Review Notes on Curves, Jacobians, and Linear Systems

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Abstract

These are my review notes for curves and Jacobians for my oral exam. The main focus will be on linear systems, curves and their Abel-Jacobi theory, and classification of curves of low genus ($g \leq 5$).

Contents

0 Basics

Theorem 0.1 (Riemann-Roch)**.** *If C is a smooth curve with genus g and L a line bundle on C with degree d, then*

$$\chi(C,L) = d + 1 - g.$$

Using vanishing theorem and Serre duality $\chi(C,L) = h^0(L) - h^1(L) = h^0(L) - h^0(K-L)$, we get the classical Riemann-Roch theorem:

$$h^0(L) - h^0(K-L) = d + 1 - g.$$

Theorem 0.2 (Adjunction Formula)**.** *Let $X \subset Y$ be a smooth hypersurface of a complex manifold, then their canonical class is related by*

$$K_X = K_Y + N_{X/Y}|_X.$$

Using the topological fact that $N_{X/Y} \cong \mathscr{O}_Y(X)|_X$, we have

$$K_X = K_Y + X|_X.$$

Theorem 0.3 (Riemann-Hurwitz Formula)**.** *If $f : C \to X$ is a nonconstant holomorphic map between smooth curves with ramification divisor $R = \sum_{p \in C}(m_p - 1)p$, then*

$$K_C = f^* K_X + R.$$

1 Linear Systems

We assume the reader is familiar with basic complex algebraic geometry, say Chapter 1-6 of [**?**]. Recall that if $L \to X$ is a (holomorphic) line bundle over a complex manifold, and let $V < H^0(X,L)$ be a vector subspace, then we can pick a basis $s_0,...,s_n$ of V, and define a rational map

$$\phi_V : X \dashrightarrow \mathbb{P}^n \qquad p \mapsto [s_0(p) : ... : s_n(p)].$$

This map is understood as follows: For any $p \in X$ we pick an open set $U \subset X$ containing p such that $L|_U \cong \mathscr{O}|_U$. Let $s \in H^0(U,L)$ be the local frame of this line bundle. Then we can write $s_i = f_i s$ for some $f_i \in H^0(U,\mathscr{O})$. Then $\phi_V(p) = [f_0(p) : ... : f_n(p)]$. Clearly this does not depend on the choice of s since any other choice only differ by a transition function, which does not matter when we projectivize. The points where ϕ_V is not defined is where all $s_0,...,s_n$ vanishes, and is called the **base locus**, which we denote by $\mathrm{Bs}(V)$.

Definition 1.1. A **linear system** $\mathscr{V}$ on a complex manifold X is a pair $\mathscr{V} = (L,V)$ where $L \to X$ is a holomorphic line bundle and V a subspace of $H^0(X,L)$. We define the **dimension** of the linear series $\dim \mathscr{V} := \dim_{\mathbb{C}} V - 1$.

If $\mathscr{V}$ is a linear system of degree d and dimension r we say that $\mathscr{V}$ is a g^r_d.

If $V = H^0(X,L)$, then we say this is a **complete linear system** and we denote $|L| := (L,H^0(X,L))$.

We first note that on a curve, there is a different interpretation of the complete linear series. If X is a complex curve, then every line bundle L is of the form $L = \mathscr{O}(D)$ for some divisor D. Then we have

Proposition 1.2. *On a complex curve X, the complete linear series $|L|$ associated to the line bundle $L = \mathscr{O}(D)$ can be identified with the set of all effective divisors*

$$|D| = \{D' \in \mathrm{Div}(C) : D' \sim D, D' \geq 0\}$$

linearly equivalent to D.

Proof. Given any section $\sigma \in H^0(X,L)$, taking its divisor gives a divisor $(\sigma) \sim D$. Moreover, since σ is a holomorphic section it has no poles, so $(\sigma) \geq 0$. Conversely, given any effective divisor $D' \sim D$, there exists a meromorphic function f on X such that $D' = D + (f)$. Now we know there exists a section $\sigma_0 \in H^0(X,L)$ such that $(\sigma_0) = D$ since $L = \mathscr{O}(D)$. Then note that $f \cdot \sigma_0$ has divisor D' and is holomorphic because $D' \geq 0$. Therefore, $f \cdot \sigma_0$ is the desired section of L. These two constructions are inverses to each other. $\square$

The importance of linear system can be illustrated by the following theorem:

Theorem 1.3. *For any complex manifold X there is a natural bijection between the set of nondegenerate morphisms $\phi : X \to \mathbb{P}^r$ mod PGL_{r+1} and basepoint free linear systems of dimension r on X up to isomorphism.*

Proof. Let $\phi : X \to \mathbb{P}^r$ be any nondegenerate morphism. If this morphism is given by a line bundle L then $L \cong \phi^*\mathscr{O}_{\mathbb{P}^r}(1)$. Therefore, we define the linear system by

$$\mathscr{V} = (\phi^*\mathscr{O}_{\mathbb{P}^r}(1), \phi^*H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1))).$$

Conversely, given a linear system (L,V) of dimension r we already have a map $\phi_V : X \to \mathbb{P}^r$. $\square$

Remark 1.4. There is actually a basis independent way of defining the map $\phi_V : X \dashrightarrow \mathbb{P}^r$ associated to the linear system $\mathscr{V} = (L,V)$. In the previous discussion where we chose a basis, we have

$$\phi_V(p) = [f_0(p) : ... : f_r(p)].$$

This actually corresponds to $\sum_{i=0}^r f_i(p)s = \sum_i s_i(p) \in L_p$ mod scalar. Therefore, this is a map $X \to \mathbb{P}(\mathrm{Hom}(V,L)) \cong \mathbb{P}(V^* \otimes L) \cong \mathbb{P}(V^*)$. Hence, the basis independent description is

$$\phi_V : X \dashrightarrow \mathbb{P}(V^*) \qquad p \mapsto \mathrm{ev}_p\,.$$

Definition 1.5. Let $\mathscr{D} = (L, V)$ and $\mathscr{E} = (M, W)$ be two linear systems on X. By the **sum** $\mathscr{D} + \mathscr{E}$ we mean the pair

$$\mathscr{D} + \mathscr{E} = (L \otimes M, U)$$

where $U \subset H^0(L \otimes M)$ is the subspace generated by the image of $V \otimes W$ under multiplication $H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$.

In the case that X is a curve, this is the subspace of complete linear system spanned by divisors of the form $D + E$ where $D \in \mathscr{D}, E \in \mathscr{E}$ both effective.

Remark 1.6. The corresponding map associated to the $\mathscr{D} + \mathscr{E}$ (assuming basepoint free) is the map

$$\phi_{\mathscr{D}+\mathscr{E}} : X \to \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$$

where the second map is the Segre embedding.

Definition 1.7. We say a linear system $\mathscr{D} = (L, V)$ **contains effective divisors** if $|\mathscr{D}| = \mathbb{P}(V^*)$, the set of all effective divisors equivalent to V by Proposition 1.2, is nonempty.

Proposition 1.8. *If $\mathscr{D}, \mathscr{E}$ are two linear series that contain effective divisors on a curve, then*

$$\dim(\mathscr{D} + \mathscr{E}) \geq \dim \mathscr{D} + \dim \mathscr{E}.$$

Proof. Saying $\dim \mathscr{D} \geq m$ is equivalent to saying we can find a divisor $D \in \mathscr{D}$ containing any given m points of C; since $\mathscr{D} + \mathscr{E}$ contains all pairwise sums $D + E$ with $D \in \mathscr{D}, E \in \mathscr{E}$, we can certainly find a divisor $D \in \mathscr{D} + \mathscr{E}$ containing any given $\dim \mathscr{D} + \dim \mathscr{E}$ points on the curve. $\square$

For a line bundle $L \to C$ on a curve, we give a useful criterion that it is basepoint free and (or) very ample:

Theorem 1.9. *Let L be a line bundle on a smooth curve C. Then:*

(a) the complete linear series $|L|$ is basepoint free iff $h^0(L - p) = h^0(L) - 1$ for all $p \in C$.

(b) $|L|$ is very ample iff $h^0(L - p - q) = h^0(L) - 2$ for all $p, q \in C$ (not necessarily distinct).

Proof. (a) We consider the short exact sequence

$$0 \to L(-p) \to L \xrightarrow{\mathrm{ev}_p} L_p \to 0.$$

The long exact sequence reads

$$0 \to H^0(L - p) \to H^0(L) \xrightarrow{\mathrm{ev}_p} H^0(L_p) \to \cdots$$

The point $p \in C$ is not a basepoint iff $\mathrm{ev_p}$ is surjective, by rank-nullity and exactness this is equivalent to

$$h^0(L) = h^0(L - p) + 1,$$

which is what we want.

(b) ϕ_L is injective iff for $p \neq q$ we have $\phi_L(p) \neq \phi_L(q)$. This is equivalent to saying that the evaluation $\mathrm{ev}_p \oplus \mathrm{ev}_q : H^0(C, L) \to L_p \oplus L_q$ is surjective. By rank-nullity theorem this is equivalent to $h^0(L - p - q) = h^0(L) - 2$.

We claim that $\mathrm{d}\phi_L : T_pC \to T_{\phi(p)}\mathbb{P}^r$ is injective on tangent space is equivalent to the condition $h^0(L - 2p) = h^0(L) - 2$. We choose a basis $s_0, \ldots, s_r$ of $H^0(L)$. Since L is basepoint free, WLOG assume $s_0(p) \neq 0$. Then in local affine coordinate

$$\phi_L = (s_1/s_0, \ldots, s_r/s_0).$$

Since $h^0(L-p) = h^0(L) - 1$ because L is basepoint free, we have that $s_1(p) = \cdots = s_r(p) = 0$. Then we have

$$\mathrm{d}\left(\frac{s_i}{s_0}\right)\bigg|_p = \frac{\mathrm{d}s_i(p) \cdot s_0(p) + s_i(p) \cdot \mathrm{d}s_0(p)}{s_0^2(p)} = \frac{\mathrm{d}s_i(p)}{s_0(p)}.$$

Since T_pC is one dimensional, we see that

$$\mathrm{d}\phi_{L,p} = (\mathrm{d}s_1(p)/s_0(p), \ldots, \mathrm{d}s_r(p)/s_0(p))$$

is injective iff it is nonzero, i.e., iff $\mathrm{d}s_i(p) \neq 0$ for some $i \in \{1, \ldots, r\}$. This holds iff the evaluation of derivative map $\mathrm{ev}'_p : H^0(L(-p)) \to H^0(L_p \otimes T_p^*C)$ is surjective. Now we have a short exact sequence

$$0 \to L(-2p) \to L(-p) \xrightarrow{\mathrm{ev}'_p} L_p \otimes T_p^*C \to 0.$$

The corresponding long exact sequence reads

$$0 \to H^0(L-2p) \to H^0(L-p) \xrightarrow{\mathrm{ev}'_p} H^0(L_p \otimes T_p^*C) \to \cdots$$

By rank nullity theorem and exactness, the map ev'_p is surjective iff

$$h^0(L-2p) + 1 = h^0(L-p) = h^0(L) - 1.$$

Hence we see that $\mathrm{d}\phi_L$ is injective at p iff $h^0(L-2p) = h^0(L) - 2$. Hence the condition $h^0(L-p-q) = h^0(L) - 2$ for any $p, q \in C$ is equivalent to ϕ_L being an injective immersion, which is an embedding since C is compact (so ϕ_L is proper). □

Using Riemann-Roch theorem, we get a useful criterion to determine whether a divisor is basepoint free and very ample:

Corollary 1.10. *Let D be a divisor of degree d on a curve C of genus g.*

(a) If $d \geq 2g$ then $\mathcal{O}(D)$ is basepoint free.

(b) If $d \geq 2g+1$ then $\mathcal{O}(D)$ is very ample.

Proof. These follows from Theorem 1.9. (a) Since $d \geq 2g$ we have $h^0(K-D+p) = 0$ for any $p \in C$. Therefore, Riemann-Roch gives

$$h^0(D) - h^0(D-p) = (h^0(K-D) + d + 1 - g) - (h^0(K-D+p) + d - 1 + 1 - g) = 1.$$

(b) If $d \geq 2g+1$ then $h^0(K-D+p+q) = 0$. Hence

$$h^0(D) - h^0(D-p-q) = (h^0(K-D) + d + 1 - g) - (h^0(K-D+p+q) + d - 2 + 1 - g) = 2.$$

This proves the claim. □

2 Applications of Riemann-Roch Theorem

2.1 The Canonical Morphism and Geometric Riemann-Roch

Theorem 2.1. *Let C be a smooth curve of genus $g \geq 2$. Then:*

(a) $|K_C|$ is basepoint free.

(b) $|K_C|$ is very ample iff C admits no map of degree 2 to $\mathbb{P}^1$ (i.e., C is not hyperelliptic).

Proof. (a) By Theorem 1.9, a point p is a basepoint of $|K_C|$ iff $h^0(K_C - p) = h^0(K_C) = g$. By Riemann-Roch, this is equivalent to $h^0(p) = 2$, which would imply that $C \cong \mathbb{P}^1$ because there is a meromorphic function on C with a pole at p.

(b) By Theorem 1.9, $|K_C|$ is very ample iff $p, q \in C$ we have $h^0(K_C - p - q) = h^0(K_C) - 2 = g - 2$. By Riemann-Roch, this fails iff $h^0(p+q) \geq 2$ for some $p, q \in C$.

Now assume $h^0(p+q) \geq 2$. Since $g \geq 2$, we conclude that there exists a meromorphic function on C with poles at p, q, which defines a double cover to $\mathbb{P}^1$, hence C is hyperelliptic.

Conversely, assume C is hyperelliptic. Then take $\pi : C \to \mathbb{P}^1$ be the hyperelliptic map, and pick a point $x \in \mathbb{P}^1$ that is unramified, so that $D := \pi^* x$ is given by $D = p + q$ for some $p, q \in C$. We claim that $h^0(D) = 2$, then by Riemann-Roch $h^0(K - p - q) = h^0(K) - 1$, concluding the proof.

It remains to show $h^0(D) = 2$. Since x is a degree 1 divisor, we see that $\mathscr{O}_{\mathbb{P}^1}(x) \cong \mathscr{O}_{\mathbb{P}^1}(1)$. Then $\mathscr{O}_C(D) \cong \pi^* \mathscr{O}_{\mathbb{P}^1}(1)$. Since π is a submersion, we have

$$H^0(C, \mathscr{O}_C(D)) = H^0(C, \pi^* \mathscr{O}_{\mathbb{P}^1}(1)) = H^0(\mathbb{P}^1, \pi_* \pi^* \mathscr{O}_{\mathbb{P}^1}(1)).$$

By projection formula, we have

$$\pi_* \pi^* \mathscr{O}_{\mathbb{P}^1}(1) \cong \mathscr{O}_{\mathbb{P}^1}(1) \otimes \pi_* \mathscr{O}_C.$$

By standard classification of double covers, $\pi_* \mathscr{O}_C \cong \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{L}$ where $\mathscr{L}$ is a line bundle such that $\mathscr{L}^2 \cong \mathscr{O}_{\mathbb{P}^1}(-B)$ where B is the branch divisor. By Riemann Hurwitz, $\chi(C) = 2\chi(\mathbb{P}^1) - \deg B$, hence $\deg B = 2g - 2 + 2\chi(\mathbb{P}^1) = 2g + 2$. Hence we see that $\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^1}(-g-1)$. Therefore,

$$\pi_* \pi^* \mathscr{O}_{\mathbb{P}^1}(1) \cong \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(-g).$$

Hence

$$h^0(D) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)) + h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-g)) = 2 + 0 = 2,$$

as desired. □

Definition 2.2. The image of the canonical morphism $\phi_{K_C} : C \to \mathbb{P}^{g-1}$ of a nonhyperelliptic curve of genus $g > 2$ is called the **canonical curve**.

Proposition 2.3. *Let $\phi : C \to \mathbb{P}^{g-1}$ be the canonical morphism. If ϕ is an embedding, then the degree of the image variety $X = \phi(C)$ is $\deg K_C = 2g - 2$.*

Proof. Let $\omega = c_1(\mathscr{O}_{\mathbb{P}^{g-1}}(1)) \in H^2(\mathbb{P}^{g-1}; \mathbb{Z})$ denote the Chern class of the hyperplane class in $\mathbb{P}^{g-1}$. Then the degree of the curve X is the intersection number $X \cdot H$ where H is a hyperplane. Therefore,

$$\deg X = \int_X \omega = \int_{\phi(C)} \omega = \langle \omega, [\phi(C)] \rangle.$$

Here, the braket is the natural pairing between homology and cohomology. Now the definition of degree of ϕ is that $\phi_*[C] = \deg \phi \cdot [\phi(C)]$. Hence we get that

$$\langle \omega, \phi_*[C] \rangle = \deg \phi \cdot \deg X.$$

Using the fact that $K_C \cong \phi^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)$, we see that after taking Chern classes we have $c_1(K_C) = \phi^* c_1(\mathscr{O}_{\mathbb{P}^{g-1}}(1)) = \phi^* \omega$. Using the property that pullback and pushforward are dual to each other, we have

$$\deg \phi \cdot \deg X = \langle \omega, \phi_*[C] \rangle = \langle \phi^* \omega, [C] \rangle = \int_C \phi^* \omega = \int_C c_1(K_C) = \deg K_C = 2g - 2.$$

Now $\phi : C \to \mathbb{P}^{g-1}$ is an embedding by assumption. Hence we see that $\deg X = \deg K_C = 2g - 2$, as claimed. □

Now we study a geometric version of Riemann-Roch theorem. Suppose $\phi : C \to \mathbb{P}^{g-1}$ is the canonical embedding. Let D be an effective divisor on C. We define

$$\overline{\phi(D)} := \bigcap_{D \subset \phi^{-1}(H)} H,$$

where H runs over all hyperplane in $\mathbb{P}^{g-1}$ containing D. Recall that we have $K_C \cong \phi^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)$. Hence, $D \subset \phi^{-1}(H)$ is equivalent to saying that the section σ_H of $\mathscr{O}_{\mathbb{P}^{g-1}}(1)$ corresponding to the hyperplane H satisfies the condition that $\phi^* \sigma_H$ vanishes on D. Therefore, there is a bijective correspondence

$$H^0(K_C - D) \stackrel{1:1}{\longleftrightarrow} \{H \in \mathscr{O}_{\mathbb{P}^{g-1}}(1) : D \subset \phi^{-1}(H)\}.$$

Now if we pick a basis $H_1, ..., H_\ell$ on the right hand side, we see that

$$\overline{\phi(D)} = H_1 \cap \cdots \cap H_\ell.$$

Therefore, we conclude that

$$h^0(K_C - D) = \operatorname{codim} \overline{\phi(D)} = (g-1) - \dim \overline{\phi(D)}.$$

Applying the Riemann-Roch theorem we get the geometric Riemann-Roch theorem:

Theorem 2.4 (Geometric Riemann-Roch)**.** *If D is an effective divisor on a smooth curve C of genus $g \geq 2$, then the dimension $r(D) = h^0(D) - 1$ of the linear system $|D|$ is given by*

$$r(D) = d - 1 - \dim \overline{\phi(D)}.$$

Now let us interpret the expression on the right hand side in a more geometric manner. Suppose $D = \sum_{i=1}^d p_i$. Then these points on C under the canonical map become points $\phi(p_1), ..., \phi(p_d)$ on the canonical curve, which lives inside $\mathbb{P}^{g-1} \cong \mathbb{P}H^0(K_C)^*$. Hence, each point $\phi(p_i)$ corresponds to a (projective) line in the vector space $H^0(K_C)^*$. What is the dimension that these line span in $H^0(K_C)^*$? Since there are d lines, the naive answer is d. However, there might be some linear relations among these lines, i.e., they may not be linearly independent. The number of linear relations r that this system have should therefore be the naive dimension d minus the actual dimension, i.e.,

$$r = d - \dim \operatorname{span}\{\phi(p_1), ..., \phi(p_d)\}.$$

But note that the span of $\{\phi(p_1), ..., \phi(p_d)\}$ is precisely the intersection of all hyperplanes in $H^0(K_C)^*$ containing them, hence

$$\dim \operatorname{span}\{\phi(p_1), ..., \phi(p_d)\} = \dim \overline{\phi(D)} + 1.$$

That is, the right hand side $d - 1 - \dim \overline{\phi(D)}$ of the geometric Riemann-Roch is precisely the number of linear relations among the points of the divisor on the canonical curve. To summarize:

Corollary 2.5 (See Figure 1)**.** *For an effective divisor D on C with genus $g \geq 2$, the dimension of the linear system $|D|$ is precisely the number of linear relations among the image points of D on the canonical image of C.*

Now we conclude with a study of the canonical morphism of the hyperelliptic curve. We first note that:

Lemma 2.6. *Let C be a hyperelliptic curve of genus $g \geq 2$, with hyperelliptic map $\pi : C \to \mathbb{P}^1$. Then*

$$K_C \cong \pi^* \mathscr{O}_{\mathbb{P}^1}(g-1).$$

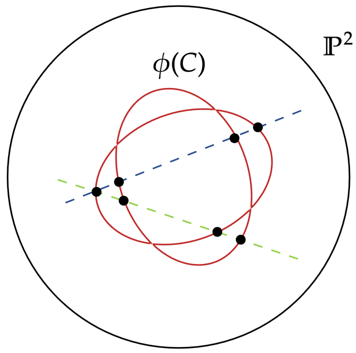

Figure 1: As an illustration of Corollary 2.5, this picture shows the canonical image of a genus 3 curve in $\mathbb{P}^2$, which is a plane quartic. The divisor D on C has 7 points, and so is the image divisor $\phi(D)$ on $\phi(C)$. There are two linear relations among these image points, hence the dimension of the linear system $|D|$ is 2.

Proof. By Riemann-Hurwitz, we have $K_C \cong \pi^* K_{\mathbb{P}^1} \otimes \mathscr{O}_C(R)$. Taking degree gives $\deg R = 2g+2$. Since every branch point $p \in B$ is simple for a double cover, the branch divisor B has degree $\deg B = 2g+2$ and satisfies $\pi^* B = 2R$. Thus $\pi^* \mathscr{O}_{\mathbb{P}^1}(B) = \pi^* \mathscr{O}_{\mathbb{P}^1}(2g+2)$, which implies $\mathscr{O}_C(R) = \pi^* \mathscr{O}_{\mathbb{P}^1}(g+1)$. Hence Riemann-Hurwitz gives

$$K_C \cong \pi^* \mathscr{O}_{\mathbb{P}^1}(-2) \otimes \pi^* \mathscr{O}_{\mathbb{P}^1}(g+1) = \pi^* \mathscr{O}_{\mathbb{P}^1}(g-1)$$

as desired. $\square$

Proposition 2.7. *Let C be a hyperelliptic curve of genus $g \geq 2$, with hyperelliptic map $\pi : C \to \mathbb{P}^1$. The canonical morphism $\phi_{K_C} : C \to \mathbb{P}^{g-1}$ is given by the composition*

$$\phi_{K_C} : C \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\nu} \mathbb{P}^{g-1},$$

where $\nu : [x : y] \mapsto [x^{g-1} : x^{g-2}y : \cdots : y^{g-1}]$ is the Veronese embedding.

Proof. We already know from the previous lemma that $K_C \cong \pi^* \mathscr{O}_{\mathbb{P}^1}(g-1)$. Thus a basis of $H^0(K_C)$ is given by pulling back degree $(g-1)$ monomials on $\mathbb{P}^1$ under the map π. Thus if x, y are homogeneous coordinates on $\mathbb{P}^1$, we know that the canonical map is given by

$$\phi(p) = [x(p)^{g-1} : x(p)^{g-2} y(p) : \cdots : y(p)^{g-1}]$$

where $x(p), y(p)$ are coordinates of $\pi(p)$ in $\mathbb{P}^1$. $\square$

2.2 Clifford's Theorem

The Riemann-Roch theorem reads $h^0(L) = h^0(K-L) + d + 1 - g$. Hence gives an upper bound $r(L) \geq d - g$. We would like a lower bound for $r(L)$ as well.

The first thing to note is that if $d = \deg L > 2g-2$, then $h^0(K-L) = 0$. Hence the Riemann-Roch inequality becomes an equality. Hence it is enough to study the case where $d \leq 2g-2$. This is

Theorem 2.8 (Clifford's Theorem)**.** *Let C be a curve of genus g and L a line bundle of degree $0 \leq d \leq 2g-2$. Then*

$$r(L) \leq \frac{d}{2}.$$

Moreover, when the equality is achieved, one of the following must be true:

- *$d = 0$ and $L = \mathscr{O}_C$;*
- *$d = 2g - 2$ and $L = K_C$;*
- *C is hyperelliptic.*

Proof. We will omit the proof of the claim when the equality is achieved.

First consider the case where L is nonspecial, i.e., when $h^1(L) = 0$. Then since $g \geq \frac{d}{2} + 1$, Riemann-Roch gives

$$r(L) = d - g \leq d - \frac{d}{2} - 1 = \frac{d}{2} - 1 < \frac{d}{2}.$$

Hence let us assume that L is special, i.e., $h^1(L) > 0$. Then by Riemann-Roch we have

$$r(K_C - L) = r(L) + g - d - 1.$$

If L does not contain any effective divisor, then $h^0(L) = 0$, then $r(L) = -1$. Since $d = \deg L \geq 0$, the Clifford equality holds trivially. Hence we may assume L contains effective divisors. If $K_C - L$ does not contain effective divisors, by Serre duality $h^0(K_C - L) = h^1(L) = 0$, hence L is nonspecial, which we have already dealt with. Hence we may assume $K_C - L$ contains effective divisors as well. Now applying Proposition 1.8 to L and $K_C - L$ gives

$$g = r(K_C) + 1 \geq r(L) + r(K_C - L) + 1 = 2r(L) + g - d$$

by Riemann-Roch. Hence we get $r(L) \leq d/2$, concluding the proof. $\square$

Remark 2.9. The assumption $d \geq 0$ is necessary. A counterexample can be given otherwise: Let C be a curve of genus 2 and let $p \in C$ be a point. Consider the line bundle $L = \mathscr{O}_C(-4p)$. Then $d = -4 \leq 2g - 2 = 2$. Since $d < 0$ we have $h^0(L) = 0$, so $r(L) = -1$, but $d/2 = -2$, so we get $-1 \leq -2$, which is a contradiction.

3 A Quick Guide to Abel-Jacobi Theory

We quickly review how we define the Jacobian $J(C)$ of a curve C. The exponential sequence give rise to a long exact sequence

$$\cdots \to H^0(\mathscr{O}) \to H^0(\mathscr{O}^*) \to H^1(C;\mathbb{Z}) \to H^1(\mathscr{O}) \to H^1(\mathscr{O}^*) \to H^2(C;\mathbb{Z}) \to H^2(\mathscr{O}) \to \cdots$$

We have $H^0(\mathscr{O}) = \mathbb{C}, H^0(\mathscr{O}^*) = \mathbb{C}^*, H^1(C;\mathbb{Z}) = \mathbb{Z}^{2g}$, and $H^2(\mathscr{O}) = 0$. The map $H^0(\mathscr{O}) \to H^0(\mathscr{O}^*)$ is given by the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$, which is surjective. Hence exactness implies $H^0(\mathscr{O}^*) \to H^1(C;\mathbb{Z})$ is the zero map, hence $H^1(C;\mathbb{Z}) \to H^1(\mathscr{O})$ is injective. Exactness also implies $H^1(\mathscr{O}^*) \to H^2(C;\mathbb{Z})$ is surjective. By definition,

$$\begin{aligned} J(C) := \operatorname{Pic}^0(C) &= \ker(H^1(\mathscr{O}^*) \to H^2(C;\mathbb{Z})) \\ &= \operatorname{im}(H^1(\mathscr{O}) \to H^1(\mathscr{O}^*)) \\ &\cong H^1(\mathscr{O})/\ker(H^1(\mathscr{O}) \to H^1(\mathscr{O}^*)) \\ &= H^1(\mathscr{O})/\operatorname{im}(H^1(C;\mathbb{Z}) \to H^1(\mathscr{O})) \\ &\cong H^1(\mathscr{O})/H^1(C;\mathbb{Z}). \end{aligned}$$

By Serre duality we have $H^1(\mathscr{O}) \cong H^0(K_C)^*$. Hence we can identify the Jacobian as

$$J(C) = H^0(K_C)^*/H_1(C;\mathbb{Z}).$$

We may view $H_1(C;\mathbb{Z})$ naturally as a subspace of $H^0(K_C)^*$, since for every cycle $\gamma \in H_1(C;\mathbb{Z})$, integration along the cycle $\int_\gamma$ defines a linear functional on $H^0(K_C)$. Now since $H^0(K_C)^* \cong \mathbb{C}^g$ and $H_1(C;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, we see that

$$J(C) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$$

is a complex torus of dimension g. Also, by definition, $J(C) = \mathrm{Pic}^0(C)$ is the space of degree 0 line bundles on C.

We can think of $q-p$ as a divisor on C of degree 0. Then the map $q-p \mapsto \int_p^q$ extends to a well-defined map μ_0 from divisors of degree 0 to $J(C)$, because if p', q' are two more points then

$$\int_p^q + \int_{p'}^{q'} - \left(\int_p^{q'} + \int_{p'}^{q} \right)$$

is the integral around a closed loop $p \to q \to p' \to q' \to p$.

The point of identifying $J(C)$ as a complex torus is that now we can define a holomorphic map, called the **Abel-Jacobi map**, by

$$\mu : C \to J(C) \qquad q \mapsto \int_p^q$$

where $p \in C$ is a fixed basepoint. Note that this is well-defined, i.e., independent of the choice of path from p to q, since the difference of integrals along any two such path is in the image of $H^1(C;\mathbb{Z})$. More generally, we have maps

$$\mu_d : C^{(d)} \to J(C) \qquad (q_1, \ldots, q_d) \mapsto \sum_{i=1}^{d} \int_p^{q_i}.$$

If we instead want to view $J(C)$ as the space of degree zero line bundles on C, then the maps are given by

$$\begin{aligned} &\mu : q \mapsto \mathscr{O}(q-p) \\ &\mu_d : (q_1, \ldots, q_d) \mapsto \mathscr{O}(q_1 + \cdots + q_d - dp). \end{aligned}$$

Theorem 3.1. *If $g \geq 1$, then the Abel Jacobi map $\mu : C \to J(C)$ given by $\mu : q \mapsto \mathscr{O}(q-p)$, where $p \in C$ is a fixed basepoint, is an embedding.*

Proof. Since C is compact it suffices to show that μ is an injective immersion. Injectivity is easy: Suppose $\mu(q) = \mu(q')$ for two distinct points q, q' on C. Then $\mathscr{O}(q-p) \cong \mathscr{O}(q'-p)$, which implies $\mathscr{O}(q-q') \cong \mathscr{O}$. Thus $q \sim q'$, which means there exists a meromorphic function f on C such that $(f) = q - q'$. But then f defines an isomorphism $C \to \mathbb{P}^1$, which is impossible since $g \geq 1$ by assumption.

Next we need to show that the differential $\mu_* : T_qC \to T_{\mu(q)}J(C)$ is injective at an arbitrary point $q \in C$. This time, it is more covinient for us to go back to the integral definition, namely we identify $\mu(q) = \int_p^q$, and $J(C) = H^0(K_C)^*/H^1(C;\mathbb{Z})$. Since $H^0(K_C)^*$ is a vector space and $J(C)$ is this vector space quotient out by the lattice $H^1(C;\mathbb{Z})$, we see that $T_{\mu(q)}J(C) \cong T_{\mu(q)}H^0(K_C)^* \cong H^0(K_C)^*$. Now the map μ_*, schematically, is given by

$$\mu_* : T_qC \to H^0(K_C)^* \qquad \frac{\partial}{\partial q} \mapsto \frac{\partial}{\partial q} \int_p^q.$$

Dually, the induced map on the cotangent space is given by

$$\mu^* : H^0(K_C)^{**} \cong H^0(K_C) \to T_q^*C \qquad \omega \mapsto \omega\left(\frac{\partial}{\partial q} \int_p^q \right) := \frac{\partial}{\partial q} \int_p^q \omega = \mathrm{ev}_q\, \omega,$$

where the last equality is given by the fundamental theorem of calculus, and the first equality is by the very construction of how we identify $H^0(K_C)^{**}$ with $H^0(K_C)$.

Now we note that K_C is basepoint free. Indeed, for any $q \in C$, we must have $h^0(q) = 1$, otherwise we have a meromorphic function with a pole at q, giving an isomorphism $C \cong \mathbb{P}^1$. Therefore, Riemann-Roch gives

$$h^0(K_C - q) = h^0(q) + (2g-3) + 1 - g = g - 1 = h^0(K_C) - 1.$$

Hence K_C is basepoint free by Theorem 1.9. Therefore, the evaluation map $\mathrm{ev}_q : H^0(K_C) \to T_q^* C$ is surjective. Hence $\mu_* : T_q C \to T_{\mu(q)} J(C)$ is injective, concluding the proof. □

4 Classification of Curves of Low Genus

In this section we classify curves with low genus, more specifically $0 \leq g \leq 5$. We mostly focus on smooth curves. The cases $g = 0, 1$ are very easy:

4.1 $g = 0, 1$: Projective Line and Complex Torus

Theorem 4.1. *Every smooth curve of genus 0 is isomorphic to $\mathbb{P}^1$.*

Proof. By Riemann-Roch, we have $h^0(p) = h^0(K_C - p) + 1 + 1 - 0 = 2$, since $\deg(K_C - p) < 0$, i.e., $h^0(K_C - p) = 0$. Thus there exists a meromorphic function f on C with exactly one pole, giving an isomorphism $f : C \to \mathbb{P}^1$. □

Theorem 4.2. *Every smooth curve of genus 1 is a complex torus. More precisely, the Abel-Jacobi map $\mu : C \to J(C)$ is an isomorphism.*

Proof. By Theorem 3.1, the Abel-Jacobi map $\mu : C \to J(C)$ is an embedding. Hence it is an isomorphism because both the domain and codomain are connected one dimensional complex manifolds and μ is holomorphic. □

Remark 4.3. In fact, every genus 1 curve is hyperelliptic. To see this, let $p, q \in C$ by any points. Riemann-Roch immediately gives $h^0(p + q) = 2$. Therefore, we have a meromorphic function with poles along p and q (note that it cannot have only one pole otherwise $C \cong \mathbb{P}^1$). Thus we conclude that C admits a 2:1 cover to $\mathbb{P}^1$.

4.2 $g = 2$: Hyperelliptic Curves

Theorem 4.4. *Every genus 2 curve is hyperelliptic and the canonical morphism $\phi_K : C \to \mathbb{P}^1$ is a 2:1 cover, i.e., the canonical morphism is a hyperelliptic map.*

Proof. By Riemann-Roch, we have $h^0(p+q) = h^0(K - p - q) + 2 + 1 - 2 = h^0(K - p - q) + 1 = 2$, since $\deg(K - p - q) = 0$ so $h^0(K - p - q) = h^0(\mathscr{O}) = 1$. Since $h^0(p + q) = 2$, there exists some meromorphic function with poles at p or q, or both p and q. But again we cannot have a meromorphic function with just one pole, otherwise $C \cong \mathbb{P}^1$. Thus we conclude that there exists a meromorphic function f on C with two poles, which gives a 2:1 map $f : C \to \mathbb{P}^1$, showing that C is hyperelliptic.

It remains to compute the degree of the canonical map $\phi : C \to \mathbb{P}^1$. Take a volume form ω of $\mathbb{P}^1$, and WLOG we may assume ω is a representative of $c_1(\mathscr{O}_{\mathbb{P}^1}(1))$. Since $K \cong \phi^* \mathscr{O}_{\mathbb{P}^1}(1)$, taking Chern class on both sides gives $c_1(K) = \phi^* c_1(\mathscr{O}_{\mathbb{P}^1}(1)) = \phi^* \omega$. Therefore,

$$\deg \phi = \int_C \phi^* \omega = \int_C c_1(K) = \deg K = 2g - 2 = 2.$$

This concludes the proof. □

Hence, every curve of genus 2 is hyperelliptic. Although the goal of this section primarily concentrates on genus 2 curve, but let us divert a little bit and study hyperelliptic curve of any genus $g \geq 2$. The next result shows that we have a very simple description on the vector space $H^0(C, K_C)$.

Proposition 4.5. *We can always write the affine part of a hyperelliptic curve C as the equation $y^2 = \prod_{i=1}^{2g+2}(x-x_i)$ in $\mathbb{C}^2$. Moreover, the differential form $\omega_0 = \mathrm{d}x/y$ is nonvanishing on the affine part of C. Then a basis of $H^0(C, K_C)$ can be given explicitly by*

$$H^0(C, K_C) = \langle \omega_0, x\omega_0, ..., x^{g-1}\omega_0 \rangle.$$

Proof. A hyperelliptic curve, by definition, is the one that admits a 2:1 branched cover $\pi : C \to \mathbb{P}^1$. Using the Riemann-Hurwitz formula $\chi(C) = 2\chi(\mathbb{P}^1) - \sum_p (e_p - 1)$, and using that $e_p - 1 = 1$ for all ramified point since the cover is degree 2, we see that the number of ramification points is $2g+2$. Call these points $x_1, ..., x_{2g+2}$.

Let B denote the divisor of branched points in $\mathbb{P}^1$. If $\tilde{\pi} : \tilde{C} \to \mathbb{P}^1$ is another 2:1 branched cover with the same branched locus B, then we have an isomorphism $\pi^{-1}(\mathbb{P}^1 \setminus B) \cong \tilde{\pi}^{-1}(\mathbb{P}^1 \setminus B)$. And this isomorphism can be extended by an isomorphism $\tilde{C} \cong C$. Hence we conclude that any two degree 2 branched cover of $\mathbb{P}^1$ with the same branched divisor are isomorphic.

Now here is one such cover: Let x, y be coordinates in $\mathbb{C}^2$ and consider the curve S

$$y^2 = \prod_{i=1}^{2g+2} (x - x_i).$$

with projection $\pi : S \to \mathbb{C}$ to the x coordinate. Clearly this is a degree 2 branched cover, branched exactly at $x_1, ..., x_{2g+2}$. Now by compactifying them, we can construct a hyperelliptic curve $\pi : C \to \mathbb{P}^1$, and we showed that this is the unique hyperelliptic curve branched at $x_1, ..., x_{2g+2}$.

Now we find a basis for $H^0(C, K_C)$. Let $j : (x, y) \mapsto (x, -y)$ be the automorphism on C given by flipping two sheets of the cover. Then j^* acts on forms so we have a map $j^* : H^0(C, K_C) \to H^0(C, K_C)$. Since $j^2 = 1$, we know $H^0(C, K_C)$ decomposes into the (± 1)-eigenspace of j^*. In fact the (+1)-eigenspace is trivial, because any form $\omega \in H^0(C, K_C)$ with $j^*\omega = \omega$ descends to a (1,0)-form on $\mathbb{P}^1$, and $H^{1,0}(\mathbb{P}^1) = 0$. Hence we conclude that $j^*\omega = -\omega$ for all holomorphic 1-forms on C.

Now consider the form

$$\omega_0 = \frac{\mathrm{d}x}{y}$$

on C. We claim that ω_0 is nowhere vanishing on C and holomorphic. Indeed, away from branched points $y \neq 0$, here $\pi : C \to \mathbb{P}^1$ is unramified, so the projection x is a valid local coordinate, and $\omega_0 = \mathrm{d}x/y$ is clearly nonvanishing and holomorphic. Near each branched point x_i, we note that since $f(x) := \prod_{i=1}^{2g+2}(x-x_i)$ satisfies $f'(x) = \sum_{i=1}^{2g+2}\prod_{j\neq i}(x-x_i)$, we have $f'(x_i) \neq 0$ since $x_1, ..., x_{2g+2}$ are distinct. Now the equation $y^2 = f(x)$ defining C has differential $2y\mathrm{d}y = f'(x)\mathrm{d}x$, so $\mathrm{d}x = \frac{2y}{f'(x)}\mathrm{d}y$. Thus near x_i we have

$$\omega_0 = \frac{\mathrm{d}x}{y} = \frac{\frac{2y}{f'(x)}\mathrm{d}y}{y} = \frac{2}{f'(x)}\mathrm{d}y$$

is holomorphic and nonzero. Hence we conclude that ω_0 is holomorphic and has no zeros anywhere on finite part of C.

Now the space $\mathscr{K}^1(C)$ of meromorphic 1-forms on C is 1-dimensional over the field $\mathscr{K}(C)$ of meromorphic functions on C, which can be checked locally since $\mathscr{K}^1(C)|_U \cong \mathscr{K}(C)|_U \otimes \Omega^1(C)|_U$. Hence any holomorphic 1-form $\omega \in H^0(C, K_C)$ is given by

$$\omega = h\omega_0$$

where h is a meromorphic function on C. Moreover, $h = \omega/\omega_0$ cannot have any poles on finite part of C, since ω is holomorphic and ω_0 is nonvanishing on finite parts of C.

We argue that h must then be a polynomial. Recall that for $\omega \in H^0(C, K_C)$ we must have $j^*\omega = -\omega$. But note that $j^*\omega_0 = -\omega_0$. Hence $j^*h = h$. This implies that h must be a function of x alone, not y. Hence we see that $h(x)$ is a meromorphic function on $\mathbb{P}^1$, and is holomorphic everywhere on finite part of C, i.e., $h(x)$ is a holomorphic function on $\mathbb{C} \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. By standard complex analysis, an entire function with at most a pole at ∞ must be a polynomial.

So let d be the degree of h. We argue that $\deg h \le g-1$. We note that if h is a polynomial of degree d, then h has $2d$ zeros on finite parts of C, so by residue theorem it must have a pole of order d at each of the points over ∞.

Now we look at $\omega_0 = \mathrm{d}x/y$. We claim that ω_0 has zeros of order $g-1$ at ∞. Indeed, let $t = 1/x$ be local coordinate on $\mathbb{P}^1$ near ∞. Then $x = 1/t$, so $\mathrm{d}x = -\frac{1}{t^2}\mathrm{d}t$. Next, we see how y behaves at ∞: We have

$$y^2 = \prod_{i=1}^{2g+2}\left(\frac{1}{t} - z_i\right) = \frac{1}{t^{2g+2}}\prod_{i=1}^{2g+2}(1 - z_i t) \sim \frac{1}{t^{2g+2}}$$

as $t \to 0$. Hence $y \sim \frac{1}{t^{g+1}}$ as $t \to 0$. Thus

$$\omega_0 = \frac{\mathrm{d}x}{y} \sim \frac{-\frac{1}{t^2}\mathrm{d}t}{\frac{1}{t^{g+1}}} = -t^{g-1}\mathrm{d}t.$$

Thus

$$\omega = h\omega_0 \sim \left(\frac{1}{t^d}\right)(t^{g-1}\mathrm{d}t) = t^{(g-1)-d}\mathrm{d}t.$$

Hence for ω to be holomorphic, we must have $d \le g-1$, as claimed. This shows that any element in $H^0(C, K_C)$ is spanned by $\omega_0, x\omega_0, \ldots, x^{g-1}\omega_0$, as desired. $\square$

4.3 $g = 3$: Generically Quartic

By Theorem 2.1, assuming that C is not hyperelliptic, then the canonical morphism $\phi : C \to \mathbb{P}^2$ embeds C as a smooth plane curve of degree 4 (Proposition 2.3). In fact, a generic genus 3 curve is not hyperelliptic, as hyperelliptic curves form a codimension 1 subspace in the moduli space M_3 of genus 3 curves.

4.4 $g = 4$: Intersection of Quadric and Cubic

Let C be a nonhyperelliptic curve of genus 4. The canonical map $\phi : C \to \mathbb{P}^3$ is an embedding with image being a degree 6 curve in $\mathbb{P}^3$, and we identify C with its image.

Theorem 4.6. *The canonical model of a nonhyperelliptic curve of genus 4 is a complete intersection of a quadric Q and a cubic S in $\mathbb{P}^3$. Conversely, any smooth curve that is the complete intersection of a quadric and cubic in $\mathbb{P}^3$ is the canonical model of a nonhyperelliptic curve of genus 4.*

Proof. We consider the restriction maps

$$\rho_m : H^0(\mathscr{O}_{\mathbb{P}^3}(m)) \to H^0(\mathscr{O}_C(m)) = H^0(mK_C).$$

Here, we used the fact that $K_C \cong \phi^*\mathscr{O}_{\mathbb{P}^3}(1)$, hence $K_C(m) = \phi^*\mathscr{O}_{\mathbb{P}^3}(m)$. Since we are identifying C with $\phi(C)$, we can identify $\phi^*\mathscr{O}_{\mathbb{P}^3} \cong \mathscr{O}_C$. In particular, let us consider the case where $m = 2$:

$$\rho_2 : H^0(\mathscr{O}_{\mathbb{P}^3}(2)) \to H^0(\mathscr{O}_C(2)).$$

We have $h^0(\mathscr{O}_{\mathbb{P}^3}(2)) = \binom{5}{3} = 10$ and $h^0(\mathscr{O}_C(2)) = h^0(2K_C) = 9$ by Riemann-Roch. Hence the restriction map of quadrics in $\mathbb{P}^3$ to C has at least 1-dimensional kernel. That is, the curve $C \subset \mathbb{P}^3$ must lie on at least one quadric surface $Q \subset \mathbb{P}^3$. The quadric must be irreducible, since any reducible quadric must be a union of planes, which cannot contain the irreducible degenerate curve C. If $Q' \subset \mathbb{P}^3$ is another quadric then, by Bézout's theorem, $Q \cap Q'$ is a curve of degree 4 which cannot contain C. We conclude that Q is unique, and ρ_2 is thus surjective.

Now consider the restriction map

$$\rho_3 : H^0(\mathscr{O}_{\mathbb{P}^3}(3)) \to H^0(\mathscr{O}_C(3)) = H^0(3K_C).$$

We have $h^0(\mathscr{O}_{\mathbb{P}^3}(3)) = \binom{6}{3} = 20$, and by Riemann-Roch $h^0(\mathscr{O}_C(3)) = h^0(3K_C) = 15$. This shows that $\dim\ker\rho_3 \geq 5$. The space of cubic surface containing the above quadric Q can be constructed by multiplying the equation of Q with a linear polynomial, hence is 4-dimensional. This shows that there exists a cubic S containing C that does not contain Q. By Bézout's theorem, $Q \cap S$ has degree 6, hence must be equal to C.

Conversely, let $C = Q \cap S$ be the intersection of a quadric Q and a cubic S. By adjunction formula

$$K_Q = K_{\mathbb{P}^3} + Q|_Q = \mathscr{O}_{\mathbb{P}^3}(-4) \otimes \mathscr{O}_{\mathbb{P}^3}(2)|_C = \mathscr{O}_Q(-2).$$

The curve C is the intersection of Q with the cubic surface S. Because S has degree 3, the divisor C on Q is cut out by a section of $\mathscr{O}_Q(3)$. We apply the adjunction formula a second time, now for $C \subset Q$:

$$K_C = K_Q + C|_Q = \mathscr{O}_Q(-2) \otimes \mathscr{O}_Q(3)|_C = \mathscr{O}_C(1).$$

This shows that the embedding $\phi : C \to \mathbb{P}^3$ satisfies $K_C = \phi^*\mathscr{O}_{\mathbb{P}^3}(1)$. Therefore C is a canonical curve. □

Now that we have classified nonhyperelliptic curve of genus 4, an interesting question to ask is whether a nonhyperelliptic curve of genus 4 is **trigonal**, that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$? We introduce the following terminology:

Definition 4.7. Let C be a curve. A divisor D with $h^0(D) - 1 = r$ and $\deg D = d$ on C is said to be a g^r_d. Equivalently, this is a degree d map $C \to \mathbb{P}^r$. When such a divisor (map) exists, we say that C has a g^r_d.

Hence the question we are asking is: whether a nonhyperelliptic curve of genus 4 has a g^1_3? How many g^1_3's does it have? This is answered by the following

Proposition 4.8 (See Fig 2)**.** *A nonhyperelliptic curve $C = Q \cap S$ of genus 4 may be expressed as a 3-sheeted cover of $\mathbb{P}^1$ in either one or two ways, depending on whether the unique quadric Q containing the canonical model of the curve is singular or smooth. If Q is smooth, then C has two g^1_3's, otherwise it has a unique g^1_3.*

Proof. Finding g^1_3's on C is equivalent to finding divisors D on C such that $r(D) = 1$ and $\deg D = 3$. Our first claim is that there is a correspondence between g^1_3's on C with $\mathbb{P}^1$-rulings on the quadric Q containing C.

First, suppose $\pi : C \to \mathbb{P}^1$ is a g^1_3. Then we can take D to be a general fiber of the map π, which has the form $D = p + q + r$ for three distinct points. By the geometric Riemann-Roch theorem (Theorem 2.4), we have $1 = r(D) = 3 - 1 - \dim\operatorname{span}\{p, q, r\}$. Hence $p, q, r \in C$ are collinear. If three points $p, q, r \in C \subset Q$ lies on a line $L \subset \mathbb{P}^3$, then Q and L meet in at least three points, hence restricting the degree 2 equation of Q on L shows that L must be contained in Q. Hence we have showed that the g^1_3 map is cut out by lines, and those lines must lie inside the quadric Q.

Conversely, if a line L is contained in Q, then the divisor $D = C \cap L = (Q \cap S \cap L) = (Q \cap L) \cap S = S \cap L$ on C has degree 3 by Bézout's theorem. Thus, if we can find a family lines completely contained in Q, it will sweep out a valid degree 3 divisor on C, thus giving us our g^1_3. This proves our claim.

Now any smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and contains two families of lines, or rulings. On the other hand, any quadric of rank 3 is a cone over a smooth plane conic (which is isomorphic to $\mathbb{P}^1$ by degree genus formula), thus has only one ruling. Applying the claim immediately concludes the proof. □

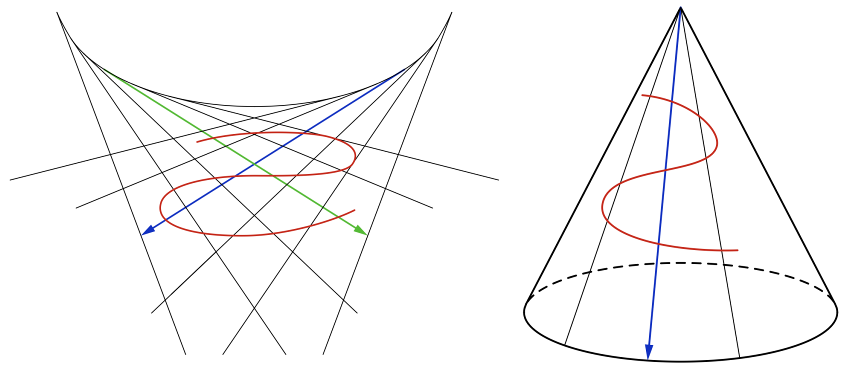

Figure 2: Left: canonical curve of genus 4 on a smooth quadric, showing two g^1_3's. Right: canonical curve of genus 4 on a quadric cone, with just one g^1_3.

4.5 $g = 5$: Generically Triple Intersection of Quadrics

For nonhyperelliptic genus 5 curve, the canonical map $\phi : C \to \mathbb{P}^4$ has degree 8 image. Hence C can be identified with a octic curve in $\mathbb{P}^4$. As in the genus 4 case, it is useful to consider the restriction map

$$\rho_2 : H^0(\mathscr{O}_{\mathbb{P}^4}(2)) \to H^0(\mathscr{O}_C(2)).$$

We have $h^0(\mathscr{O}_{\mathbb{P}^4}(2)) = \binom{6}{4} = 15$. By Riemann-Roch we have $h^0(\mathscr{O}_C(2)) = 12$. So $\dim\ker\rho_2 \geq 3$. We deduce that C lies on at least 3 independent quadrics. If the intersection of the three quadric is 1-dimensional, then by Bézout's theorem we have

$$C = Q_1 \cap Q_2 \cap Q_3$$

is a complete intersection of quadrics. This is the generic case. The discussion for detailed classification of genus 5 curve is a bit long, and in the generic case similar argument as in the genus 4 case can be used. We will merely present the result:

Theorem 4.9. *Let $C \subset \mathbb{P}^4$ be a canonical curve of genus 5. Then C lies on exactly three quadrics. Either:*

(a) C is the intersection of these quadrics, in which case C is not trigonal (i.e., has no g^1_3), but has a one-dimensional family of g^1_4's.

(b) The quadrics containing C intersect in a cubic surface that is a nonsingular rational normal scroll. In this case, C has a unique g^1_3.

References

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