Tianyi Wang Oral Exam Transcript

Given by Ron Donagi and Tony Pantev

Exam Date: April 30, 10AM (It lasted for about 90min)

Before the Oral (Skip to next section if you just want to see the questions)

Emmerson just finished his oral, Tony Pantev (TB) and Jonathan Block (JB) was his examiner. While Emmerson was waiting for the result outside, I congratulated him and we chatted a little bit. Tony walked out. Tony shook Emmerson's hand and told him that he passed, and he saw me.

TP: Where are we doing the exam?

Me: 4E19.

TP: (To JB) Hi Jonathan, we are doing it at 4E19.

Me: But I am doing a minor thesis with him, so technically he doesn't need to come.

TP: (To JB) Alright, but you want to come, right?

JB: Sure...

(At 4E19) TP and JB sat down. Ron Donagi (RD) enters as well.

Me: (To TP) Oh I forgot to bring syllabus, I can grab it real quick.

TP: No need. We will ask EVERYTHING anyway :). I need to make coffee, (to RD and JB) do you want coffee?

RD and JB nodded.

TP: (To me) do you want coffee?

Me: I had 3 cups already this morning... I'm good.

TP: Ok I'll be right back. Meanwhile you two start tourturing him.

JB: I always wanted to know how Witten proved the Atiyah-Singer index theorem using supergeometry (My minor thesis topic). Can you summarize it for me?

Me: I wrote that thesis months ago, I can't remember anything now.

JB: Okay... Instead, let me ask you a supergeometry problem since it is related to your thesis. Consider the space of maps $\varphi : \mathbb{R}^{0|1} \to M$ from the odd line to a manifold. What is this space?

TW: Ok. Let θ be the odd coordinate of $\mathbb{R}^{0|1}$. Then we can Taylor expand $\varphi(\theta) = \varphi(0) + \varphi'(0)\theta$ since $\theta^2 = 0$, so this space is parameterized by two parameters $\varphi(0), \varphi'(0)$. These are two complex numbers, so...

JB: I don't think so. I think the answer is ... (forgot what he said)

RD: (To JB) Are you sure? Suppose $M = \mathbb{R}^1$, then... (I didn't hear the rest)

JB: Oh. Then never mind, forget what I asked.

Then Tony came back and the actual exam started. I pretty much forgot Jonathan was there for the remaining part of the exam since he didn't ask anything (which I greatly appreciate).

Question 1 (Tony Pantev)

Let X be a smooth quadric in $\mathbb{P}^3$.

1. Does it contain a line?
2. Suppose it contains a line $L \subset X$. Let B be a line disjoint from L and consider the rational map
$$\mathrm{pr} : \mathbb{P}^3 \dashrightarrow B \qquad p \mapsto \mathrm{span}(p, L) \cap B.$$
This map is undefined if $p \in L$. Algebraically, if we view $\mathbb{P}^3 = \mathbb{P}(V)$ for a 4-dimensional vector space V and the line $L = \mathbb{P}(W)$ where $W < V$ is a 2-dimensional subspace, then $B = \mathbb{P}(V/W)$ and the map is simply the induced quotient map $\mathrm{pr} : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/W)$, which is undefined for a point in $\mathbb{P}(W)$. By construction, its restriction
$$\mathrm{pr}|_X : X \dashrightarrow B$$
is not defined on L. Show that $\mathrm{pr}|_X$ extends to a morphism $\phi : X \to B$.
3. Describe all fibers of ϕ. How many singular fibers does it have?

Remark

This one is fairly easy so I did it quickly, also Tony gave me the question in advance. See my file of problem collections given by Tony Pantev for the solution.

Question 2 (Ron Donagi)

1. Given a variety X and a line bundle $L \to X$, define the map $\varphi_L : X \dashrightarrow \mathbb{P}^N$.
2. Suppose $X = \mathbb{P}^1$ and $L = \mathcal{O}(d)$. What is φ_L in this case? How does N relates to d?
3. Suppose $X = \mathbb{P}^2$ and $L = \mathcal{O}(2)$, what is N? What is the degree of the image variety $\varphi_L(X)$?

Remark

This is all homework problem so nothing special. 1. Given a basis $s_0, ..., s_N \in H^0(X, L)$, the map is $\varphi_L(x) = [s_0(x) : ... : s_N(x)]$. 2. $N = h^0(\mathcal{O}_{\mathbb{P}^1}(d)) - 1 = \binom{d+1}{1} - 1 = d$. Then $\varphi_L : [x, y] \mapsto [x^d : x^{d-1}y : ... : y^d]$. 3. $N = 5$. To compute the degree of $\varphi(\mathbb{P}^2)$, note that since φ is an embedding $\dim \varphi(\mathbb{P}^2) = 2$. So its degree is intersection with a generic $\mathbb{P}^3$. Let $H = c_1(\mathcal{O}_{\mathbb{P}^5}(1)) \in H^2(\mathbb{P}^5; \mathbb{Z})$ be the hyperplane class.

$$\deg \varphi(\mathbb{P}^2) := \varphi(X) \cap H^2 = \int_{\varphi(\mathbb{P}^2)} H^2 = \int_{\mathbb{P}^2} \varphi^* c_1(\mathcal{O}_{\mathbb{P}^5}(2))^2 = \int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(2))^2 = 2^2 \int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2$$

because $L = \mathcal{O}_{\mathbb{P}^2}(2) \cong \varphi^* \mathcal{O}_{\mathbb{P}^5}(1)$ and we take the Chern class. Finally $c_1(\mathcal{O}_{\mathbb{P}^1}(1))^2 = \omega_{\text{FS}}$ so integrates to 1 on $\mathbb{P}^2$.

Ron: Very good.

Question 3 (Ron Donagi)

Let C be a curve of genus $g \geq 2$, what is the smallest a such that $\varphi_{K_C^{\otimes a}} : C \to \mathbb{P}^N$ is an embedding?

Remark

When I started this question, we crossed the one hour mark. This is where I really screwed up. This is a homework question but I think the whole class did it wrong. We thought he meant find the smallest a such that $\varphi_{K_C^{\otimes a}}$ is an embedding *for all* $g \geq 2$. But apparently he meant the genus is fixed, and we need to discuss by cases depending on what g is.

The idea is still the same. We quickly established why $a = 0, 1$ does not work for dimension reasons. For any line bundle L on a curve, L is very ample (φ_L is an embedding) iff

$$h^0(L - p - q) = h^0(L) - 2$$

for all $p, q \in C$. Then just discuss the two cases: $g = 2$ and $g > 2$. Find the minimal a by using Riemann-Roch to see what is the value of a for both sides of the equation to match. But my brain just froze.

At first, Tony was trying to help, but Ron said: "I think Tianyi can do it, just give him some time." This immediately stressed me up because I thought if I could not do it then I would really disappoint him. This was the only part of the exam where I screwed up.

Tony led me to the answers eventually, but I still don't know how I did it. It was a lot of case work and I forgot every case immediately after I finished the calculation (and I am stupid enough to erase it after I am done). After we concluded this problem:

Me: Sorry. That was embarrassing.

RD: Very good.

Me: (Thinking to myself) Very good??? Excuse me??? I just f*cked up that problem so bad.

Question 4 (Tony Pantev)

Let $X \subset \mathbb{C}^4$ be the quadric 3-fold defined by $X = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\}$.

1. Where is X singular?
2. Show that X contains a (linear) plane Π in $\mathbb{C}^4$. Find what this plane is.
3. Let Y be the strict transform of X in $\mathrm{Bl}_\Pi \mathbb{C}^4$. Show that Y is smooth. Let $\phi : Y \dashrightarrow X$ be the restriction of the blow down map $\mathrm{Bl}_\Pi \mathbb{C}^4 \to \mathbb{C}^4$. Find all fibers of ϕ (generically the fiber will be a point because ϕ is an isomorphism away from the exceptional points, so the interesting fibers are the fibers of positive dimension).

Remark

He gave me this question before. Again, see the file of problems given by Tony to see the solution. I went through this very quickly.

Question 5 (Tony Pantev)

Let X be a hypersurface of degree n in $\mathbb{P}^4$. Suppose X contains a hyperplane H. Is X smooth?

Remark

This is technically the only new question I haven't seen. But Tony gave me something similar early on during our training. Let the homogeneous coordinate on $\mathbb{P}^4$ be $x_0, ..., x_4$. WLOG suppose $H = \{x_4 = 0\} \subset \mathbb{P}^4$. Then since $H \subset X$ we immediately know the equation of X is given by

$$x_4 f(x) = 0$$

for some polynomial f of degree $d - 1$. Taking derivative gives

$$\nabla X = (\nabla x_4) f + x_4 (\nabla f) = (x_4 f_0, ..., x_4 f_3, f + x_4 f_4).$$

When $x_4 = 0$, i.e., when we restrict to H, note that

$$\nabla X = (0, 0, 0, 0, f).$$

So X is singular if $f = 0$ somewhere on $H \cong \mathbb{P}^3$. A nonzero homogeneous polynomial of degree $d - 1$ on $\mathbb{P}^3$ has a zero iff it is nonconstant. So X is not smooth unless f is a nonzero constant, in which case $X = H$ and X is smooth.

Finally...

TP: I think I'm done. Ron?

RD: I'm good. Jonathan?

JB: Oh I never started. (Then everybody laughed.)

Me: Should I wait outside?

Tony nodded. I walked outside and waited. Still couldn't believe that it's over and I was upset about that problem. This is literally a homework problem last semester and I felt screwing up like that was unacceptable. But I think for the rest of the exam I did pretty okay so I was expecting a pass. Several minute later Tony walked out, and Ron also came out.

TP: (Shake my hand) Congratulations!

RD: (Shake my hand) Congratulations!

Me: Thank you so much. Sorry for screwing up that problem (Question 3).

TP: You didn't screw up anything.

Me: (To RD and TP) Now that I passed my oral, may I ask the two of you to coadvise me?

TP: I'm fine with it.

RD: Sure.

Tony left. I also shook Jonathan's hand and I thanked him for sparing me. He sent me his congratulations. Ron went back to the room to grab his stuff.

Me: (To RD) I think our whole class did that problem wrong. We thought the genus was not fixed, and we need to find the minimal a for all genus $g \geq 2$.

RD: Ohh, there is an easier way to do it. What you wrote ($h^0(L-p-q) = h^0(L)-2$) is equivalent to $h^1(L-p-q) = h^1(L)$, and this is easier to compute.

Me: Ok? I understand that $h^1(L)$ is easier to compute, since $L = K_C^{\otimes a}$. But what about $h^1(L-p-q)$? Seems difficult to compute.

RD: (After a moment of silence) Yeah it's probably equivalent to what you did.

Some Reflections and Advice (More in the last part of my webpage)

Besides Question 3, I think I did fine for the other questions.

As for Question 3, one big mistake I made was writing on the last 1/4 of the entire board all the time. I don't know why I did it, as I always divide the board into 4s and write from the left to the right and repeat, but somehow when I was stuck, I panicked and forgot to switch boards. This was a big mistake because this problem has a lot of case works, and I erased my previous board work after I did it because I didn't have space. So when Tony asked "So what have we proved?" I didn't remember what I erased. In the end he had to tell me exactly what I proved sentence by sentence.

So one advice I would give: Divide the board into 4s, write from left to right, and do not erase anything until you finish the last 1/4 of the board and erase the earliest board. Its good habit!!! And not just because nice board work is important, but also because in this way you do not lose any of your immediate prior work and you can quickly remind yourself by looking at it.