve are nodes and cusps. For such C, determine the degree and number of nodes and cusps of the dual curve. Show that if $d > 2$ there are non-singular C whose dual has singularities worse than nodes and cusps. What happens if C is singular?

Problem Set 4

Problem 18 (Hyperelliptic curves)**.** Let C be a hyperelliptic curve of genus g. Is the natural multiplication map

$$H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(C, (K_C)^2)$$

surjective? What is the dimension of its image?

Problem 19 (Canonical map)**.** Let C be a curve of genus g for which the map

$$H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(C, (K_C)^2)$$

is surjective, and let $\phi : C \to \mathbb{P}^{g-1}$ be the canonical map. Use Riemann–Roch to calculate the dimension of the linear system of quadrics in $\mathbb{P}^{g-1}$ containing the image $\phi(C)$. What is the base locus of this linear system? Does it depend continuously on the curve C? What happens if C is trigonal? What happens if C is hyperelliptic?

Problem 20 (Abel map)**.** Consider the Abel map $\mathcal{M}_g \to \mathcal{A}_g$. Is it an immersion at hyperelliptic curves? Is it an immersion at curves as in Problem 19?

Problem 21 (Fubini–Study Pullbacks)**.**

(a) Show that restricting the Fubini–Study Kähler form ω_{FS} on $\mathbb{P}^n$ to the standard hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ yields the Fubini–Study form on $\mathbb{P}^{n-1}$.

(b) Let $A \in \text{PGL}(n+1, \mathbb{C})$, and let $F_A : \mathbb{P}^n \to \mathbb{P}^n$ be the induced automorphism. Show that $F_A^* \omega_{\text{FS}} = \omega_{\text{FS}}$ if and only if A is in the image of $\text{U}(n+1)$.

Problem 22 (Harmonicity of the Kähler Form)**.** Let (X, g) be a Kähler manifold with Kähler form ω. Show that ω is harmonic with respect to the Laplacian $\Delta = dd^* + d^*d$.

Problem Set 5

Problem 23. Let S be a non-singular quadric surface. We saw that $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. For $C \subset S$ a non-singular curve, define its bidegree (a, b) by intersecting C with the lines $\mathbb{P}^1 \times \{\text{pt}\}$ and $\{\text{pt}\} \times \mathbb{P}^1$. Compute the genus of C in terms of a, b.

Problem 24. Let S be a non-singular quadric surface and $p \in S$. Show that $\text{Bl}_p(S)$ is a del Pezzo surface dP_2.

Problem 25. Prove that a non-singular cubic surface S is rational, i.e. there is a rational map $S \dashrightarrow \mathbb{P}^2$ which is an isomorphism away from some curves in S and $\mathbb{P}^2$. Hint: you may want to prove first that S contains two disjoint lines.

Problem 26. The Veronese variety is the image of $\mathbb{P}^n$ under the embedding given by the complete linear series $\mathcal{O}(d)$. The Segre variety is the image of $\mathbb{P}^m \times \mathbb{P}^n$ under the map $(x_i), (y_j) \mapsto (x_i y_j)$. Compute the degrees of the Veronese and Segre varieties.

Problem Set 6

Problem 27. Find the smallest integer a such that every curve C of genus ≥ 2 is embedded via sections of $K_C^{\otimes a}$.

Problem 28. Compute the Euler characteristic of a rational elliptic surface. Assume each singular fiber has a single node. How many such fibers are there? What changes if some of the fibers have cusps? Tacnodes?

Problem 29. Show that the Fermat quartic surface is elliptic. How many singular fibers does it have? Show that not all smooth quartic surface are elliptic. (Hint: The period map for K3 surfaces goes to a quadric in $\mathbb{P}^{21}$. Show that the image of the locus of smooth quartic surfaces is an open subset of a hyperplane section, which is a quadric in $\mathbb{P}^{20}$. What is the image of the elliptic quartic surfaces?)

Problem 30. Let X be the smooth complete intersection of two quadrics in $\mathbb{P}^5$. Show that X determines a smooth curve C of genus 2. Show that X contains a line ℓ. Describe the set of lines in X that intersect this ℓ. Describe the set of all lines in X. Generalize to the smooth complete intersection of two quadrics in higher dimensional projective spaces.

Problem Set 7 (All from Huybrechts Chapter 4 Except the First Problem)

Problem 31. Prove a holomorphic version of the splitting principle.

Problem 32. Let ∇_i be connections on vector bundles E_i, $i = 1, 2$. Change both connections by one-forms $a_i \in A^1(X, \mathrm{End}(E_i))$ and compute the new connections on the associated bundles $E_1 \oplus E_2$, $E_1 \otimes E_2$, and $\mathrm{Hom}(E_1, E_2)$.

Problem 33. (a) Show that $F_{E_1 \oplus E_2} = F_{\nabla_1} \oplus F_{\nabla_2}$.

(b) Show that $F_{\nabla^*} = -F^t_\nabla$ (curvature on dual bundles).

(c) Compute the connection and curvatures of the determinant bundle.

Problem 34. Find an example of two connections ∇_1 and ∇_2 on a vector bundle E, such that F_{∇_1} is positive and F_{∇_2} is negative.

Problem 35. Show that for a base-point free line bundle L on a compact complex manifold X the integral $\int_X c_1(L)^n$ is non-negative.

Problem 36. Show that $\mathrm{td}(E_1 \oplus E_2) = \mathrm{td}(E_1) \cdot \mathrm{td}(E_2)$.

Problem 37. Compute the Chern classes of (the tangent bundle of) $\mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m$. Try to interpret $\int_{\mathbb{P}^n} c_n(\mathbb{P}^n)$ and $\int_{\mathbb{P}^n \times \mathbb{P}^m} c_n(\mathbb{P}^n \times \mathbb{P}^m)$.

Problem 38. Let E be a vector bundle and L a line bundle. Show

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{\mathrm{rk}(E) - j}{i - j} c_j(E) c_1(L)^{i-j}.$$

Problem 39. Show that $c_1(\mathrm{End}(E)) = 0$ on the form level and compute $c_2(\mathrm{End}(E))$ in terms of $c_i(E)$, $i = 1, 2$. Compute $(4c_2 - c_1^2)(L \oplus L)$ for a line bundle L. Show that $c_{2k+1}(E) = 0$, if $E \cong E^*$.

Problem 40. Let L be a holomorphic line bundle on a compact Kähler manifold X. Show that for any closed real $(1,1)$-form α with $[\alpha] = c_1(L)$ there exists an hermitian structure on L such that the curvature of the Chern connection ∇ on L satisfies $(i/2\pi)F_\nabla = \alpha$. (Hint: Fix an hermitian structure on h_0 on L. Then any other is of the form $e^f \cdot h_0$. Compute the change of the curvature.)

Problem 41. Let X be a compact Kähler manifold and let E be a holomorphic vector bundle admitting a holomorphic connection $D : E \to \Omega_X \otimes E$. Show that $c_k(E) = 0$ for all $k > 0$.

1 Solutions to PSet 1

Solution 1. The simplest way is probably to use the CW complex structure on $\mathbb{P}^1$: It has a 2-cell e^2 and a 0-cell $\{*\}$, and the attaching map is given by $\varphi : \partial e^2 \cong S^1 \to \{*\}$, and there is only one such map. This is also the cell structure on S^2, hence we conclude that $\mathbb{P}^1 \cong S^2$.

Alternatively, we can use the stereographic projection. Note that $\mathbb{C} \hookrightarrow \mathbb{P}^1$ via $z \mapsto [z : 1]$. Hence we can identify $\mathbb{C} \cup \{\infty\}$ with $(S^2 \setminus \{N\}) \cup \{N\}$ stereographic projection. Concretely, one checks that the map $\Phi : S^2 \to \mathbb{P}^1$ given by

$$\Phi(x, y, z) = \begin{cases} [x + iy : 1 - z] & (x, y, z) \neq (0, 0, 1) \\ [1 : 0] & (x, y, z) = (0, 0, 1) \end{cases}$$

is a continuous bijection with continuous inverse.

Solution 2. Recall that the algebraic dimension of a complex manifold X is defined by $a(X) := \operatorname{trdeg}_{\mathbb{C}} K(X)$. Siegel theorem shows $a(\mathbb{P}^n) \leq n$ and $a(\mathbb{T}) \leq 1$. We show that these upper bounds are actually achieved: $a(\mathbb{P}^n) = n$ and $a(\mathbb{T}) = 1$.

To show that $a(\mathbb{P}^n) = n$, we note that every ratio of homogeneous polynomial with same degree in $\mathbb{C}[z_0, ..., z_n]$ defines a meromorphic function on the entire $\mathbb{P}^n$, and such ratios form a subfield $\mathcal{Q}$ of $K(\mathbb{P}^n)$. Moreover, it can be seen that

$$\mathcal{Q} = \mathbb{C}(z_1/z_0, ..., z_n/z_0).$$

The is true because we can recover any expression z_i/z_j from $(z_i/z_0)/(z_j/z_0)$, and all elements of $\mathcal{Q}$ are generated by all the z_i/z_j's. One can check that $z_1/z_0, ..., z_n/z_0$ are transcendental and linearly independent, hence $\operatorname{trdeg}_{\mathbb{C}} \mathcal{Q} = n$. Therefore, $a(\mathbb{P}^n) = n$.

To show that $a(\mathbb{T}) = 1$, we recall that from complex analysis, we showed that $K(\mathbb{T}) = \mathbb{C}(\wp, \wp')$. But in fact we have the identity $2\wp' = 12\wp - g_2$. Hence in fact $K(\mathbb{T}) = \mathbb{C}(\wp)$ which has transcendence degree 1 over $\mathbb{C}$, hence $a(\mathbb{T}) = 1$.

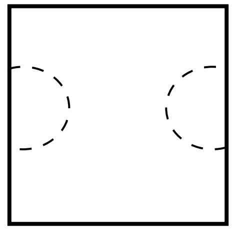
Solution 3. We identify the complex torus $\mathbb{T}$ with $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ (The same argument works for any lattice $\Lambda \subset \mathbb{C}$). In fact, this is just a square with boundary identified in the obvious way.

Since $\mathbb{T} = S^1 \times S^1$, $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$ is the universal cover of $\mathbb{T}$, and $\mathbb{P}^1 \cong S^2$ is simply connected. Hence by algebraic topology, any holomorphic map $f : \mathbb{P}^1 \to \mathbb{T}$ factors through the universal cover, meaning there is a constant map (we only know it is continuous) $g : \mathbb{P}^1 \to \mathbb{C}$ such that the diagram below commutes:

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[ur, "g"] \arrow[r, "f"'] & \mathbb{T} \end{tikzcd}$$

where the vertical map is projection $\pi : \mathbb{C} \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. The commutativity of the diagram shows that the function $g : \mathbb{P}^1 \to \mathbb{C}$ can be obtained by $f : \mathbb{P}^1 \to \mathbb{T}$ by extending the value of f periodically on the entire complex plane. This also shows that g is holomorphic:

Indeed, inside the square, by assumption g is clearly holomorphic because f is. On the boundary of the square, by definition of holomorphicity, we can find neighborhoods U on the square as shown in the figure below such that $f : f^{-1}(U) \to U$ is holomorphic.



But for sufficiently small $U \subset \mathbb{T}$, the set U is biholomorphic to $\pi^{-1}(U)$. Thus $g : f^{-1}(U) \to \pi^{-1}(U)$ is holomorphic. Since homomorphicity is a local condition, it follows that $g : \mathbb{P}^1 \to \mathbb{C}$ is a globally holomorphic function, which must be constant. Hence $f = \pi g$ must also be constant.

Solution 4. The Hopf manifold X is given by $\mathbb{C}^*/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{C}^*$ by $k \cdot z = \lambda^k z$. Let Γ be the lattice generated by $\log \lambda$ and $2\pi i$. Then I claim that there exists a well-defined holomorphic map $f : \mathbb{C}/\Lambda \to X$ that is a biholomorphism. To see this, we first note that there exists a well-defined $f : \mathbb{C}/\Lambda \to X$ making the following diagram commute:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\exp"] \arrow[d] & \mathbb{C}^* \arrow[d] \\
\mathbb{C}/\Lambda \arrow[r, "f"'] & X
\end{tikzcd}$$

In the above the vertical maps are quotient maps by the group action. We define the map as follows: Let $[z] \in \mathbb{C}/\Lambda$ represented by $z \in \mathbb{C}$. Then send it to $[\exp(z)] \in X$. The diagram clearly commutes if f is well-defined. To see f is well-defined, suppose that $z' \in \mathbb{C}$ also represents the equivalence class $[z]$ in $\mathbb{C}/\Lambda$. Then we must have

$$z' = z + 2\pi i m + n \log \lambda$$

for some $m, n \in \mathbb{Z}$. But note that

$$\exp(z + 2\pi i m + n \log \lambda) = \lambda^n e^z.$$

Therefore, under the quotient $\mathbb{C}^* \to X$, this will be mapped to $[\exp(z)]$. Thus f is well-defined. Also, f has a well-defined inverse, by taking logarithm (well-defined because elements in $\mathbb{C}/\Lambda$ is well-defined up to a shift of $2\pi i m$ for any integer m). Finally, we note that f is a biholomorphism, because the quotient maps are local biholomorphisms (in fact we only know that f is a holomorphic bijection, but then the inverse function theorem tells us that its inverse is also holomorphic), and the claim follows from commutativity of the diagram above. Hence, we conclude that $X \cong \mathbb{C}/\Lambda$ where Λ is generated by $2\pi i, \log \lambda$ on the complex plane.

Solution 5. Given a vector bundle $E \to X$, for any open set $U \subset X$, we can define the sheaf of sections $\mathcal{F} : U \mapsto \Gamma(U, E)$, where $\Gamma(U, E)$ denotes holomorphic (smooth, continuous...depending on which category we want to work with) sections of E over U. By construction $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_X$-modules, because of the existence of local frames on E.

Conversely, suppose we are given a locally free sheaf $\mathcal{F}$, then we have an open cover $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$ via some Ψ_i. We define the vector bundle E by gluing together $\mathcal{O}_{U_i}^{\oplus r}$'s on U_i via some set of transition functions. On the intersection $U_{ij} := U_i \cap U_j$, we consider

$$\Psi_{ij} := \Psi_i \circ \Psi_j^{-1}|_{U_{ij}} : \mathcal{O}_{U_{ij}}^{\oplus r} \xrightarrow{\cong} \mathcal{O}_{U_{ij}}^{\oplus r}.$$

One can check that the above two constructions are inverses of each other.

Finally, we give a definition of $\mathcal{O}(k)$ on $\mathbb{P}^n$. We define

$$\mathcal{O}(d)(U) = \left\{ \frac{g}{f} : g \in \mathbb{C}[z_0, ..., z_n]_{d+m}, f \in \mathbb{C}[z_0, ..., z_n]_m \text{ for some } m \in \mathbb{Z}, f(p) \neq 0 \text{ for all } p \in U \right\}.$$

Using this definition, it clearly follows that $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$. Hence it suffices to check that $\mathcal{O}(1)$ agrees with the vector bundle $\tau^\vee$, where τ is the tautological bundle on $\mathbb{P}^n$ defined by

$$\tau = \{(\ell, z) : z \in \ell\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}.$$

Therefore,

$$\tau^\vee = \{(\ell, \varphi_\ell) : \varphi_\ell \in \mathrm{Hom}(\ell, \mathbb{C})\}.$$

So for any $U \subset \mathbb{P}^n$, an element of $\varphi \in \Gamma(U, \tau^\vee)$ is given by an element of the form

$$\varphi : \bigsqcup_{\ell \in U} \ell \to \mathcal{O}_{\mathbb{P}^n}$$

that is linear on each fiber ℓ. Clearly, any element in $\mathcal{O}(1)(U)$ satisfies this requirement. We claim that these are the only ones.

To see this, we first study the transition functions on $\tau^\vee$. We consider the open sets $U_i \subset \mathbb{P}^n$ defined by $U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\}$. Let $\ell \in U_i$, then there is a unique vector $e(\ell) \in \ell \in \mathbb{C}^{n+1}$ with i-th coordinate being 1. In fact, this is a basis of the line ℓ. This provides local trivilizations of the vector bundle $\pi : \tau \to \mathbb{P}^n$, by

$$\Psi_i : \tau|_{\pi^{-1}(U_i)} \xrightarrow{\cong} U_i \times \mathbb{C} \qquad (\ell, z) = (\ell, \lambda(z) \cdot e(\ell)) \mapsto (\ell, \lambda(z))$$

where $\lambda(z) \in \mathbb{C}$ is the component of the vector $z \in \ell$ under the basis $e(\ell)$. Now suppose $\ell \in U_i \cap U_j$. Let $e_i(\ell), e_j(\ell)$ be the unique representative of ℓ, whose i-th and j-th coordinate is 1, respectively. Then note that the i-th coordinate of $e_j(\ell)$ is given by z_i/z_j. Hence we must have

$$e_i(\ell) = \frac{1}{z_i/z_j} e_j(\ell) = \frac{z_j}{z_i} e_i(\ell).$$

Therefore, we conclude that the transition function $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_{\mathbb{P}^n}^*)$ of τ on $U_i \cap U_j$ is given by

z_j/z_i. Hence the transition function on $\tau^\vee$ must be given by

$$g_{ij}^{-1} = z_i/z_j. \tag{1}$$

On the other hand, since the charts $\xi_i : U_i \to \xi_i(U_i) \cong \mathbb{C}^n$ on $\mathbb{P}^n$ is given by $[z_0, ..., z_n] \mapsto (z_0/z_i, ..., \widehat{z_i/z_i}, ..., z_n/z_i)$, the chart transition maps $\xi_j(U_i \cap U_j) \to \xi_i(U_i \cap U_j)$ is given by multiplication by z_i/z_j. Hence the transition map

$$\Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j}) \xrightarrow{\cong} \Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$$

must also be given by mutiplication by z_i/z_j. This means the vector bundle defined by $\mathcal{O}(1)$ has the same transition functions as (1), the transition function of $\tau^\vee$. Hence they must be isomorphic: $\tau^\vee \cong \mathcal{O}(1)$. This concludes the proof.

Solution 6. We first define the Plücker embedding and show that it is indeed an embedding:

Theorem. *Let* $\mathrm{Gr}(k, W)$ *be the complex Grassmannian of* k*-planes in a vector space* W*. Then the Plücker embedding*

$$i : \mathrm{Gr}(k, W) \to \mathbb{P}(\textstyle\bigwedge^k W)$$

defined by $i(e_1, ..., e_k) = [e_1 \wedge \cdots \wedge e_k]$ *is an embedding.*

Proof of Theorem. First, we note that $i : \mathrm{Gr}(k, W) \to \mathbb{P}(\bigwedge^k W)$ is an injection, because given a line $[e_1 \wedge \cdots \wedge e_k]$ in $\bigwedge^k W$, we can recover $K = \mathrm{span}(e_1, ..., e_k)$. Namely,

$$K = \{w \in W : w \wedge e_1 \wedge \cdots \wedge e_k = 0\}.$$

Also note that $di_K : T_K \mathrm{Gr}(k, W) \to T_{i(K)}\mathbb{P}(\bigwedge^k W)$ is injective (hence i is an immersion). Indeed, recall that a neighborhood of $K \in \mathrm{Gr}(k, W)$, and suppose we choose a complement V with $V \oplus K = W$, then an open chart of $\mathrm{Gr}(k, W)$ at K is given by the open sets

$$U_V = \{Z \in \mathrm{Gr}(k, W); Z \cap V = 0\}.$$

Any such vector space Z will be isomorphic to K via the projection $\pi_K : W \to V$. If we denote the projection to V by $\pi_V : W \to V$, any such Z can be identified with the graph of the linear map $h_Z : \pi_V \circ \pi_K^{-1}|_Z : K \to V$. Therefore, the open neighborhood U_V is identified (via the graphs h_Z) with the set $\mathrm{Hom}(K, V)$, where $K \in \mathrm{Gr}(k, W)$ corresponds to the trivial element $0 \in \mathrm{Hom}(K, V)$. Since $V \cong W/K$, we have

$$T_K \mathrm{Gr}(k, W) \cong \mathrm{Hom}(K, W/K).$$

Now choose a basis $e_1, ..., e_k$ of K. Then a small deformation of these basis will be given by

$$v_i(t) = e_i + tw_i \qquad w_i \in W/K.$$

Then under the Plücker map the tangent space become

$$v_1(t) \wedge \cdots \wedge v_k(t) = e_1 \wedge \cdots \wedge e_k + t \sum_{i=1}^{k} e_1 \wedge \cdots \wedge w_i \wedge \cdots \wedge e_k + O(t^2).$$

Taking derivative of t and set $t = 0$ gives the linear term of the expression above, which vanishes iff each $w_i \in K$. Therefore, the map

$$\mathrm{d}i_K : T_K \operatorname{Gr}(k, W) \to T_{i(K)} \mathbb{P}(\bigwedge^k W)$$

is injective. Hence the Plücker embedding $i : \operatorname{Gr}(k, W) \to \mathbb{P}(\bigwedge^k W)$ is an immersion. To show that i is an embedding, we note that $\operatorname{Gr}(k, W)$ is compact and the Plücker map is clearly a continuous injection, so it is a homeomorphism onto its image. $\square$

The Plücker embedding gives an embedding $\operatorname{Gr}(2, 4) \to \mathbb{P}^5 = \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$, by sending $(v, w) \in \operatorname{Gr}(2, 4)$ to the class of

$$v \wedge w = \sum_{i,j} v^i w^j e_i \wedge e_j = \sum_{i<j} (v^i w^j - v^j w^i) e_i \wedge e_j.$$

in $\mathbb{P}^5$. In Plücker coordinate $[z_{12} : z_{13} : z_{14} : z_{23} : z_{24} : z_{34}]$, where z_{ij} is the component scalar in front of $e_i \wedge e_j$ for $i < j$, the point $v \wedge w$ maps to

$$[v_1w_2 - v_2w_1 : v_1w_3 - v_3w_1 : v_1w_4 - v_4w_1 : v_2w_3 - v_3w_2 : v_2w_4 - v_4w_2 : v_3w_4 - v_4w_3].$$

Hence one can check that these points satisfies the polynomial equation

$$\mathcal{Q} : z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0.$$

Hence it remains to show that every point in $\mathcal{Q} \subset \mathbb{P}^5$ comes from some point in $\operatorname{Gr}(2, 4)$ under the Plücker embedding. This is entirely a matter of linear algebra. Suppose we have a point where the equation above is satisfied, then suppose without loss of generality that we are on the chart where $z_{12} \neq 0$. Hence say $z_{12} = -1$. Consider

$$v = (1, 1, z_{23} - z_{13}, z_{24} - z_{14}) \qquad w = (1, 0, z_{23}, z_{24}).$$

Then one simply check that $v \wedge w$ maps onto the given point. This shows that $\operatorname{Gr}(2, 4) \cong \mathcal{Q}$, hence is a 4-dimensional quadric since $\mathcal{Q}$ is.

Solution 7. Let $F(V, k_1, ..., k_\ell)$ denote the flag variety where $\dim V = n + 1$. Then a point in $F(V, k_1, ..., k_\ell)$ is the flag $W_1 \subset W_2 \subset \cdots \subset W_\ell \subset V$ with $\dim W_i = k_i$ and $0 \leq k_1 \leq \cdots \leq k_\ell \leq n+1$. Then this gives a natural map

$$\phi : F(V, k_1, ..., k_\ell) \to \operatorname{Gr}(k_1, n + 1) \times \operatorname{Gr}(k_2, n + 2) \times \cdots \times \operatorname{Gr}(k_\ell, n + 1).$$

We first note that

Lemma. *ϕ is a closed immersion.*

Proof of Lemma. It suffices to show that its image is a closed subvariety of the product of the Grassmannians, and that ϕ is an isomorphism onto its image. Because ϕ simply maps the flag $(W_1 \subset \cdots \subset W_\ell)$ to the tuple $(W_1, \ldots, W_\ell)$, it is clearly injective. Furthermore, the flag variety $F(V, k_1, \ldots, k_\ell)$ is typically constructed locally by taking sub-charts of this exact product, making ϕ a local isomorphism onto its image. It remains to show ϕ has closed image.

Over an affine open set, W is spanned by the rows of a $k \times (n+1)$ matrix A of full rank, and W' is spanned by the rows of a $k' \times (n+1)$ matrix B of full rank.The condition $W \subset W'$ means that the row space of A is contained in the row space of B. Equivalently, the combined $(k+k') \times (n+1)$ block matrix:

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

must have rank exactly equal to k' (since the rows of A don't add any new dimensions to the span of B).Since $k < k'$, the rank of this block matrix is $\leq k'$ if and only if all $(k'+1) \times (k'+1)$ minors of the block matrix vanish. These minors are just polynomial equations in the local coordinates of the Grassmannians. Since the condition is defined locally by the vanishing of polynomials, the image is a closed set. $\square$

Then we also have the Plücker embedding

$$\mathrm{Gr}(k_i, n+1) \to \mathbb{P}(\Lambda^{k_i} V) = \mathbb{P}^{\binom{n+1}{k_i}-1}.$$

Also the Segre embedding embeds products of projective space into large enough projective space:

$$\mathbb{P}^{\binom{n+1}{k_1}-1} \times \cdots \times \mathbb{P}^{\binom{n+1}{k_\ell}-1} \to \mathbb{P}^N.$$

Hence the composition of this sequence of closed immersions gives a closed immersion

$$F(V, k_1, ..., k_\ell) \to \mathbb{P}^N.$$

We conclude that $F(V, k_1, ..., k_\ell)$ is a closed subset of a projective variety, hence also a projective variety.

2 Solutions to PSet 2

Solution 8. First assume $d \geq 0$. We see from Proposition 2.4.1 of Huybrechts that global sections of $\mathcal{O}(d)$ are degree d homogeneous polynomials on $\mathbb{P}^n$. And p.86 of the textbook shows that

$$\varphi_L(z_0 : \cdots : z_n) = (z_0^{i_0} \cdots z_n^{i_n})_{i_0, \ldots, i_n} \tag{2}$$

where $I = (i_1, ..., i_n)$ satisfies $\sum_k i_k = d$, and I runs through all such multi-index.

Since $h^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d}$, and $\varphi_L : \mathbb{P}^n \to \mathbb{P}^N$ where $N = h^0(\mathbb{P}^n, \mathcal{O}(d)) - 1$, we see that when $d = 1$, $\varphi_L : \mathbb{P}^n \to \mathbb{P}^n$ is just the identity map, hence is an isomorphism. Moreover, this is the only case where φ_L is an isomorphism because in all other cases the dimension of the domain and codomain does not match.

When $d < 0$, we have $h^0(\mathbb{P}^n, \mathcal{O}(d)) = 0$. Since φ_L is defined through basis of $H^0(X, L)$, we see that φ_L is only defined iff $d \geq 0$.

Now we prove that (2) is an embedding when $d \geq 1$. Clearly (2) is smooth, and we further note that it is an injection: To see this, we can go to open sets $U_k \subset \mathbb{P}^n$ defined by $U_k = \{(z_0 : \cdots z_n) : z_k \neq 0\}$. Suppose $(z_0 : \cdots z_n) \in U_k$, then as long as there exists any $z_i \neq 0$, for $i \neq k$, we can recover z_j/z_k because $z_j/z_k = z_j z_i^{d-1}/z_k z_i^{d-1}$, for every j. Hence this recovers the point $(z_0 : \cdots z_n)$. If no such z_i exists, then $(z_0 : \cdots z_n)$ must only have a 1 in the k-th component (WLOG we take z_k=1). Moreover, this procedure actually constructed an inverse $\varphi_L(\mathbb{P}^n) \to \mathbb{P}^n$ which is obviously continuous. Moreover, a standard computationg shows that $\mathrm{d}\varphi_L$ has full rank: Indeed, up to a permutation of rows, we have

$$\varphi_L(z_0, ..., z_n) = (z_0^d, z_1^d, ..., z_n^d, ...)$$

hence the holomorphic Jacobian is given by the block matrix

$$J = \left(\frac{\partial \varphi_L^I}{\partial z_j}\right) = \begin{pmatrix} M & A \\ 0 & B \end{pmatrix} \qquad M = d\begin{pmatrix} z_0^{d-1} & & & \\ & z_1^{d-1} & & \\ & & \ddots & \\ & & & z_n^{d-1} \end{pmatrix}$$

which is clearly of full rank. Then since the total differential $\mathrm{d}\varphi_L$ is given by

$$\mathrm{d}\varphi_L = \begin{pmatrix} J & 0 \\ 0 & \bar{J} \end{pmatrix},$$

it is also of full rank. Hence we conclude that φ_L is an embedding, because it is an injective immersion (since $\mathrm{d}\varphi_L$ is injective) and the domain $\mathbb{P}^n$ is compact.

Finally, let $H \subset \mathbb{P}^N$ be a hyperplane. Then it is cut out by a linear equation, without loss of generality we may assume it is $w_0 = 0$, where $(w_0 : \cdots : w_N)$ are coordinates of $\mathbb{P}^N$. Then $\varphi_L^{-1}(H)$ is given by the equation $z_0^d = 0$.

Solution 9. Let $p \in \mathbb{P}^n$ and $V < H^0(\mathbb{P}^n, \mathcal{O}(d))$ be the linear system of sections of $\mathcal{O}(d)$ that vanish at p. Then we define φ_V similarly: Let $s_0, ..., s_k$ be a basis of V, where $k = \dim V$. Then define

$$\begin{aligned} \varphi_V : \mathbb{P}^n \setminus \mathrm{Bs}(V) &\to \mathbb{P}^k \\ (z_0 : \cdots : z_n) &\mapsto (s_0(z) : \cdots : s_k(z)). \end{aligned}$$

Note that $\mathrm{Bs}(V) := Z(s_0) \cap \cdots \cap Z(s_k) = \{p\}$, hence by definition φ_V is no longer a morphism on $\mathbb{P}^n$, only on $\mathbb{P}^n \setminus \{p\}$.

We need to describe the closure of the image $\operatorname{cl}\varphi_V(\mathbb{P}^n - p)$. Let $\pi : X \to \mathbb{P}^n$ be the blow up of $\mathbb{P}^n$ at p. We first claim that blowing up resolves the indeterminacy of φ_V, giving a well-defined map $\widetilde{\varphi_V} : X \to \mathbb{P}^N$ such that the diagram

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\widetilde{\varphi_V}"] & \\ \mathbb{P}^n \arrow[r, dashed, "\varphi_V"'] & \mathbb{P}^N \end{tikzcd}$$

commutes. Indeed: Let $E \subset X$ denote the exceptional divisor. On $X \setminus E$ of course we define $\widetilde{\varphi_V}$ by $\varphi_V \circ \pi$. To determine how to define $\widetilde{\varphi_V}$ on $E \cong \mathbb{P}(T_p\mathbb{P}^n)$, we argue as follows:

In order to extend the map $\varphi_V : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$, we should somehow include the "correct" value of $\varphi_V(p)$. This value should be well-approximated by the following procedure: We take a curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{P}^n$ such that $\gamma(0) = p$. Then we should have $\varphi_V(p) = \lim_{t\to 0}\varphi_V(\gamma(t))$. Suppose $\gamma'(0) = v \in T_p\mathbb{P}^n$. Let $s_0, ..., s_N$ be a basis of V. Then Taylor expanding at p should give

$$s_i(\gamma(t)) \simeq 0 + \mathrm{d}s_i(v)t$$

to first order in t. Therefore, we should define the limit $\lim_{t\to 0}\varphi_V(\gamma(t))$ as

$$\lim_{t\to 0}\varphi_V(\gamma(t)) := \lim_{t\to 0}(t\mathrm{d}s_0(v) : ... : t\mathrm{d}s_N(v)) = (\mathrm{d}s_0(v) : ... : \mathrm{d}s_N(v)).$$

Moreover, we take all curves approaching p from different tangent directions under consideration. Hence, from this argument, it is clear that we should define $\widetilde{\varphi_V}$ on E by

$$\widetilde{\varphi_V}(v) = (\mathrm{d}s_0(v) : ... : \mathrm{d}s_N(v)) \qquad v \in T_p\mathbb{P}^n \cong E.$$

Moreover, it is clear from the description above that we have

$$\operatorname{cl}\varphi_V(\mathbb{P}^n - p) = \widetilde{\varphi_V}(\mathrm{Bl}_p\,\mathbb{P}^n).$$

Next, we claim that for $d > 1$ the map $\widetilde{\varphi_V}$ is an embedding, so that $\operatorname{cl}\varphi_V(\mathbb{P}^n - p) \cong \mathrm{Bl}_p\,\mathbb{P}^n$ (When $d = 1$ take p to be the point $(0 : ... : 0 : 1)$ we see easily that φ_V is a projection $\mathbb{P}^n \to \mathbb{P}^{n-1}$ and is a closed map, so the closure is still $\mathbb{P}^{n-1}$).

This can be checked in local coordinates. We first describe the rational map $\varphi_V : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ locally. Let $x_1, ..., x_{n+1}$ be homogeneous coordinates on $\mathbb{P}^n$, and let $p = (0 : ... : 0 : 1)$. Let us look at the chart $x_{n+1} = 1$. Then the map $\varphi_V : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ is locally given by

$$\varphi_V(x_1, ..., x_n) = (x_1^{a_1}\cdots x_n^{a_n})_{a_1,...,a_n} \qquad 0 < a_1 + \cdots + a_n \leq d.$$

Indeed, before we dehomogenize by setting $x_{n+1} = 1$, the map φ_V sends a point x to the tuple of all possible monomials $x_1^{a_1}\cdots x_{n+1}^{a_{n+1}}$ where $\sum_i a_i = d$, except x_{n+1}^d. Since $d \geq 2$, at least some a_i for $1 \leq i \leq n$ is nonzero. Now let us look at the blow up at zero of this chart. The blow up is given by

$$\{(x, u) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : x_i u_j = x_j u_i\},$$

where $u_1, ..., u_n$ are homogeneous coordinates on $\mathbb{P}^{n-1}$. Now look at the chart $u_n = 1$. Then this tells us that the coordinate in this chart satisfies

$$\begin{aligned} x_i u_j &= x_j u_i \quad i, j \leq n-1, \\ x_i &= x_n u_i. \end{aligned}$$

Hence locally the map $\varphi_V : \mathbb{C}^n \dashrightarrow \mathbb{P}^N$ in the blow up becomes

$$\varphi_V : (u_1, ..., u_{n-1}, x_n) = (u_1^{a_1} \cdots u_{n-1}^{a_{n-1}} x_n^{|a|})_a$$

with $0 < |a| \leq d$, where $|a| = \sum_{i=1}^n a_i$. But the right hand side lives in $\mathbb{P}^N$ and is divisible by x_n since $|a| \geq 1$. Thus the rational map $\varphi_V : \mathbb{C}^n \dashrightarrow \mathbb{P}^N$ can be resolved by the map

$$\widetilde{\varphi_V}(u_1, ..., u_{n-1}, x_n) = (u_1^{a_1} \cdots u_{n-1}^{a_{n-1}} x_n^{|a|-1})_a.$$

Note that this map is an embedding: Indeed, if $a_n = 1, a_i = 1, a_j = 0$ for $j \neq i$ we recover u_i $(1 \leq i \leq n-1)$. If $a_i = 0$ for all i and $a_n = 2$, we recover x_n. This concludes the proof.

Solution 10. We first find the points where the curve $C = Z(y^2 z^2 - x^4)$ is singular. Denote $F(x, y, z) = y^2 z^2 - x^4$. Solving for x, y, z in $\mathbb{P}^2$ that makes $F = \partial_x F = \partial_y F = \partial_z F = 0$, we see that the curve is singular at two points $P = [0 : 0 : 1]$ or $Q = [0 : 1 : 0]$. We are blowing up $\mathbb{P}^2$ at the point P.

Since the blow up will produce isomorphic copy away from the blow up point P, it suffices to just study the blow up at the open set $U = \{[x : y : z] \in \mathbb{P}^2 : z \neq 0\} \subset \mathbb{P}^2$. Without loss of generality, we may set $z = 1$ on U. Then the curve becomes

$$C|_U = Z(y^2 - x^4).$$

This curve is singulart at $x = y = 0$, i.e., at the point $P = [0 : 0 : 1]$. Blowing up this curve gives

$$\widehat{C|_U} = \{([x : y : 1], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 : y^2 = x^4, xv = yu\}.$$

This is the total transform of $C|_U$. Note that the fiber of $\widehat{C|_U}$ at the exceptional divisor $x = y = 0$ is given by a copy of $\mathbb{P}^1$, because the defining equations are automatically satisfied. To find the strict transform, we must eliminate the exceptional point $x = y = 0$ and take the closure. In practice this means solve the system of equations defining the set $\widehat{C|_U}$, and we are allowed to divide by x, y. On the chart $u = 1$, the equations become

$$y^2 = x^4 \qquad xv = y.$$

Squaring the second equation then substituting in the first gives $x^2 v^2 = x^4$, dividing by x^2 gives

$$v^2 = x^2$$

This is singular at $x = v = 0$. However, on the chart $v = 1$, the equations become

$$y^2 = x^4 \qquad x = yu.$$

Squaring the second equation then substituting in the first and dividing by y^2 gives

$$1 = y^2 u^4$$

This curve is smooth.

Now we blow up a second time on this chart. the second blow up will be given by

$$\mathcal{Q} = \{((x, v), [a : b]) \in \mathbb{C}^2 \times \mathbb{P}^1 : x^2 = v^2, xb = va\}.$$

Now we study the strict transform. On the chart $a = 1$, we have $xb = v$ and $v^2 = x^2$. Squaring the first equation and substituting in the second, we see that $b^2 = 1$, which is nonsingular. On the chart $b = 1$, we have $x = va$ and $v^2 = x^2$. Similarly we get $a^2 = 1$, which is again nonsingular. Hence we see that blowing up a second time completely resolved the singularity of C (on its intersection with the open set $U \subset \mathbb{P}^2$).

Solution 11. The first statement is a corollary of the following result:

Lemma. *Let M be a compact Kähler manifold with Kähler form ω. For every $0 \leq k \leq n$ where $n = \dim M$, the operator $L_\omega^k : \mathcal{A}^{n-k}(M) \to \mathcal{A}^{n+k}(M)$ is an isomorphism.*

In particular, set $k = n - 1$, then $L_\omega^{n-1} : \mathcal{A}^1(X) \to \mathcal{A}^{2n-1}(X)$ is an isomorphism, hence it immediately follows that $L_\omega : \mathcal{A}^1(X) \to \mathcal{A}^3(X)$ is injective.

Proof. We note that $L_\omega^k : \mathcal{A}^{n-k}(M) \to \mathcal{A}^{n+k}(M)$ is a smooth bundle homomorphism between vector bundles of the same rank. Hence it suffices to show that L_ω^k is injective. Writting $k = n - j$, we will show by induction on j that $L_\omega^{n-j} : \mathcal{A}^j(M) \to \mathcal{A}^{2n-j}(M)$ is injective.

The case $j = 0$ is immediate, for L_ω^n is just multiplication by $\omega^n = n!(\mathrm{d}\,\mathrm{Vol}_g)$. Assume now that L_ω^{n-j+1} is injective on $(j-1)$-forms, and suppose $\alpha \in \mathcal{A}^j(M)$ such that $L_\omega^{n-j}\alpha = 0$. Then $L_\omega^{n-j+1}\alpha = 0$. Since interior multiplication is an antiderivation, direct computation shows that for every complex vector field X,

$$0 = X \lrcorner L_\omega^{n-j+1}\alpha = X \lrcorner (\omega \wedge \cdots \wedge \omega \wedge \alpha) = (n - j + 1)(X \lrcorner \omega) \wedge L_\omega^{n-j}\alpha + L_\omega^{n-j+1}(X \lrcorner \alpha).$$

The expression on the right hand side is equal to $L_\omega^{n-j+1}(X \lrcorner \alpha)$ because $L_\omega^{n-j}\alpha = 0$ by assumption. Hence we conclude that

$$0 = L_\omega^{n-j+1}(X \lrcorner \alpha).$$

Therefore, the inductive hypothesis shows that $X \lrcorner \alpha = 0$. Since this is true for every complex vector field X, we conclude that $\alpha = 0$, as desired. $\square$

Finally, let $\varphi \in C^\infty(X)$ be a (real-valued) smooth function, and h is the Kähler metric. Then $h = g - i\omega$ where ω is Kähler. Hence taking the imaginary part of φh we conclude that $\varphi\omega$ is another

Kähler form on X. It must be closed, hence

$$0 = \mathrm{d}(\varphi\omega) = \mathrm{d}\varphi \wedge \omega \pm \underbrace{\varphi \wedge \mathrm{d}\omega}_{=0 \text{ because } \mathrm{d}\omega=0}.$$

Since we already showed L_ω is injective on 1-forms, we must have $\mathrm{d}\varphi = 0$. Hence φ is locally constant. Since X is connected, we conclude that φ is constant.

Solution 12. Let $E \to X$ be a rank r holomorphic vector bundle and $L \to X$ a holomorphic line bundle. We first show that $\mathbb{P}(E \otimes L) \cong \mathbb{P}(E)$.

One way to do this is through transition functions: Both are $\mathbb{P}^r$-bundles over X, and the transition function of E and the transition function of $E \otimes L$ differ by a factor of the transition function of L, which goes away under projectivization, so the resulting bundles must be isomorphic.

Another way to do this is to construct an actual isomorphism $\Phi : \mathbb{P}(E) \to \mathbb{P}(E \otimes L)$ over X. A point $V_x \in \mathbb{P}(E)$ over $x \in X$ is a line $V_x < E_x$. This uniquely corresponds to the line $V_x \otimes L_x < (E \otimes L)_x$. Hence define $\Phi : V_x \mapsto V_x \otimes L_x$. Conversely, every line in $(E \otimes L)_x$ is given by $W_x \otimes L_x$ for some line $W_x < E_x$, hence define $\Psi : \mathbb{P}(E \otimes L) \to \mathbb{P}(E)$ via $\Psi : W_x \otimes L_x \mapsto W_x$. It is easy to check $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{id}$ hence we have the desired bundle isomorphism over X.

Next, we prove $\mathcal{O}_{\mathbb{P}(E^* \otimes L^*)}(1) \cong \mathcal{O}_{\mathbb{P}(E^*)}(1) \otimes \pi^* L$, where $\pi : \mathbb{P}(E^*) \to X$ is the projection map. Replacing E^*, L^* by E, L and taking the dual it suffices to prove that

$$\mathcal{O}_{\mathbb{P}(E \otimes L)}(-1) \cong \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^* L^*.$$

An element on the right hand side is given by $(V_x, v \otimes w)$, where $V_x \in \mathbb{P}(E)$ is a line in E_x, and $v \in V_x, w \in L_x$. We define a map

$$\mathfrak{f} : \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^* L^* \to \mathcal{O}_{\mathbb{P}(E \otimes L)}(-1) \qquad \mathfrak{f} : (V_x, v \otimes w) \mapsto (V_x \otimes L_x, \mathrm{span}(v \otimes w)).$$

and extend it bilinearly. The inverse is given by

$$\mathfrak{g} : \mathcal{O}_{\mathbb{P}(E \otimes L)}(-1) \to \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^* L^* \qquad \mathfrak{g} : (W_x \otimes L_x, v \otimes w) \mapsto (W_x, v \otimes w).$$

It is easy to check that $\mathfrak{f}\mathfrak{g} = \mathfrak{g}\mathfrak{f} = \mathrm{id}$. This concludes the proof.

Solution 13. We remark that any plane cubic is of the form $y^2 z = x^3 + axz^2 + bz^3$. They are topologically tori but two such curves are isomorphic iff they have the same j-invariant, defined by

$$j = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Moreover, a cubic is singular iff the discriminant $\Delta = -16(4a^3 + 27b^2) = 0$. Hence smooth cubic is a Zariski open set in the moduli of plane cubics. The classification of cubics is given by Figure 1 (with sample equations in that representative), and the closure relations are given by Fig 2. The idea is that in the moduli space of cubics, we can deform one cubic continuously to obtain the others.

We start by asking whether the cubic equation can be factored into equations of smaller degrees. Then there are three cases: (i) $3 = 1 + 1 + 1$, in which case the cubic is give by 3 lines (case 4,6,7,8);

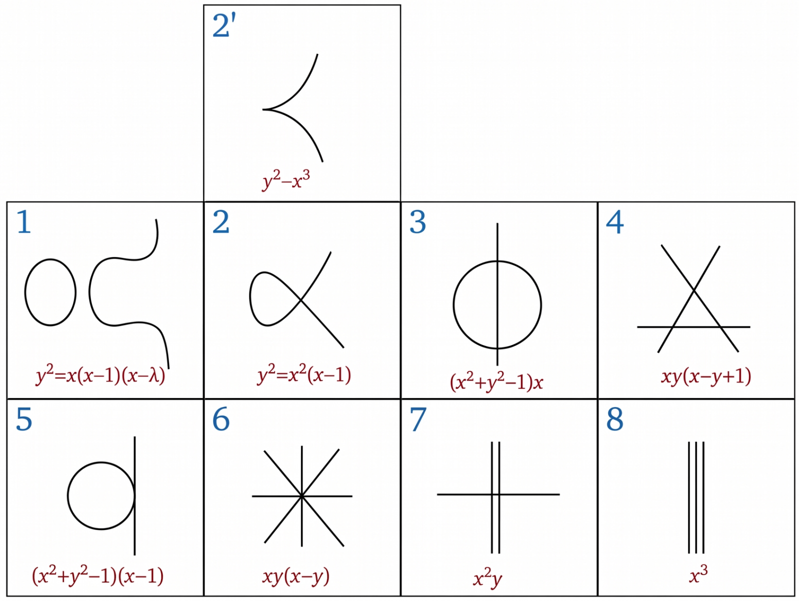


Fig 1.

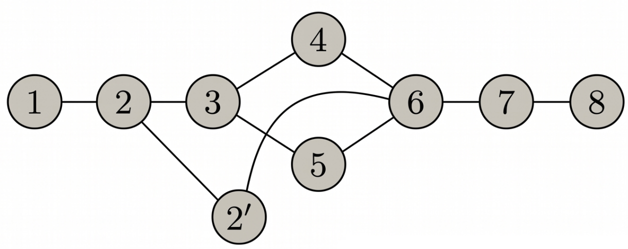


Fig 2.

(ii) $3 = 2 + 1$, in which case the cubic is given by a line and a conic (3,5), (iii) $3 = 3$, in which case we have elliptic curves, smooth or degenerate (1,2,2').

To see that (2) is in the closure of (1): we see that by setting $\lambda \to 0$ in $y^2 = x(x-1)(x-\lambda)$ gives (2); To go from (2) to (3), we can deform (2) such that the two lines coming out of (2) becomes straight (which now becomes 5), then then move the line into the circle, that is, use $y^2 = x^2(y+t)$; To go from (3) to (4), we can shrink the circle and the hyperbola using $(x^2+y^2+t)(x-1)$, as $t \to 0$ the smooth conic degenerates into a product of two lines; To go from (3) to (5) we simply move the line using $(x^2+y^2-1)(x-1+t)$; To go from (4) to (6) we again move the line, in other words the homotopy $xy(x-y+t)$ gives a deformation from (4) to (6) when t goes from 0 to 1; To go from (5) to (6), we use use $(x^2+y^2-t^2)(x-t)$, as $t \to 0$ the smooth conic collapses into two lines again; To go from (6) to (7) and from (7) to (8) we move the line using $xy(x-ty)$ and $x^2(x-ty)$ respectively. Finally, to go from (2) to (2') we just collapse the circle, and to go from (2') to (6) we use the use the parameter t in $x^3 - y^3 - ty^2$ to do the deformation: When $t = 0$ this is a product of three lines intersecting at the origin, which is (6), when $t > 0$ this becomes $x^3 = y^2(y+t)$. A change of coordinates $Y = x$ and $X = y\sqrt{(y+t)}$ gives $X^2 = Y^3$, which is (2').

To show that any two cubics in each of these non-open stratum are isomorphic to each other, one can show that PGL_3 acts transtively on each of these stratum.

3 Solutions to PSet 3

Solution 14. Part (c) is slightly long and is perhaps the most difficult.

(a) We argue by contradiction: Assume that μ is $\bar{\partial}$-exact. Then there exists $\alpha \in \Omega^{1,0}(X)$ such that $\mu = \bar{\partial}\alpha$. Now note that $\partial\alpha \in \Omega^{2,0}(X)$, so $\partial\alpha = 0$ because $\dim X = 1$. Hence we see that $\mathrm{d}\alpha = (\partial + \bar{\partial})\alpha = \bar{\partial}\alpha$. Since μ is the volume form on X, we have that

$$\mathrm{vol}(X) = \int_X \mu = \int_X \mathrm{d}\alpha = \int_{\partial X} \alpha = 0$$

by Stokes' theorem, which is the desired contradiction.

Now we show that $H^1(X, K_X) \neq 0$. By Dolbeault theorem, $H^{p,q}_{\bar{\partial}}(X) \cong H^q(X, \Omega^p_X)$. Hence $H^1(X, K_X) \cong H^{1,1}_{\bar{\partial}}(X)$, which is nonzero since we just showed that μ is a nontrivial element in it.

Alternatively, we could also use Hodge theory. Since X is a Riemann surface, it is Kähler. Suppose $H^1(X, K_X) = 0$. Then $h^{1,1} = 0$. For a Riemann surface $H^2(X;\mathbb{Z}) = \mathbb{Z}$, so its second Betti number is $b_2 = 1$. Using Hodge decomposition we have $b_2 = h^{2,0} + h^{1,1} + h^{0,2}$. Now $h^{2,0} = h^{0,2}$ by conjugation symmetry and $h^{1,1} = 0$ by assumption, so $1 = 2h^{2,0}$. But $h^{2,0}$ should be an integer, again we have a contradiction, as desired.

Finally, we have a surjective map

$$\mathrm{Tr} : H^1(X, K_X) \to \mathbb{C} \qquad \omega \mapsto \int_X \omega,$$

because $\int_X \mu = \mathrm{vol}(X) \neq 0$, hence all scalar multiple of μ surjects onto $\mathbb{C}$.

(b) To prove that the sequence

$$0 \to K_X \to K_X(D) \xrightarrow{\mathrm{Res}} \bigoplus_i \mathbb{C}_{x_i} \to 0$$

is exact, it suffices to prove the the stalk is exact at every point $p \in X$. When p is not any of the point in D, the stalk satisfies $K_X(D)|_p = K_X|_p$, and $\bigoplus_i \mathbb{C}x_i|_p = 0$. Hence the sequence of stalks is certainly exact away from D. Now it remains to check the case that $p = x_i$ is one of in D.

Let $\omega \in K_X(D)$, and let us choose a sufficiently small open disk U around p, and local holomorphic chart $\varphi : U \to \varphi(U) \subset \mathbb{C}$, such that $\varphi(p) = 0, \varphi(U) = B_\epsilon(0)$ for some $\epsilon > 0$, and locally

$$\varphi^*\omega = \phi(z)\frac{\mathrm{d}z}{z},$$

for some holomorphic ϕ on $B_\epsilon(0)$. Then by definition of integration of a differential form, we have

$$\operatorname{Res}_i \omega = \frac{1}{2\pi i}\int_{\partial B_\epsilon(0)} \frac{\phi(z)}{z}\mathrm{d}z.$$

Now suppose $\omega \in \Gamma(U, K_X)$. Then $\phi(z)/z$ must be holomorphic, and the integral above vanishes by Cauchy's theorem, hence $\operatorname{Res}\circ\iota = 0$, where $\iota : \Gamma(U, K_X) \to \Gamma(U, K_X(D))$ is the inclusion map. Therefore, $\operatorname{im}(\iota) \subset \ker(\operatorname{Res})$. Conversely, let $\omega \in \ker(\operatorname{Res})$ whose coordinate expression is given by $\phi(z)\mathrm{d}z/z$ in a holomorphic chart. Then locally we may expand $\phi(z)$ as a Taylor series (we can assume U is sufficiently small such that the power series converges inside U):

$$\phi(z) = \phi(0) + \phi'(0)z + \frac{\phi''(0)}{2!}z^2 + \cdots .$$

Then substitute this expression into $\int_{\partial B_\epsilon(0)} \phi(z)\mathrm{d}z/z$, all the other terms in the expansion except the first vanishes because they are all holomorphic after dividing by z. In other words, we have

$$0 = \operatorname{Res}_i \omega = \frac{1}{2\pi i}\int_{\partial B_\epsilon(0)} \frac{\phi(0)}{z}\mathrm{d}z = \phi(0).$$

Hence we see that $\phi(0) = 0$, which means $\phi(z)/z$ is holomorphic, thus $\omega \in \Gamma(U, K_X)$. This shows that $\ker(\operatorname{Res}) \subset \operatorname{im}(\iota)$. Finally, the map $\operatorname{Res} : \Gamma(U, K_X(D)) \to \mathbb{C}$ is surjective because by the calculation above we know $\operatorname{Res}(\omega) = \phi(0)$. Hence we can just pick a family of $\omega \in \Gamma(U, K_X(D))$ whose holomorphic function ϕ has arbitrary value of $\phi(0)$, which certainly surjects onto $\mathbb{C}$. To summarize, we have shown that for any sufficiently small neighborhood U around $p = x_i$, we have an exact sequence

$$0 \to \Gamma(U, K_X) \xrightarrow{\iota} \Gamma(U, K_X(D)) \xrightarrow{\operatorname{Res}} \Gamma(U, \bigoplus_i \mathbb{C}_{x_i}) \to 0.$$

Taking direct limit, this induces an exact sequence on stalks. This concludes the proof that the sequence

$$0 \to K_X \to K_X(D) \xrightarrow{\operatorname{Res}} \bigoplus_i \mathbb{C}_{x_i} \to 0$$

is exact.

(c) Let i be fixed. We first find an explicit description of $\delta(1_{x_i}) \in H^1(X, K_X)$ as a Čech 1-cocycle. Let U_0 be a small open set around x_i and contains no other x_j's and let $U_1 = X \setminus \{x_i\}$. We consider the cover $\mathscr{U} = \{U_0, U_1\}$ of X. We consider the following commutative diagram of chain complexes with exact rows, and columns are given by the Čech coboundary map.

$$\begin{array}{ccccc} C^0(\mathscr{U}, K_X) & \longrightarrow & C^0(\mathscr{U}, K_X(D)) & \xrightarrow{\text{Res}} & C^0(\mathscr{U}, \oplus_i \mathbb{C}_{x_i}) \\ \downarrow & & \downarrow & & \downarrow \\ C^1(\mathscr{U}, K_X) & \longrightarrow & C^1(\mathscr{U}, K_X(D)) & \xrightarrow{\text{Res}} & C^1(\mathscr{U}, \oplus_i \mathbb{C}_{x_i}) \end{array}$$

We have $1_{x_i} \in C^0(\mathscr{U}, \oplus_i \mathbb{C}_{x_i})$ which is given by $(1,0)$ with respect to the cover $\mathscr{U}$. Consider the class $(\omega_0, \omega_1) \in C^0(\mathscr{U}, K_X(D))$ given by

$$\omega_0 = \frac{\mathrm{d}z_i}{z_i} \in \Gamma(U_0, K_X(D)) \qquad \omega_1 = 0 \in \Gamma(U_1, K_X(D)).$$

Clearly $\text{Res}(\omega_0, \omega_1) = (1,0)$. Moreover, the Čech boundary map $C^0(\mathscr{U}, K_X(D)) \to C^1(\mathscr{U}, K_X(D))$ applied to this class is just the difference

$$\omega_{01} = \omega_0 - \omega_1 = \frac{\mathrm{d}z_i}{z_i} \in \Gamma(U_0 \cap U_1, K_X(D)).$$

But note that since $U_0 \cap U_1$ does not contain the point x_i, this class also lives in $\Gamma(U_0 \cap U_1, K_X)$, and we have run through the classic snake lemma connecting homomorphism construction, which shows that the class representing $\delta(1_{x_i})$ in $C^1(\mathscr{U}, K_X)$ is precisely ω_{01}. Note that this is indeed a Čech 1-cocycle in $C^1(\mathscr{U}, K_X)$, since there are no three-fold intersections of distinct open sets in $\mathscr{U} = \{U_0, U_1\}$.

Next, we consider a partition of unity $\{\rho_0, \rho_1\}$ subordinate to $\mathscr{U} = \{U_0, U_1\}$. We consider the (1,0)-forms

$$\begin{aligned} \eta_0 &= -\rho_1 \omega_{01} = -\rho_1 \frac{\mathrm{d}z_i}{z_i} \qquad \text{defined on } U_0 \\ \eta_1 &= \rho_0 \omega_{01} = \rho_0 \frac{\mathrm{d}z_i}{z_i} \qquad \text{defined on } U_1. \end{aligned}$$

Then on $U_0 \cap U_1$, we have

$$\eta_1 - \eta_0 = (\rho_0 + \rho_1)\omega_{01} = \omega_{01}.$$

Moreover, since ω_{01} is holomorphic on $U_0 \cap U_1$, we have that $\bar{\partial}\omega_{01} = 0$. Hence $\bar{\partial}\eta_0 = \bar{\partial}\eta_1$. Hence $\bar{\partial}\eta_0, \bar{\partial}\eta_1$ glue together to a globally well-defined (1,1)-form. Note that near x_i we have $\rho_1 = 0$, hence $\bar{\partial}\eta_0 = -\bar{\partial}(\rho_1 \mathrm{d}z_i/z_i) = 0$ near x_i. Now, we define

$$\mu_i = \eta_1 = \rho_0 \frac{\mathrm{d}z_i}{z_i}.$$

Of course, outside of $\text{Supp}(\rho_0) \subset U_0$ we view $\mu_i = 0$. Note that μ_i is C^∞ away from x_i, and in a neighborhood of x_i it is identically $\mathrm{d}z_i/z_i$.

It remains to check that $\delta(1_{x_i}) = \bar{\partial}\mu_i$ under the identification $H^1(X, K_X) \cong H^{1,1}_{\bar{\partial}}(X)$. To be more specific, let us recall how this identification arises: We consider the short exact sequence of sheaves

$$0 \to K_X \to \mathcal{A}_X^{1,0} \xrightarrow{\bar{\partial}} \mathscr{Z}^{1,1} \to 0$$

where $\mathscr{Z}^{1,1}$ is the sheaf of $\bar{\partial}$-closed (1,1)-forms. The corresponding long exact sequence reads in part

$$H^0(X, \mathcal{A}_X^{1,0}) \xrightarrow{\bar{\partial}} H^0(X, \mathscr{Z}^{1,1}) \xrightarrow{\delta'} H^1(X, K_X) \to H^1(X, \mathcal{A}_X^{1,0}) = 0$$

where $H^1(X, \mathcal{A}_X^{1,0}) = 0$ because $\mathcal{A}_X^{1,0}$ is an acyclic sheaf (due to the existence of partition of unity). Therefore by the first isomorphism theorem, we have

$$H^1(X, K_X) \cong H^0(X, \mathscr{Z}^{1,1})/\bar{\partial} H^0(X, \mathcal{A}_X^{1,0}) = H_{\bar{\partial}}^{1,1}(X).$$

Now the question is, given $[\omega_{01}] \in H^1(X, K_X)$, we want to find a representative $\Theta \in H^0(X, \mathscr{Z}^{1,1})$ such that $\delta'(\Theta) = [\omega_{01}]$.

Since we have a trivial inclusion $K_X \to \mathcal{A}_X^{1,0}$, we can view $\omega_{01} \in \Gamma(U_0 \cap U_1, \mathcal{A}_X^{1,0})$. Because $H^1(X, \mathcal{A}_X^{1,0}) = 0$, every cocycle must be a coboundary. This means there exists smooth 0-chains (smooth local forms η_0 on U_0 and η_1 on U_1) such that their difference gives back our 1-cocycle:

$$\eta_1 - \eta_0 = \omega_{01}$$

on $U_0 \cap U_1$. This is exactly what we constructed previously using partition of unity. Now we apply the map $\bar{\partial} : H^0(X, \mathcal{A}_X^{1,0}) \to H^0(X, \mathscr{Z}^{1,1})$ to get the form $(\bar{\partial}\eta_0, \bar{\partial}\eta_1)$. From the classical construction of connecting homomorphism in the snake lemma, we see the global form represented by $(\bar{\partial}\eta_0, \bar{\partial}\eta_1) \in C^0(\mathscr{U}, \mathscr{Z}^{1,1})$ is precisely the class that get mapped to $[\omega_{01}]$ via δ', which is the global form $\bar{\partial}\mu_i = \bar{\partial}\eta_1$ on U_1 (recall that $\bar{\partial}\eta_1 = \bar{\partial}\eta_0$ on $U_0 \cap U_1$).

(d) We show that for the form $\bar{\partial}\mu_i = \delta(1_{x_i})$ given in (c), one has

$$\int_X \bar{\partial}\mu_i = -2\pi i.$$

Again we first note that μ_i is of type (1,0), hence $\mathrm{d}\mu_i = (\partial + \bar{\partial})\mu_i = \bar{\partial}\mu_i$. We cannot directly apply Stokes' theorem to $\mathrm{d}\mu_i = \bar{\partial}\mu_i$ on X because μ_i is not well-defined at x_i. But, we can take a disk D around x_i, and let its radius shrink to zero. Thus let us compute:

$$\int_{X \setminus D} \bar{\partial}\mu_i = \int_{X \setminus D} \mathrm{d}\mu_i = \int_{\partial(X \setminus D)} \mu_i$$

by Stoke's theorem (Technically, stokes' theorem require $\mathrm{d}\mu_i$ to be compactly supported on $X \setminus D$. But we note that integrating on $X \setminus \bar{D}$, on which $\mathrm{d}\mu_i$ is compactly supported, is the same as integrating on $X \setminus D$ because the difference of these two sets is the boundary of a disk, which has measure zero). Now note that we have a change of orientation

$$\partial(X \setminus D) = -\partial D,$$

as the outward pointing normal on $X \setminus D$ points toward the interior of the disk, while the outward pointing normal on D points toward the exterior of the disk. Thus we conclude that

$$\int_{\partial(X\setminus D)} \mu_i = -\int_{\partial D} \mu_i = -\int_{\partial D} \frac{\mathrm{d}z}{z} = -2\pi i.$$

Now shrinking the radius of D to zero, we see that

$$\int_X \bar{\partial}\mu_i = -2\pi i,$$

as desired.

Finally, to deduce the residue theorem on X, we note that the long exact sequence induced by the short exact sequence in (b) reads in part

$$\cdots \to H^0(X, K_X(D)) \xrightarrow{\text{Res}} H^0(X, \bigoplus_i C_{x_i}) \xrightarrow{\delta} H^1(X, K_X) \to \cdots.$$

Exactness implies $\delta \circ \text{Res} = 0$. Hence given any $\omega \in H^0(X, K_X(D))$, we have that

$$\delta(\operatorname{Res}\omega) = \delta\left(\sum_i (\operatorname{Res}_i \omega) 1_{x_i}\right) = \sum_i (\operatorname{Res}_i \omega)\bar{\partial}\mu_i.$$

Now integrating and using $\int_X \bar{\partial}\mu_i = -2\pi i$ we get

$$-\sum_i \operatorname{Res}_i \omega = 0.$$

Solution 15. Let $C^{(d)} = C^d/S_d$ be the d-fold symmetric product. We show that $C^{(d)}$ is smooth iff $\dim C \leq 1$.

When $\dim C = 0$, $C^{(d)}$ is just a point, so there is nothing to prove. Let $\dim C = 1$. Since smoothness is a local question, WLOG we can assume $C \cong \mathbb{C}$. Then $C^d = \mathbb{C}^d = \operatorname{Spec}\mathbb{C}[z_1, ..., z_d]$. $C^{(d)}$ is by definition the quotient of C^d by the group S_d acting on C^d by permuting the coordinates.

In general, if $X = \operatorname{Spec} A$ is an affine scheme and A is the ring of coordinate functions, then $X/G = \operatorname{Spec} A^G$, where A^G is the element of A fixed by the G-action. This is just the geometric statement that functions on X/G are in bijection with the G-equivariant functions on X.

Now, $\mathbb{C}[z_1, ..., z_d]^{S_d} = \mathbb{C}[\sigma_1, ..., \sigma_d]$, where $\sigma_1, ..., \sigma_d$ are elementary symmetric polynomials: $\sigma_1 = \sum_i z_i, \sigma_2 = \sum_{i<j} z_i z_j, ..., \sigma_d = \prod_i z_i$. Thus, we see that

$$C^{(d)} = \operatorname{Spec}\mathbb{C}[z_1, ..., z_d]^{S_2} = \operatorname{Spec}\mathbb{C}[\sigma_1, ..., \sigma_d] \cong \mathbb{C}^d$$

if we make holomorphic change of coordinates $(z_1, ..., z_d) \to (\sigma_1, ..., \sigma_d)$.

Next, we show that if $\dim C \geq 2$ then $C^{(d)}$ is not smooth. We note that it suffices to prove this in the case where $d = 2$. Indeed, we note that if $C^{(d)}$ is not smooth, then neither is $C^{(d+1)}$: To see this, we pick disjoint open sets $U, V \subset C$. Since U, V are disjoint, we have

$$\operatorname{im}(U^d \times V \xrightarrow{/S_{d+1}} C^{(d+1)}) \cong \operatorname{im}(U^d \xrightarrow{/S_d} C^{(d)}) \times V.$$

Now we show that $C^{(2)}$ is not smooth if $m = \dim C \geq 2$. Again, this is a local question, so assume $C^{(2)} = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m]$, where the S_2 action swaps x_j and y_j. We perform a linear change of coordinates to diagonalize the S_2 involution. Define:

$$\begin{aligned} u_j &= x_j + y_j \quad \text{(invariants)} \\ v_j &= x_j - y_j \quad \text{(anti-invariants)} \end{aligned}$$

The coordinate ring is now $\mathbb{C}[u_1, \ldots, u_m, v_1, \ldots, v_m]$. Under the S_2 action, $\sigma(u_j) = u_j$ and $\sigma(v_j) = -v_j$. The invariant ring A^{S_2} is generated by the u_j coordinates and all quadratic combinations of the v_j coordinates.

Let $w_{j,k} = v_j v_k$ for $1 \leq j \leq k \leq m$. The invariant ring is generated by:

$$A^{S_2} = \mathbb{C}[u_1, \ldots, u_m, w_{1,1}, w_{1,2}, \ldots, w_{m,m}]$$

Because $m \geq 2$, we have at least two anti-invariant coordinates, v_1 and v_2, which yield the generators $w_{1,1} = v_1^2$, $w_{2,2} = v_2^2$, and $w_{1,2} = v_1 v_2$. These generators are not algebraically independent (whereas in 1-dimensional case the elementary symmetric polynomials $\sigma_1, ..., \sigma_d$ *are* independent). They satisfy the relation:

$$w_{1,1} w_{2,2} - w_{1,2}^2 = (v_1^2)(v_2^2) - (v_1 v_2)^2 = 0$$

Therefore, A^{S_2} is not a free polynomial ring. It is isomorphic to a quotient ring:

$$A^{S_2} \cong \mathbb{C}[u_j, W_{j,k}]/I,$$

where $W_{j,k}$ are formal variables and I is an ideal containing the polynomial $F = W_{1,1} W_{2,2} - W_{1,2}^2$. To see that this defines a singular space, we compute the Jacobian of F with respect to the variables $(W_{1,1}, W_{2,2}, W_{1,2})$:

$$\nabla F = (W_{2,2}, W_{1,1}, -2W_{1,2})$$

This gradient vanishes identically at the origin $(0, 0, 0)$. Pulling back to our original coordinates, the origin corresponds to $v_1 = v_2 = 0$, which is the locus where $x_1 = y_1$ and $x_2 = y_2$ (precisely the diagonal of $C \times C$). Because the Jacobian matrix loses rank at the diagonal, the quotient space $\operatorname{Spec}(A^{S_2})$ is singular.

Solution 16. Note that $\sum_{n=1}^{50} n = 1275$. Hence taking the homogenization of C we see that

$$C = Z\left(z^{1273} y^2 - \prod_{n=1}^{50} (x - nz)^n\right),$$

which has singularity at $(n : y : 1)$ for $2 \leq n \leq 50$, and $(0 : 1 : 0)$. Because $\deg C = 1275$, the arithmetic genus is given by

$$g = \frac{(d-1)(d-2)}{2} = \mathbf{810901}.$$

Now consider the singularities. We first work on the chart U_z, where we have $C \cap U_z = Z(y^2 - \prod_{n=1}^{50}(x-n)^n)$. For each isolated singular point $(n, 0)$ $(2 \leq n \leq 50)$. Locally at $x = n$ the equation looks like $y^2 = x^n$, so by the formula for general double point of the form $y^2 = x^n$, we have $\delta_p = \lfloor \frac{n}{2} \rfloor$ (Slides p.5). Thus δ-invariant is given by $\delta_{(n,0)} = \lfloor \frac{n}{2} \rfloor$. Hence in total this contributes

$$\sum_{n=2}^{50} \lfloor \frac{n}{2} \rfloor = \mathbf{625}$$

to the δ-invariant.

For the singularity at $(0 : 1 : 0)$, we go to the chart U_y. Here $C_y := C \cap U_y = Z(z^{1273} - \prod_{n=1}^{50}(x - nz)^n)$ has only one singularity at $(0, 0)$ with multiplicity 1273 (The multiplicity k of a singularity at the origin is simply the degree of the lowest-degree term in its polynomial. Here, the lowest degree term is z^{1273}, so $k = 1273$). By the formula for general algorithm (Slides p.5), when we blow up a singularity of multiplicity k, it contributes $\frac{k(k-1)}{2}$ to the total δ-invariant of that point. For $k = 1273$, this gives exactly $\mathbf{809628}$.

The blow-up of C_y at $(0, 0)$ is given by $\pi : \mathrm{Bl}_{(0,0)}\mathbb{C}^2 \to \mathbb{C}^2$ with the strict transform

$$Z\left(t^{1273} - x^2 \prod_{n=1}^{50}(1 - nt)^n\right).$$

which only meets the exceptional divisor at $(0, 0)$, where $(0, 0)$ is a double point: Indeed, locally around $(0, 0)$, the product $\prod(1 - nt)^n$ evaluates to 1. Thus, the local equation looks like $t^{1273} - x^2 = 0$. Once again, this matches the slide's formula for a general double point $y^2 = x^n$ (just with the variables flipped and $n = 1273$). Its δ contribution is $\lfloor 1273/2 \rfloor = \mathbf{636}$. This strict transform is now smooth after this local resolution, so the algorithm terminates.

Putting everything together, the geometric genus is given by:

$$\begin{aligned} g_{\text{geom}} &= g - \sum_{p \in \mathrm{Sing}(C)} \delta_p \\ &= 810901 - \underbrace{625}_{U_z\text{singularities}} - \underbrace{809628}_{\text{(0:1:0) first blow up}} - \underbrace{636}_{\text{(0:1:0) second blow up}} = 12. \end{aligned}$$

Solution 17. Let $C = Z(f) \subset \mathbb{P}^2$ be a smooth projective curve of degree d. Since the dual curve $C^\vee$ consists of all lines meeting C in fewer than d distinct points, the lines must be tangent to C at some points. Hence the dual curve is simply the collection of tangent lines of C. More precisely, it is the image of the Gauss map

$$\mathcal{G} : C \to \mathbb{P}^2 \qquad p \mapsto (F_x(p) : F_y(p) : F_z(p)).$$

The Gauss map $\mathcal{G}$ is a well-defined morphism between projective varieties. Hence it has a closed image, so $C^\vee = \mathcal{G}(C)$ is closed in $\mathbb{P}^2$. Moreover, for nontrivial C (not a line), $C^\vee$ parameterizes all the tangent directions of C, hence is 1-dimensional, i.e., a curve. We will give a more explicit parameterization of $C^\vee$ in local coordinates next, which also allows us to describe possible singularities of $C^\vee$.

Let us choose local holomorphic coordinates in $\mathbb{P}^2$ near a point p such that $p = (0,0)$ in this coordinate and the curve locally takes the form

$$f(x,y) = 0 \qquad \frac{\partial f}{\partial y}(0,0) \neq 0.$$

Then by implicit function theorem, locally one can write $y = g(x)$. The tangent line at $(x, g(x))$ has slop $g'(x)$ and is given by the equation

$$Y = g'(x)(X - x) + g(x).$$

Putting this equation into general form $\alpha X + \beta Y + \gamma = 0$ (remember $Z = 1$ in our chart), we have that

$$\alpha = g'(x) \qquad \beta = -1 \qquad \gamma = g(x) - xg'(x).$$

So locally the gauss map $\mathcal{G}$ is given by

$$\mathcal{G}(x) = (g'(x), g(x) - xg'(x)),$$

where we ignored β because $\beta = -1$ constantly. Hence we see that the image of the Gauss map, i.e., $C^\vee$, is parameterized by x and has coordinate $(g'(x), g(x) - xg'(x))$, which is a curve.

Now let us analyze the type of singularities on $C^\vee$. If $g'(x)$ and $g(x) - xg'(x)$ vary smoothly with x and the gauss map $\mathcal{G}$ is locally injective with nonzero derivative, then $C^\vee$ will be smooth. There are two possible types of singularities:

- If C has a bitangent, i.e., if there is a line that is tangent to C at two distinct points $x_1 \neq x_2$. Then $g'(x_1) = g'(x_2)$ and $g(x_1) - x_1 g'(x_1) = g(x_2) - x_2 g'(x_2)$. Then we see that in the image of the Gauss map, this produces two branches of $C^\vee$ crossing each other transversely. Hence $C^\vee$ has a ordinary double point (node).

- Suppose $p \in C$ is an inflection point. We quickly define what this means: We expand $g(x)$ as a Taylor series near $x = 0$. Then the expansion looks like $g(x) = a_m x^m + a_{m+1} x^{m+1} + \cdots$, where $a_m \neq 0$ is the first nonzero coefficient. Since $g(0) = 0$ by assumption, we always have $m > 0$. If $m = 1$, then $g'(0) \neq 0$, and we see that the tangent of C to p has some nonzero slope. If $m = 2$, then $g'(0) = 0$ but $g''(0) \neq 0$. Then the tangent at p is the x-axis, but p is still a smooth point and C behaves like the parabola $y = x^2$ near the origin. When $m \geq 3$, this is what we call a *inflection point*, where C looks flatter than a parabola, think $y = x^3$.

  Now suppose $p \in C$ is an inflection point. Then locally we can write $g(x) = ax^3 + O(x^4)$. So

$g'(x) = 3ax^2 + O(x^3)$. Then

$$\alpha(x) = g'(x) = 3ax^2 + O(x^3) \qquad \gamma(x) = g(x) - xg'(x) = -2ax^3 + O(x^4).$$

Hence we see that locally

$$\gamma \sim \alpha^{3/2}.$$

Therefore near $p = (0,0)$, the curve $C^\vee$ behaves like $\gamma^2 = \alpha^3$, which means this is a cusp of $C^\vee$.

In the parameter space of plane curves of degree d the locus where more complicated contact (e.g. higher-order flex or tangent with higher multiplicity) occurs is a proper closed subvariety, so a general smooth C avoids it. Concretely: the Gauss map $\mathcal{G}$ is an immersion except at simple inflection points, and for a general C the ramification is simple and bitangents are all ordinary (two simple tangency points). Hence the only singularities of $C^\vee$ for a generic smooth C are ordinary nodes and ordinary cusps.

Now we determine the number of nodes and cusps on $C^\vee$. Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree d whose only singularities are

- δ: the number of ordinary nodes of C,
- κ: the number of ordinary cusps of C.

and set

- $d^* = \deg(C^\vee)$,
- δ^*: the number of ordinary nodes of $C^\vee$,
- κ^*: the number of ordinary cusps of $C^\vee$.

Then the classical *Plücker formulas* (for a proof, see Griffiths and Harris, p.277-280) are:

$$d^* = d(d-1) - 2\delta - 3\kappa, \tag{P1}$$

$$2\delta^* + 3\kappa^* = d^*(d^*-1) - d, \tag{P2}$$

$$\kappa^* = 3d(d-2) - 6\delta - 8\kappa. \tag{P3}$$

Eliminating κ^* between (P2) and (P3) gives an explicit expression for δ^*:

$$\delta^* = \frac{1}{2}\Big(d^*(d^*-1) - d - 3\kappa^*\Big) = \frac{1}{2}\Big(d^*(d^*-1) - d - 9d(d-2) + 18\delta + 24\kappa\Big). \tag{P4}$$

If C is non-singular (so $\delta = \kappa = 0$), then

$$\begin{aligned} d^* &= d(d-1), \\ \kappa^* &= 3d(d-2), \\ \delta^* &= \tfrac{1}{2}d(d-2)(d^2-9). \end{aligned}$$

Hence for a general smooth plane curve of degree d, the dual curve $C^\vee$ has only nodes and cusps, with degree $d^* = d(d-1)$, $3d(d-2)$ cusps, and $\frac{1}{2}d(d-2)(d^2-9)$ nodes.

For $d > 2$ there do exist smooth plane curves C whose dual $C^\vee$ has singularities worse than ordinary nodes and ordinary cusps. The simplest source of such worse singularities are *hyperflexes* (a tangent line meeting C to order ≥ 4) and tangents with higher-than-simple ramification (e.g. a tangent line having contact of order ≥ 3 at one point and also tangent elsewhere, or bitangents with non-ordinary contact). Locally, such higher contact produces ramification of the Gauss map of order > 1, and the image singularity on the dual is a ramphoid cusp, tacnode, or more complicated singularity.

If C is singular, then the dual becomes more complicated: the Plücker formulas acquire correction terms (the degree d^* decreases, and the counts κ^*, δ^* change by local contributions), and the dual can acquire more severe singularities, multiple components, or multiplicities coming from the tangent cone at singular points.

4 Solutions to PSet 4

Solution 18. A hyperelliptic curve C is a degree 2 branched cover $\pi : C \to \mathbb{P}^1$. We will use two facts:

(i) For any line bundle $\mathcal{L}$ on $\mathbb{P}^1$, the map $H^0(\mathbb{P}^1, \mathcal{L}) \to H^0(C, \pi^*\mathcal{L})$ is injective.

Proof. The pullback line bundle $\pi^*\mathcal{L}$ on C is defined so that for each $x \in C$, the fibre satisfies $(\pi^*\mathcal{L})_x \cong \mathcal{L}_{\pi(x)}$. Given a section $s \in H^0(\mathbb{P}^1, \mathcal{L})$, its pullback section $\pi^*s \in H^0(C, \pi^*\mathcal{L})$ is defined pointwise by $(\pi^*s)(x) := s(\pi(x)) \in (\pi^*\mathcal{L})_x \cong \mathcal{L}_{\pi(x)}$. Equivalently, in a local trivialization of $\mathcal{L}$ over an open set $U \subset \mathbb{P}^1$, if s corresponds to a holomorphic function f on U, then π^*s corresponds to the holomorphic function $f \circ \pi$ on $\pi^{-1}(U)$. Thus we obtain a natural linear map

$$\pi^* \colon H^0(\mathbb{P}^1, \mathcal{L}) \longrightarrow H^0(C, \pi^*\mathcal{L}), \quad s \longmapsto \pi^*s.$$

Now we show that this natural map π^* is injective. Suppose $s \in H^0(\mathbb{P}^1, \mathcal{L})$ satisfies $\pi^*s = 0$. Then for every $x \in C$,

$$(\pi^*s)(x) = s(\pi(x)) = 0 \in \mathcal{L}_{\pi(x)}.$$

Since π is surjective (as a branched covering of $\mathbb{P}^1$), this implies $s(p) = 0$ for all $p \in \mathbb{P}^1$. Hence $s \equiv 0$, proving that π^* is injective. $\square$

(ii) We have $K_C \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(g-1)$ where g is the genus of C. This can be proven using Riemann-Hurwitz.

Proof. Let $B \subset \mathbb{P}^1$ denote the branch divisor of π and let $R \subset C$ denote the ramification divisor. By the Riemann–Hurwitz formula applied to the degree 2 map π we have as divisors

$$K_C = \pi^*K_{\mathbb{P}^1} + R.$$

Taking degrees in this equality gives

$$2g-2=\deg(K_C)=\deg(\pi^*K_{\mathbb{P}^1})+\deg(R)=2\cdot(-2)+\deg(R),$$

hence

$$\deg(R)=2g+2.$$

Since every branch point $p\in B$ is simple for a double cover, the branch divisor B has degree $\deg(B)=2g+2$ and satisfies $\pi^*B=2R$.

A (standard) description of degree 2 covers shows that there exists a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ with

$$\mathcal{L}^{\otimes 2}\cong\mathcal{O}_{\mathbb{P}^1}(B)\cong\mathcal{O}_{\mathbb{P}^1}(2g+2),$$

and such that the ramification divisor satisfies

$$\mathcal{O}_C(R)\cong\pi^*\mathcal{L}.$$

Since $\mathrm{Pic}(\mathbb{P}^1)=\mathbb{Z}$, so we conclude that

$$\mathcal{L}\cong\mathcal{O}_{\mathbb{P}^1}(g+1).$$

Hence $\mathcal{O}_C(R)\cong\pi^*\mathcal{O}_{\mathbb{P}^1}(g+1)$. Now substitute into the Riemann–Hurwitz formula:

$$K_C\cong\pi^*K_{\mathbb{P}^1}\otimes\mathcal{O}_C(R)\cong\pi^*\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big)\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(g+1)\cong\pi^*\mathcal{O}_{\mathbb{P}^1}(g-1),$$

as required. □

There is a multiplication map

$$H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(g-1))\otimes H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(g-1))\to H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2g-2)).$$

Note that the dimensions of the left hand side is $g\cdot g=g^2$ and the dimension of the right hand side is $2g-1$. Now by (i) and (ii), this fits into a commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(g-1))\otimes H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(g-1)) \arrow[r] \arrow[d, hook] & H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2g-2))) \arrow[d, hook] \\ H^0(C,K_C)\otimes H^0(C,K_C) \arrow[r, "\mu"'] & H^0(C,K_C^2) \end{tikzcd}$$

where μ is the multiplication map in question. But since $h^0(C,K_C)=g$, the left vertical map is actually an isomorphism. Hence we conclude that the

$$\dim\operatorname{im}\mu=h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2g-2))=2g-1.$$

To compute $h^0(K_C^2)$, we use the Riemann-Roch formula for line bundles

$$h^0(L) - h^0(K \otimes L^{-1}) = \deg L + 1 - g$$

where we set $L = K_C^2$. Note that $h^0(K \otimes L^{-1}) = h^0(K_C^{-1})$. Since $\deg K_C^{-1} = 2 - 2g < 0$ when $g > 1$ (which is true because C is hyperelliptic), we have $h^0(K_C^{-1}) = 0$. Hence

$$h^0(K_C^2) = 3g - 3.$$

Hence we conclude that μ is surjective iff $2g - 1 = 3g - 3$, i.e., when $g = 2$. Otherwise, μ will never be surjective.

Solution 19. If $g = 1$, then $\mathbb{P}^{g-1}$ is trivial. Hence, the space of quadrics in $\mathbb{P}^{g-1}$ containing the image $\phi(C)$ is zero-dimensional. Now assume $g \geq 2$. Let

$$\mu : H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(C, K_C^{\otimes 2})$$

be the multiplication map. The canonical morphism $\phi : C \to \mathbb{P}^{g-1}$ is given by

$$p \mapsto [\alpha_1(p) : \ldots : \alpha_g(p)],$$

where $\{\alpha_i\}$ is a basis of $\Omega^1(C) = H^0(C, K_C)$. Rather, we should understand $\alpha_1, ..., \alpha_g \in \mathcal{O}_C$ under a trivialization of K_C, which is well-defined everywhere because projectivization do not care about choices of trivialization, which differ by elements in $\mathcal{O}_C^*$.

Let us compute the dimension of the linear system of quadrics in $\mathbb{P}^{g-1}$ that contain $\phi(C)$. We claim that these quadrics correspond to $\ker \mu$. Let $V = H^0(C, K_C)$, so $\mathbb{P}^{g-1} = \mathbb{P}(V^\vee)$. Suppose $\{x_i\}$ is the dual basis of $\{\alpha_i\}$. Quadrics in $\mathbb{P}^{g-1}$ correspond to symmetric bilinear forms on $V^\vee$, i.e. elements of $\mathrm{Sym}^2(V^\vee)$, which map to

$$\phi^*(q) = \sum_{i,j} \lambda_{ij} \alpha_i \alpha_j \in H^0(C, 2K_C)$$

under the multiplication map.

For any $p \in C$, we have $\phi(p)$ lies on $q = \sum \lambda_{ij} x_i x_j$ if and only if $\phi^*(q)$ vanishes at p, i.e.

$$\sum_{i,j} \lambda_{ij} x_i x_j [\alpha_1(p) : \ldots : \alpha_g(p)] = \sum_{i,j} \lambda_{ij} \alpha_i(p) \alpha_j(p) = 0.$$

Hence, q vanishes on $\phi(C)$ precisely when $\sum \lambda_{ij} \alpha_i \alpha_j = 0$ in $H^0(C, 2K_C)$. Thus, the relevant space of quadrics is $\ker \mu$. However, since multiplication in $H^0(\mathbb{P}^{g-1}, \mathcal{O}(2))$ is symmetric and bilinear, we must interpret $\ker \mu$ as a subspace of $\mathrm{Sym}^2 H^0(C, K_C)$. That is, we need to consider the multiplication map $\mu : \mathrm{Sym}^2 H^0(C, K_C) \to H^0(C, K_C^2)$.

Now we compute its dimension. Since $\mathrm{Sym}^2 H^0(C, K_C)$ has basis $\alpha_i \odot \alpha_j$ with $1 \leq i \leq j \leq g$, we have

$$\dim \mathrm{Sym}^2 H^0(C, K_C) = \frac{g(g+1)}{2}.$$

By Riemann-Roch, $h^0(2K_C) = 3g-3$. Therefore,

$$\dim\ker\mu = \frac{g(g+1)}{2} - (3g-3) = \frac{(g-2)(g-3)}{2}.$$

The base locus of this linear system is the set

$$\{x \in \mathbb{P}^{g-1} : q(x) = 0 \text{ for all } q \in \ker\mu\}.$$

We first examine the hyperelliptic case. For a hyperelliptic curve C, we have the following result:

Proposition. *We can always write the affine part of a hyperelliptic curve C as the equation $y^2 = \prod_{i=1}^{2g+2}(x - x_i)$ in $\mathbb{C}^2$. Moreover, the differential form $\omega_0 = \mathrm{d}x/y$ is nonvanishing on the affine part of C. Then a basis of $H^0(C, K_C)$ can be given explicitly by*

$$H^0(C, K_C) = \langle \omega_0, x\omega_0, ..., x^{g-1}\omega_0\rangle.$$

Proof. See my notes on curves, Jacobians, and linear system notes, Proposition 4.5 for a proof. □

Let $\{z_i\}$ be the dual basis. For a quadric $q = \sum_{0\le i\le j\le g-1} \lambda_{ij} z_i z_j \in \ker\mu$, the corresponding element in $H^0(C, 2K_C)$ is

$$\sum_{0\le i\le j\le g-1} \lambda_{ij} \frac{x^{i+j}\mathrm{d}x\otimes\mathrm{d}x}{y^2} = \sum_{k=0}^{2g-2}\left(\sum_{i+j=k}\lambda_{ij}\right)\frac{x^k\mathrm{d}x\otimes\mathrm{d}x}{y^2} = 0.$$

Thus, $q \in \ker\mu$ if and only if $\sum_{i+j=k}\lambda_{ij} = 0$. Therefore, the base locus is $Z(I)$ where I is generated by the union of the ideals of quadrics such that $\sum_{i+j=k}\lambda_{ij} = 0$. In particular:

- when $g = 2$, $I = \varnothing$, so the base locus is $\mathbb{P}^{g-1}$;
- when $g = 3$, $I = \langle z_1^2 - z_0z_2\rangle$, and the base locus $Z(z_1^2 - z_0z_2)$ is a conic;
- when $g = 4$, $I = \langle z_1^2 - z_0z_2,\ z_0z_3 - z_1z_2,\ z_2^2 - z_0z_4\rangle$, and the base locus $Z(I)$ is the twisted cubic.

Next, consider non-hyperelliptic cases:

For $g = 3$, $\phi(C)$ is a plane quartic (as discussed in *curves and ppavs*, p. 21), so no quadric vanishes on $\phi(C)$, otherwise $\phi(C) \subset Q$ which means $4 = \deg\phi(C) \le \deg Q = 2$. So in this case $\ker\mu = 0$, and the base locus is $\mathbb{P}^{g-1}$. For $g = 4$, $\phi(C) = Q \cap S_3$, where Q is a quadratic surface and S_3 is a cubic surface. If Q is smooth, it is a quadric, and the base locus is Q. If Q is singular, it must be a cone, and $\phi(C)$ is a trigonal curve.

Solution 20. The Abel map $J : \mathcal{M}_g \to \mathcal{A}_g$ has differential $\mathrm{d}J : T_C\mathcal{M}_g \to T_{J(C)}\mathcal{A}_g$ at a point $C \in \mathcal{M}_g$ is given by

$$\mathrm{d}J : H^1(C, T_C) \to \operatorname{Sym}^2 H^1(C, \mathcal{O}_C).$$

Using Serre duality, this is exactly the dual of the multiplication map

$$\mu : \mathrm{Sym}^2 H^0(C, K_C) \to H^0(C, K_C^2).$$

Thus $\mathrm{d}J$ is injective iff μ is surjective. So in the hyperelliptic case, J is an immersion when $g = 2$, and not an immersion in any other hyperelliptic case. In Problem 19, the multiplication map is assumed to be surjective, so J is an immersion for all cases in Problem 19.

Solution 21. In an affine coordinate chart $[z^1 : \cdots : 1 : \cdots : z^n]$ of $\mathbb{P}^n$, identified with $(z^1, ..., z^n)$ in $\mathbb{C}^n$, the Fubini-Study metric $\omega_{\mathrm{FS}} = \frac{i}{2}\Theta$, where

$$\Theta = \partial\bar{\partial}\log\bigl(1 + |z|^2\bigr). \tag{*}$$

In explicit coordinate form, this is given by

$$\Theta = \frac{1}{1+|z|^2}\sum_j \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^j - \frac{1}{(1+|z|^2)^2}\sum_{j,k} \bar{z}^j z^k \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k. \tag{**}$$

(a) We can identify $\mathbb{P}^{n-1}$ in this local chart of $\mathbb{P}^n$ as the hypersurface $z^n = 0$. Then $\mathrm{d}z^n = \mathrm{d}\bar{z}^n = 0$ identically, and we see immediately from (**) that $\frac{i}{2}\Theta|_{\mathbb{P}^{n-1}}$ is the Fubini-Study metric on $\mathbb{P}^{n-1}$.

(b) Suppose $A \in \mathrm{U}(n+1)$. Then A preserves the norm $|z|^2$. Hence we see immediately from (*) that $F_A^*\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$.

Conversely, suppose $F_A^*\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$. We first remark that pullback commutes with $\partial, \bar{\partial}$, because they are just d composed with projections to $\Omega^{p,q}$'s and pullback commutes with d. Therefore, we have

$$\partial\bar{\partial}\log|Az|^2 = \partial\bar{\partial}\log|z|^2.$$

Now we put

$$u = \log|Az|^2 - \log|z|^2.$$

This is a real function satisfying $\partial\bar{\partial}u = 0$. We first need a lemma:

Lemma. *Suppose u is a real valued function satisfying $\partial\bar{\partial}u = 0$ on $\mathbb{P}^n$. Then there exists a holomorphic function $f : \mathbb{P}^n \to \mathbb{C}$ such that $u = \mathrm{Re}\, f$. Hence u is a constant.*

Proof of Lemma. Write $f = u + iv$ for some real valued v. So we want to find v such that f is holomorphic, i.e., $\bar{\partial}f = 0$. In other words,

$$\bar{\partial}v = i\bar{\partial}u.$$

Since u, v are both real valued, we can take the complex conjugate of the equation above and get

$$\partial v = -i\partial u.$$

Adding the two equation up gives

$$\mathrm{d}v = i(\bar{\partial}u - \partial u). \tag{3}$$

So the problem is equivalent to finding the solution v to the PDE (3). We define $\alpha = \bar{\partial}u - \partial u$. Note that α is closed because

$$\mathrm{d}\alpha = (\partial + \bar{\partial})(\bar{\partial}u - \partial u) = \partial\bar{\partial}u - \partial^2 u + \bar{\partial}^2 u - \bar{\partial}\partial u = 0$$

by $\partial\bar{\partial}u = 0$, conjugation, and $\partial^2 = \bar{\partial}^2 = 0$. Moreover, note that $H^1_{\mathrm{dR}}(\mathbb{P}^n) = 0$, so in $\mathbb{P}^n$, every closed form is exact. Thus the solution v exists on $\mathbb{P}^n$. Also note that since $i(\bar{\partial}u - \partial u)$ is real, v can be taken to be real. So a holomorphic $f = u + iv$ exists on $\mathbb{P}^n$, as desired. $\square$

Therefore, we conclude that $u = C$ for some constant C. Then $\log|Az|^2 - \log|z|^2 = C$. So $\langle Az, Az\rangle = e^C\langle z, z\rangle$ for all z which implies $AA^* = A^*A = e^C I$, so $A \in \mathrm{PU}(n+1)$.

Solution 22. To show that ω is Harmonic, we need to show that $\Delta\omega = 0$. Since ω is Kähler, $\mathrm{d}\omega = 0$. Hence $\Delta\omega = \mathrm{d}\mathrm{d}^*\omega$.

Lemma. *On a Kähler manifold of complex dimension n, one has the identity $*\omega = \lambda\omega^{n-1}$, where λ is a constant that depends only on the dimension.*

Proof of Lemma. Recall that for α, β of the same degree, $*\beta$ is the *unique* form such that $\alpha \wedge *\beta = \langle\alpha, \beta\rangle\, \mathrm{dVol}$. On a Kähler manifold $\mathrm{dVol} = \omega^n/n!$. Hence

$$\omega \wedge *\omega = \frac{\|\omega\|^2}{n!}\omega^n,$$

which implies that

$$*\omega = \frac{\|\omega\|^2}{n!}\omega^{n-1}.$$

This concludes the proof. $\square$

Therefore,

$$\mathrm{d}^*\omega = \pm * \mathrm{d} * \omega = \pm * \mathrm{d}\big(\lambda\omega^{n-1}\big) = 0.$$

Hence we conclude that $\Delta\omega = 0$, as desired.

5 Solutions to PSet 5

Solution 23. The claim is that for a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a, b), the genus is given by

$$g(C) = (a-1)(b-1).$$

Let $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to the i-th coordinate, where $i = 1, 2$. Since C is a section of $\mathcal{O}(a,b) = \pi_1^*\mathcal{O}(a) \otimes \pi_2^*\mathcal{O}(b)$, adjunction formula gives

$$K_C \cong K_{\mathbb{P}^1\times\mathbb{P}^1}|_C \otimes \mathcal{O}(a,b)|_C = \mathcal{O}(-2,-2)|_C \otimes \mathcal{O}(a,b)|_C = \mathcal{O}(a-2,b-2)|_C.$$

Taking the degree gives

$$\deg K_C = 2g-2 = \int_C c_1(\mathcal{O}(a-2,b-2)) = \int_C \pi_1^* c_1(\mathcal{O}(a-2)) + \int_C \pi_2^* c_1(\mathcal{O}(b-2)).$$

Now let $\mathbb{P}^1 \times \mathbb{P}^1$ have coordinates $([u:v],[s:t])$. Let $H_1 = \{v=0\} \subset \mathbb{P}^1_{[u:v]}$ and $H_2 = \{t=0\} \subset \mathbb{P}^1_{[s:t]}$. Then they generate the cohomologies $H^2(\mathbb{P}^1;\mathbb{Z})$, and $\pi_1^* H_1 = \{v=0\} = [1:0] \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi_2^* H_2 = \{t=0\} = \mathbb{P}^1 \times [1:0] \subset \mathbb{P}^1 \times \mathbb{P}^1$. Hence

$$\deg K_C = 2g-2 = (a-2)\int_C \pi_1^* H_1 + (b-2)\int_C \pi_2^* H_2 = (a-2)b + (b-2)a.$$

Solving $2g-2 = (a-2)b + (b-2)a$ gives $g = (a-1)(b-1)$, as desired.

Solution 24. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ be a nonsingular quadratic surface, and denote $B = \mathrm{Bl}_p S$ be the blow up of S at a point p. We want to show that $B \cong \mathrm{dP}_2$. By definition, this means that we need to show that the anticanonical divisor $-K_B$ is ample and $K_B^2 = K_B \cdot K_B = 7$.

We recall the intersection number of two divisors: If X is a smooth surface and C_1, C_2 are two divisors (complex codim 1, so real codim 2) on X. Let $\eta_{C_1}, \eta_{C_2} \in H^2(S,\mathbb{Z})$ denote the Poincaré dual of C_1, C_2 respectively. Then

$$C_1 \cdot C_2 = \int_X \eta_{C_1} \wedge \eta_{C_2}.$$

Since $\eta_{C_1} = c_1(\mathcal{O}_X(C_1))$ and similarly $\eta_{C_2} = c_1(\mathcal{O}_X(C_2))$, we also have that

$$C_1 \cdot C_2 = \deg(\mathcal{O}_X(C_1)|_{C_2}) = \deg(\mathcal{O}_X(C_2)|_{C_1}) = C_2 \cdot C_1.$$

We will first show that $K_B^2 = 7$. Let $\pi : B \to S$ be the blow down map. Then by Huybrechts Proposition 2.5.5, we have

$$K_B = \pi^* K_S + E,$$

where $E \cong \mathbb{P}(T_p S) \cong \mathbb{P}^1$ is the exceptional divisor. We first study how the intersection numbers of these pieces interact with each other. We claim that:

- $E^2 = E \cdot E = -1$: This follows from Huybrechts Corollary 2.5.6. We have $\mathcal{O}_B(E)|_E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence by definition of intersection,

$$E \cdot E = \deg(\mathcal{O}_B(E)|_E) = \int_E c_1(\mathcal{O}_B(E)) = \int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -1.$$

- $\pi^* K_S \cdot E = 0$. In fact, we will prove a stronger statement: Let D be any divisor on S, we have $\pi^* D \cdot E = 0$. To see this, note that we can always perturb D such that it does not intersect p, Then pulling back preserves intersection number since $\pi : B \setminus E \to S - \{p\}$ is an isomorphism.

- $\pi^*K_S \cdot \pi^*K_S = 8$: First, we show that $K_S \cdot K_S = 8$ in S. Indeed, since $S = \mathbb{P}^1 \times \mathbb{P}^1$, we have

$$K_S = \mathcal{O}(-2,-2) = p_1^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(-2).$$

  A simple computation shows that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong H^2(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$, and by Künneth formula we know that $H^\cdot(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z}) = \mathbb{Z}[H_1, H_2]/(H_1^2, H_2^2)$, where the two generators are $H_1, H_2 \in H^2(\mathbb{P}^1; \mathbb{Z})$. Viewing H_1, H_2 as divisors in $S = \mathbb{P}^1 \times \mathbb{P}^1$, we have $\eta_{H_1} = p_1^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ for the first component $\mathbb{P}^1$ in S, and $\eta_{H_2} = p_2^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ for the second component of $\mathbb{P}^1$ in S, where $p_1, p_2 : S \to \mathbb{P}^1$ are projections. Therefore,

$$H_1 \cdot H_2 = \int_{\mathbb{P}^1 \times \mathbb{P}^1} \eta_{H_1} \wedge \eta_{H_2} = \int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \cdot \int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = 1$$

  by Fubini theorem. Now the canonical bundle, under this correspondence, is given by

$$K_S = -2H_1 - 2H_2.$$

  Hence

$$K_S^2 = (-2H_1 - 2H_2)^2 = 8H_1 \cdot H_2 = 8$$

  because $H_1^2 = H_2^2 = 0$ in $H^\cdot(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z})$. Now it remains to show that pullback along π preserves the intersection number. Essentially, this is because $\pi : B - E \to S - p$ is a diffeomorphism, and both E and p are measure zero submanifolds in B and S. Concretely, this is because

$$\begin{aligned}\pi^*K_S \cdot \pi^*K_S = \int_B \eta_{\pi^*K_S} \wedge \eta_{\pi^*K_S} = \int_B \pi^*(\eta_{K_S} \wedge \eta_{K_S}) = \int_{B \setminus E} \pi^*(\eta_{K_S} \wedge \eta_{K_S}) \\ = \int_{S \setminus p} \eta_{K_S} \wedge \eta_{K_S} = \int_S \eta_{K_S} \wedge \eta_{K_S} = K_S \cdot K_S = 8.\end{aligned}$$

Therefore, combining our 3 pieces of calculations, we have

$$K_B^2 = (\pi^*K_S + E)^2 = \pi_*K_S^2 + \pi^*K_S \cdot E + E^2 = 8 + 0 - 1 = 7,$$

as desired.

Next, we show that $-K_B$ is ample. We use the following result:

Lemma (Nakai-Moishezon). *A line bundle $\mathcal{L}$ on a smooth surface S is ample if and only if the corresponding divisor D with $\mathcal{O}_S(D) = \mathcal{L}$ satisfies*

(i) $D^2 > 0$;

(ii) For any $C \subset S$ irreducible curves, we have $D \cdot C > 0$.

Partial Proof of Lemma. We will only prove that $\mathcal{L}$ is positive implies (i) and (ii). The converse is highly nontrivial and we will omit it.

Suppose $\mathcal{L}$ is ample. By Kodaira's theorem, $\mathcal{L}$ is positive, meaning it admits a Hermitian metric whose curvature form $\omega \in c_1(\mathcal{L})$ is a strictly positive, closed $(1,1)$-form (a Kähler form) on S. The self-intersection of D is given by the integral $D^2 = \int_S \omega \wedge \omega$. Since ω is a Kähler form, $\omega \wedge \omega$ is a strictly positive volume form on the surface S, ensuring $D^2 > 0$. Furthermore, for any irreducible curve $C \subset S$, the intersection number is $D \cdot C = \int_C \omega$. The restriction of ω to the smooth locus of the complex curve C is a strictly positive volume form, meaning $\int_C \omega > 0$. This establishes (i) and (ii). $\square$

In our case $D = -K_B$. We already calculated $D^2 = -K_B \cdot -K_B = K_B^2 = 7 > 0$, so (i) is satisfied. To check (ii), we need to discuss the cohomology ring of B. The cohomology of $S = \mathbb{P}^1 \times \mathbb{P}^1$ is given by $H^{\cdot}(S) = \mathbb{Z}[H_1, H_2]/(H_1^2, H_2^2)$. Since π is an isomorphism away from the exceptional divisor $E \subset B$, we have that

$$H^{\cdot}(B) = \mathbb{Z}[F_1, F_2, E]/(F_1^2, F_2^2, F_1 \cdot E, F_2 \cdot E, E^2 + 1, F_1 \cdot F_2 + 1),$$

where $F_1 = \pi^* H_1$ and $F_2 = \pi^* H_2$. To see the relations above, we note that $F_1^2 = F_2^2 = 0$ directly follows from $H_1^2 = H_2^2 = 0$, and $E^2 = -1$ follows from our calculation above, same for $F_1 \cdot E = F_2 \cdot E = 0$. Finally, to see $F_1 \cdot F_2 = 0$, we recall from our calculation that $H_1 \cdot H_2 = 1$, and since π is an isomorphism away from the exceptional locus, we can perturb H_1, H_2 so that they are disjoint from $p \in S$ so F_1, F_2 is disjoint from E.

Now let us check that (ii) holds. Let $C \subset B$ be an irreducible curve. There are two cases:

- $C = E$: In this case, $-K_B \cdot E = -(\pi^* K_S + E) \cdot E = -E^2 = 1 > 0$. Done.
- C is the strict transform of a curve on $S = \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a, b), and has multiplicity $m \geq 0$ at the blown up point p. Note that the multiplicity m condition states that the strict transform intersect E at m points, hence C must be given by

$$C = aF_1 + bF_2 - mE.$$

  Then it follows from a straightforward calculation that

$$-K_B \cdot C = -(-2F_1 - 2F_2 + E) \cdot (aF_1 + bF_2 - mE) = 2b + 2a - m.$$

  But note that $m \leq a + b$, as the multiplicity of the curve at a point cannot exceed its total degree, which is $a + b$. Therefore we conclude that $-K_B \cdot C > 0$.

This concludes the proof. In our class we identified dP_2 as the blow up of $\mathbb{P}^2$ at 2 points. We briefly discuss why we can identify $\mathrm{Bl}_p(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Bl}_{p_1,p_2} \mathbb{P}^2$. The cleanest geometric proof that I know of uses toric geometry [1]. The picture is Figure 1.

Both $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are toric varieties (i.e., variety that contains an algebraic torus as an open dense set). The shapes drawn are moment polytopes (or equivalently, the duals of the toric fans) that correspond to these toric varieties. In toric geometry, there is a dictionary between the geometric features of the variety and the combinatorial features of the polytope: The polygon itself corresponds to the complex surface; The edges correspond to torus-invariant divisors (curves); The vertices (corners) correspond to torus-fixed points where these divisors intersect.

[1] This was explained to me by Anson Law.

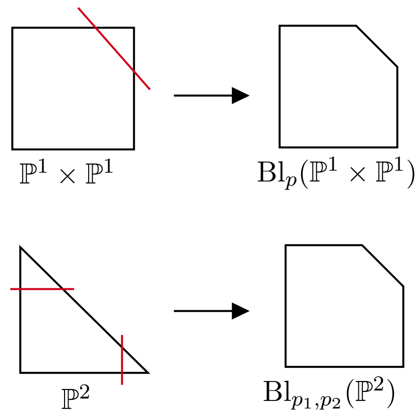


Figure 1: The toric polytopes of $\mathrm{Bl}_p(\mathbb{P}^1 \times \mathbb{P}^1)$ at one point is equivalent to that of $\mathrm{Bl}_{p,q}(\mathbb{P}^2)$.

The Cartesian product of two projective lines is represented by a square. The four vertices correspond to the four torus-fixed points of $\mathbb{P}^1 \times \mathbb{P}^1$ (the intersections of the rulings $0 \times \mathbb{P}^1$, $\infty \times \mathbb{P}^1$, $\mathbb{P}^1 \times 0$, and $\mathbb{P}^1 \times \infty$). The complex projective plane is represented by a right triangle. The three vertices are the three standard torus-fixed points $[0:0:1], [0:1:0], [1:0:0]$.

In toric geometry, blowing up a torus-fixed point corresponds exactly to "chopping off" a vertex of the polytope. The new edge created by this truncation corresponds to the exceptional divisor. Take the square $\mathbb{P}^1 \times \mathbb{P}^1$ and blow up one point. You chop off one corner. A square with one corner removed becomes a pentagon. Take the triangle $\mathbb{P}^2$ and blow up two points. You chop off two corners. A triangle with two corners removed also becomes a pentagon.

Because both operations yield the exact same polytope, up to affine transformations in $\mathrm{GL}(2,\mathbb{Z})$, the fundamental theorem of toric geometry dictates that the resulting blown-up surfaces are isomorphic.

Solution 25. Let S be a nonsingular cubic surface in $\mathbb{P}^3$. We use the following lemma:

Lemma. *S contains two disjoint lines l, m.*

Proof of Lemma. We first assume that $S = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\}$ is the Fermat cubic. Then consider the lines given by $l = \{x_0 + x_1 = 0, x_2 + x_3 = 0\}$ and $m = \{x_0 + \omega x_1 = 0, x_2 + \omega x_3 = 0\}$ where ω is a 3rd root of unity, i.e., $\omega^3 = 1$, but we require $\omega \neq 1$. Then clearly l, m are contained in S. We show that they are disjoint.

Suppose not, and let $p = [x_0 : x_1 : x_2 : x_3]$ be in the intersection of l, m. Then it must satisfy both sets of equations. So we have $x_0 = -x_1$ and $x_0 = -\omega x_1$. Thus $x_1 = \omega x_1$, or equivalently $x_1(1-\omega) = 0$. Since $\omega \neq 1$, we must have $x_1 = 0$. This immediately forces $x_0 = 0$. Similarly, $x_2 = x_3 = 0$. But this cannot happen in projective space. So we conclude that l, m are disjoint.

Now, let $\mathbb{P}^{19} = \mathbb{P}^{\binom{3+3}{3}-1}$ be the parameter space of homogeneous polynomials of degree 3 in variables x_0, x_1, x_2, and let $U \subset \mathbb{P}^{19}$ be the open set of smooth polynomials.

Let $\mathrm{Gr}(1,3)$ be the Grassmannian of lines in $\mathbb{P}^3$. We can construct an incidence correspondence—the relative Fano scheme of lines—defined as:

$$\mathcal{F} = \{(S, L) \in U \times \mathrm{Gr}(1, 3) : L \subset S\}.$$

The projection map $\pi : \mathcal{F} \to U$ is a finite, étale morphism. For smooth cubic surfaces, the fibers are always reduced and 0-dimensional. Because U is connected and π is an unbranched topological covering space, the number of lines is constant across all fibers (which happens to be 27). More importantly, any continuous path in the base space U lifts uniquely to the covering space. If you continuously deform the Fermat surface into another smooth cubic surface, its lines deform smoothly along with it.

Finally, we note that disjointness is a topological invariant: We showed that on the Fermat surface we have $l \cdot m = 0$. Hence if $t \in U$ then $l_t \cdot m_t$ vary continuously with respect to t. Hence we conclude that $l_t \cdot m_t = 0$ for all $t \in U$, concluding the proof. $\square$

Using this lemma, we construct a birational map $\Phi : S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with inverse $\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S$. From Problem 2, we know that dP_2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at a point, but dP_2 is just $\mathbb{P}^2$ blown up at two points, and since blow up preserves rationality, we have that $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$, and we are done.

Let l, m be two disjoint lines in S as given as in the Lemma, note that both are isomorphic to $\mathbb{P}^1$. Now if $p \in \mathbb{P}^3 - (l \cup m)$, then there exists a unique line n through p that intersects both l, m at a point (If a line n passes through p and intersect l, then it must lie entirely on the plane H spanned by p and l. But a plane H intersects another line m at exactly one point). Therefore, we define

$$\Phi : S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \qquad \Phi(p) = (l \cap n, m \cap n).$$

Conversely, let $(p, q) \in l \times m$. Let n be the line $n = \overline{pq} \subset \mathbb{P}^3$. Then there are only finitely many lines of S that meets l again by the above lemma. For almost all values of $(p, q) \in l \times m$, the line n intersects S at three points p, q, r (three points because $\deg S = 3$ by assumption, so $S \cdot n = 3$). Then we define

$$\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S \qquad \Psi(p, q) = r.$$

One can check that Φ, Ψ are inverses of each other when restricted to the common intersection of their respective domain and codomains. Moreover, both maps are rational by writting out coordinates. This concludes the proof.

Solution 26. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension k. Then its **degree** is defined to be the intersection number $\deg X = X \cdot L$ where L is a linear subspace $L \cong \mathbb{P}^{N-k}$ of complementary dimension. Short answer:

$$\deg(v(\mathbb{P}^n)) = d^n \qquad \deg(\sigma(\mathbb{P}^m \times \mathbb{P}^n)) = \binom{m+n}{m}.$$

To prove these two claims, we will first need the following lemma:

Lemma. *Let $f : X \to \mathbb{P}^N$ be the morphism defined by the basepoint free linear system $|\mathcal{L}|$. Then we* *have*

$$f^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{L}.$$

Proof of Lemma. This is Huybrechts Proposition 2.3.26. $\square$

Now we first look at the Veronese embedding $v : \mathbb{P}^n \to \mathbb{P}^N$ given by

$$v(z) = (z^I)_I,$$

where $I = (i_0, ..., i_n)$ runs through all multi-index with $|I| = \sum_k i_k = d$, and $z^I = \prod_k z_k^{i_k}$. The corresponding linear system is $\mathcal{O}_{\mathbb{P}^n}(d)$. The degree of the variety $v(\mathbb{P}^n)$ is by definition the intersection number $v(\mathbb{P}^n) \cdot \mathbb{P}^{N-n}$. A general linear subspace $\mathbb{P}^{N-n}$ is the intersection of n hyperplanes. In $\mathbb{P}^N$, a hyperplane H is given by the equation $z_0 = 0$, which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ by Proposition 2.3.18 of Huybrechts. Hence the Poincaré dual η_H of H corresponds to $c_1(\mathcal{O}_{\mathbb{P}^N}(1)) \in H^2(\mathbb{P}^N;\mathbb{Z})$. Therefore, the Poincare dual of a general linear subspace $\mathbb{P}^{N-n}$ which corresponds to intersection of n hyperplanes is given by

$$\eta_{H\cap\cdots\cap H} = \eta_H \wedge \cdots \wedge \eta_H = (\eta_H)^n.$$

Therefore, using the previous lemma, we have

$$\deg X = X \cdot \mathbb{P}^{N-n} = \int_{v(\mathbb{P}^n)} (\eta_H)^n = \int_{\mathbb{P}^n} (v^*\eta_H)^n = \int_{\mathbb{P}^n} (v^* c_1(\mathcal{O}_{\mathbb{P}^N}(1)))^n = \int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(d))^n.$$

As discussed above, the Poincare dual η_h of a hyperplane $h \subset \mathbb{P}^n$ is given by $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence $c_1(\mathcal{O}_{\mathbb{P}^n}(d)) = d\eta_h$. Therefore

$$\deg X = \int_{\mathbb{P}^n} (d\eta_h)^n = d^n \int_{\mathbb{P}^n} \eta_h^n = d^n$$

because n hyperplanes in $\mathbb{P}^n$ intersects at a single point: for example, take $h_i = Z(z_i - 1)$. Therefore, $\int_{\mathbb{P}^n} \eta_h^n = 1$.

Next, we consider the Segre embedding

$$\sigma : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N \qquad \sigma(z, w) = (z_i w_j)_{0 \le i \le m, 0 \le j \le n}.$$

In this case $N = (m+1)(n+1) - 1$. In fact, we can define a line bundle $\mathcal{O}(1,1)$ over $\mathbb{P}^m \times \mathbb{P}^n$ given by

$$\mathcal{O}(1,1) = p_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^n}(1).$$

Then $\sigma : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ is precisely the canonical map determined by the complete linear system $\mathcal{O}(1,1)$.

Let $a = c_1(\mathcal{O}_{\mathbb{P}^m}(1,0))$ and $b = c_1(\mathcal{O}_{\mathbb{P}^n}(0,1))$. In $\mathbb{P}^N$, a hyperplane H is given by the equation $z_0 = 0$, which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ by Proposition 2.3.18 of Huybrechts. Hence the Poincaré dual η_H of H corresponds to $c_1(\mathcal{O}_{\mathbb{P}^N}(1)) \in H^2(\mathbb{P}^N;\mathbb{Z})$. Therefore, by the previous lemma again,

$$\sigma^*\eta_H = \sigma^*(c_1(\mathcal{O}_{\mathbb{P}^N}(1))) = c_1(\mathcal{O}(1,1)) = c_1(\mathcal{O}(1,0)) + c_1(\mathcal{O}(0,1)) = a + b.$$

Therefore,

$$\deg X = \int_{\sigma(\mathbb{P}^m\times\mathbb{P}^n)} (\eta_H)^{m+n} = \int_{\mathbb{P}^m\times\mathbb{P}^n} (a+b)^{m+n} = \binom{m+n}{m}\int_{\mathbb{P}^m\times\mathbb{P}^n} a^m b^n = \binom{m+n}{m},$$

where in the second to last step, we used the observation that only $a^m b^n$ survives the integration among all the terms in the expansion $(a+b)^{m+n}$. This is because $a = p_1^* c_1(\mathcal{O}_{\mathbb{P}^m}(1)) =: p_1^*\alpha$ and $b = p_2^* c_1(\mathcal{O}_{\mathbb{P}^n}(1)) =: p_2^*\beta$, and $\alpha^{m+1} = 0, \beta^{n+1} = 0$. The last equality follows from Fubini's theorem:

$$\int_{\mathbb{P}^m\times\mathbb{P}^n} a^m b^n = \int_{\mathbb{P}^m} \alpha^m \cdot \int_{\mathbb{P}^n} \beta^n = 1.$$

6 Solutions to PSet 6

Solution 27 (I interpreted this problem wrong. See my oral exam transcript for how Ron expects us to solve it.)**.** Answer: $a = 3$. The question is equivalent to asking the minum a such that $K_C^{\otimes a}$ is very ample, i.e., the map $\varphi_{K_C^{\otimes a}} : C \to \mathbb{P}^N$ is an embedding. We use the following criterion:

Theorem (Hartshorne Proposition 3.1)**.** *A divisor D on a curve C is is very ample if and only if for every two points $P, Q \in C$ (including the case $P = Q$), we have*

$$h^0(D - P - Q) = h^0(D) - 2.$$

We use this criterion to check very ampleness, with $D = aK_C = aK$. By Riemann-Roch, we have

$$h^0(aK) - h^1(aK) = h^0(aK) - h^0((1-a)K) = a(2g-2) + 1 - g = (2a-1)(g-1).$$

Let us assume $a > 1$, we have $h^0((1-a)K) = 0$ because $\deg K^{\otimes(1-a)} = (1-a)(2g-2) < 0$. Hence we get that

$$h^0(aK) = (2a-1)(g-1). \tag{4}$$

Now we compute $h^0(aK - P - Q)$. Again by Riemann-Roch and Serre duality, we have

$$h^0(aK - P - Q) - h^0((1-a)K + P + Q) = a(2g-2) - 2 + 1 - g.$$

Now let $L = (1-a)K + P + Q$, then

$$\deg(L) = (1-a)(2g-2) + 2 < 0$$

unless $a = 2, g = 2$. So when $a > 2$, we have $h^0(L) = 0$. Therefore,

$$h^0(aK - P - Q) = a(2g - 2) - 1 - g. \tag{5}$$

Comparing equation (4) and (5), we see that we always have

$$h^0(aK - P - Q) = h^0(aK) - 2.$$

Hence, we conclude that for all $a \geq 3$, the divisor aK (hence the line bundle $K^{\otimes a}$) is very ample. Now it remains to show that $a = 3$, so we need to show that $a = 2$ is not ample for some genus $g \geq 2$.

Let us consider the case where $a = 2, g = 2$. Using Riemann-Roch gives

$$h^0(2K - P - Q) = h^0(-K + P + Q) + 2(2g - 2) - 2 + 1 - g = h^0(-K + P + Q) + 1$$

while

$$h^0(2K) = 2(2g - 2) - 1 = 3.$$

Hence $2K$ is very ample iff $h^0(-K + P + Q) = 0$. Or equivalently, $2K$ is not very ample iff $h^0(-K + P + Q) = 1$. But $-K + P + Q$ is a degree 0 divisor, so its h^0 is 1 if $-K + P + Q \sim 0$ and is 0 otherwise. Hence $2K$ is not very ample iff there exist points $P, Q \in C$ such that $P + Q \sim K$.

But note that C is a genus 2 curve so is hyperelliptic, and the 2:1 branch cover to $\mathbb{P}^1$ is given precisely by $\varphi_K : C \to \mathbb{P}^1$. Now we claim that if P, Q are the fibers of φ_K at a point $[a : b] \in \mathbb{P}^1$ then $P + Q \sim K$. Indeed, let $s_0, s_1 \in H^0(C, K)$ be a basis, then $\varphi_K(P) = [s_0(P) : s_1(P)]$ and $\varphi_K(Q) = [s_0(Q) : s_1(Q)]$. To say that these two points is $[a : b] \in \mathbb{P}^1$ is to say that P, Q are zeros of the $s := bs_0 - as_1 \in H^0(C, K)$. That is, $(s) = P + Q$. Note that there cannot be any more points because $\deg K = 2$. But since $s \in H^0(C, K)$ we have $(s) \sim K$. So $P + Q \sim K$. As a result, $2K$ is not very ample.

Therefore we conclude that the smallest integer such that aK is ample is $a = 3$.

Solution 28. Since S is a rational elliptic surface, we know that S is the blow up of $\mathbb{P}^2$ at 9 points $p_1, ..., p_9$. Therefore,

$$S \cong (\mathbb{P}^2 - \{p_1, ..., p_9\}) \sqcup \bigsqcup_{i=1}^{9} \mathbb{P}(T_{p_i}\mathbb{P}^2) \cong (\mathbb{P}^2 - \{p_1, ..., p_9\}) \sqcup \bigsqcup_{i=1}^{9} \mathbb{P}^1.$$

Since $\chi(A \sqcup B) = \chi(A) + \chi(B)$, we have that

$$\chi(S) = \chi(\mathbb{P}^2) - 9\chi(\text{pt}) + 9\chi(\mathbb{P}^1) = 3 - 9 + 9 \cdot 2 = 12.$$

We also now that $S \to \mathbb{P}^1$ is a fibration with generic fiber an elliptic curve, hence homeomorphic to a torus T^2. Assume each singular fiber has a single node. Let n be the number of singular fibers, and let the singular locus be $q_1, ..., q_n$, with singular fibers $S_1, ..., S_n$ respectively. Then we know that

$$S - \{S_1, ..., S_n\} \to \mathbb{P}^1 - \{q_1, ..., q_n\}$$

is a fiber bundle. For a fiber bundle $F \to E \to B$, we have $\chi(E) = \chi(F)\chi(B)$. Using the additivity of Euler characteristic on disjoint sets, we get that

$$\chi(S) - n\chi(S_1) = (\chi(\mathbb{P}^1) - n\chi(\mathrm{pt}))\chi(T^2) = 0 \tag{6}$$

because $\chi(T^2) = 0$. Now it remains to calculate the Euler characteristic of the singular fiber $S_1 = F$ which is homeomorphic to a pinched torus, as it has a single node, as shown in the figure below:

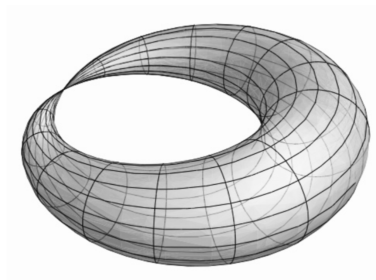


Figure 2: A singular fiber F which is a pinched torus.

Clearly $H_0(F) = \mathbb{Z}$. If we are pinching the torus, then we are killing one generator in $H_1(T^2) = \mathbb{Z}^2$, so $H_1(F) = \mathbb{Z}$. To compute $H_2(F)$, we use the Mayer-Vietoris sequence, and we choose two open sets $U \cup V = F$, with $V \simeq *$ a small neighborhood around the pinched point, $U \simeq S^1$ a neighborhood complementary to V, such that $U \cap V \simeq S^1 \sqcup S^1$. The Mayer-Vietoris sequence reads

$$\cdots \to H_2(U) \oplus H_2(V) \to H_2(F) \xrightarrow{\alpha} H_1(U \cap V) \xrightarrow{\beta} H_1(U) \oplus H_1(V) \to \cdots$$

which becomes

$$\cdots \to 0 \to H_2(F) \xrightarrow{\alpha} \mathbb{Z}^2 \xrightarrow{\beta} \mathbb{Z} \to \cdots$$

So we know α is injective. By exactness $\operatorname{im} \alpha = \ker \beta$. But $\beta : H_1(U \cap V) \to H_1(U) \oplus H_1(V) = H_1(U)$ is the inclusion of two copies of S^1 in $U \cap V$ into U, hence $\beta(a) = \beta(b) = x \in H^1(U)$ which is the generator. Therefore $\ker \beta \cong \mathbb{Z}$. Hence

$$H_2(F) = \mathbb{Z}.$$

Therefore, for a singular fiber, $\chi(F) = 1 - 1 + 1 = 1$. Hence (6) becomes

$$12 = \chi(S) = n.$$

Hence we conclude that there are 12 singular fibers if the fiber contains a single node. If the fiber has cusps, then $F \cong \mathbb{P}^1$ since it is a pinched torus with the other generator pinched, so it becomes a S^2. Then $\chi(F) = 2$. If the singular fiber T has on tacnode, then we can also calculate $\chi(T)$. In general, by the above precedure, we always have

$$12 = \chi(S) = n + 2c + \chi(T)t,$$

where n is the number of a fiber with a single node, c the number of fiber with a single cusp, and t

the number of fiber with a tacnode, and $\chi(T)$ the Euler characteristic of the tacnode fiber.

Solution 29. The Fermat equation is given by $S = \{x^4 + y^4 + z^4 + w^4 = 0\} \subset \mathbb{P}^3$. Note that the equation factors

$$x^4 + y^4 + z^4 + w^4 = (x^2 - iy^2)(x^2 + iy^2) + (z^2 - iw^2)(z^2 + iw^2).$$

Hence on the Fermat quartic, we have $\frac{x^2-iy^2}{z^2-iw^2} = -\frac{z^2+iw^2}{x^2+iy^2}$. Thus we can define a map $\Phi : S \to \mathbb{P}^1$ given by

$$\Phi : [x, y, z, w] \mapsto [x^2 - iy^2, z^2 - iw^2] = [z^2 + iw^2, -x^2 - iy^2].$$

It is not hard to show that Φ is surjective, and its fiber over a generic point $[s : t] \in \mathbb{P}^1$ is given by the system of equations

$$t(x^2 - iy^2) - s(z^2 + iw^2) = 0 \qquad t(z^2 - iw^2) + s(x^2 + iy^2) = 0.$$

Hence a generic fiber C is the intersection of two quadric in $\mathbb{P}^3$ (which is a curve). Next, we show that the fiber C are elliptic curves (i.e., of genus 1).

Proposition. *Let $C = Q \cap Q'$ where Q, Q' are two generic quadrics in $\mathbb{P}^3$. Then C is an elliptic curve (i.e., has genus 1).*

Proof of Proposition. Let $C = Q \cap Q'$ where Q, Q' are two quadrics in $\mathbb{P}^3$. Then Q, Q' corresponds to two sections of $\mathcal{O}(2)$. Moreover, we note that the normal bundle decomposes:

$$N_{Q\cap Q'/\mathbb{P}^3} \cong N_{Q/\mathbb{P}^3} \oplus N_{Q'/\mathbb{P}^3}$$

because for transversal intersections $Q \cap Q'$, the tangent bundle satisfies $T_{Q\cap Q'} = T_Q \cap T_{Q'}$, and we have an exact sequence

$$0 \to T_{Q\cap Q'} \to T_{\mathbb{P}^3} \to N_{Q/\mathbb{P}^3} \oplus N_{Q'/\mathbb{P}^3} \to 0$$

where $T_{\mathbb{P}^3} \to N_{Q/\mathbb{P}^3} \oplus N_{Q'/\mathbb{P}^3}$ is the map $\chi \mapsto (\chi \bmod T_Q, \chi \bmod T_{Q'})$. Therefore,

$$\det\bigl(N_{Q\cap Q'/\mathbb{P}^3}\bigr) = \det\bigl(N_{Q/\mathbb{P}^3} \oplus N_{Q'/\mathbb{P}^3}\bigr) = \det(\mathcal{O}(2) \oplus \mathcal{O}(2)) = \mathcal{O}(2) \otimes \mathcal{O}(2) = \mathcal{O}(4),$$

where the second to last equality can be checked by considering transition functions. Hence by adjunction,

$$K_C = K_{\mathbb{P}^3}|_C \otimes \det\bigl(N_{C/\mathbb{P}^3}\bigr) = \mathcal{O}(-4)|_C \otimes \mathcal{O}(4)|_C = \mathcal{O}|_C = \mathcal{O}_C.$$

Therefore, $2g - 2 = \deg K_C = \deg \mathcal{O}_C = 0$, so $g = 1$. $\square$

Now we need to determine the singular fibers. For $\lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1$, we denote $C_\lambda = \Phi^{-1}(\lambda)$ the fiber of S over λ, and

$$Q_\lambda = \{\lambda_1(x^2 - iy^2) - \lambda_0(z^2 - iw^2) = 0\} \qquad Q'_\lambda = \{\lambda_1(z^2 + iw^2) + \lambda_0(x^2 + iy^2) = 0\}.$$

the two quadrics whose intersection is the fiber C_λ.

Lemma. *A point $[x, y, z, w] \in \mathbb{P}^3$ on the fiber C_λ is singular iff the Jacobian matrix of Q_λ, Q'_λ (together) has rank strictly less than 2 at that point.*

Proof of Lemma. Indeed, $C_\lambda = Q_\lambda \cap Q'_\lambda$ is smooth at a point $p \in \mathbb{P}^3$ iff Q_λ and Q'_λ intersect transversally at p. This means that the tangent planes $T_pQ_\lambda, T_pQ'_\lambda$ intersects transversally at p, or equivalently the normal vectors $\nabla Q_\lambda, \nabla Q'_\lambda$ of these two planes must be linearly independent at p. Therefore, the Jacobian matrix $(\nabla Q_\lambda, \nabla Q'_\lambda)^T$ must have rank strictly less than 2 (obviously it can never be more than 2) if p is singular. □

Now ignoring the overall factor of 2, the Jacobian matrix $J = (\nabla Q_\lambda, \nabla Q'_\lambda)^T$ is given by

$$J = \begin{pmatrix} \lambda_1 x & -i\lambda_1 y & -\lambda_0 z & i\lambda_0 w \\ \lambda_0 x & i\lambda_0 y & \lambda_1 z & i\lambda_1 w. \end{pmatrix}$$

For the rank to be less than 2, all six of the 2×2 minors of J must vanish simultaneously. Let's compute them:

$$\begin{aligned}
\Delta_{xy} &= (\lambda_1 x)(i\lambda_0 y) - (\lambda_0 x)(-i\lambda_1 y) = 2i\lambda_0\lambda_1 xy \\
\Delta_{xz} &= (\lambda_1 x)(\lambda_1 z) - (\lambda_0 x)(-\lambda_0 z) = \left(\lambda_0^2 + \lambda_1^2\right) xz \\
\Delta_{xw} &= (\lambda_1 x)(i\lambda_1 w) - (\lambda_0 x)(i\lambda_0 w) = i\left(\lambda_1^2 - \lambda_0^2\right) xw \\
\Delta_{yz} &= (-i\lambda_1 y)(\lambda_1 z) - (i\lambda_0 y)(-\lambda_0 z) = -i\left(\lambda_1^2 - \lambda_0^2\right) yz \\
\Delta_{yw} &= (-i\lambda_1 y)(i\lambda_1 w) - (i\lambda_0 y)(i\lambda_0 w) = \left(\lambda_0^2 + \lambda_1^2\right) yw \\
\Delta_{zw} &= (-\lambda_0 z)(i\lambda_1 w) - (\lambda_1 z)(i\lambda_0 w) = -2i\lambda_0\lambda_1 zw.
\end{aligned}$$

Let's assume $\lambda_0\lambda_1(\lambda_0^4 - \lambda_1^4) \neq 0$. If this is the case, then $\lambda_0 \neq 0$, $\lambda_1 \neq 0$, $\lambda_0^2 + \lambda_1^2 \neq 0$, and $\lambda_0^2 - \lambda_1^2 \neq 0$. Looking at our minors, this assumption forces:

$$xy = xz = xw = yz = yw = zw = 0.$$

This means the product of any two coordinates is zero, which is only possible if at least three of the variables $\{x, y, z, w\}$ are zero. However, no point with three zero coordinates can satisfy $Q_\lambda = Q'_\lambda = 0$ since all squared terms have non-zero coefficients. Thus, for a singular point to exist, λ must be a root of $\lambda_0\lambda_1(\lambda_0^4 - \lambda_1^4) = 0$. This polynomial yields exactly 6 points in $\mathbb{P}^1$:

$$\lambda \in \{[0:1], [1:0], [1:1], [1:-1], [1:i], [1:-i]\}. \tag{7}$$

We claim that these are indeed singular points. Since the argument is mostly similar, we will only illustrate this for $\lambda = [0:1], [1:1]$. In this case,

$$\begin{aligned}
Q_\lambda &= (x^2 - iy^2) = (x - \sqrt{i}y)(x + \sqrt{i}y) \\
Q'_\lambda &= (z^2 + iw^2) = (z - \sqrt{-i}w)(z + \sqrt{-i}w).
\end{aligned}$$

Both quadrics split into a pair of planes. Their intersection is the intersection of two reducible quadrics, which forms a cycle of 4 projective lines (a Kodaira I_4 fiber). The singular points of this curve are the 4 "corners" where the lines meet, which occur when both planes in either quadric vanish simultaneously.

When $\lambda = [1 : 1]$, the conditions $\lambda_1^2 - \lambda_0^2 = 0$ force $\Delta_{xw} = \Delta_{yz} = 0$. The remaining minors dictate that either ($x = 0$ or $y = 0$) AND ($z = 0$ or $w = 0$). Setting $x = 0$ and $w = 0$, the quadrics evaluate to

$$Q_\lambda = -iy^2 - z^2 = (\sqrt{-i}y - z)(\sqrt{-i}y + z)$$
$$Q'_\lambda = iy^2 + z^2 = (\sqrt{i}y - iz)(\sqrt{i}y + iz).$$

Hence we have found the singular fibers, which are fibers over the points in (7). As a sanity check, S is a K3 surface so its Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

so $\chi(S) = 24$. Moreover, each I_4 has Euler characteristics $\chi(\#) = 4$. Thus $\chi(S) = 6\chi(\#)$, where 6 is precisely the number of singular fibers given in (7). Compare this with the previous problem to see that $\chi(S)$ is the product of number of singular fibers and the Euler characteristics of the singular fiber.

To conclude, we show that not all smooth quartic surfaces are elliptic. The period map for K3 surfaces goes to the period domain $\mathcal{M}_{\text{K3}} \to \mathcal{D}$, where $\mathcal{D}$ is the period domain.

More specifically, For any K3 surface X, the middle cohomology $H^2(X;\mathbb{Z})$ is a lattice of rank 22. The Hodge decomposition tells us that $H^{2,0}(X)$ is 1-dimensional, generated by a non-vanishing holomorphic 2-form Ω_X.The period map assigns the K3 surface to the complex line spanned by this 2-form:

$$X \mapsto [\Omega_X] \in \mathbb{P}(H^2(X;\mathbb{C})) \cong \mathbb{P}^{21}$$

Because Ω_X is a $(2,0)$-form, the Riemann bilinear relations dictate two constraints:

- Orthogonality: $\int_X \Omega_X \wedge \Omega_X = 0$.
- Positivity: $\int_X \Omega_X \wedge \bar{\Omega}_X > 0$.

The space of all lines in $\mathbb{P}^{21}$ satisfying these two relations is the period domain $\mathcal{D}$:

$$\mathcal{D} = \{[\alpha] \in \mathbb{P}(H^2(X;\mathbb{C})) = \mathbb{P}^{21} : (\alpha, \alpha) = 0, (\alpha, \bar{\alpha}) > 0\}$$

This space has complex dimension 20. Consequently, the moduli space of all K3 surfaces, $\mathcal{M}_{\text{K3}}$, is 20-dimensional.

We claim that each inclusion

$$\mathcal{M}_{\text{Elliptic quartic}} \subset \mathcal{M}_{\text{quartic}} \subset \mathcal{M}_{\text{K3}}$$

has image one dimension less. Let X be a quartic surface in $\mathbb{P}^3$. Then, there is a hyperplane class

h which is the restriction of the hyperplane class $\mathcal{O}_{\mathbb{P}^3}(1)$.

Lemma. $h^2 = 4$.

Proof of Lemma. The class $h \in H^2(X;\mathbb{Z})$ is the Poincaré dual of a hyperplane section. Geometrically, it is the curve H defined by intersecting the quartic surface X with a generic hyperplane $\Pi \cong \mathbb{P}^2$ in $\mathbb{P}^3$.To compute h^2, we look at the intersection of two generic, distinct hyperplane sections H_1 and H_2 on X:

$$H_1 \cap H_2 = (X \cap \Pi_1) \cap (X \cap \Pi_2) = X \cap (\Pi_1 \cap \Pi_2)$$

In $\mathbb{P}^3$, the intersection of two generic planes Π_1 and Π_2 is a line $L \cong \mathbb{P}^1$.Now we are just asking for the intersection of a line L with a surface X of degree 4 in $\mathbb{P}^3$. By Bézout's theorem, a general line intersects a degree 4 surface in exactly 4 points.Thus, $h^2 = 4$. $\square$

Now $h^2 = 4$ implies $h \neq 0$. By Lefschetz (1,1)-class theorem, $c_1 : \mathrm{Pic}(X) \to H^2(X;\mathbb{C})$ has image $H^{1,1}(X) \cap H^2(X;\mathbb{Z})$. Since h is defined by a divisor, we have $h \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$. By Hodge theory (p,q)-forms are orthogonal to (p',q')-forms under the intersection pairing unless $p + p' = 2$ and $q + q' = 2$. Therefore, any class in $H^{1,1}$ is strictly orthogonal to $H^{2,0}$. This gives us a linear condition $(h,\alpha) = 0$ for all $\alpha \in H^{2,0}(X)$.

Now let X be an elliptic quartic surface. That is, there is an elliptic fibration $X \to \mathbb{P}^1$. Let f be the cohomology class of a generic fiber of this fibration. Again $f \in H^{1,1}(X) \cap H^2(X;\mathbb{Z})$.

Lemma. $f^2 = 0$.

Proof of Lemma. To find f^2, we intersect two generic fibers F_1 and F_2. Because they are generic, we can choose them to lie over two distinct points $t_1, t_2 \in \mathbb{P}^1$. Since $t_1 \neq t_2$, their preimages are completely disjoint in X. Since the curves literally do not share any points, their intersection number is 0. Therefore, $f^2 = 0$. $\square$

Since $f^2 = 0$, we conclude that f is not a nonzero multiple of h. Also, we note that f must intersect the hyperplane class:

Lemma. $(f,h) > 0$.

Proof of Lemma. The pairing (f,h) is the intersection number of the fiber curve F with the hyperplane section H. Algebraically, the class h is the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$ to X. Because $\mathcal{O}(1)$ is an ample line bundle, its restriction to X remains ample. By the Nakai-Moishezon criterion, an ample divisor must have a strictly positive intersection number with any effective curve. Since the fiber F is an effective curve, we immediately get $(f,h) > 0$. $\square$

Thus $f \neq 0$. It follows from the exact same Hodge orthogonality logic that we have an additional condition $(f,\alpha) = 0$ for all $\alpha \in H^{2,0}(X)$.

Solution 30. Let $X = Q_0 \cap Q_1$ be the complete intersection of two quadrics in $\mathbb{P}^5$.

We first construct the smooth curve C of genus 2 determined by X. Since every quadric in $\mathbb{P}^5$ corresponds to (and is uniquely determined by) a 6×6 positive symmetric matrix, we have $Q_0 = x^T M_0 x$ and $Q_1 = x^T M_1 x$, where $x \in \mathbb{P}^5$. Then consider the set

$$\Delta = \{[t_0 : t_1] \in \mathbb{P}^1 : \det(\lambda_0 M_0 + \lambda_1 M_1) = 0\}.$$

The geometric picture is the following: for every $t = [t_0 : t_1] \in \mathbb{P}^1$, we consider the pencil $Q_t = t_0 Q_0 + t_1 Q_1$ in $\mathbb{P}^5$. This way, we get a family $Q \to \mathbb{P}^1$ with fibers being quadrics in $\mathbb{P}^5$, and $\Delta \subset \mathbb{P}^1$ is precisely the point where the fibers of the family $Q \to \mathbb{P}^1$ is singular. Since the defining equation of Δ is a degree 6 equation, we should expect 6 points in Δ, say $p_1, ..., p_6$. Then we define the curve C to be the branch cover $C \to \mathbb{P}^1$ of degree 2 branched at $p_1, ..., p_6$. Since this a degree 2 branch cover branched at six points, the Riemann-Hurwitz formula tells us

$$\chi(C) = 2\chi(\mathbb{P}^1) - \sum_{p \in C} (e_p - 1).$$

Hence

$$2 - 2g = 4 - 6 = -2,$$

which means $g = 2$.

Next, we show that X contains a line ℓ. We first recall the following construction: Let $\mathcal{I}$ be the incidence variety defined by $\mathcal{I} = \{(\ell, Q) : \ell \subset Q\}$ where ℓ is a line in $\mathbb{P}^5$ and Q a quadric in $\mathbb{P}^5$. The space parameterizing the projective lines in $\mathbb{P}^5$ (which is in bijection with planes in $\mathbb{C}^6$) is $\mathrm{Gr}(1,5)$, which has dimension 8. The space parameterizing all quadrics in $\mathbb{P}^5$ is $\mathbb{P}(\mathrm{Sym}^2 \mathbb{C}^6) = \mathbb{P}^{\binom{7}{2}-1} = \mathbb{P}^{20}$. Now $\mathcal{I}$ comes with natrual projection maps

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Gr}(1,5) & & \mathbb{P}(\mathrm{Sym}^2 \mathbb{C}^6) = \mathbb{P}^{20} \end{tikzcd}$$

with $p : \mathcal{I} \to \mathrm{Gr}(1,5)$ being a fiber bundle, because the group PGL_6 acts both on $\mathrm{Gr}(1,5)$ and on $\mathcal{I}$ in an equivariant way. We study the dimension of the fiber of $p : \mathcal{I} \to \mathrm{Gr}(1,5)$. Let $\ell \in \mathrm{Gr}(1,5)$ be given by $x_2 = x_3 = \cdots = x_5 = 0$ in $\mathbb{P}^6$. Then the quadric Q contains the line ℓ iff $Q(x_0, x_1, 0, ..., 0) = 0$ for all x_0, x_1. This means that Q is spanned by the degree 2 monomials in $x_0, ..., x_6$ except for $x_0^2, x_0 x_1, x_1^2$. Hence the dimension of the fiber is $\dim p^{-1}(\ell) = 20 - 3 = 17$. Therefore, $\dim \mathcal{I} = 17 + 8 = 25$. Next we consider the map $q : \mathcal{I} \to \mathbb{P}^{20}$, which maps to the space of quadrics in $\mathbb{P}^5$. If we could show that the fiber $q^{-1}(Q)$ has positive dimension, then we would be able to show that a quadric in $\mathbb{P}^5$ contains a line. Indeed, since $\dim \mathcal{I} = 25$ and $\dim \mathbb{P}^{20} = 20$, the fiber has dimension 5.

But, $\mathcal{I}$ is not exactly what we want. What we want is the incidence variety

$$\mathscr{I} = \{(\ell, Q_0, Q_1) : \ell \subset Q_0 \cap Q_1\}.$$

But this is just the subset of $\mathcal{I} \times \mathcal{I}$ such that the ℓ-component is the same. In other words, we have

$$\mathscr{I} = \mathcal{I} \times_{\mathrm{Gr}(1,5)} \mathcal{I}.$$

which fits into the diagram

$$\begin{tikzcd} \mathscr{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{I} \arrow[d, "p"] \\ \mathcal{I} \arrow[r, "p"'] & \mathrm{Gr}(1,5) \end{tikzcd}$$

Over $\ell \in \mathrm{Gr}(1,5)$, we have $\mathscr{I}_\ell = \mathcal{I}_\ell \times \mathcal{I}_\ell$ which is of dimension $17 + 17 = 34$. Hence $\dim \mathscr{I} = \dim \mathrm{Gr}(1,5) + \dim \mathscr{I}_\ell = 8 + 34 = 42$. We also have natural projection maps

$$\begin{tikzcd} & \mathscr{I} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathrm{Gr}(1,5) & & \mathbb{P}^{20} \times \mathbb{P}^{20} \end{tikzcd}$$

where $\Psi : \mathscr{I} \to \mathbb{P}^{20} \times \mathbb{P}^{20}$ is given by $\Psi(\ell, Q_0, Q_1) = (Q_0, Q_1)$. Since $\dim \mathscr{I} = 42$ and $\dim \mathbb{P}^{20} \times \mathbb{P}^{20} = 40$, we conclude that $\dim \Psi^{-1}(Q_0, Q_1) = 2$. Therefore, for any two quadrics Q_0, Q_1 that intersects transversely, the space of lines in $Q_0 \cap Q_1$ is 2-dimensional (positive dimension). Therefore, we conclude that $X = Q_0 \cap Q_1$ contains a line ℓ.

We claim that the set of lines in X that intersects ℓ is isomorphic to C, while the set of all lines in X is isomorphic to $\mathrm{Jac}(C)$. We prove the first statement first:

Proposition. *The set of lines in X that intersects ℓ is isomorphic to C.*

Proof of Proposition. Let ℓ' be a line that intersects ℓ at a single point. Let $\Pi = \langle \ell, \ell' \rangle \cong \mathbb{P}^2 \subset \mathbb{P}^5$ be the plane these two lines span.

If for some $t^* \in \mathbb{P}^1$, the quadric Q_{t^*} restricts to zero on Π, then this means the plane Π is entirely contained in the quadric Q_{t^*}.

Conversely, suppose there is a plane Π containing ℓ such that $\Pi \subset Q_{t^*}$. Then for any quadric Q_s in the pencil, its restriction to Π is a conic containing ℓ, which must split into two lines ℓ, ℓ'. We note that:

- $Q_s \cap \Pi$ cannot be identically zero for $s \neq t^*$. Otherwise $\Pi \subset Q_s$ as well. Since $s \neq t^*$, these two quadric generates X in the sense that $X = Q_s \cap Q_{t^*}$, hence $\Pi \subset X$. However, a smooth complete intersection of two quadrics in $\mathbb{P}^5$ cannot contain a plane, which is a consequence of the Lefschetz Hyperplane Theorem.

- Moreover, the line ℓ' in fact does not depend on the parameter s. In fact, we claim that $\ell \cup \ell' = Q_s \cap \Pi = X \cap \Pi$. Indeed, since $\Pi \subset Q_{t^*}$, we have $Q_s \cap \Pi = Q_s \cap (\Pi \cap Q_t) = (Q_s \cap Q_{t^*}) \cap \Pi = X \cap \Pi$ since $X = Q_s \cap Q_{t^*}$ for any $s \neq t^*$ in $\mathbb{P}^1$.

Therefore, we have shown that there is a bijective correspondence between lines $\ell' \subset X$ intersecting ℓ and the plane Π containing ℓ that is entirely contained in Q_t for some $t \in \mathbb{P}^1$. Therefore, the set of lines in X that intersects ℓ is isomorphic to the incidence variety

$$Y = \{(t, [\Pi]) \in \mathbb{P}^1 \times \mathrm{Gr}(2,5) \mid \ell \subset \Pi \subset Q_t\}.$$

There is a natural projection $Y \to \mathbb{P}^1$ which is a branched cover. Moreover, it is a classical result that for every smooth quadric Q_t there are exactly two planes in Q_t that contains a given line ℓ. Hence for generic $t \in \mathbb{P}^1$ the fiber consists of two points, hence the covering map $Y \to \mathbb{P}^1$ is degree 2. Moreover, the cover is branched at precisely the points $t \in \mathbb{P}^1$ where Q_t is singular, i.e., at Δ. This is precisely the curve C we constructed. Hence $Y \cong C$, as desired. □

Proposition. *The set of all lines in X is isomorphic to the Jacobian variety* $\mathrm{Jac}(C)$.

Proof of Proposition. This is a result in the paper "Ron Donagi, *Group law on the intersection of two quadrics*, 1980" and M. Ried's Cambridge thesis which is unpublished. (Are you kidding me?) □

7 Solutions to PSet 7

Solution 31. Let P be the principal GL_n-bundle associated with E.

Lemma. *The space of flags is the quotient P/B, where B is the Borel subgroup of* GL_n *consisting of upper triangular matrices.*

Proof. Indeed, if E is a rank n holomorphic vector bundle over X, its frame bundle P is the space of all ordered bases for the fibers of E. The general linear group GL_n acts transitively on these frames. A "flag" in a vector space is a nested sequence of subspaces (a line inside a plane inside a 3-space, etc.). The Borel subgroup B (upper triangular matrices) is exactly the subgroup of GL_n that preserves a standard flag. Therefore, modding out the frame bundle P by B gives you P/B, which is the bundle of all possible flags for the fibers of E. □

Because E is holomorphic and B is a complex Lie subgroup, P/B is a complex manifold (a holomorphic fiber bundle over X).

Set $Z = P/T$, where T is the maximal torus consisting of diagonal matrices. A point of Z is a point of X, plus n independent 1-dimensional linear subspaces of the fiber of E. Over this space Z, the pullback of the bundle E splits holomorphically into a direct sum of n line bundles.

Lemma. *The projection $Z = P/T \to P/B$ (induced by quotient since $T < B$) is a fibration with contractible fibers.*

Proof. The Borel group B (upper triangular) can be written as a semidirect product of the diagonal matrices T and the unipotent matrices U (strictly upper triangular matrices with 1's on the diagonal). Therefore, the quotient B/T is diffeomorphic to U. The unipotent group U is just a complex vector space $\mathbb{C}^{n(n-1)/2}$. □

Therefore the pullback morphism $H^*(P/B) \to H^*(Z)$ is an isomorphism. Since the pullback $H^*(X) \to H^*(P/B)$ is an injection by the standard splitting principle, the composition of pullbacks $H^*(X) \to H^*(P/B) \to H^*(Z)$ is also an injection, as desired.

Solution 32. Denote the new connection by ∇' and the old connection by ∇. For the direct sum connection, one sees that

$$(\nabla' - \nabla)(s_1 \oplus s_2) = a_1(s_1) \oplus a_2(s_2).$$

Hence $\nabla' = \nabla + a_1 \oplus a_2$. Similarly, on the tensor product one can see that

$$\nabla' = \nabla + a_1 \otimes 1_{E_2} + 1_{E_1} \otimes a_2.$$

Finally, for the hom bundle one can see that

$$\nabla'(f)(s_1) = (\nabla_2 + a_2)(f(s_1)) - f((\nabla_1 + a_1)(s_1)) = \nabla(f)(s_1) + a_2 f(s_1) - f(a_1(s_1)).$$

Solution 33.

(i) Since $\nabla = \nabla_1 \oplus \nabla_2$, we have

$$F = \nabla^2 = (\nabla_1 \oplus \nabla_2)(\nabla_1 \oplus \nabla_2) = \nabla_1^2 \oplus \nabla_2^2 = F_1 \oplus F_2.$$

(ii) Let $s \in \Gamma(E)$ be a smooth local section of E and $f \in \Gamma(E^*)$ be a smooth local section of the dual bundle E^*. The natural pairing is defined as $\langle f, s\rangle = f(s)$. We are given the definition of the dual connection:

$$\mathrm{d}\langle f, s\rangle = \langle \nabla^* f, s\rangle + \langle f, \nabla s\rangle.$$

To compute the curvature $F_{\nabla^*} = (\nabla^*)^2$, we must extend the Leibniz rule to bundle-valued differential forms. For $\alpha \in \Omega^k(E^*)$ and $\beta \in \Omega^l(E)$, the exterior derivative of their pairing reads:

$$\mathrm{d}\langle \alpha, \beta\rangle = \langle \nabla^* \alpha, \beta\rangle + (-1)^k \langle \alpha, \nabla\beta\rangle.$$

Hence we have (Note that the first equation has a plus sign because $\nabla^* f$ is a 1-form so $(-1)^k = -1$ and when we move it to the other side of the equation we get a plus sign):

$$\begin{aligned}
\langle F_\nabla f, s\rangle = \langle \nabla^{*2} f, s\rangle &= \mathrm{d}\langle \nabla^* f, s\rangle + \langle \nabla^* f, \nabla s\rangle \\
&= \mathrm{d}(\mathrm{d}\langle f, s\rangle - \langle f, \nabla s\rangle) + \langle \nabla^* f, \nabla s\rangle \\
&= -\mathrm{d}\langle f, \nabla s\rangle + \langle \nabla^* f, \nabla s\rangle \\
&= -\langle \nabla^* f, \nabla s\rangle - \langle f, \nabla^2 s\rangle + \langle \nabla^* f, \nabla s\rangle \\
&= -\langle f, \nabla^2 s\rangle = -\langle f, F_\nabla s\rangle.
\end{aligned}$$

Therefore, with respect to the natural pairing, we see that $F_{\nabla^*} = -F_\nabla^t$.

Now we given a connection ∇ on a vector bundle E of rank r, we can define the connection D on the determinant bundle $\det E = \bigwedge^r E$ to be

$$\nabla^{\det}(s_1 \wedge \cdots \wedge s_r) = \sum_{j=1}^{r} s_1 \wedge \cdots \wedge (\nabla s_j) \wedge \cdots \wedge s_r.$$

Differentiating twice gives

$$(\nabla^{\det})^2(s_1 \wedge \cdots \wedge s_r) = \sum_{i \neq j} s_1 \wedge \cdots \wedge \nabla s_i \wedge \cdots \wedge \nabla s_j \wedge \cdots \wedge s_r + \sum_i s_1 \wedge \cdots \wedge \nabla^2 s_i \wedge \cdots \wedge s_r.$$

Now the first sum is zero by pairwise cancellation: $\nabla s_i \wedge \nabla s_j = -\nabla s_j \wedge \nabla s_i$. The second term becomes

$$(\nabla^{\det})^2(s_1 \wedge \cdots \wedge s_r) = \sum_i (F_\nabla)^i_i s_1 \wedge \cdots \wedge s_r = (\operatorname{Tr} F_\nabla) s_1 \wedge \cdots \wedge s_r.$$

Hence, we conclude that $F_{\nabla^{\det}} = \operatorname{Tr} F_\nabla$.

Solution 34. Consider the trivial line bundle $\mathcal{O}$ on a complex curve X. We give two Hermitian metric h_+, h_- on $\mathcal{O}$, defined by the smooth functions $h_\pm = \exp\bigl(\pm|z|^2\bigr)$, where $|z|^2$ is the norm of the fiber coordinates of $\mathcal{O}$. Then let $\nabla_\pm$ be the Chern connections of $h_\pm$. That is, $\nabla_+ = \mathrm{d} + A_+$ where

$$A_+ = \partial \log\bigl(\exp\bigl(|z|^2\bigr)\bigr) = \partial(z\bar{z}) = \bar{z}\mathrm{d}z.$$

And the curvature is given by

$$F_+ = \bar{\partial}\partial \log\bigl(\exp\bigl(|z|^2\bigr)\bigr) = \bar{\partial}(\bar{z}\mathrm{d}z) = -\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Similarly, we have

$$F_- = \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

So that the curvature forms are

$$\omega_\pm = \mp \frac{i}{2\pi} \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

And it is easy to check that ω_+ is negative and ω_- is positive.

Solution 35. Since L is basepoint free, there is a holomorphic map $\varphi_L : X \to \mathbb{P}^N$ where $N := h^0(X, L) - 1$ such that $\varphi_L \mathcal{O}_{\mathbb{P}^N}(1) \cong L$. Thus

$$\int_X c_1(L)^n = \int_X \varphi_L^* c_1(\mathcal{O}_{\mathbb{P}^N}(1))^n = \int_X \varphi_L^*(\omega_{\mathrm{FS}})^n \geq 0$$

since the pullback of a positive form is semipositive.

Solution 36. Let $B = \frac{i}{2\pi} F_\nabla$ and B_1, B_2 be the same for ∇_1 on E_1 and ∇_2 on E_2 respectively. By splitting principle we may assume that both E_1, E_2 are direct sums of line bundles. Thus B is diagonal and so are B_1, B_2, and furthermore $B = B_1 \oplus B_2$. Then note that

$$\frac{\det(tB)}{\det(I - e^{-tB})} = \frac{\det(tB_1)}{\det(I - e^{-tB_1})} \frac{\det(tB_2)}{\det(I - e^{-tB_2})}.$$

Evaluating at $t = 1$ gives

$$\operatorname{td}(E_1 \oplus E_2) = \operatorname{td}(E_1) \cdot \operatorname{td}(E_2).$$

Solution 37. Recall the Euler sequence on $\mathbb{P}^n$ is given by

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0.$$

Thus

$$c(T_{\mathbb{P}^n}) = c(T_{\mathbb{P}^n})c(\mathcal{O}) = c(\mathcal{O}(1))^{n+1} = (1+\omega_{\mathrm{FS}})^{n+1}.$$

Thus

$$c_n(\mathbb{P}^n) = (n+1)\omega_{\mathrm{FS}}^n.$$

We have another exact sequence

$$0 \to T_{\mathbb{P}^n} \to T_{\mathbb{P}^n\times\mathbb{P}^m} \to T_{\mathbb{P}^m} \to 0.$$

Hence let α, β denote the Fubini-Study metric on $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, we then have

$$c(T_{\mathbb{P}^n\times\mathbb{P}^m}) = c(T_{\mathbb{P}^n})c(T_{\mathbb{P}^m}) = (1+\alpha)^{n+1}(1+\beta)^{m+1}.$$

Taking the top degree gives

$$c_{n+m}(\mathbb{P}^n \times \mathbb{P}^m) = (n+1)(m+1)\alpha^n\beta^m.$$

The integral of these top degree Chern classes are precisely the Euler characteristics of $\mathbb{P}^n, \mathbb{P}^n \times \mathbb{P}^n$ by Gauss-Bonnet formula.

Solution 38. By splitting priciple, one may assume that $E = \bigoplus_{k=1}^r L_k$ with $c_1(L_k) = \gamma_k$ and $c_1(L) = \gamma$. Then

$$c(E \otimes L) = \prod_{k=1}^r c(L_k \otimes L) = \prod_{k=1}^r (1+\gamma+\gamma_k).$$

The idea is that taking the degree i-part directly gives the desired formula. First, recall the standard formula

$$\prod_{k=1}^r (x+\gamma_k) = \sum_{j=0}^r \sigma_j(\gamma_1, ..., \gamma_r)x^{r-j},$$

where $\sigma_j(\gamma_1, ..., \gamma_r)$ are the elementary symmetric polynomials in variables $\gamma_1, ..., \gamma_r$. Since $\gamma_1, ..., \gamma_r$ are Chern roots of E, the j-th elementary symmetric polynomial of these roots is precisely $c_j(E)$. Hence,

$$c(E \otimes L) = \sum_{j=0}^r c_j(E)(1+\gamma)^{r-j}.$$

Next, by binomial theorem we have

$$(1+\gamma)^{r-j} = \sum_{m=0}^{r-j} \binom{r-j}{m}\gamma^m.$$

Hence, we see that

$$c(E \otimes L) = \sum_{j=0}^{r} c_j(E) \sum_{m=0}^{r-j} \binom{r-j}{m} \gamma^m = \sum_{j=0}^{r} \sum_{m=0}^{r-j} \binom{r-j}{m} c_j(E) \gamma^m.$$

Taking the degree i piece (i.e., the piece where $m = i - j$) gives the desired formula.

Solution 39. Note that $\operatorname{End} E = \operatorname{Hom}(E, E) \cong E \otimes E^*$. Hence we have

$$c_1(\operatorname{End} E) = \frac{i}{2\pi} \operatorname{Tr}\big(F_{\nabla_{E \otimes E^*}}\big) = \frac{i}{2\pi} \operatorname{Tr}\big(F_{\nabla_E} \otimes 1 - 1 \otimes F^t_{\nabla_E}\big) = 0$$

where we used $F_{\nabla_{E^*}} = -F^t_{\nabla_E}$.

To calculate $c_2(\operatorname{End} E) = c_2(E \otimes E^*)$, we use the splitting priciple: Assume $E = \bigoplus_{i=1}^r L_i$. Then $E \otimes E^* = \bigoplus_{1 \le i,j \le r} L_i \otimes L_j^*$. Let $x_i = c_1(L_i)$. Then note that

$$c_1(E) = \sum_i x_i, c_2(E) = \sum_{i<j} x_i x_j.$$

Now the Chern roots of $\operatorname{End} E = E \otimes E^*$ is given by $y_{ij} := c_1(L_i \otimes L_j^*) = x_i - x_j$ for $1 \le i, j \le r$. Therefore, using Greek letter α, β to denote double indices (i, j) for $1 \le i, j \le r$, we have

$$c_2(\operatorname{End} E) = \sum_{\alpha,\beta} y_\alpha y_\beta = \frac{1}{2}\left(\left(\sum_\alpha y_\alpha\right)^2 - \sum_\alpha y_\alpha^2\right).$$

Since

$$\sum_\alpha y_\alpha = \sum_{i,j} x_i - x_j = r\sum_i x_i - r\sum_j x_j = 0,$$

we have that

$$c_2(\operatorname{End} E) = -\frac{1}{2}\sum_{i,j}(x_i - x_j)^2.$$

Now we compute:

$$\begin{aligned}
\sum_{i,j}(x_i - x_j)^2 &= \sum_{i,j} x_i^2 + x_j^2 - 2x_i x_j = 2r\sum_i x_i^2 - 2\left(\sum_i x_i\right)^2 \\
&= 2r\left(\left(\sum_i x_i\right)^2 - 2\sum_{i<j} x_i x_j\right) - 2\left(\sum_i x_i\right)^2 \\
&= 2r(c_1(E)^2 - 2c_2(E)) - 2c_1(E)^2.
\end{aligned}$$

Therefore, we have

$$c_2(\operatorname{End} E) = -r(c_1(E)^2 - 2c_2(E)) + c_1(E)^2 = 2rc_2(E) - (r-1)c_1(E)^2.$$

To compute the quantity

$$(4c_2 - c_1^2)(L \oplus L)$$

we substitute $r = 2$ to the $c_2(\operatorname{End} E)$ formula above and find that the desired quantity is precisely

$$c_2(\operatorname{End} L^{\oplus 2}) = c_2(L^{\oplus 2} \otimes L^{*\oplus 2}) = c_2(\mathcal{O}^{\oplus 4}) = 0$$

since $c(\mathcal{O}^{\oplus 4}) = c(\mathcal{O})^4 = (1)^4 = 1$, which has no degree 2 parts.

Finally, to show that $c_{2k+1}(E) = 0$ if $E \cong E^*$, we note that if E has Chern roots $x_1, ..., x_r$, then $c_m(E) = \sum_{i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m}$, while

$$c_m(E^*) = \sum_{i_1 < \cdots < i_m} (-x_{i_1}) \cdots (-x_{i_m}) = (-1)^m c_m(E).$$

Therefore, when $m = 2k+1$ is odd, by assumption we have

$$c_{2k+1}(E) = c_{2k+1}(E^*) = (-1)^{2k+1} c_{2k+1}(E) = -c_{2k+1}(E).$$

Hence we conclude that $c_{2k+1}(E) = 0$, as desired.

Solution 40. We choose an arbitrary Hermitian metric h_0 on L, and let Θ_0 be the curvature form of its Chern connection. Both α and $\frac{i}{2\pi}\Theta_0$ are real (1,1)-forms representing $c_1(L)$. Hence they differ by an exact form. Since X is compact Kähler, by the global $\partial\bar{\partial}$-lemma, we can find a smooth real function u such that

$$\frac{i}{2\pi}\partial\bar{\partial}u = \frac{i}{2\pi}\Theta_0 - \alpha.$$

Now we define a new fiber metric $h = e^u h$ and let Θ be the curvature form of its Chern connection. Then we see that in terms of any holomorphic local frame we can write

$$\Theta = \bar{\partial}\partial \log |s|_h^2 = \bar{\partial}\partial \log\big(e^u |s|_h^2\big) = \partial\bar{\partial}u + \Theta = -2\pi i \alpha.$$

This concludes the proof.

Solution 41. We know from Proposition 4.2.19 that a holomorphic vector bundle E admits a holomorphic connection $E \to \Omega_X \otimes E$ iff its Atiyah class

$$A(E) \in H^1(X, \Omega_X \otimes \operatorname{End} E) \cong H^{1,1}(X; \operatorname{End} E)$$

vanishes. Moreover, from the textbook (Proposition 4.3.10) we know that $A(E) = [F_\nabla]$, hence we have equality in the cohomology ring

$$c(E) = \det\left(1 + \frac{i}{2\pi}F_\nabla\right) = \det\left(1 + \frac{i}{2\pi}A(E)\right) = 1.$$

Therefore, we see that $c_k(E) = 0$ for all $k > 0$.