SMOOTH MANIFOLDS AND RIEMANNIAN GEOMETRY: NOTES AND HOMEWORK

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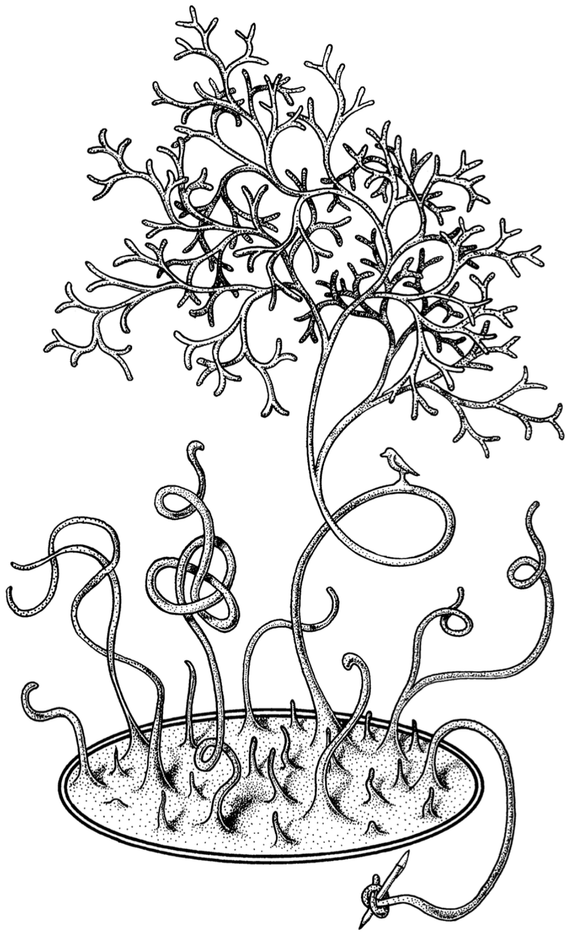

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Preface

This is a collection of notes for math 240B at UCSB, winter 2023, taught by Prof. Rugang Ye. The first part of this course will focus on tensor fields and differential forms, the references are

- John Lee, *Introduction to Smooth Manifolds*, [1]
- William Boothby, *An Introduction to Differential Manifolds and Riemannian Geometry*, [3]

For the second part of the course, we will study Riemannian geometry, and the textbook is:

- Manfredo Perdigão do Carmo, *Riemannian Geometry*, [2]

I also refered to the following book quite frequently:

- John Lee, *Introduction to Riemannian Manifolds*, [4]

Homework for the class are also contained inside the notes. Some homeworks may be omitted if they were already proven in the theorems or in main text. Sorry for the inconvienience.

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1.1 The Cotangent Bundle

Throughtout this note, we use M^n to denote a smooth manifold of dimension n. Recall that a ***derivation*** at a point $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ if for all $f, g \in C^\infty(M)$, we have

$$v(fg) = f(p)vg + g(p)vf.$$

Then the ***tangent space*** at p, denoted by T_pM, is the set of all derivations at p. This is a vector space that should be thought of as the geometric tangent plane of M. Now let (U, φ) be a smooth chart of M. Then the set of tangent vectors

$$\left.\frac{\partial}{\partial x^i}\right|_p = (\varphi_*)^{-1}_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) \qquad i = 1, ..., n$$

is a natural basis for T_pM. Here, $(\varphi_*)_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$ is the differential (or pushforward) of the diffeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$, and $\frac{\partial}{\partial x^i}|_{\varphi(p)}$ are ordinary directional derivative operators in Euclidean space. Expanding the expression above using nothing but definitions, one sees that for $f \in C^\infty(M)$

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}(f \circ \varphi^{-1})\right|_{\varphi(p)}.$$

A more detailed treatment of tangent vectors can be found in [1, Chapter 3].

Definition 1.1. The ***cotangent space*** T_p^*M is the dual of T_pM.

The following lemma immediately shows that $\dim T_p^*M = \dim T_pM$.

Lemma 1.2. *For a finite dimensional vector space V, we have* $\dim V^* = \dim V$.

Proof. Let $v_1, ..., v_n$ be a basis for V. Define $\varepsilon^1, ..., \varepsilon^n \in V^*$ by $\varepsilon^i(v_j) = \delta^i_j$. Now let $\ell \in V^*$. Then one can check that $\ell = \sum_i \ell(v_i)\varepsilon^i$. Moreover, this expression is unique. □

Now we want to find a natural basis for T_p^*M, using the natural basis $\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p$ of T_pM. We need the following definition

Definition 1.3. Let U be an open set of M and $p \in U$. Let $f \in C^\infty(U)$. The ***differential*** df_p of f at p is defined as follows: for any $v \in T_pM$,

$$df_p(v) = vf.$$

This is clearly a linear map, so $df_p \in T_p^*M$ for any $f \in C^\infty(U)$ where U is any open set containing p. Let $x^i : \mathbb{R}^n \to \mathbb{R}$ be the ith coordinate function, i.e., $x^i(\xi^1, ..., \xi^n) = \xi^i$. Let (U, φ) be a chart on M containing p. Then

Proposition 1.4. The covectors $d(x^1 \circ \varphi)_p, ..., d(x^n \circ \varphi)_p$ is a dual basis of T_p^*M to the basis $\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p$ of T_pM.

Proof. Note that we have

$$d(x^i \circ \varphi)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \frac{\partial}{\partial x^j}\left(x^i \circ \varphi \circ \varphi^{-1}\right)\Big|_{\varphi(p)} = \delta^i_j.$$

This concludes the proof. □

We usually commit the following crime of abusing notation by simply writting $d(x^i \circ \varphi)_p$ as dx^i_p. Then we refer to $dx^1_p, ..., dx^n_p$ as a basis of T_p^*M.

Recall that we can construct the ***tangent bundle*** TM of M^n by

$$TM = \coprod_{p\in M} T_pM.$$

This is a smooth manifold of dimension $2n$ in a canonical way. Let (U, φ) be a smooth chart of M, then define $\pi : TM \to M$ by $\pi(p, v) = p$. Then the charts on TM will be $\pi^{-1}(U) = \coprod_{p\in U} T_pM$. The induced chart $\tilde{\varphi}$ of TM is given by $\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$, where

$$\tilde{\varphi}(p, v) = (\varphi(p), v^1, ..., v^n)$$

where $v = v^i \frac{\partial}{\partial x^i}|_p$.

Homework 1.5. Let (U, φ) and (V, ψ) be two intersecting charts on M. Let $\tilde{\varphi}$ and $\tilde{\psi}$ be the corresponding induced charts on TM. Prove that $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth.

Solution. Based on the proof of [1, Prop 3.18], we can explicitly write out the transition map formula:

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x^1, ..., x^n, v^1, ..., v^n) = \left(y^1(x), ..., y^n(x), \frac{\partial y^1}{\partial x^j}v^j, ..., \frac{\partial y^n}{\partial x^j}v^j\right)$$

which is clearly smooth. □

Note that we can define another map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, given by simply

$$\Phi(p, v) = (p, v^1, ..., v^n).$$

Then note that Φ' is a diffeomorphism, and $\Phi'(\pi^{-1}(p)) = p \times \mathbb{R}^n$. One can check that if we define $\pi_U : U \times \mathbb{R}^n \to U$ to be the projection on U, then the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^n \arrow[d, "\pi_U"] \\
& U
\end{tikzcd}$$

Moreover, the map $\Phi := \Phi|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \mathbb{R}^n$ is a linear isomorphism. This map Φ is called a ***local trivialization***.

One can do similar constructions for to get the ***cotangent bundle***. This is defined by

$$T^*M = \coprod_{p\in M} T_p^*M.$$

Again, given a chart (U,φ), there is a natural chart on T^*M. One define the projection map $\pi : T^*M \to M$ similarly by

$$\pi(p,\alpha) = p \qquad \alpha \in T_p^*M,\ p \in U.$$

Then for every $\alpha \in T_p^*M$, one can write

$$\alpha = \xi_i dx^i|_p$$

for some real $\xi_1, ..., \xi_n$. Here we define $\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ by

$$\tilde{\varphi}(p,\alpha) = (\varphi(p), \xi_1, ..., \xi_n).$$

Then we have a natural chart $(\pi^{-1}(U), \tilde{\varphi})$ on T^*M. Similarly one can define the local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi(p,\alpha) = (p, \xi_1, ..., \xi_n).$$

Then once again one can verify that $\pi_U \circ \Phi = \pi$. The tangent bundle and cotangent bundle are examples of a vector bundle, which we will define in the next section.

1.2 Vector Bundles

Definition 1.6. A smooth ***vector bundle*** of rank k over a smooth manifold M^n is a triple (E, M, π), where E is the ***total space*** of the vector bundle (we often say that E is the vector bundle) which is a C^∞-manifold of dimension $n+k$, and $\pi : E \to M$ is a smooth surjective map, such that

(I) For each $p \in M$, the set $E_p := \pi^{-1}(p)$ is a vector space of dimension k.

(II) For each $p \in M$, there is an open set U of M containing p and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\pi = \pi_U \circ \Phi$, i.e., the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^n \arrow[d, "\pi_U"] \\ & U \end{tikzcd}$$

commutes, and

$$\Phi|_{E_p} : E_p \to p \times \mathbb{R}^k \cong \mathbb{R}^k$$

is a linear isomorphism for each $p \in U$.

As mentioned before, (U, Φ) is called a ***local trivialization***. This is because a smooth vector bundle E is locally diffeomorphic to the product $U \times \mathbb{R}^k$ via Φ. Therefore, a vector bundle is something that locally looks like a product. One can also define vector bundle (not necessarily smooth) in a topological sense, by replacing every word "smooth" in the definition above to "continuous", and "diffeomorphism" to "homeomorphism", "smooth manifold" to "topological manifold".

Example 1.7. Let M^n be a smooth manifold. Then $M \times \mathbb{R}^k$ is a vector bundle with global trivialization $\Phi : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ being the identity map.

Example 1.8 (Whitney Sum)**.** Let M be a smooth manifold and let $E \to M$ and $E' \to M$ be smooth vector bundles of rank k and k' respectively. The ***Whitney sum*** $E \oplus E' \to M$ is a new smooth vector bundle over M of rank $k + k'$, constructed as follows: The total space is defined by

$$E \oplus E' = \coprod_{p \in M} E_p \oplus E'_p.$$

There is an obvious projection $\pi : E \oplus E' \to M$. For each p, choose a neighborhood U of p small enough that there exists local trivializations (U, Φ) of E and (U, Φ') of E'. We then define $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^{k+k'}$ by

$$\Psi(v, v') = (\pi(v), (\pi_{\mathbb{R}^k} \circ \Phi(v), \pi_{\mathbb{R}^{k'}} \circ \Phi'(v'))).$$

This actually makes $E \oplus E'$ into a smooth vector bundle. See [1, Example 10.7] for details.

Example 1.9 (Dual Bundle)**.** Let $E \xrightarrow{\pi} M$ be a smooth vector bundle over M. Define the ***dual bundle*** to E to be the bundle $E^* \xrightarrow{\pi^*} M$ as follows: The total space is given by $E^* = \coprod_{p \in M} E_p^*$ with $\pi^*(E_p^*) = p$ the obvious projection. For each local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, we define $\Phi^* : (\pi^*)^{-1}(U) \to U \times (\mathbb{R}^k)^*$ by

$$\Phi^*(p, \omega) = (p, (\Phi|_{E_p}^{\top})^{-1}\omega).$$

Note that $\Phi|_{E_p} : E_p \to \mathbb{R}^k$ is a linear isomorphism, so $\Phi|_{E_p}^{\top} : (\mathbb{R}^k)^* \to E_p^*$ is also a linear isomorphism, hence we can take its inverse. Note that we can identify $\mathbb{R}^k$ with $(\mathbb{R}^k)^*$ by identifying the standard basis $e_1, ..., e_k$ of $\mathbb{R}^k$ with the dual basis $\varepsilon^1, ..., \varepsilon^k$ of the standard basis. Hence Φ^* is a local trivialization of E^*. More information can be found here and here.

Homework 1.10. Show that Φ^* is a diffeomorphism.

Solution. We check the conditions for Lemma 10.6 of Lee. Note that we have constructed the local trivializations Φ^*s by choosing a set of open covers U of M first. So (i) is satisfied. For (ii), note that $(\Phi^*)|_{E_p^*} = (\Phi|_{E_p}^{\top})^{-1}$, which is a linear isomorphism between vector spaces since $\Phi|_{E_p}$ is. So (ii) is satisfied. For (iii), Suppose (V, Ψ) is another local trivialization such that $U \cap V \neq \emptyset$. Then $(\Psi^*) \circ (\Phi^*)^{-1} : (U \cap V) \times (\mathbb{R}^k)^* \to (U \cap V) \times (\mathbb{R}^k)^*$ is given by

$$[(\Psi^*) \circ (\Phi^*)^{-1}](p, \eta) = \Psi^*(p, (\Phi|_{E_p}^{\top})\eta) = (p, (\Psi|_{E_p}^{\top})^{-1} \circ (\Phi|_{E_p}^{\top})\eta)$$

and note that $(\Psi|_{E_p}^{\top})^{-1} \circ (\Phi|_{E_p}^{\top})$ is a composition of linear isomorphisms that depends smoothly on p just because both $\Phi|_{E_p}$ and $\Psi|_{E_p}$ are according to Lemma 10.5 of Lee. Therefore (iii) is again satisfied. Hence by Lemma 10.6, E^* has a unique topology and smooth structure making it into a smooth manifold, with $\pi^* : E^* \to M$ as projection, and the set of (U, Φ^*)s as local trivializations. Hence in particular, Φ^* is a diffeomorphism. □

Example 1.11 (Trivial Line Bundle)**.** Let $\hat{\mathbb{R}}^n = \mathbb{R}^n - \{0\}$. Let $E \to \hat{\mathbb{R}}^n$ be the bundle defined by $E = \coprod_{p \in \hat{\mathbb{R}}^n} E_p$, where E_p is the line passing through p and the origin. Then for every $v \in E_p$, we have $v = \xi p$ for some real number ξ. Hence a global trivialization of E can be given by

$$\Phi(v) = \Phi(\xi p) = (p, \xi) \in \hat{\mathbb{R}}^n \times \mathbb{R}.$$

Example 1.12 (Tautological Bundle)**.** Let n be fixed and let $1 \leq k \leq n$. Then the ***Grassmannian*** $G(k, n)$ is defined by all k-dimensional subspace of $\mathbb{R}^n$. This is studied in detail in [1, Example 1.36]. The ***tautological bundle*** $E \to G(k, n)$ is defined as follows: The total space is given by the disjoint union of all pairs

$$E = \coprod_{V \in G(k,n), v \in V} (V, v)$$

with the projection map being $\pi(V, v) = V$.

Homework 1.13. Show that E is a smooth vector bundle.

Solution. Let $X \in G(k, n)$ be a point. Let U be the set of all $V \in G(n, k)$ such that the orthogonal projection p onto X maps V isomorphically to X. Then define the local trivialization $\Phi : \pi^{-1}(U) \to U \times X$ by $\Phi(V, v) = (V, p(v))$. Then one can check that conditions in Lemma 10.6 of Lee are satisfied. Hence E is a smooth vector bundle. □

1.3 Möbius Band and Fiber Bundle

The most intuitive definition of a ***Möbius band*** is defined as $M = ([0, 1] \times [-1, 1])/\sim$, where $(0, y) \sim (1, -y)$. The space M is endowed with the quotient topology. Let $q : [0, 1] \times [-1, 1] \to M$ denote the quotient map. The disadvantage of this definition is that it is hard to find a local chart for a point $q(x) \in M$, where $x \in (0 \times [-1, 1]) \cup (1 \times [-1, 1])$, thus making it difficult for us to give M a manifold structure. Therefore, a more convenient way is to define

$$M = (\mathbb{R} \times [-1, 1])/\sim$$

where

$$(x, y) \sim (x + n, (-1)^n y) \qquad n \in \mathbb{Z}.$$

In this way, for every $x \in \mathbb{R} \times [-1, 1]$, we can take a small enough chart (U, φ) of x, such that $q : U \to q(U)$ is a homeomorphism. Then this gives a chart around $q(x)$, and the homeomorphism is given by

$$q(U) \xrightarrow{q|_U^{-1}} U \xrightarrow{\varphi} \mathbb{R}^2.$$

This makes M into a manifold.

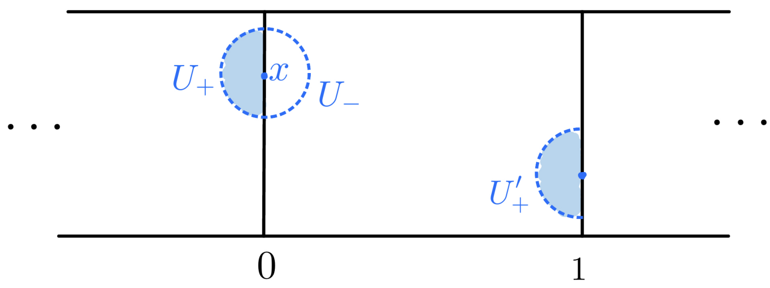


Figure 1.1: With previous definition it is hard to construct a chart near $q(x)$ because we can only find a chart $\varphi_- : U_- \to \varphi_-(U_-) \subset \mathbb{R}^2$ and a chart $\varphi'_+ : U_- \to \varphi'_+(U'_+) \subset \mathbb{R}^2$. But when we identify the space, the chart is no longer injective. In this case, our problem is solved since U'_+ is equivalent to U_+. Hence we have a nice open set $U_+ \cup U_-$ around x.

Note that now the projection $\hat{\pi} : \mathbb{R} \times [-1, 1] \to S^1$ defined by $\hat{\pi}(x, 0) = \varepsilon \circ \pi_{\mathbb{R}}(x, 0) = e^{2\pi i x} \in S^1$ induces a projection $\pi : M \to S^1$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R} \times [-1,1] \arrow[r, "q"] \arrow[d, "\pi_{\mathbb{R}}"'] & M \arrow[d, dashed, "\pi"] \\
\mathbb{R} \arrow[r, "\varepsilon"'] & S^1
\end{tikzcd}$$

This map $\pi : M \to S^1$ actually leads to another important definition.

Definition 1.14. Let E, M, F be topological spaces and $\pi : E \to M$ be a continuous surjective map such that for every $p \in M$, there exists a neighborhood of p and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\pi_U"] \\
& U
\end{tikzcd}$$

then (M, E, F, π) is called a ***fiber bundle***.

If we properly replace the word "continous" by "smooth", "topological space" by "smooth manifold", and "homeomorphism" by "diffeomorphism" in the definition above, we get the definition of a ***smooth fiber bundle***.

Homework 1.15. Show that the Möbius strip $(M, S^1, [-1, 1], \pi)$ is a smooth fiber bundle.

Solution. Since $\varepsilon \circ \pi_1$ is constant on each equivalence classes, it descends to a continuous map $\pi : M \to S^1$. Since $\varepsilon : \mathbb{R} \to S^1$ defined by $\varepsilon(x) = e^{2\pi i x}$ is a smooth covering map, for every $x \in S^1$, let us take $U \subset S^1$ that is evenly covered by ε, and $\tilde{U} \subset \mathbb{R}$ be a component of $\varepsilon^{-1}(U)$. Since $q : \mathbb{R} \times [-1, 1] \to M$ is a local homeomorphism, by choosing U small enough we can assume that q restricts to a homeomorphism from $\tilde{U} \times [-1, 1]$ to $\pi^{-1}(U)$. Now, define $\Phi = (\varepsilon \times \mathrm{id}_{[-1,1]}) \circ q|^{-1}_{\pi^{-1}(U)}$. This will serve as our local trivialization. Then it is a rountine (and tedious) check that the diagram in the definition of fiber bundle above commutes. $\square$

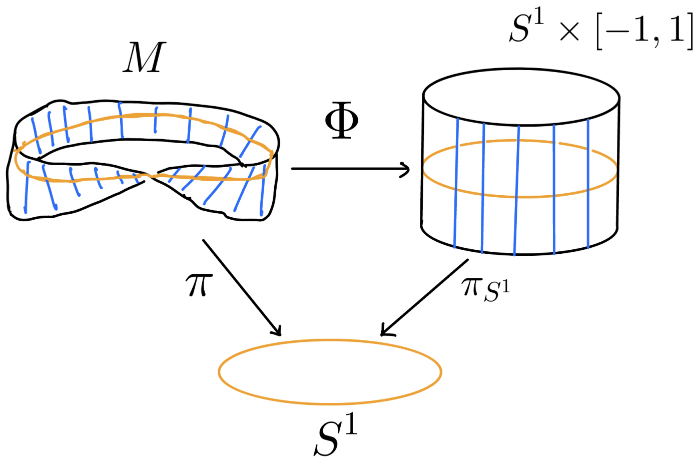


Figure 1.2: The Möbius band is a smooth fiber bundle. A smooth fiber bundle is just locally diffeomorphic to a product (in this case, locally diffeomorphic to $S^1 \times [-1, 1]$ via the local trivialization Φ).

2 | Tensors

2.1 Tensor Product of Vector Spaces

Let V^n, W^m be finite-dimensional real vector spaces of dimension n, m respectively. Let V^*, W^* denote their respective dual vector spaces. Then we define the ***tensor product*** $V^* \otimes W^*$ to be the set of all bilinear functions $\sigma : V \times W \to \mathbb{R}$. This naturally gives $V^* \otimes W^*$ a vector space structure.

Note that if we choose $\omega \in V^*, \eta \in W^*$, then we can construct a bilinear map $\omega \otimes \eta : V \times W \to \mathbb{R}$ by

$$(\omega \otimes \eta)(v, w) = \omega(v)\eta(w)$$

for $v \in V, w \in W$. This is clearly a bilinear map, hence indeed $\omega \otimes \eta \in V^* \otimes W^*$. The next proposition shows that every element in $V^* \otimes W^*$ can be written as a linear combination of these tensor products.

Proposition 2.1. Let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_m$ be a basis of W. Let $v^1, ..., v^n$ denote the dual basis of V^*, i.e., $v^i(v_j) = \delta^i_j$, and similarly let $w^1, ..., w^m$ denote that dual basis of W^*. Then $v^i \otimes w^j$ is a basis for $V^* \otimes W^*$, where $1 \le i \le n, 1 \le j \le m$. In particular, $\dim(V^* \otimes W^*) = nm$.

Proof. Let $\sigma \in V^* \otimes W^*$. Then it is easy to check that

$$\sigma = \sum_{i,j} \sigma(v_i, w_j) v^i \otimes w^j.$$

Hence we see that $v^i \otimes w^j$ form a spanning set of $V^* \otimes W^*$. To check linear independence, suppose $\sum_{i,j} a_{ij} v^i \otimes w^j = 0 \in V^* \otimes W^*$. Then note that $\sum_{i,j} a_{ij} v^i \otimes w^j(v_k, w_l) = a_{kl} = 0$ for all $1 \le k \le n, 1 \le l \le m$. This shows linear independence of $v^i \otimes w^j$, completing the proof. □

Similarly, for real vector space $V_1, ..., V_m$, one can define $V_1^* \otimes \cdots \otimes V_m^*$ to be the set of all multilinear functions $\sigma : V_1 \times \cdots \times V_m \to \mathbb{R}$.

One might wonder how do we define tensor product of vector spaces $V \otimes W$? This can still be defined in terms of multilinear maps, via the following lemma from linear algebra:

Lemma 2.2. *For a finite dimensional vector space V, we have a canonical isomorphism $(V^*)^* \cong V$.*

Proof. The isomorphism $\Phi : V \to (V^*)^*$ is given by $\Phi_V(\omega) = \omega(v)$ for $v \in V$ and $\omega \in V^*$. The proof that Φ is an isomorphism is left as an exercise. □

Therefore, if we identify V with $(V^*)^*$, via $v \mapsto \Phi_V$, we can define ***tensor product of vector space*** $V \otimes W$ to be all bilinear functions $V^* \times W^* \to \mathbb{R}$. This can also be extended to tensor product of multiple vector spaces.

Homework 2.3. Let V^n be a finite dimensional real vector space, and let $e_1, ..., e_n$ be one of its basis, and $e_{1'}, ..., e_{n'}$ be another basis of V. For any $\omega \in V^*$, we can write $\omega = \omega_i \theta^i = \omega_{j'} \theta^{j'}$ where $\theta^i, \theta^{j'}$s are the corresponding dual basis. Find a relation between the numbers ω_i and $\omega_{j'}$.

Solution. In this problem we study how vector and covectors transforms under change of basis. Let Λ be a linear transformation that takes the ordered basis e_i to the ordered basis $e_{j'}$. In index notation, we can write

$$e_{j'} = \Lambda^i_{j'} e_i.$$

Now, for any vector $u \in V$, we have that

$$u = u^i e_i = u^{j'} e_{j'} = u^{j'} \Lambda^i_{j'} e_i.$$

Hence by linear independence, we get that

$$u^{j'} = \Lambda^{j'}_i u^i$$

where $\Lambda^{j'}_i \Lambda^i_{k'} = \delta^{j'}_{k'}$. Now, for covector we have

$$\omega(u) = \omega_i \theta^i (u^k e_k) = \omega_i u^i = \omega_{j'} u^{j'} = \omega_{j'} \Lambda^{j'}_i u^i.$$

Therefore, we get that (equating the third and the last expression and for each k let $u^i = 1$ for $i = k$ and 0 for all $i \neq k$) $\omega_i - \omega_{j'} \Lambda^{j'}_i = 0$, which gives

$$\omega_{j'} = \Lambda^i_{j'} \omega_i.$$

□

There is an alternative way of defining tensor products $V \otimes W$ of vector spaces V, W. Let $\mathscr{F}(V \times W)$ be the free vector space generated by $V \times W$. That is, a general element in this vector space is of the form $\sum_i a_i (v_i, w_i)$ for a_i real numbers and $v_i \in V, w_i \in W$. Then define an equivalence relation $\sim$ on $\mathscr{F}(V \times W)$ as follows

$$\begin{aligned}
(av_1 + bv_2, w) &\sim a(v_1, w) + b(v_2, w)\\
(v, cw_1 + dw_2) &\sim c(v, w_1) + d(v, w_2)\\
a(v, w) &\sim (av, w) \sim (v, aw)
\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$ and $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$. Then we define

$$V \otimes W = \mathscr{F}(V \times W)/\sim.$$

Note that using this definition we can directly make sense of $V^* \otimes W^*$. To see that the multilinear function definition is equivalent to this definition, consider a map $F : \mathscr{F}(V^* \otimes W^*)/\sim \to V^* \otimes W^*$, where the RHS is the multilinear function space defined previously, by mapping an equivalence class $[\alpha, \beta]$ to $F(\alpha \otimes \beta) \in V^* \otimes W^*$, defined as follows:

$$F(\alpha \otimes \beta)(v, w) = \alpha(v)\beta(w).$$

Of course, one needs to check this is well-defined and is an isomorphism, which is easy to do.

2.2 Symmetric and Alternating Tensors

Now we can define mixed tensors. Denote

$$T^{r,s}(V) = \mathscr{T}_s^r(V) = \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}.$$

Then $T^{0,k}(V) = \otimes^k V^*$ is the set of multilinear maps $\prod^k V \to \mathbb{R}$. Then we have two types of important tensors: If $\alpha \in T^{0,k}(V)$, then α is said to be ***symmetric*** if

$$\alpha(v_1, ..., v_k) = \alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

for any $(v_1, ..., v_k) \in \prod^k V$ and $\sigma \in S_k$. Equivalently, one can define α to be the unchanged by interchanging any pair of arguments. We say α is ***alternating*** if

$$\alpha(v_1, ..., v_k) = \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

for any $(v_1, ..., v_k) \in \prod^k V$ and $\sigma \in S_k$. Equivalently, one can define α to be the tensor that takes the negative of the original value by interchanging any pair of arguments.

Now, given any $\alpha \in T^{0,k}(V)$, we can define its ***symmetrization*** $\mathscr{S}(\alpha)$ to be

$$\mathscr{S}(\alpha)(v_1, ..., v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

It is clear that $\mathscr{S}(\alpha)$ is a symmetric tensor. One can also define the ***antisymmetrization*** $\mathscr{A}(\alpha)$ to be

$$\mathscr{A}(\alpha)(v_1, ..., v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)}).$$

Proposition 2.4. For any $\alpha \in T^{0,k}(V)$, its antisymmetrization $\mathscr{A}(\alpha)$ is alternating.

Proof. Let $\tau \in S_k$ and $(v_1, ..., v_k) \in \prod^k V$. Then we calculate

$$\begin{aligned}
\mathscr{A}(\alpha)(v_{\tau(1)}, ..., v_{\tau(k)}) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\alpha(v_{\sigma\circ\tau(1)}, ..., v_{\sigma\circ\tau(k)}) \\
&= \frac{1}{k!}\sum_{\eta \in S_k} \frac{\operatorname{sgn}(\eta)}{\operatorname{sgn}(\tau)}\alpha(v_{\eta(1)}, ..., v_{\eta(k)}) \\
&= \operatorname{sgn}(\tau)\mathscr{A}(\alpha)(v_{\tau(1)}, ..., v_{\tau(k)})
\end{aligned}$$

where we made the substitution $\eta = \sigma \circ \tau$ and we used $\operatorname{sgn}(\eta) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$. But since $\operatorname{sgn}(\tau) = \pm 1$, so dividing by $\operatorname{sgn}(\tau)$ is the same as multiplying by it. □

Now we may define

$$\wedge^r V^* = \mathscr{A}(\otimes^r V^*) = \{\sigma \in \otimes^r V^* : \sigma \text{ alternating}\}.$$

Theorem 2.5. *Let V^n be a n-dimensional real vector space. Then* $\dim \wedge^r V^* = \binom{n}{r}$.

Proof. Let $v_1, ..., v_n$ be a basis of V and $v^1, ..., v^n$ be the corresponding dual basis of V^*. Then we claim that the set

$$\mathcal{A} = \left\{\mathscr{A}(v^{i_1} \otimes \cdots \otimes v^{i_r}) : 1 \leq i_1 < i_2 < \cdots < i_r \leq n\right\}$$

is a basis of $\wedge^r V^*$, which proves the claim since this set has $\binom{n}{r}$ elements. Let $\sigma \in \wedge^r V^* \subset \otimes^r V^*$. One can check easily that $\left\{v^{i_1} \otimes \cdots \otimes v^{i_r} : 1 \leq i_1, i_2, \cdots, i_r \leq n\right\}$ is a basis for $\otimes^r V$ (See Lee Prop.12.8). Therefore one can write

$$\sigma = \sum a_{i_1,...,i_r} v^{i_1} \otimes \cdots \otimes v^{i_r}.$$

Since σ is alternating, we have $\sigma = \mathscr{A}(\sigma)$. Therefore,

$$\sigma = \sum a_{i_1,...,i_r} \mathscr{A}(v^{i_1} \otimes \cdots \otimes v^{i_r}).$$

Now, if $i_k = i_j$ for some $k \neq j$, then $\mathscr{A}(v^{i_1} \otimes \cdots \otimes v^{i_r}) = 0$. Therefore, the summand above will only be nonzero for those terms with distinct indices $i_1, ..., i_r$. Then we can make all indices to be in increasing orders with costs of a plus or minus sign in front. Hence this shows that $\mathcal{A}$ is a spanning set for $\wedge^r V^*$. To show that $\mathcal{A}$ is linearly independent, suppose

$$\sum a_{i_1,...,i_r} v^{i_1} \otimes \cdots \otimes v^{i_r} = \sum_I a_I v^I = 0$$

where I runs over all r-tuples with increasing indices. Then apply both sides of the identity let $J = (j_1, ..., j_r)$ be any increasing index. Then apply both sides of the identity above to the vectors $(v_{j_1}, ..., v_{j_r})$ gives

$$\sum_I a_I v^I(v_J) = a_J = 0.$$

Hence $a_J = 0$ for all increasing J. This shows linearly independence. □

2.3 Wedge Product and Orientations

Definition 2.6. The ***wedge product*** $\wedge$ is a bilinear map $\wedge : \wedge^r V^* \times \wedge^s V^* \to \wedge^{r+s} V^*$ by

$$\wedge(\alpha, \beta) = \alpha \wedge \beta = \frac{(r+s)!}{r!s!} \mathscr{A}(\alpha \otimes \beta).$$

Then we have

Theorem 2.7. *The wedge product $\wedge$ is associative: $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$.*

Proof. This is Lemma 14.10 and Lemma 14.11-(b) of Lee. □

Now if we define $\wedge^* V = \oplus_{r=0}^{\infty} \wedge^r V^*$, then $(\wedge^* V, \wedge)$ is an associative algebra.

Remark 2.8. Note that if V^n is of dimension n, then $\wedge^k V^* = \{0\}$ for any $k > n$. To see this, if we pick a basis $v_1, ..., v_n$ of V, then $\alpha(v_{i_1}, ..., v_{i_k}) = 0$ for any $\alpha \in \wedge^k V^*$ since α is alternating and since a tuple $(i_1, ..., i_k)$ with values in just 1 to n must contain a repeated index. Then by multilinearity, $\alpha \equiv 0$.

Theorem 2.9. *If $\sigma \in \wedge^r V^*$ and $\tau \in \wedge^s V^*$, then $\sigma \wedge \tau = (-1)^{rs} \tau \wedge \sigma$.*

Proof. This is Proposition 14.11-(c) of Lee. □

Definition 2.10. Let V^n be a vector space and $L : V \to V$ be a linear map. Then this induces a map $L(v_1 \wedge \cdots \wedge v_n) = (Lv_1) \wedge \cdots \wedge (Lv_n)$.

Now, suppose that $Lv_i = \sum_j a_i^j v_j$. Then we see that

$$L(v_1 \wedge \cdots \wedge v_n) = \sum_J (a_1^{j_1} \cdots a_n^{j_n})(v_{j_1} \wedge \cdots v_{j_n})$$

where $J = (j_1, ..., j_n)$ runs over all n-tuples where each entry takes value between $1, ..., n$. However, the previous theorem tells us that if J contains any repeated indices, then $v_{j_1} \wedge \cdots \wedge v_{j_n} = 0$. Therefore we can in fact take J to run over all permutations of $(1, ..., n)$. Therefore the equation above actually becomes

$$\begin{aligned} L(v_1 \wedge \cdots \wedge v_n) &= \sum_{\sigma \in S_n} (a_1^{\sigma(1)} \cdots a_n^{\sigma(n)})(v_{\sigma(1)} \wedge \cdots v_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(a_1^{\sigma(1)} \cdots a_n^{\sigma(n)})(v_1 \wedge \cdots \wedge v_n) \\ &= \det\left(a_i^j\right)(v_1 \wedge \cdots \wedge v_n). \end{aligned}$$

Definition 2.11. Let V^n be a vector space and let $v_1, ..., v_n$ and $w_1, ..., w_n$ be two basis for V. Then these two basis are said to be ***equivalent*** if the linear map $A : V \to V$ such that $w_j = Av_j$ has positive determinant, i.e., $\det A > 0$.

Corollary 2.12. *The two basis $v_1, ..., v_n$ and $w_1, ..., w_n$ of V are equivalent iff*

$$w_1 \wedge \cdots \wedge w_n = \lambda(v_1 \wedge \cdots \wedge v_n)$$

for some $\lambda > 0$.

3 | Differential Forms

3.1 Differential Forms on Manifolds

Definition 3.1. Let $\pi : E \to M$ be a vector bundle. A map $\sigma : M \to E$ is called a ***section*** of E if $\pi \circ \sigma = \mathrm{id}_M$. The section σ is called said to be a ***smooth section*** if $\sigma : M \to E$ is a smooth map between smooth manifolds.

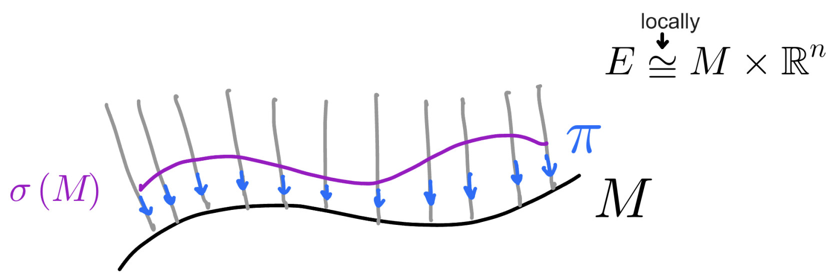


Figure 3.1: Image of a smooth section $\sigma : M \to E$.

As an example, it is clear that a smooth vector field $X \in \mathfrak{X}(M)$ is a smooth section of the tangent bundle $X : M \to TM$.

Now let M^n be a smooth manifold. Define $T^kT^*M = \coprod_{p\in M} \otimes^k(T_p^*M)$ to be the bundle of ***covariant*** k***-tensors on*** M (More details in Chapter 12, p.316 of Lee). There is a natural induced chart $(\tilde{U}, \tilde{\varphi})$ of T^kT^*M associated to the chart (U, φ). There is an obvious projection $\pi : T^kT^*M \to M$, so let $\tilde{U} = \pi^{-1}(U)$. Then write every $\sigma \in T^kT^*M$ as

$$\sigma = \sum \xi_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

where the indices are summed over. Then the map

$$\tilde{\varphi}(\sigma) = (\varphi(\pi(\sigma)), \xi_{i_1\cdots i_k}) \in \mathbb{R}^n \times \mathbb{R}^{n^k}$$

will be a chart map. It can be shown that such collections $(\tilde{U}, \tilde{\varphi})$ satisfies the smooth manifold chart lemma (Lee Lemma 1.35), hence this is a smooth manifold structure on T^kT^*M.

Now, we consider a subset of T^kT^*M consisting of alternating tensors:

$$\wedge^k T^*M = \coprod_{p\in M} \wedge^k(T_p^*M).$$

Then one can verify the condition of Lemma 10.32 of Lee, which implies that $\wedge^k T^*M$ is a smooth subbundle of $T^k T^*M$, hence is a smooth vector bundle of rank $\binom{n}{k}$ over M.

Definition 3.2. A section of $\wedge^k T^*M$ is called a ***differential*** ***k-form***. The integer k is called the ***degree*** of the form. We denote the space of smooth k-forms by $\Omega^k(M)$.

Proposition 3.3. Let M be a smooth manifold and let (U, φ) be a chart of M with coordinates x^i. Then a form $\omega \in \Omega^k(U)$ can be written as

$$\omega = \sum \xi_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where $\xi_{i_1 \cdots i_k} : U \to \mathbb{R}$. Then ω is smooth iff all the coodinate functions $\xi_{i_1 \cdots i_k}$ are smooth.

Proof. The chart Ψ from $\wedge^k T^*U \to \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ is defined by

$$\Psi(\sum \xi_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}(p)) = (\varphi(p), \xi_{i_1 \cdots i_k}) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Therefore we have a commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\omega} & \wedge^k T^*U \\ {\scriptstyle \varphi^{-1}} \uparrow & & \downarrow {\scriptstyle \Psi} \\ \varphi(U) & \xrightarrow[\hat{\omega}]{} & \varphi(U) \times \mathbb{R}^{\binom{n}{k}} \end{array}$$

Therefore, we see that

$$\hat{\omega}(\mathbf{x}) = \Psi \circ \omega \circ \varphi^{-1}(\mathbf{x}) = (\mathbf{x}, \xi_{i_1 \cdots i_k}(\varphi^{-1}(\mathbf{x}))).$$

Since φ^{-1} is a diffeomorphism, we see that ω is smooth iff $\hat{\omega}$ is smooth, which is equivalent to smoothness of coordinate functions $\xi_{i_1 \cdots i_k} : U \to \mathbb{R}$. □

Corollary 3.4. *If (U, φ, x^i) and (V, ψ, y^i) are two overlapping smooth coordinate charts on M^n, then the following identity holds on $U \cap V$:*

$$dy^1 \wedge \cdots \wedge dy^n = \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n = J(\psi \circ \varphi^{-1}) \circ \varphi dx^1 \wedge \cdots \wedge dx^n.$$

Proof. p.281 of [1] gives a formula of the differential dy^j in terms of dx^i, which is $dy^j = \frac{\partial y^j}{\partial x^i} dx^i$. Then the discussion in the previous chapter gives the desired formula. □

3.2 Orientations of Manifolds

Definition 3.5. A smooth atlas $\mathcal{A}$ of a smooth manifold M is called an ***orientation atlas*** if the following holds true: For all (U, φ, x^i) and (V, ψ, y^j) in $\mathcal{A}$ such that $U \cap V \neq \emptyset$, we have

$$J(\psi \circ \varphi^{-1}) = \det\left(\frac{\partial y^i}{\partial x^j}\right) > 0$$

for all $x \in U \cap V$. A maximal orientation atlas is called an ***orientation***. A manifold M is said to be ***orientable*** if an orientation exists. An ***oriented manifold*** is a manifold with a chosen orientation.

Theorem 3.6. *An n-dimensional manifold M is orientable iff M admits a nowhere vanishing n-form ω.*

Proof. Let ω be a nowhere vanishing n-form on M. Then for each chart (U, φ, x^i) there exists a nowhere vanishing function $f \neq 0$ such that $\omega = f dx^1 \wedge \cdots \wedge dx^n$. We can always take such a chart near each point so that $f > 0$, otherwise we can replace x^1 by $-x^1$. Now suppose $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two such charts. Then suppose $\omega = f dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = g dx_\beta^1 \wedge \cdots \wedge dx_\beta^n$ where $f, g > 0$. Then

$$0 < g = \omega(\partial_1^\beta, ..., \partial_n^\beta) = J(\varphi_\beta \circ \varphi_\alpha^{-1})\omega(\partial_1^\alpha, ...\partial_\alpha^n) = J(\varphi_\beta \circ \varphi_\alpha^{-1}) f.$$

So $J(\varphi_\beta \circ \varphi_\alpha^{-1}) = g/f > 0$.

Conversely, let $\mathcal{A}$ be an orientation. For each local chart U_α, we consider the form $\omega_\alpha = dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$. Pick a partition of unity ρ_α suboridnate to the open cover U_α. Then consider the n-form

$$\omega = \sum_\alpha \rho_\alpha \omega_\alpha.$$

We claim that ω is nowhere vanishing. For each $p \in M$, there is a neighborhood U of p such that the sum $\sum_\alpha \rho_\alpha \omega_\alpha$ reduces to a finite sum $\sum_{i=1}^k \rho_i \omega_i$. It follows that near p, one has

$$\omega(\partial_1^1, ..., \partial_n^1) = \sum_i J(\varphi_i \circ \varphi_1^{-1})\rho_i > 0.$$

This shows that ω is nowhere vanishing. □

With this theorem in mind, we can formulate the following (equivalent) definition:

Definition 3.7. We shall say M is ***orientable*** if it is possible to define a smooth n-form ω on M which is nonvanishing, in which case M is said to be ***oriented*** by the choice of ω in the following sense: Two nonvanishing n-forms ω and η on M induces two orientations as shown by the previous theorem. We call those two orientations equivalent iff $\omega = f\eta$ for some smooth function f that is positive everywhere.

Theorem 3.8. *Every connected orientable smooth manifold has exactly two orientations.*

Proof. Let M be a connected smooth orientable manifold. By the previous theorem, there exists a nonvanishing n-form ω on M. Any other choice of orientation will induce a nonvanishing n-form η on M. Since $\omega = f\eta$ for some smooth function $f : M \to \mathbb{R}$ such that $f(p) \neq 0$ for all $p \in M$, then $f(M) \subset \mathbb{R}$ is a connected subset which does not contain 0, hence f is either always positive or always negative. If $f > 0$, then ω and η induces the same orientation at T_pM for each $p \in M$, hence they induce the same orientation. Therefore assume $f < 0$ and ω, η induces different orientations on M. Then any other orientation will induce a non-vanishing n-form τ such that $\tau = g\omega$ for some smooth function $g \in C^\infty(M)$. So $g > 0$ or $g < 0$, in which case τ induces the same orientation as ω or η does. □

3.3 Pullback and Exterior Differentiation

Let M^n be a smooth manifold, and let $\Omega^k(M)$ denote the set of smooth k-form on M, or equivalently $\Omega^k(M) = C^\infty(\wedge^k T^*M)$, which is the set of all smooth sections of $\wedge^k T^*M$.

Definition 3.9. Let $F : M \to N$ be a smooth map. Recall that $F_{*p} : T_pM \to T_pN$ is given by $F_{*p}(v)(f) = v(f \circ F)$ for $b \in T_pM, f \in C^\infty(N)$. Then its dual map $F_p^* : T^*_{F(p)}N \to T^*_pM$ is called the ***pullback*** of F at p.

If $\omega \in \Omega^k(N)$, then we can also define the pullback $F^*\omega \in \Omega^k(M)$ given by

$$(F^*\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(F_{*p}(v_1), ..., F_{*p}(v_k)).$$

Using this definition, we have the following easy observation:

Lemma 3.10. *Suppose $F : M \to N$ is smooth. Then*

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ *is linear over* $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge F^*(\eta)$.

(c) In any smooth chart, we have that

$$F^*\left(\sum_I \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

The ***exterior derivative*** d is a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for each k, constructed as follows: Let $\omega \in \Omega^k(M)$. Then in a local chart (U, φ), we may write

$$\omega = \sum_{I : i_1 < ... < i_k} \omega_I dx^I$$

where ω_I's are smooth functions $U \to \mathbb{R}$. Then we define

$$d\omega = \sum_{I : i_1 < ... < i_k} d\omega_I \wedge dx^I = \sum_{I : i_1 < ... < i_k} \left(\sum_j \frac{\partial \omega_I}{\partial x^j} dx^j\right) \wedge dx^I \tag{3.1}$$

$$= \sum_{i_1 < ... < i_k} \left(\sum_j \frac{\partial \omega_{i_1 \cdots i_k} \circ \varphi^{-1}}{\partial x^j} dx^j\right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{3.2}$$

Note that this definition (potentially) depends on the chart we choose. But it turns out that this definition is independent of charts.

Proposition 3.11. Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be an operator that satisfies the following properties:

(i) d is linear over $\mathbb{R}$.

(ii) For any $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

(iii) $d^2 = 0$.

(iv) For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.

Then d exists and is unique. Moreover, in any smooth coordinate chart, d is given by 3.1.

Proof. First, suppose $M = \mathbb{R}^n$. Then for $\omega \in \Omega^k(\mathbb{R}^n)$ we define its exterior derivative to be

$$d\left(\sum_J \omega_J dx^J\right) = \sum_J d\omega_J dx^J = \sum_J \left(\sum_i \frac{\partial \omega_J}{\partial x^i} dx^i\right) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Then one can check that d defined in $\mathbb{R}^n$ satisfies the properties (i)-(iv). Moreover, one can check that if $F : U \to V$ is a smooth map in Euclidean space, then we have

$$F^*(d\omega) = d(F^*\omega) \tag{3.3}$$

the detailed calculation is in [1, Proposition 14.23]. Now for a general manifold M, we first show the existence of d. Let $\omega \in \Omega^k(M)$, and we define d on chart (U, φ) by

$$d\omega = \varphi^* d(\varphi^{-1*}\omega).$$

This is well-defined, since for any other chart (V, ψ), the map $\varphi \circ \psi^{-1}$ is a diffeomorphism. Hence by 3.3 we have

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) = d((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega) = d(\psi^{-1*}\omega).$$

Therefore, multiplying both sides by ψ^*, we get that $\varphi^* d(\varphi^{-1*}\omega) = \psi^* d(\psi^{-1*}\omega)$. It follows quite easily from the calculation in $\mathbb{R}^n$ that this definition satisfies (i)-(iv).

Next, we prove uniqueness. Suppose d is any operator satisfying (i)-(iv). We claim that if $\omega_1, \omega_2 \in \Omega^k(M)$ such that $\omega_1 = \omega_2$ on an open set $U \subset M$, then $d\omega_1 = d\omega_2$ on U. To see this, let $p \in U$ and $\eta = \omega_1 - \omega_2$. Let $\psi \in C^\infty(M)$ be a bump function that is identically 1 on some neighborhood of p and supported in U. Then $\psi\eta$ is identically zero. So (i)-(iv) implies that

$$0 = d(\psi\eta) = d\psi \wedge \eta + \psi d\eta.$$

Evaluating this at p using $\psi(p) = 1$ and $d\psi_p = 0$, we have that $d\omega_1|_p - d\omega_2|_p = d\eta_p = 0$.

Now let $\omega \in \Omega^k(M)$ be arbitrary, and (U, φ) be a smooth chart on M. Write $\omega = \sum_I \omega_I dx^I$ on U. For any $p \in U$, by means of a bump function we can construct global smooth functions $\hat{\omega}_I, \hat{x}^i$ that agrees with ω_I, x^i in a neighborhood of p. By virtual of (i)-(iv), we see that $d\omega$ has to have the expression of 3.1 at p: More detailed explanation can be found in [3, p.221]. Since p is arbitrary, this d must be everywhere equal to the one we defined locally by 3.1. $\square$

Proposition 3.12. If $F : M \to N$ is a smooth map, then for each k, the pullback map $F^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d: $F^*(d\omega) = d(F^*\omega)$. In other words, the diagram below commutes:

$$\begin{tikzcd}
\Omega^k(N) \arrow[r, "F^*"] \arrow[d, "d"'] & \Omega^k(M) \arrow[d, "d"] \\
\Omega^{k+1}(N) \arrow[r, "F^*"'] & \Omega^{k+1}(M)
\end{tikzcd}$$

Proof. We first claim that for any $f \in C^\infty(N)$, we have $F^*(df) = d(f \circ F)$. This is a simple calculation:

$$F_p^*(df_{F(p)})(v) = df_{F(p)}(F_{*p}(v)) = (F_{*p}(v))(f) = v(f \circ F) = d(f \circ F)_p(v).$$

Next, since d is local, for any $\omega \in \Omega^k(N)$, we can restrict to local charts (V, ψ, x^i) of M and (U, φ, y^j) of N such that $F(V) \subset U$. Let

$$\omega = \sum_J \omega_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Then

$$d\omega = \sum_J d\omega_{j_1 \cdots j_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

Hence

$$\begin{aligned} F^*(d\omega) &= \sum_J F^*(d\omega_{j_1 \cdots j_k}) \wedge F^*(dx^{j_1}) \wedge \cdots \wedge F^*(dx^{j_k}) \\ &= \sum_J d(\omega_{j_1 \cdots j_k} \circ F) \wedge d(x^{j_1} \circ F) \wedge \cdots \wedge d(x^{j_k} \circ F). \end{aligned}$$

Next,

$$\begin{aligned} F^*\omega &= \sum_J (\omega_{j_1 \cdots j_k} \circ F) \wedge F^* dx^{j_1} \wedge \cdots \wedge F^* dx^{j_k} \\ &= \sum_J (\omega_{j_1 \cdots j_k} \circ F) \wedge d(x^{j_1} \circ F) \wedge \cdots \wedge d(x^{j_k} \circ F). \end{aligned}$$

Therefore,

$$d(F^*\omega) = \sum_J d(\omega_{j_1 \cdots j_k} \circ F) \wedge d(x^{j_1} \circ F) \wedge \cdots \wedge d(x^{j_k} \circ F) = F^*(d\omega).$$

This completes the proof. □

Theorem 3.13. *Let $F : N^n \to M^n$ be a smooth map, and let $\omega \in \Omega^n(M)$. Let $(U, \varphi, x^i), (V, \psi, y^j)$ be two charts. Suppose $\omega = h\, dy^1 \wedge \cdots \wedge dy^n$ on (V, y^j). Then the following holds on $U \cap F^{-1}(V)$:*

$$F^*\omega = F^*(u\, dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det F_*) dx^1 \wedge \cdots \wedge dx^n.$$

Proof. We have shown in the previous proof that

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F).$$

Now how does the one form $d(y^j \circ F)$ relate to dx^i's? By the way how 1-forms transform (p.281 of Lee's Smooth manifold), we have that

$$d(y^j \circ F) = d(y^j \circ \psi \circ F) = \frac{\partial (y^j \circ \psi \circ F \circ \varphi^{-1})}{\partial x^i} dx^i = \frac{\partial \hat{F}^j}{\partial x^i} dx^i.$$

Therefore, we see that

$$\begin{aligned} F^*(u dy^1 \wedge \cdots \wedge dy^n) &= (u \circ F) \left(\frac{\partial \hat{F}^1}{\partial x^{j_1}} \cdots \frac{\partial \hat{F}^n}{\partial x^{j_n}} \right) dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\ &= (u \circ F) \det \left(\frac{\partial \hat{F}^j}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n \\ &= (u \circ F)(\det F_*) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

This completes the proof. □

We conclude with an invariant formulation of exterior differentiation:

Proposition 3.14. Let $\omega \in \Omega^1(M)$. Then there holds, for smooth vector fields, $X, Y \in \mathfrak{X}(M)$,

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]). \tag{3.4}$$

Proof. Since ω is local and linear and the commutator is multilinear, it suffices to prove the statement for $X = \partial/\partial x^i, Y = \partial/\partial x^j$ on a chart. Write $\omega = \sum_k f_k dx^k$, then

$$d\omega = \sum_k df_k \wedge dx^k = \sum_{k,l} \frac{\partial f_k}{\partial x^l} dx^l \wedge dx^k.$$

Therefore,

$$\begin{aligned} d\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) &= \sum_{k,l} \frac{\partial f_k}{\partial x^l} dx^l \wedge dx^k \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \\ &= \sum_{k,l} \frac{\partial f_k}{\partial x^l} (\delta_i^l \delta_j^k - \delta_i^k \delta_j^i) \\ &= \frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j}. \end{aligned}$$

One can check that this is equal to the right hand side of 3.4 applies to the vector fields $X = \partial_i, Y = \partial_j$. This concludes the proof. □

4 | Integration of Differential Forms

4.1 Basic Definitions and Properties

Let M^n be an oriented smooth manifold (i.e. orientable and has a chosen orientation). We want to define integral of an n-form on M. First, assume ω is a compactly supported n-form, i.e., $\omega \in \Omega^n_c(M)$. Further assume that the support of ω is contained in a single chart (U, φ) of M. Then we define the ***integral of*** ω ***over*** M as

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega \tag{4.1}$$

where the expression on the right hand side is in $\mathbb{R}^n$. And if φ is ***positively oriented***, i.e., the coordinate frame $(\partial/\partial x^i)$ is a positively oriented basis for each tangent space, then we choose the plus sign in the expression above. If φ is negatively oriented, then we choose the minus sign in the expression above.

We should be more explicit about the definition above. Let $\eta \in \Omega^n(D)$ where $D \subset \mathbb{R}^n$. Then using the standard coordinate $(x^1, ..., x^n)$ of $\mathbb{R}^n$, one can write

$$\eta = g(x^1, ..., x^n) dx^1 \wedge \cdots \wedge dx^n$$

for some continuous function $g : D \to \mathbb{R}$. Then we define the ***integral of*** η ***over*** D to be

$$\int_D \eta = \int_D g(x^1, ..., x^n) dx^1 \wedge \cdots \wedge dx^n := \int_D g(x^1, ..., x^n) dx^1 \cdots dx^n.$$

Then, we can now write (4.1) more explicitly. Suppose that on the chart (U, φ, x^i), we have that

$$\omega = f dx^1 \wedge \cdots \wedge dx^n$$

for some smooth function $f : \operatorname{Supp} \omega \subset M \to \mathbb{R}$. Then the pullback formula gives (recall that on manifolds, the coordinate function x^i is really $x^i \circ \varphi$):

$$\begin{aligned}
(\varphi^{-1})^* \omega &= (\varphi^{-1})^* (f dx^1 \wedge \cdots \wedge dx^n) \\
&= (f \circ \varphi^{-1}) d(x^1 \circ \varphi \circ \varphi^{-1}) \wedge \cdots \wedge d(x^n \circ \varphi \circ \varphi^{-1}) \\
&= (f \circ \varphi) dx^1 \wedge \cdots \wedge dx^n
\end{aligned}$$

where the last expression is a differential form in a subset of $\mathbb{R}^n$, and x^i's are coordinate functions in $\mathbb{R}^n$. Therefore, using the definition of integral in $\mathbb{R}^n$ above, we can rewrite (4.1) as

$$\int_M \omega = \pm \int_M (\varphi^{-1})^* (f dx^1 \wedge \cdots \wedge dx^n) := \pm \int_{\varphi(U)} (f \circ \varphi^{-1}) dx^1 \cdots dx^n.$$

One needs to check that this definition is indepent of charts chosen. First, recall the change of variable formula in $\mathbb{R}^n$:

Lemma 4.1. *Let $F : U \to F(U)$ be a diffeomorphism between subsets of $\mathbb{R}^n$. Let $J(F) = (\partial F^i/\partial x^j)$ denote the Jacobian of F. Let (y^j) denote the coordinate systems in $F(U)$ and let (x^i) denote that of U. Then we have, for an integrable function f,*

$$\int_{F(U)} f dy^1 \cdots dy^n = \int_U (f \circ F)|J(F)|dx^1 \cdots dx^n.$$

Proposition 4.2. The definition of integral above does not depend on which chart we choose.

Proof. We only need to check the case for positively oriented charts, since the other cases will follow exact same calculation, except we will get a minus sign when computing Jacobian determinants. So suppose (U, φ, x^i) and (V, ψ, y^j) are two intersecting charts such that $\operatorname{Supp}\omega \subset U \cap V$. And suppose

$$\omega = f dx^1 \wedge \cdots \wedge dx^n = h dy^1 \wedge \cdots \wedge dy^n.$$

Then the transformation for dx^i into dy^j gives the relation

$$\begin{aligned} \omega &= h dy^1 \wedge \cdots \wedge dy^n \\ &= h\left(\frac{\partial y^1}{\partial x^{i_1}} \cdots \frac{\partial y^n}{\partial x^{i_n}}\right) \circ \varphi dx^{i_1} \wedge \cdots \wedge dx^{i_n} \\ &= h \det\left(\frac{\partial y^i}{\partial x^j}\right) \circ \varphi dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

where in the last step, we reorder every $dx^{i_1} \wedge \cdots dx^{i_n}$ into $dx^1 \wedge \cdots \wedge dx^n$, at a cost of a $\operatorname{sgn}(\sigma)$ where $\sigma(i_1, ..., i_n) = (1, ..., n)$. But that with the partial derivatives combined is exactly the determinant of the matrix $\left(\partial y^i/\partial x^j\right)$. Therefore, the expression above gives

$$f = \det\left(\frac{\partial y^i}{\partial x^j}\right) \circ \varphi.$$

Now, since $\psi \circ \varphi^{-1}$ is a positively oriented chart, we have that $J(\psi \circ \varphi^{-1}) > 0$. Therefore, we have that

$$\begin{aligned} \int_{\psi(V)} (h \circ \psi^{-1}) dy^1 \cdots dy^n &= \int_{\varphi(U)} h \circ \psi^{-1} \circ (\psi \circ \varphi^{-1})|J(\psi \circ \varphi^{-1})| dx^1 \cdots dx^n \\ &= \int_{\varphi(U)} h \circ \psi^{-1} \circ (\psi \circ \varphi^{-1}) J(\psi \circ \varphi^{-1}) dx^1 \cdots dx^n \\ &= \int_{\varphi(U)} \left(h \cdot \det\left(\frac{\partial y^i}{\partial x^j}\right) \circ \varphi\right) \circ \varphi^{-1} dx^1 \cdots dx^n \\ &= \int_{\varphi(U)} (f \circ \varphi^{-1}) dx^1 \cdots dx^n. \end{aligned}$$

Therefore, this definition is indeed independent of coordinate charts. Note that we may assume $(\psi \circ \varphi^{-1})(\varphi(U \cap V)) = \psi(V)$ since $\operatorname{Supp}\omega$ is contained in $U \cap V$, so outside of $\psi(U \cap V)$ the integration gives zero. $\square$

Now suppose that $\omega \in \Omega^n_c(M)$. Let (U_i, φ_i) be positive charts and let (ρ_i) be a partition of unity suboridnate to the open cover (U_i) of M. Then we may write

$$\omega = \sum_i \rho_i \omega =: \sum_i \omega_i.$$

Since $\operatorname{Supp} \omega$ is compact, it intersects finitely many open cover (U_i)'s. Hence we really have a finite sum

$$\omega = \sum_{i=1}^m \omega_i.$$

Then we define the ***integral of*** ω ***over*** M to be

$$\int_M \omega = \sum_{i=1}^m \int_M \omega_i.$$

Proposition 4.3. This definition is independent of which partition of unity we choose.

Proof. Let (η_j) be a partition of unity subordinate to another set of charts (V_j, ψ_j). So $\omega = \sum \eta_j \omega$. Note that we can write $\omega\rho_i = \sum_j (\omega\rho_i)\eta_j$. Thus we can swap the sums since they are finite. Therefore, we have that

$$\sum_i \int_M \omega\rho_i = \sum_i \sum_j \int_M \omega\rho_i\eta_j = \sum_j \int_M \left(\sum_i \omega\eta_j\right)\rho_i = \sum_j \int_M \omega\eta_j.$$

Hence this definition is indeed independent of the choice of partition of unity. □

Some properties are immediate:

Lemma 4.4. $\int_M (a\omega + b\eta) = a\int_M \omega + b\int_M \eta$. *Also, if* ω *is a positively oriented orientation form, then* $\int_M \omega > 0$.

Proof. See [1], Proposition 16.6. □

Theorem 4.5 (Integration and Orientations)**.** *Let* M^n *be a smooth manifold and* $\omega \in \Omega^n_c(M)$.

(a) If $-M$ *denotes* M *with the opposite orientation, then*

$$\int_{-M} \omega = -\int_M \omega.$$

(b) If $F : N \to M$ *is a diffeomorphism, then*

$$\int_M \omega = \pm \int_N F^*\omega$$

where the $\pm$ *depends on whether* F *is orientation preserving or reversing. We say* F *is* ***orientation preserving*** *if for each* $p \in N$*, the isomorphism* $F_{*p} : T_pN \to T_pM$ *takes positively oriented bases of* T_pN *to positively oriented bases of* T_pM*, and say* F *is* ***orientation reversing*** *if it is not orientation preserving.*

Proof. (a) Let (U, φ, x^i) be a positively oriented chart of M containing $\operatorname{Supp}\omega$. Then since $-M$ has opposite orientation of M, this would be a negatively oriented chart. A positively oriented chart on $-M$ containing $\operatorname{Supp}\omega$ would be (U, ψ, y^j), where $\psi = T \circ \varphi$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $T(x^1, x^2, ..., x^n) = (x^2, x^1, ..., x^n)$. Then suppose $\omega = f dx^1 \wedge \cdots \wedge dx^n = h dy^1 \wedge \cdots dy^n$. By the previous calculation, we have that

$$f = h \cdot \det\left(\frac{\partial y^i}{\partial x^j}\right) \circ \varphi = h \cdot \det\left(\frac{\partial(\psi \circ \varphi^{-1})^i}{\partial x^j}\right) \circ \varphi = h \cdot \det\left(\frac{\partial T^i}{\partial x^j}\right) \circ \varphi = -h.$$

Then by the change of variable formula, one sees that

$$\begin{aligned}
\int_{-M} \omega &= \int_{\psi(U)} (\psi^{-1})^* \omega \\
&= \int_{\psi(U)} (h dy^1 \wedge \cdots dy^n) \\
&= \int_{\psi(U)} (h \circ \psi^{-1}) dy^1 \cdots dy^n \\
&= \int_{\varphi(U)} (h \circ \psi^{-1} \circ \psi \circ \varphi^{-1}) |J(\psi \circ \varphi^{-1})| dx^1 \cdots dx^n \\
&= \int_{\varphi(U)} (h \circ \varphi^{-1}) dx^1 \cdots dx^n \\
&= -\int_{\varphi(U)} (f \circ \varphi^{-1}) dx^1 \cdots dx^n \\
&= -\int_{\varphi(U)} (\varphi^{-1})^* \omega = -\int_M \omega.
\end{aligned}$$

Note that we have used the fact that $|J(\psi \circ \varphi^{-1})| = |J(T)| = 1$. This proves (a).

(b) Suppose F is orientation preserving, and (U, φ) is a positive chart of M such that $\operatorname{Supp}\omega \subset U$. Then we see that $(F^*\omega)_x(v_1, ..., v_n) \neq 0$ implies $\omega_{F(x)}(F_*(v_1), ..., F_*(v_n)) \neq 0$, which implies that $\omega \neq 0$ at $F(x)$. Hence $F(\operatorname{Supp} F^*\omega) \subset \operatorname{Supp}\omega \subset U$. Therefore $\operatorname{Supp} F^*\omega \subset F^{-1}(U)$. Since F is orientation preserving, then $(F^{-1}(U), \varphi \circ F)$ is a positively oriented chart of N containing $\operatorname{Supp} F^*\omega$. Thus

$$\begin{aligned}
\int_N F^*\omega &= \int_{\varphi(U)} [(\varphi \circ F)^{-1}]^* F^*\omega \\
&= \int_{\varphi(U)} (F \circ F^{-1} \circ \varphi^{-1})^* \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega = \int_M \omega.
\end{aligned}$$

If F is orientation reversing, then $(F^{-1}(U), \varphi \circ F)$ is a negatively oriented chart of N containing $\operatorname{Supp} F^*\omega$. Hence by the definition of the integral, we have

$$\int_N F^*\omega = -\int_{\varphi(U)} [(\varphi \circ F)^{-1}]^* F^*\omega = -\int_{\varphi(U)} (\varphi^{-1})^* \omega = -\int_M \omega$$

from the previous calculation. This proves (b).

□

Remark 4.6. The previous theorem says that if $F : N \to M$ is a diffeomorphism, then we have $\int_N F^*\omega = \pm \int_M \omega$. Using the pullback formula we obtained from Theorem 3.13, we see that when N and M are Euclidean spaces, then this statement is just a reformulation of the change of variable formula Lemma 4.1. This formulation is more concise and elegant. However, we should not think that Lemma 4.1 as a consequence of this statement. Remember that the whole theory of integration on manifolds is built on integration on $\mathbb{R}^n$ at the first place. Also, recall that in the proof of the previous theorem, we used the change of variable formula. It would be useful, however, to use this more concise statement as an aid to remember the change of variable formula in $\mathbb{R}^n$.

4.2 Manifolds with Boundary and Boundary Orientation

We denote the ***n-dimensional upper half-space*** $H^n = \{(x^1, ..., x^n) \in \mathbb{R}^n : x^n \geq 0\}$. An n-dimensional ***manifold with boundary*** is a second-countable Hausdorff space M in which every point has a neighborhood U homeomorphic to an open subset in H^n via a chart map $\varphi : U \to \varphi(U) \subset H^n$. A point $p \in M$ is called an ***interior point*** of M if it is in the domain of some chart (U, φ) such that $\varphi(U) \cap \partial H^n = \emptyset$. We call such a chart an ***interior chart***. We say $p \in M$ is a ***boundary point*** if it is in the domain of some ***boundary chart*** (V, ψ), i.e., a chart with $\psi(V) \cap \partial H^n \neq \emptyset$. The boundary ∂M of M consists of all boundary points of M.

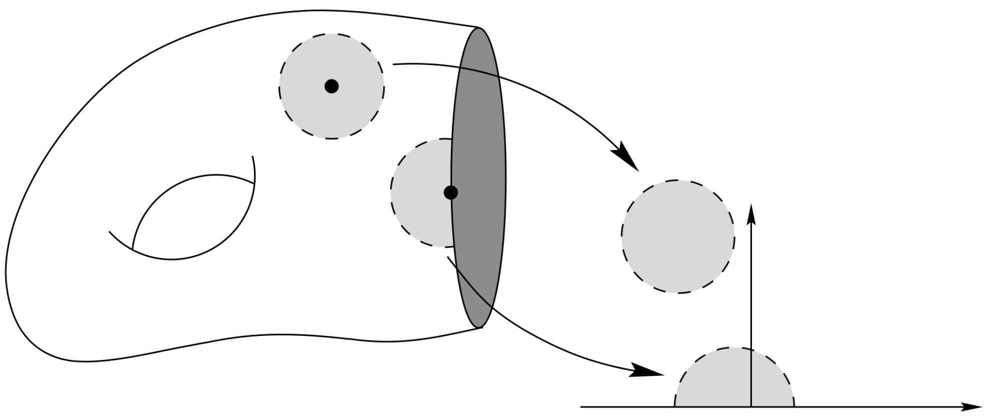


Figure 4.1: A manifold with boundary. An interior point and a boundary point is shown.

A smooth structure on a manifold M with boundary is defined similarly to our previous construction of smooth structure on a manifold. A map $f : U \to \mathbb{R}^k$ where U is a subset of H^n is said to be ***smooth*** if it admits a smooth extension to some open subset $V \subset \mathbb{R}^n$ that contains U. We define a smooth structure on M to be a maximal smooth atlas, smooth in the sense that the chart transition maps are smooth as defined above.

The following theorem is Theorem 5.11 of [1], whose proof is omitted here:

Theorem 4.7. *If M is a smooth manifold with boundary, then with subspace topology, ∂M is a topolpgical $(n-1)$-manifold without boundary, and has a smooth structure such that it is a properly embedded submanifold of M.*

Definition 4.8. Let M be a manifold with boundary and let $p \in \partial M$. A vector $v \in T_pM$ is said to be ***inward-pointing*** if $v \notin T_p(\partial M)$, and there exists some $\varepsilon > 0$, a smooth curve $\gamma : [0, \varepsilon] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. A vector $v \in T_pM$ is said to be ***outward-pointing*** if $-v$ is inward pointing.

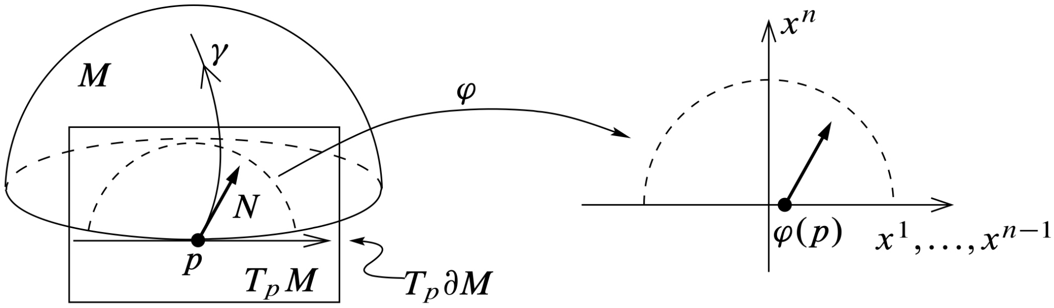


Figure 4.2: An inward-pointing vector $N \in T_pM$.

Lemma 4.9. *Suppose M is a smooth manifold with boundary, $p \in \partial M$, and (U, φ, x^i) is a chart containing p. Then the inward-pointing vectors $v = v^i \partial/\partial x^i$ in T_pM are precisely those with $v^n > 0$, the outward-pointing ones are those with $v^n < 0$.*

Proof. Let $p \in \partial M$. Suppose $v \in T_pM$ is an inward-pointing vector. Then $v \notin T_p(\partial M)$ and there exists a curve $\gamma : [0, \varepsilon] \to M$ such that $\gamma(0) = p, \gamma'(0) = v$. Then under the coordinate basis, we have that

$$v = v^i \frac{\partial}{\partial x^i} = \gamma'(0) = \frac{d(\varphi \circ \gamma)^i}{dt}(0) \frac{\partial}{\partial x^i}.$$

We claim that since $v \notin T_p(\partial M)$, then $v^n \neq 0$. In fact, we will show that $v \in T_p(\partial M)$ iff $v^n = 0$: Suppose $v \in T_p(\partial M)$. Then by [1] Proposition 3.23, there exists a curve $\eta : [0, \varepsilon] \to M$ such that $\eta \in \partial M$ and $\eta'(0) = v$. Then under the chart (U, φ), we see that a portion of $\eta([0, \varepsilon])$ near p is mapped to ∂H^n. Hence $(\varphi \circ \eta)^n(t) = 0$ for sufficiently small t by definition. Therefore we see that $v^n = d(\varphi \circ \eta)^n/dt|_{t=0} = 0$. Conversely, if $v^n = 0$, then consider $\hat{\eta} : [0, \varepsilon] \to H^n$ defined by $\hat{\eta}^i(t) = v^i(t) + \varphi^n(p)$. Then it is easy to check that the curve $\eta = \varphi^{-1} \circ \hat{\eta} : [0, \varepsilon] \to M$ satisfies $\eta(0) = p, \eta'(0) = v$. Since $\hat{\eta} \subset \partial H^n$, we have $\eta \subset \partial M$. Therefore $v \in T_p(\partial M)$.

Now that we argued since $v \notin T_p(\partial M)$, we have $v^n \neq 0$, then we must have $(\varphi \circ \gamma)^n(t) > 0$ for all $t > 0$, otherwise $d(\varphi \circ \gamma)^n(t)/dt|_{t=0} = v^n$ will be zero, which is a contradiction. But then this shows that $v^n > 0$ since

$$\frac{d(\varphi \circ \gamma)^n(t)}{dt}|_{t=0} = \lim_{t \to 0^+} \frac{(\varphi \circ \gamma)^n(t) - \varphi^n(p)}{t} = \lim_{t \to 0^+} \frac{(\varphi \circ \gamma)^n(t)}{t} > 0.$$

Conversely, suppose that $v = v^i \partial/\partial x^i$ with $v^n > 0$. Then by our argument above $v \notin T_p(\partial M)$. Consider the curve $\hat{\gamma} : [0, \varepsilon] \to H^n$ defined by $\hat{\gamma}^i(t) = v^i t + \varphi^i(p)$. Then it is easy to check that $\gamma = \varphi^{-1} \circ \hat{\gamma} : [0, \varepsilon] \to M$ satisfies $\gamma(0) = p, \gamma'(0) = v$. Hence v is inward-pointing. The outward-pointing statement follows immediately from definition. □

Theorem 4.10. *If M is any smooth manifold with boundary, there is a smooth outward pointing vector field along ∂M.*

Proof. Cover a neighborhood of ∂M by smooth boundary charts $(U_\alpha, \varphi_\alpha)$. In each such chart,

$$\nu_\alpha = -\frac{\partial}{\partial x^n}|_{\partial M \cap U_\alpha}$$

is a smooth vector field along $\partial M \cap U_\alpha$, and is outward-pointing by the previous lemma. Let (ρ_α) be a partition of unity subordinate to the cover $(\partial M \cap U_\alpha)_\alpha$ of ∂M, and define a global vector field ν along ∂M by

$$\nu = \sum_\alpha \rho_\alpha \nu_\alpha.$$

Clearly ν is a smooth vector field along ∂M. To show that ν is outward-pointing, let (y^j) be any smooth boundary coordinate in a neighborhood of $p \in \partial M$. Because each ν_α is outward pointing, it satisfies $dy^n(v_\alpha) < 0$. Hence

$$\nu^n = dy^n(\nu) = \sum_\alpha \rho_\alpha dy^n(\nu_\alpha) < 0.$$

This concludes the proof. □

Now we define, for an oriented manifold M, its ***boundary orientation*** on ∂M. In non-rigorous terms, an orientation on ∂M is a consistent choice of orientation at each $T_p(\partial M)$. Let $N \in T_pM$ be an outward pointing vector at ∂M. Let $(E_1, ..., E_{n-1})$ be an ordered basis of $T_p(\partial M)$. We define this basis to have positive orientation iff $(N, E_1, ..., E_{n-1})$ is a positively oriented basis of T_pM.

More rigorously, let $\omega \in \Omega^n(M)$, and let N be an outward-pointing vector field on ∂M. Then define $\iota^*_{\partial M}(N \lrcorner \omega) \in \Omega^{n-1}(\partial M)$ by

$$\iota^*_{\partial M}(N \lrcorner \omega)_p(v_1, ..., v_{n-1}) = \omega((\iota_{\partial M})_*(N_p), (\iota_{\partial M})_*(v_1), ..., (\iota_{\partial M})_*(v_{n-1}))$$

where $\iota_{\partial M} : \partial M \to M$ is the inclusion map, and $v_1, ..., v_{n-1} \in T_p(\partial M)$.

Proposition 4.11. If ω is an orientation form on M, then $\iota^*_{\partial M}(N \lrcorner \omega)$ is an orientation form on ∂M.

Proof. Denote $\sigma = \iota^*_{\partial M}(N \lrcorner \omega)$. Then $\sigma \in \Omega^{n-1}(\partial M)$. It will follow that σ is an orientation form on ∂M if we can show it never vanishes. Given a basis $(E_1, ..., E_{n-1})$ of $T_p(\partial M)$, the fact that $N_p \notin T_p(\partial M)$ implies $(N_p, E_1, ..., E_{n-1})$ is a basis for T_pM. Therefore

$$\sigma_p(E_1, ..., E_{n-1}) = \omega_p(N_p, E_1, ..., E_{n-1}) \neq 0$$

since ω_p is non-vanishing at every p, and if ω_p is zero on a basis, then a multilinear-argument would show that $\omega_p \equiv 0$. We also used the fact that $\iota^*_{\partial M}$ is just restriction to vectors tangent to ∂M. Since σ_p does not vanish and p is arbitrary, σ is an orientation form. Moreover, since $\sigma(E_1, ...E_{n-1}) > 0$ iff $\omega_p(N_p, E_1, ..., E_{n-1}) > 0$, the orientation determined by σ is exactly as described in our informal statement.

It remains to show that this orientation is independent of the choice of N. Let (x^i) be coordinate of a boundary chart near $p \in \partial M$. If N and $\tilde{N}$ are two different outward-pointing vector field on ∂M, then both $(N_p, \partial_1, ..., \partial_{n-1})$ and $(\tilde{N}_p, \partial_1, ..., \partial_{n-1})$ are bases for T_pM, and the transition matrix between them has determinant given by $N^n(p)/\tilde{N}^n(p) > 0$. Thus both bases determine the same orientation for T_pM. Hence both $N, \tilde{N}$ determines the same orientation for $T_p(\partial M)$. □

Corollary 4.12. *Let ∂H^{n-1} be equipped with the induced orientation when H^n itself has the standard orientation inherited from $\mathbb{R}^n$. Then we can identify ∂H^n with $\mathbb{R}^{n-1}$. But the induced orientation agrees with the standard orientation on $\mathbb{R}^{n-1}$ only when n is even. That is,*

$$\partial H^n = (-1)^n \mathbb{R}^{n-1}.$$

Proof. The standard orientation on $\mathbb{R}^{n-1}$ is positively oriented for ∂H^n iff the ordered basis $[-\partial_n, \partial_1, ..., \partial_{n-1}]$ is a positively oriented basis for $\mathbb{R}^n$, where $-\partial_n$ is an outward-pointing vector field of H^n. Note that

$$[-\partial_n, \partial_1, ..., \partial_{n-1}] = -[\partial_n, \partial_1, ..., \partial_{n-1}] = (-1)^n[\partial_1, ..., \partial_n].$$

This concludes the proof. □

4.3 Stokes's Theorem

Theorem 4.13 (Stokes's Theorem)**.** *Let M^n be an oriented manifold with boundary, and $\omega \in \Omega_c^{n-1}(M)$. Then let ∂M to have the induced boundary orientation. We then have that*

$$\int_M d\omega = \int_{\partial M} \iota_{\partial M}^* \omega.$$

More concisely, we can write

$$\int_M d\omega = \int_{\partial M} \omega. \tag{4.2}$$

Proof. First suppose $M = H^n$. Since ω is of compact support, choose $R > 0$ such that $\operatorname{Supp} \omega \subset [-R, R]^{n-1} \times [0, R] =: A$. Write ω in standard coordinate as

$$\omega = \sum_{i=1}^{n} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where the hat means that dx^i is omitted. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

We can change the order of integration in each term so as to do the x^i integration first. By the fundamental theorem of calculus, the terms for which $i \neq n$ reduce to

$$\begin{aligned}
&\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R\frac{\partial\omega_i}{\partial x^i}(x)dx^1\cdots dx^n\\
&=\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R\frac{\partial\omega_i}{\partial x^i}(x)dx^i dx^1\cdots\widehat{dx^i}\cdots dx^n\\
&=\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R[\omega_i(x)]_{x^i=-R}^{x^i=R}\,dx^1\cdots\widehat{dx^i}\cdots dx^n=0,
\end{aligned}$$

because we have chosen R large enough such that $\omega = 0$ when $x^i = \pm R$. The only term that might not be zero is the one for which $i = n$. For that term we have

$$\begin{aligned}
\int_{H^n} d\omega &= (-1)^{n-1}\int_{-R}^R\cdots\int_{-R}^R\int_0^R\frac{\partial\omega_n}{\partial x^n}(x)dx^n dx^1\cdots dx^{n-1}\\
&=(-1)^{n-1}\int_{-R}^R\cdots\int_{-R}^R[\omega_n(x)]_{x^n=0}^{x^n=R}\,dx^1\cdots dx^{n-1}\\
&=(-1)^n\int_{-R}^R\cdots\int_{-R}^R\omega_n\left(x^1,\ldots,x^{n-1},0\right)dx^1\cdots dx^{n-1}
\end{aligned}$$

because $\omega_n = 0$ when $x^n = R$. To compare this to the other side of (4.2), we compute as follows:

$$\int_{\partial H^n}\omega=\sum_i\int_{A\cap\partial H^n}\omega_i\left(x^1,\ldots,x^{n-1},0\right)dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n.$$

Because x^n vanishes on ∂H^n, the pullback of dx^n to the boundary is identically zero. Thus, the only nonzero term is the one for which $i = n$, which becomes

$$\begin{aligned}
\int_{\partial H^n}\omega &= \int_{A\cap\partial H^n}\omega_n\left(x^1,\ldots,x^{n-1},0\right)dx^1\wedge\cdots\wedge dx^{n-1}\\
&=(-1)^n\int_{A\cap\partial H^n}\omega_n(x^1,...,x^{n-1},0)dx^1\cdots dx^{n-1}
\end{aligned}$$

which is equal to $\int_{H^n} d\omega$. Note that in the last step we used Corollary 4.12.

Next we consider another special case: $M = \mathbb{R}^n$. In this case, the support of ω is contained in a cube of the form $A = [-R, R]^n$. Exactly the same computation goes through, except that in this case the $i = n$ term vanishes like all the others, so the left-hand side of (4.2) is zero. Since M has empty boundary in this case, the right-hand side is zero as well.

Now let M be a smooth manifold with boundary, $\omega \in \Omega_c^{n-1}(M)$ whose support is contained in a single chart (U, φ). WLOG assume this chart is positively oriented. The definition yields

$$\int_M d\omega=\int_{H^n}(\varphi^{-1})^*d\omega=\int_{H^n}d(\varphi^{-1*}\omega)=\int_{\partial H^n}(\varphi^{-1})^*\omega$$

from the calculation above, where ∂H^n has the induced orientation. Since φ_* takes outward-pointing vectors on ∂M to outward-pointing vectors on H^n, it follows that $\varphi|_{U\cap\partial M}$ is an orientation preserving diffeomorphism on $\varphi(U)\cap\partial H^n$, and the expression above is equal to $\int_{\partial M}\omega$. For an interior chart, we get the same computations with H^n replaced by $\mathbb{R}^n$. This proves the theorem in this case.

Finally, let $\omega\in\Omega_c^{n-1}(M)$ be arbitrary. Choose a cover of $\operatorname{Supp}\omega$ by finitely many domains of positively oriented smooth charts $(U_i)_i$, and choose a subordinate smooth partition of unity $(\psi_i)_i$. We can apply the argument to $\psi_i\omega$ for each i and obtain

$$\begin{aligned}\int_{\partial M}\omega &= \sum_i\int_{\partial M}\psi_i\omega=\sum_i\int_M d\left(\psi_i\omega\right)=\sum_i\int_M d\psi_i\wedge\omega+\psi_i d\omega\\ &=\int_M d\left(\sum_i\psi_i\right)\wedge\omega+\int_M\left(\sum_i\psi_i\right)d\omega=0+\int_M d\omega\end{aligned}$$

because $\sum_i\psi_1\equiv 1$. This completes the proof of Stoke's theorem. □

Example 4.14. Read Example 16.12 and Theorem 16.17 of [1] to see how Stokes's theorem can be applied to obtain the fundamental theorem of line integrals and Green's theorem. This site also contains a derivation of the classical Stokes's theorem and the divergence theorem in $\mathbb{R}^3$ using Stokes's theorem.

4.4 De Rham Cohomology

Now that we have defined exterior differentiation operator $d:\Omega^k(M)\to\Omega^{k+1}(M)$, we recall that $d^2=0$. This means that $\operatorname{Im} d|_{\Omega^{k+1}(M)}\subset\operatorname{Ker} d|_{\Omega^k(M)}$ for all k. Therefore, we have a sequence of ***cochain complex***, i.e. a sequence of abelian groups with ascending index with a sequence of maps d connecting them such that $d^2=0$:

$$\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\Omega^2(M)\xrightarrow{d}\cdots\Omega^k(M)\xrightarrow{d}\Omega^{k+1}(M)\xrightarrow{d}\cdots$$

Then we define the kth ***de Rham cohomology group*** of M by

$$H^k_{dR}(M;\mathbb{R})=\frac{\operatorname{Ker} d|_{\Omega^k(M)}}{\operatorname{Im} d|_{\Omega^{k-1}(M)}}=\frac{Z^k(M)}{B^k(M)}.$$

Therefore, this group is a quotient of ***closed*** k-forms $Z^k(M)$, i.e., $d\omega=0$, by the ***exact*** k-forms $B^k(M)$, i.e., $\omega=d\eta$ for some $\eta\in\Omega^{k-1}(M)$. Then

$$H^*_{dR}(M)=\bigoplus_{0\le k\le\dim M}H^k_{dR}(M)$$

is a ring with wedge product as multiplication operation. Now we calculate H^0_{dR} and H^1_{dR} of S^1:

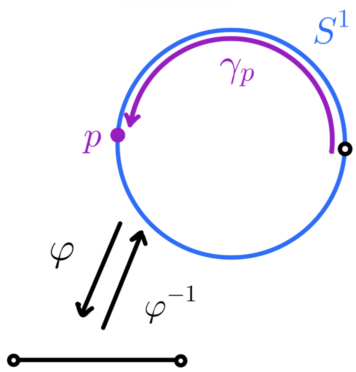
Proposition 4.15. $H^0_{dR}(S^1)\cong\mathbb{R}$.

Proof. Since $H^0_{dR}(S^1) = Z^0(S^1)/B^0(S^1) = Z^0(S^1)$ which is the set of closed 0-forms on S^1: i.e., $f \in C^\infty(S^1)$ such that $df = 0$. Hence these are all constant functions with value in reals. □

To calculate $H^1_{dR}(S^1)$, we need the following lemma:

Lemma 4.16. *Let $\omega \in \Omega^1(S^1)$. If $\int_{S^1} \omega = 0$, then ω is exact.*

Proof. Assume $\int_{S^1} \omega = 0$. We define, for every $p \in S^1 - \{(1,0)\}$, the function $f(p) = \int_{\gamma_p} \omega$, where $\gamma_p : (0, \Theta) \to S^1$ is the curve that starts at $(1,0) \in S^1$ and ends at p, where we take U to be the open set $S^1 - \{(1,0)\}$ and consider the chart (U, φ, θ) where $\varphi^{-1}(\phi) = (\cos\phi, \sin\phi)$ for $\phi \in (0, 2\pi)$. Then we denote $\varphi^{-1}(\Theta) = p$. We can even be more specific and define $\gamma_p : (0, \Theta)$ by $\gamma_p(\phi) = \varphi^{-1}(\phi) = (\cos\phi, \sin\phi)$ for any $\phi \in (0, \Theta)$. This is shown in the figure below:



It can be shown that we can extend f to a function $f : S^1 \to \mathbb{R}$ that is well-defined and smooth (exercise below) by setting $f(1,0) = 0$. In particular, f is smooth at (1,0). Then we claim that $\omega = df$, hence proving exactness of ω. Write $\omega = \eta d\theta$ with chart (φ, θ). Then note that $\eta \circ \gamma(\phi) = \eta \circ \varphi^{-1}(\phi)$ and $\theta \circ \gamma_p(\phi) = \theta \circ \varphi \circ \varphi^{-1}(\phi) = \theta(\phi) = \phi$ where we untangled the abuse of notation where $\theta : S^1 \to \mathbb{R}$ is really $\theta \circ \varphi^{-1}$ for $\theta : (0, \Theta) \to \mathbb{R}$ which is the coordinate function on $(0, \Theta)$ by $\Theta(\phi) = \phi$.

$$\int_{\gamma_p} \omega = \int_{(0,\Theta)} \gamma_p^*(\eta d\theta) = \int_{(0,\Theta)} (\eta \circ \gamma_p) d(\theta \circ \gamma_p) = \int_0^\Theta (\eta \circ \varphi^{-1})(\phi) d\phi.$$

Therefore, we see that

$$(f \circ \varphi^{-1})(\Theta) = \int_0^\Theta (\eta \circ \varphi^{-1})(\phi) d\phi. \tag{4.3}$$

Fundamental theorem of calculus applied to (4.3) and the definition of differential (exterior derivative) gives

$$df = \frac{\partial f}{\partial \theta} d\theta = \left(\frac{\partial (f \circ \varphi^{-1})}{\partial \theta} \circ \varphi \right) d\theta = (\eta \circ \varphi^{-1} \circ \varphi) d\theta = \eta d\theta = \omega.$$

This shows that $df = \omega$ on $U = S^1 - \{(1,0)\}$. The next exercise shows that this must be true globally for a function $f : S^1 \to \mathbb{R}$ that is a smooth extension of U, hence the proof of the lemma will be completed. □

Homework 4.17. Show that f defined in the previous lemma is well-defined and smooth.

Solution. In the proof of Lemma.4.16, we have already seen how to define f on $U = S^1 - \{(1,0)\}$. Smoothness of f on U follows from (4.3), fundamental theorem of calculus, and the fact that $\eta \circ \varphi^{-1}$ is smooth, for it is the composition of two smooth maps. Similarly, we can consider the chart $V = S^1 - \{(-1,0)\}$ together with the chart ψ on $V \to (-\pi, \pi)$ given by $\psi^{-1}(\phi) = (\cos\phi, \sin\phi)$ for $\phi \in [-\pi, \pi]$. We can define g on V by $g(p) = \int_{\gamma_p'} \omega$ for $p \in V$ where $\gamma_p'(\phi) = \psi^{-1}(\phi)$ is a curve from (1,0) to p. Performing similar chart calculation, we get a similar expression of $g \circ \psi^{-1}$ as we did in (4.3). This shows smoothness on V.

Now we need to show that f, g agree on $U \cap V$. Note that $\gamma_p'(\phi) = \psi^{-1}(\phi) = \varphi^{-1} \circ \varphi \circ \psi^{-1}(\phi) = \gamma_p \circ (\varphi \circ \psi^{-1})(\phi)$. Thus our goal is to show that that the line integral $\int_\gamma \omega$ is independent of ***reparametrization***, i.e., invariant under composing $\gamma : (a,b) \to S^1$ with a diffeomorphism $\Psi : (c,d) \to (a,b)$. Denoting $\gamma' = \gamma \circ \Psi$. First assume that Ψ is orientation preserving (i.e., increasing), then by the previous results of integration, we have that

$$\int_{\gamma'} \omega = \int_{(c,d)} (\gamma \circ \Psi)^* \omega = \int_{(c,d)} \Psi^* \gamma^* \omega = \int_{(a,b)} \gamma^* \omega = \int_\gamma \omega.$$

If Ψ is orientation reversing, then there will be a minus sign involved, hence we would get $\int_{\gamma'} \omega = -\int_\gamma \omega$, which is consistent with our definition of integration. This shows that $f = g$ on $U \cap V$.

Now to extend $f : U \to \mathbb{R}$ to a globally smooth function $f : S^1 \to \mathbb{R}$, set $f(1,0) = 0$. Since f is smooth on U by (4.3), we only need to check smoothness of f at $(1,0)$. But note that in this definition, $f|V \equiv g$, hence by smoothness of g, we know f is smooth at $(1,0)$, completing the proof. □

Theorem 4.18. $H^1_{dR}(S^1) \cong \mathbb{R}$.

Proof. We construct an isomorphism $\lambda : H^1_{dR}(S^1) \to \mathbb{R}$ defined by

$$\lambda([\omega]) = \lambda_{[\omega]} = \frac{1}{2\pi} \int_{S^1} \omega$$

for $[\omega] \in H^1_{dR}(S^1)$. It is clear that λ defined this way is a homomorphism under addition. To see that λ is well-defined, let $[\omega'] = [\omega]$. Then we must have $\omega' = \omega + d\eta$ for some $\eta \in \Omega^0(S^1)$. Hence by Stoke's theorem we have

$$\lambda_{[\omega']} = \frac{1}{2\pi} \int_{S^1} \omega' = \frac{1}{2\pi} \int_{S^1} \omega + \frac{1}{2\pi} \int_{S^1} d\eta = \frac{1}{2\pi} \int_{S^1} \omega + \frac{1}{2\pi} \int_{\partial S^1} \eta = \frac{1}{2\pi} \int_{S^1} \omega = \lambda_{[\omega]}$$

since $\partial S^1 = \emptyset$. To see that λ is injective, suppose $\lambda_{[\omega]} = \lambda_{[\omega']}$. Then we must have

$$\int_{S^1} (\omega - \omega') = 0.$$

Then by Lemma.4.16, $\omega - \omega'$ is exact, hence $[\omega] = [\omega']$ in $H^1_{dR}(S^1)$. To see that λ is surjective, suppose $\alpha \in \mathbb{R}$. Then define a form $\omega \in \Omega^1(S^1)$ on charts U, V by $\omega|_U = \alpha d\theta$ and $\omega|_V = \alpha d\theta'$ where θ', θ are coordinate functions on V, U. Then we have

$$d\theta' = \frac{\partial \theta'}{\partial \theta} d\theta = \frac{\partial(\theta' \circ \psi^{-1})}{\partial \theta} d\theta = d\theta$$

on $U \cap V$, since $(\theta' \circ \psi^{-1})(\theta) = \theta$ for $\theta \in (0, \pi)$ and $(\theta' \circ \psi^{-1})(\theta) = \theta - 2\pi$ for $\theta \in (\pi, 2\pi)$. Therefore we see that ω is well-defined. We can replace integration on S^1 by integration on U since $S^1 - U$ is a set of measure zero. Therefore,

$$\lambda_{[\omega]} = \frac{1}{2\pi} \int_{S^1} \omega = \frac{1}{2\pi} \int_U \omega = \frac{1}{2\pi} \int_U \alpha d\theta = \alpha$$

for $\int_U d\theta = 2\pi$. This shows that λ is indeed an isomorphism, concluding the proof. $\square$

4.5 The De Rham Theorem

In the previous section, we have seen how to define de Rham cohomology groups of a smooth manifold using differential forms on M. Now we come to one of the most beautiful result in the theory of differential forms and smooth manifolds, namely the de Rham theorem, which says that we can use differential forms to obtain topological informations about manifolds. However, most of the proofs will be rather technical. Due to limited time, we won't cover most of the proofs. Interested readers can consult [1, Chapter 17-18], where all the proofs are given.

To begin with, we need to further extend our definition of integration on manifolds. A topological space M^n is called an n-dimensional ***manifold with corners*** if for every $p \in M$, there exists a pair (U, φ), where U is an open set in M containing p, and φ is a homeomorphism from U to an open set of

$$\bar{\mathbb{R}}^n_+ := \left\{(x_1, ..., x_n) \in \mathbb{R}^n : x^i \geq 0 \text{ for all } 1 \leq i \leq n\right\}.$$

A ***smooth structure with corners*** on M is a maximal collection of smoothly compactible charts, where two charts with corners $(U, \varphi), (V, \psi)$ are smoothly compatible if $\varphi \circ \psi : \psi(U \cap V) \to \varphi(U \cap V)$ is smooth.

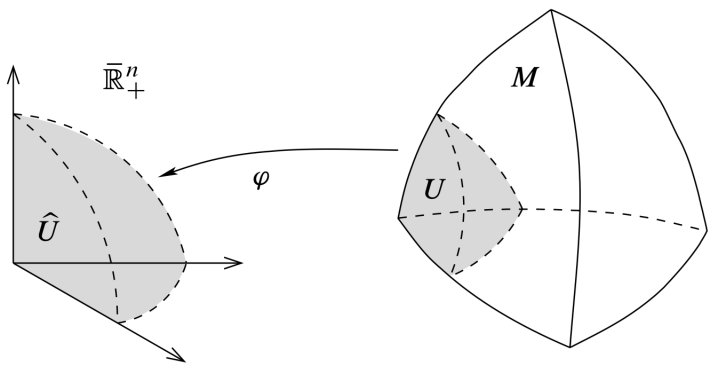


Figure 4.3: A manifold with corners and a chart (U, φ) with corners is shown in the picture.

It turns out that Stokes's theorem also holds for manifold M^n with corners, although the definition of ∂M and their orientations are a little tricky. Details are given in [1, p.415-421]. Let us simply take $\int_M d\omega = \int_{\partial M} \omega$ for $\omega \in \Omega^{n-1}(M)$ as a fact.

Now some more definitions: A ***p*-simplex** Δ_p is the set defined by

$$\Delta_p = \left\{(t_0, ..., t_n) \in \mathbb{R}^{p+1} : \Sigma_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\right\}.$$

It is easy to see that a p-simplex is a p-dimensional manifold with corners. Let M be a topological space. A ***singular p-simplex*** in M is a continuous map $\sigma : \Delta_p \to M$. From here we can define singular homology groups: Let $C_p(M)$ denote the free abelian groups generated by all continuous maps $\sigma : \Delta_p \to M$ with $\mathbb{Z}$-coefficients, there is a map $\partial_p : C_p(M) \to C_{p-1}(M)$ by

$$\partial_p(\sigma) = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_p].$$

Then it can be shown that $\partial^2 = 0$, hence one can define the p^{th} ***singular homology groups*** to be

$$H^p(M;\mathbb{Z}) = \frac{\operatorname{Ker} \partial|_{C^p(M)}}{\operatorname{Im} \partial|_{C^{p-1}(M)}}.$$

For details, we invite the reader to consult [5]. We remark that by [1, Theorem 18.7], we can take $\sigma : \Delta_p \to M$ to be smooth, in the sense that it has a smooth extension to a neighborhood of each point. Now, we are only concerned with ***singular cohomology groups with coefficients in*** $\mathbb{R}$ for a finite-dimensional manifold M, denoted by $H^p(M;\mathbb{R})$. By the universal coefficient theorem [5, Theorem 3.2], this is

$$H^p(M;\mathbb{R}) \cong \operatorname{Hom}(H_p(M);\mathbb{R}).$$

Suppose now M is a smooth manifold, $\omega \in \Omega^p(M)$ is closed. Let $\sigma : \Delta_p \to M$ be a smooth p-simplex. Then we define the ***integral of*** ω ***over*** σ to be

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^* \omega.$$

Note that this makes sense since the right hand side is integration on a subset of $\mathbb{R}^{n+1}$. Then we extend this definition linearly to a general element $\sigma = \sum_i c_i \sigma_i$.

Theorem 4.19 (Stokes's Theorem for Chains)**.** *Let $C \in C_p(M)$ be a smooth p-chain in a smooth manifold M, i.e., formal linear combinations of smooth maps $\sigma : \Delta_p \to M$. Then let ∂C denote its boundary where ∂ is defined above. Let $\omega \in \Omega^{p-1}(M)$. Then*

$$\int_{\partial C} \omega = \int_C d\omega.$$

Proof. See [1, Theorem 18.12]. □

Then we state the key result of this section:

Theorem 4.20 (The De Rham Theorem)**.** *There is an isomorphism, for every smooth manifold M and nonnegative integer p,*

$$H^p_{dR}(M) \cong H^p(M;\mathbb{R}).$$

More specifically, the isomorphism $\mathcal{D} : H^p_{dR}(M) \to H^p(M;\mathbb{R}) \cong \operatorname{Hom}(H_p(M);\mathbb{R})$ is given as follows: for $[\omega] \in H^p_{dR}(M)$, $[C] \in H_p(M)$, define $\mathcal{D}_{[\omega]}$ via

$$\mathcal{D}_{[\omega]}([C]) = \int_C \omega.$$

The proof of the De Rham theorem is quite technical. We refer the readers to [1, Theorem 18.14] for details. Here we remark why $\mathcal{D}$ is well-defined. This is ensured by Stokes's theorem for chains. Suppose $[\omega] = [\omega']$, then $\omega - \omega' = d\eta$ for some $\eta \in \Omega^{p-1}(M)$. Then

$$(\mathcal{D}_{[\omega]} - \mathcal{D}_{[\omega']})([C]) = \int_C (\omega - \omega') = \int_C d\eta = \int_{\partial C} \eta = 0$$

since $\partial C = \emptyset$, as C is a cycle representing the equivalence class $[C]$. Similarly, if $[C] = [C']$, then $C - C' = \partial B$ for some $B \in C_{p+1}(M)$. Note that by Theorem 18.7 of [1], we can assume all chains are smooth chains. Then again by Stokes's theorem, we have

$$(\mathcal{D}_{[\omega]})([C]) - (\mathcal{D}_{[\omega]})([C']) = \int_C \omega - \int_{C'} \omega = \int_{\partial B} \omega = \int_B d\omega = 0$$

since ω is closed, being a representative of $[\omega] \in H^p_{dR}(M)$, by assumption. This shows that $\mathcal{D}$ is indeed well-defined.

Therefore, now it is clear that

Corollary 4.21. *For homotopy equivalent (in particular, homeomorphic) smooth manifolds M, N, we have $H^p_{dR}(M) \cong H^p_{dR}(N)$. In other words, the de Rham cohomology group is a topological invariant.*

5 | Basic Riemannian Geometry

5.1 Riemannian Metrics

Definition 5.1. A ***Riemannian metric*** g on M is a smooth symmetric covariant 2-tensor field that is positive-definite at each point, i.e., $g_p(v, v) > 0$ for all $p \in M$, and for all $v \in T_pM$ nonzero.

A ***Riemannian manifold*** (M, g) is a pair where M is a smooth manifold together with a Riemannian metric g

If $\alpha \in \otimes^k V^*$ and $\beta \in \otimes^l V^*$, then even if α, β are symmetric, the tensor product $\alpha \otimes \beta$ is in general not symmetric. Therefore we can define a ***symmetric product*** $\alpha\beta \in \otimes^{k+l} V^*$ by

$$\alpha\beta(v_1, ..., v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})\beta(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)}).$$

For example, if α, β are covectors, then

$$\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha).$$

Then in any local coordinates (U, φ, x^i), a Riemannian metric can be written by

$$\begin{aligned} g = g_{ij} dx^i \otimes dx^j &= \frac{1}{2}(g_{ij} dx^i \otimes dx^j + g_{ji} dx^i \otimes dx^j) \\ &= \frac{1}{2}(g_{ij} dx^i \otimes dx^j + g_{ij} dx^j \otimes dx^i) \\ &= g_{ij} dx^i dx^j. \end{aligned}$$

For example, the Euclidean metric in $\mathbb{R}^n$ in standard coordinate is given by

$$E = \delta_{ij} dx^i dx^j.$$

Proposition 5.2. Every smooth manifold with or without boundary admits a Riemannian metric.

Proof. Let M be a smooth manifold with or without boundary, covered by a collection of coordinate charts $(U_\alpha, \varphi_\alpha)$. Then in each U_α, it is easy to check that $g_\alpha = \varphi_\alpha^* E$ is a Riemannian metric. Let (ψ_α) be a partition of unity subordinate to the cover (U_α), and define

$$g = \sum_\alpha \psi_\alpha g_\alpha.$$

Each term is interpreted to be zero outside of Supp ψ_α. Since at a neighborhood of every point g becomes just a finite sum, we see that g is smooth. Clearly g is symmetric. Therefore, we only need to check positive definiteness at every point. Let $v \in T_pM$ be nonzero. Then

$$g_p(v,v) = \sum_\alpha \psi_\alpha(p)(g_\alpha)_p(v,v) > 0.$$

This concludes the proof. □

Usually another conventional notation is given by $\langle v,w\rangle_p = g_p(v,w)$. And if $X, Y \in \mathfrak{X}(M)$, then $\langle X, Y\rangle$ is the smooth function $g(X,Y)$.

If $\gamma : [a,b] \to M$ is piecewise smooth cirve, then define the ***length*** of γ to be

$$L(\gamma) = \int_a^b |\gamma'(t)|dt = \int_a^b g(\gamma'(t),\gamma'(t))^{1/2}dt = \int_a^b \langle\gamma'(t),\gamma'(t)\rangle^{1/2}dt.$$

Proposition 5.3. Let (M,g) be a Riemannian manifold with or without boundary and let $\gamma : [a,b] \to M$ be a piecewise smooth curve segment. Then if $\tilde{\gamma}$ is a reparametrization of γ, i.e. $\tilde{\gamma} = \gamma\circ\varphi$ where $\varphi : [c,d] \to [a,b]$ is a diffeomorphism, then $L(\gamma) = L(\tilde{\gamma})$.

Proof. Using the definition of tangent vectors of curves, if f is a smooth function on the suitable domain, then

$$(\gamma\circ\varphi)'(t)f = (f\circ\gamma\circ\varphi)'(t) = (f\circ\gamma)'(t)\varphi'(t) = \varphi'(t)(\gamma'(\varphi(t)))f.$$

Hence we see that $(\gamma\circ\varphi)'(t) = \varphi'(t)\gamma'(\varphi(t))$. First suppose $\gamma' > 0$ (i.e., orientation preserving), then

$$\begin{aligned} L(\tilde{\gamma}) &= \int_c^d |\tilde{\gamma}'(t)|dt \\ &= \int_c^d |\varphi'(t)\gamma'(\varphi(t))|dt \\ &= \int_c^d |\gamma'(\varphi(t))|\varphi'(t)dt \\ &= \int_a^b |\gamma'(s)|ds = L(\gamma). \end{aligned}$$

In the next-to-last inequality, we used the change of variable formula for integrals. If now $\gamma' < 0$, then we just need to introduce two minus signs, one for $|\varphi'(t)| = -\varphi(t)$, another when we do change of variables since φ is orientation reversing. The two minus signs cancel with each other, still giving the same result. □

Next, we define, for two points $p, q \in M$ on a Riemannian manifold, their ***distance*** to be

$$d(p,q) = \inf_\gamma\{L(\gamma) : \gamma \text{ from } p \text{ to } q \text{ piecewise smooth}\}.$$

We will show that d is actually a metric, and the metric topology on M is the same as the manifold topology, hence making a Riemannian manifold a metric space. The proof requires the following lemma:

Lemma 5.4. *Let g be a Riemannian metric on an open subset $U \subset \mathbb{R}^n$. Given a compact subset $K \subset U$, there exist constants $c, C > 0$, such that for all $x \in K, v \in T_x\mathbb{R}^n$,*

$$c|v|_E \leq |v|_g \leq C|v|_E. \tag{5.1}$$

In other words, every Riemannian metric is locally comparable to Euclidean metric in coordinates.

Proof. Let $K \subset U$ be a compact set, and let $L \subset T\mathbb{R}^n$ be the set

$$L = \{(x, v) \in T\mathbb{R}^n : x \in K, |v|_E = 1\} \cong K \times S^{n-1}.$$

Therefore L is compact. Since the norm $|v|_g$ is continuous and strictly positive on L, there are positive constants $c, C > 0$ such that $c \leq |v|_g \leq C$, whenever $(x, v) \in L$. Then if $x \in K$ and $v \in T_x\mathbb{R}^n$ nonzero, set $\lambda = |v|_E$. We note that $(x, \lambda^{-1}v) \in L$. So we have

$$|v|_g = \lambda|\lambda^{-1}v|_g \leq \lambda C = C|v|_E.$$

Similar argument shows that $|v|_g \geq c|v|_E$. The same equality is trivially true when $v = 0$. □

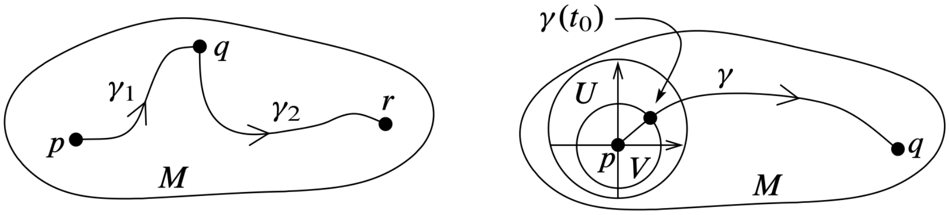


Figure 5.1: Proofs that d defines a metric. Left: triangle inequality; Right: positivity of d.

Theorem 5.5. *The distance function d defined above is a metric.*

Proof. It is clear that $d(p, q) \geq 0$ and $d(p, p) = 0$ from definition. $d(p, q) = d(q, p)$ follows from the fact that any curve can be reversely parametrized and we have shown that this does not change the length of the curve. Now we prove the triangle inequality: Let $p, q, r \in M$ and suppose γ_1 is a curve from p to q and γ_2 a curve from q to r. Then let $\gamma = \gamma_1 \sharp \gamma_2$ be the curve segment that first follows γ_1 then γ_2.[1] We have

$$d(p, r) \leq L(\gamma) = L(\gamma_1) + L(\gamma_2).$$

Taking inf over all such γ_1, γ_2 gives $d(p, r) \leq d(p, q) + d(q, r)$. We still need to show that $d(p, q) > 0$ if $p \neq q$. Let $p, q \in M$ be distinct points, let U be a smooth coordinate domain that contains p but not q. Let us identify U with $\varphi(U) \subset \mathbb{R}^n$, where $\varphi : U \to \varphi(U)$ is a chart. Let $V = \varphi^{-1}(B_\varepsilon(\varphi(p)))$ be a regular coordinate ball of radius ε centered at p such that $\overline{V} \subset U$. Then Lemma 5.4 says that whenever $x \in \overline{V}, v \in T_xM$, we have

$$c|v|_E \leq |v|_g \leq C|v|_E$$

[1] This shows that in the definition of d, it is necessary to take piecewise smooth curves.

for some $c, C > 0$. Integrating, we find that

$$cL_E(\gamma) \leq L_g(\gamma) \leq CL_E(\gamma)$$

for any piecewise smooth $\gamma \subset \overline{V}$. Now suppose $\gamma : [a,b] \to M$ is a curve from p to q. Let

$$t_0 = \inf_{t\in[a,b]} \{t : \gamma(t) \notin \overline{V}\}.$$

Then it follows that $\gamma(t_0) \in \partial V$ and $\gamma(t) \in \overline{V}$ for $a \leq t \leq t_0$ by continuity. Then

$$L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) \geq cL_E(\gamma|_{[a,t_0]}) \geq cd_E(p, \gamma(t_0)) = c\varepsilon.$$

Taking inf over all such γ, we conclude that $d_g(p,q) \geq c\varepsilon > 0$. This concludes the proof. □

Theorem 5.6. *The topology induced on M by the metric d_g coincides with the topology on M as a topological space.*

Proof. First suppose that $U \subset M$ is open in the manifold topology. Let $p \in U$ and V a regular coordinate ball with radius ε around p such that $\overline{V} \subset U$ as in the previous proof. The argument in the previous proof shows that $d_g(p,q) \geq c\varepsilon$ whenever $q \notin \overline{V}$. The contrapositive of this statement is that $d_g(p,q) \leq c\varepsilon$ implies $q \in \overline{V} \subset U$. This shows that U is open in the metric topology.

Conversely, let W be open in the metric topology, let $p \in W$. Let

- V be a regular coordinate ball of radius r around p;
- E be the Euclidean metric on $\overline{V}$ determined by given coordinates on $\overline{V}$;
- $c, C > 0$ be such that (5.1) is satisfied for $v \in T_qM$, $q \in \overline{V}$;
- $\varepsilon < r$ be a positive number small enough such that $B^{d_g}_{C\varepsilon}(p) \subset W$ (by previous paragraph);
- $V_\varepsilon(p) = \{x \in \overline{V} : d_E(p,x) < \varepsilon\}$. Note that since $d_E(p,\cdot) : M \to \mathbb{R}$ is continuous, $V_\varepsilon(p)$ is open in the manifold topology.

If $q \in V_\varepsilon(p)$, let γ be the straight line segment in coordinates from p to q. Using Lemma 5.4 as above, we conclude that

$$d_g(p,q) \leq L_g(\gamma) \leq CL_E(\gamma) < C\varepsilon.$$

This shows that $q \in B^{d_g}_{C\varepsilon}(p) \subset W$. Thus $V_\varepsilon(p) \subset W$. Since, as we remarked, $V_\varepsilon(p)$ is open in the manifold topology, we see that $W = \bigcup_{p\in W} V_\varepsilon(p)$ is open in the manifold topology. This concludes the proof. □

Definition 5.7. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Then a diffeomorphism $F : M_1 \to M_2$ is called an ***isometry***, if $F^*g_2 = g_1$.

Let (M, g) be a Riemannian manifold, and let $i_N : N \to M$ be a smooth immersion. Set
$$h = i_N^* g.$$
Then h is a smooth Riemannian metric on N, defined at every point by
$$h_p(v, w) = (i_N^* g)_p(v, w) = g_{i_N(p)}((i_N)_{*p}(v), (i_N)_{*p}(w)).$$

Definition 5.8. Let (M^n, g) be an *oriented* Riemannian manifold. Then the ***volume form*** of (M, g) is defined to be
$$dV_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$$
on each orientation chart (U, φ, x^i).

Lemma 5.9. *The volume form dV_g agrees on the intersection of two consistenly oriented charts (U, φ, x^i) and (V, ψ, y^j), i.e.,*
$$\sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det \tilde{g_{ij}}}\, dy^1 \wedge \cdots \wedge dy^n.$$

Proof. We know that
$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}, \frac{\partial y^l}{\partial x^j}\frac{\partial}{\partial y^l}\right) = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\tilde{g_{kl}}.$$
Therefore,
$$\det(g_{ij}) = \det\left(\frac{\partial y^k}{\partial x^i}\right)^2 \det(\tilde{g_{kl}}).$$
Thus the volume form agree by Corollary 3.4, where we left the remaining calculation as an exercise below. Note that since M is oriented, we have $\det(\partial y^k/\partial x^i) = J_{\psi\circ\varphi} > 0$. Therefore
$$\left[\det\left(\frac{\partial y^k}{\partial x^i}\right)^2\right]^{1/2} = \left|\det\left(\frac{\partial y^k}{\partial x^i}\right)\right| = \det\left(\frac{\partial y^k}{\partial x^i}\right).$$
This proves the claim. □

Homework 5.10. Finish the calculation in the previous lemma and show that the volume form agrees on the intersection.

Solution. Continuing from above, we have that (also using Corollary 3.4):
$$\begin{aligned}\sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n &= \left[\det\left(\frac{\partial y^k}{\partial x^i}\right)^2 \det \tilde{g_{kl}}\right]^{1/2} \cdot \det\left(\frac{\partial x^i}{\partial y^l}\right) dy^1 \wedge \cdots \wedge dy^n \\ &= \sqrt{\det \tilde{g_{kl}}}\, \det\left(\frac{\partial y^k}{\partial x^i}\right) \det\left(\frac{\partial x^i}{\partial y^l}\right) dy^1 \wedge \cdots \wedge dy^n \\ &= \sqrt{\det \tilde{g_{kl}}}\, dy^1 \wedge \cdots \wedge dx^n.\end{aligned}$$

This shows that dV_g is indeed well-defined. □

The notion of a volume form is actually quite reasonably well-motivated: Consider a positive orthonormal basis $e_1, ..., e_n$ of T_pM and write the coordinate basis $\frac{\partial}{\partial x^i}(p) = \sum_{ij} a_{ij} e_j$. Then the volume $\mathrm{Vol}(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$ of the parallelepiped formed by the coordniate basis is equal to $\mathrm{Vol}(e_1, ..., e_n) = 1$ multiplied by $\det(a_{ij})$. And since

$$g_{ik}(p) = g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) = \sum_{jl} a_{ij} a_{kl} \langle e_j, e_l\rangle = \sum_j a_{ij} a_{kj}$$

we have that

$$\mathrm{Vol}\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\right) = \det(a_{ij}) = \sqrt{(\det g_{ij})(p)}.$$

By multilinearity, it is reasonable to define volume form as above. Therefore, for a subset $R \subset M$, suppose R is contained in a single positive chart (U, φ, x^i), then we define the ***volume*** of R to be

$$\mathrm{Vol}(R) = \int_R dV_g = \int_{\varphi^{-1}(R)} \sqrt{\det(g_{ij}) \circ \varphi^{-1}} dx^1 \cdots dx^n.$$

Note that $\mathrm{Vol}(R)$ is well-defined by the previous lemma.

5.2 Covariant Derivative

Definition 5.11. A ***connection*** ∇ on a smooth manifold M is a mapping $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is given by $(X, Y) \mapsto \nabla_X Y$, such that for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$, we have

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$.

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

Lemma 5.12. *The value $\nabla_X Y|_p$ at a point $p \in M$ only depends on the value of X at p and the value of Y in a neighborhood of p. That is, if $X(p) = X'(p)$ and $Y = Y'$ in a neighborhood of p, then $\nabla_X Y|_p = \nabla_{X'} Y|_p$ and $\nabla_X Y|_p = \nabla_X Y'|_p$.*

Proof. Suppose that $X(p) = X'(p)$ for $X, X \in \mathfrak{X}(M)$. Denote $U = X - X'$. Then take a chart (U, φ, x^i) containing p. Write $U = U^i(x)\partial_i$ for $x \in U$. Then by (i), we have that $\nabla_X Y - \nabla_{X'} Y = \nabla_U Y = U^i \nabla_{\partial_i} Y$. But since $U^i(p) = 0$ for all i, we have that

$$\nabla_X Y|_p - \nabla_{X'} Y|_p = 0.$$

Next, suppose $Y, Y' \in \mathfrak{X}(M)$ agrees in a neighborhood of p. By shrinking U if necessary, we may assume that $Y|_U = Y'|_U$. Let ρ be a bump function supported on U. Then the vector field $\rho(Y - Y') \equiv 0$ on M. Therefore, by (ii) and (iii), we have that

$$\nabla_X \rho(Y - Y') = \nabla_X \rho Y - \nabla_X \rho Y' = \rho(\nabla_X Y - \nabla_X Y') + X(\rho)(Y - Y').$$

Evaluating the equation above at p, noting that $\rho(p) = 1$, and $(Y - Y')(p) = 0$. Therefore, this gives that

$$\nabla_X Y|_p - \nabla_X Y'|_p = 0.$$

This concludes the proof. $\square$

Remark 5.13. By the lemma, it makes sense for us to talk about $\nabla_v Y$ just for some vector $v \in T_p M$. To find the value of this expression, extend v to any vector field $X \in \mathfrak{X}(M)$ and evaluate $\nabla_X Y$ at p. By the previous lemma, this is a well-defined quantity, which we may call $\nabla_v Y$. Therefore, it also make sense to talk about $\nabla_X Y$ even if X is not completely defined on M, but is extendable to a smooth vector field on M. In fact, in the following proposition, part (c) is justified exactly by this reason.

Proposition 5.14. Let M be a smooth manifold with connection ∇. Then there exists a unique correspondence which associates to a vector field V along the smooth curve $c : I \to M$ another vector field $\frac{DV}{dt}$ along c, called the ***covariant derivative of*** V ***along*** c, such that

(a)
$$\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}.$$

(b)
$$\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$$

where W is a vector field along c and f is a differentiable function on I.

(c) If V is induced by a vector field $Y \in \mathfrak{X}(M)$, i.e., $V(t) = Y(c(t))$, then
$$\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y.$$

Proof. Suppose such a correspondence exists. Choose a chart (U, φ, x^i) such that $U \cap c(I) \neq \emptyset$. Then we may express the vector field V locally as $V = V^j \frac{\partial}{\partial x^j}$. By (a) and (b), we have that
$$\frac{DV}{dt} = \sum_j \left(\frac{dV^j}{dt} \frac{\partial}{\partial x^j} + V^j \frac{D}{dt} \frac{\partial}{\partial x^j} \right)$$
where $v^j = v^j(t)$ and $\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}(c(t))$. Now by part (c) and definition part (i), we have
$$\frac{D}{dt}\frac{\partial}{\partial x^j} = \nabla_{dc/dt} \frac{\partial}{\partial x^j} = \nabla \frac{dx^i}{dt} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_i \frac{dx^i}{dt} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$$
where $\varphi \circ c = (x^1(t), ..., x^n(t))$. Therefore, we arrive at
$$\frac{DV}{dt} = \frac{dV^j}{dt}\frac{\partial}{\partial x^j} + \frac{dx^i}{dt} V^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \tag{5.2}$$
where the index i, j is summed over using Einstein summation convention. (5.2) suggests that if such a correspondence exists, then it is unique. To show existence, define DV/dt in U by (5.2). It is easy to verify that (5.2) has the desired properties, and if (V, ψ, y^j) is another coordinate neighborhood with $U \cap V \neq \emptyset$, then two definitions of DV/dt agrees in $U \cap V$ by uniqueness of DV/dt in U. It follows that this definition can be extended to all of M, concluding the proof. $\square$

Definition 5.15. Let M be a smooth manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called ***parallel*** when $DV/dt = 0$ for all $t \in I$.

Remark 5.16. If we set

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

then the vector V being parallel along c is just the condition

$$\frac{DV}{dt} = \left(\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k} = 0.$$

This is really just a sysytem of n ODEs given by

$$\frac{dV^k}{dt} + \frac{dx^i}{dt} V^j \Gamma^k_{ij} = 0$$

for $k = 1, ..., n$. Given a set of initial conditions, i.e., $V^k(t_0) = v^k$, then by existence and uniqueness theorem in ODE, we have a unique solution. Moreover, since the system in linear, any solution is defined for all $t \in I$. This proves existence and uniqueness of parallel vectors.

For future convenience, we may abbreviate $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ by $\nabla_i \partial_j$.

Definition 5.17. Let M be a smooth manifold with an affine connection and a Riemannian metric $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. Then a connection is sais to be ***compatible*** with the metric g when for any smooth curve c and pair of parallel vector fields V, W along c, we have

$$\frac{d}{dt} \langle V, W \rangle = 0.$$

Proposition 5.18. ∇ is compatible with a metric g iff for any pair of vector fields V, W along the differentiable curve $c : I \to M$ one has

$$\frac{d}{dt} \langle V, W \rangle = \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle. \tag{5.3}$$

Proof. Obviously (5.3) implies compatibility with the metric. Let us prove the other direction. Choose an orthonormal basis $P_1(t_0), ..., P_n(t_0)$ of $T_{c(t_0)} M$, where $t_0 \in I$. By the remark above, we can extend the vectors $P_i(t_0)$ along c so that they remain parallel. Because ∇ is compatible with the metric, $P_1(t), ..., P_n(t)$ is an orthonormal basis of $T_{c(t)} M$ for any $t \in I$. Therefore, we can write $V = v^i P_i$ and $W = w^i P_i$ where v^i, w^j are differentiable functions on I. By (5.2) and replacing ∂_j by P_j, we obtain that

$$\frac{DV}{dt} = \frac{dv^i}{dt} P_i + v^i \frac{DP_i}{dt} = \frac{dv^i}{dt} P_i$$

since P_i is parallel. Similarly $DW/dt = (dw^i/dt) P_i$. Therefore we have

$$\langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle = \sum_i \left(\frac{dv^i}{dt} w^i + \frac{dw^i}{dt} v^i \right) = \frac{d}{dt} \left(\sum_i v^i w^i \right) = \frac{d}{dt} \langle V, W \rangle.$$

This concludes the proof. □

Corollary 5.19. *A connection ∇ on a Riemannian manifold M is compatible with the metric iff*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{5.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof. Suppose that ∇ is compatible with the metric. Let $p \in M$ and let $c : I \to M$ be a differentiable curve with $c(t_0) = p$ where $t_0 \in I$ and $(dc/dt)(t_0) = X(p)$. Then we have

$$X(p)\langle Y, Z\rangle = \frac{d}{dt}\langle Y, Z\rangle|_{t=t_0} = \langle \frac{DY}{dt}, Z\rangle_p + \langle Y, \frac{DZ}{dt}\rangle_p = \langle \nabla_X Y, Z\rangle_p + \langle Y, \nabla_X Z\rangle_p.$$

In the last equality we used part (c) of Proposition 5.14. Since $p \in M$ is arbitrary, (5.4) follows. The converse is obvious if we take $X = d/dt$ to be the tangent vector along a curve $c : I \to M$ and Y, Z be two induced vector along c that are parallel, and appeal to (c) of Proposition 5.14 again. □

Definition 5.20. An affine connection on a smooth manifold M is sais to be symmetric if for all $X, Y \in \mathfrak{X}(M)$, one has

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Theorem 5.21 (Levi-Civita)**.** *Given a Riemannian manifold (M, g), there exists a unique affine connection ∇ on M satisfying the conditions:*

(a) ∇ is symmetric;

(b) ∇ is compatible with g.

Proof. Assuming the existence of such a ∇. Then we have three equations

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{5.5}$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \tag{5.6}$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \tag{5.7}$$

Now (5.5)+(5.6)-(5.7) and symmetry gives

$$\begin{aligned} &X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\qquad = \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle + 2\langle Z, \nabla_Y X\rangle . \end{aligned}$$

Therefore

$$\begin{aligned} \langle Z, \nabla_Y X\rangle =& \frac{1}{2}\{X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &- \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\}. \end{aligned}$$

This expression shows that ∇ is uniquely determined from the metric. Therefore, if such ∇ exists, then it must be unique. To prove existence, define ∇ using the expression above, and it is easy to verify that ∇ is well-defined, and satisfies the desired conditions. This completes the proof. □

Now we wish to find the expression of $\nabla_X Y$ in a coordinate chart (U, φ, x^i). By multilinearity, all we need is an expression for $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_i \partial_j$. Previously, we defined

$$\nabla_i \partial_j = \Gamma_{ij}^k \partial_k.$$

Hence it remains to express Γ_{ij}^k in terms of the metric g. By compatibility, we have

$$\begin{aligned}\partial_i g(\partial_j, \partial_k) &= g(\nabla_i \partial_j, \partial_k) + g(\partial_j, \nabla_i \partial_k) \\ &= g(\Gamma_{ij}^\ell \partial_\ell, \partial_k) + g(\partial_j, \Gamma_{ik}^\ell \partial_\ell) = \Gamma_{ij}^\ell g_{\ell k} + \Gamma_{ik}^\ell g_{j\ell}.\end{aligned}$$

Now we introduce a new notation $\Gamma_{ij}^\ell g_{\ell k} = \Gamma_{ijk}$ and $\Gamma_{ik}^\ell g_{j\ell} = \Gamma_{ikj}$. Then the equation above gives

$$\Gamma_{ijk} + \Gamma_{ikj} = \partial_i g_{jk}. \tag{5.8}$$

Cyclicly permuting the indices gives two more equations:

$$\Gamma_{jki} + \Gamma_{jik} = \partial_j g_{ki}. \tag{5.9}$$

$$\Gamma_{kij} + \Gamma_{kji} = \partial_k g_{ij}. \tag{5.10}$$

Note that Γ_{ijk} is symmetric in the first 2 indices, to see this note that since $[\partial_i, \partial_j = 0]$, we have

$$\nabla_i \partial_j - \nabla_j \partial_i = \Gamma_{ij}^k \partial_k - \Gamma_{ji}^k \partial_k = (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k = 0.$$

Hence $\Gamma_{ij}^k = \Gamma_{ji}^k$. This shows that Γ_{ijk} is symmetric in the first 2 indices. Now (5.8)+(5.2)-(5.10) gives

$$\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} = 2\Gamma_{ijk} = 2\Gamma_{ij}^\ell g_{\ell k}.$$

Since g_{ij} has an inverse $g^{k\ell}$ with $g^{k\ell} g_{\ell m} = \delta_m^k$, multiplying both sides by the inverse matrix and (possibly) renaming the indices, we get that

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}). \tag{5.11}$$

5.3 Curvature

Definition 5.22. The ***curvature*** R of a Riemannian manifold M is a corresponding that associates to every pair $X, Y \in \mathfrak{X}(M)$ a mapping $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Proposition 5.23. The Riemann curvature tensor R satisfies the following properties:

(i) $R(\cdot, \cdot)$ is bilinar: $R(fX_1 + gX_2, Y) = fR(X_1, Y) + g(X_2, Y)$, for $f, g \in C^\infty(M)$, and similarly for the second argument.

(ii) The curvature operator $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is linear, i.e., $R(X, Y)(Z + W) = R(X, Y)Z + R(X, Y)W$, and $R(X, Y)(fZ) = fR(X, Y)Z$ for $f \in C^\infty(M)$.

(iii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$. This is called the ***Bianchi identity***.

Proof. The full proof is given in [2], section 4.2 (p.90-91). Hence we omit the proof here. □

Again we wish to express R in coordinate frame (U, φ, x^i). Again by multilinearity we only need to know $R(\partial_i, \partial_j)\partial_k$. We write

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}^{\ell}\partial_\ell.$$

Then by definition we have

$$\begin{aligned} R(\partial_i, \partial_j)\partial_k &= R_{ijk}^{\ell}\partial_\ell \\ &= -\nabla_i\nabla_j\partial_k + \nabla_j\nabla_i\partial_k + \nabla_{[\partial_i,\partial_j]}\partial_k \\ &= -\nabla_i\Gamma_{jk}^{\ell}\partial_\ell + \partial_j\Gamma + \nabla_j\Gamma_{ik}^{\ell}\partial_\ell + 0 \\ &= -(\partial_i\Gamma_{jk}^{\ell})\partial_\ell - \Gamma_{jk}^{\ell}\Gamma_{i\ell}^{m}\partial_m + (\partial_j\Gamma_{ik}^{\ell})\partial_\ell + \Gamma_{ik}^{\ell}\Gamma_{j\ell}^{m}\partial_m \\ &= \Big(\partial_j\Gamma_{ik}^{\ell} - \partial_i\Gamma_{jk}^{\ell} + \Gamma_{ik}^{m}\Gamma_{jm}^{\ell} - \Gamma_{jk}^{m}\Gamma_{im}^{\ell}\Big)\partial_\ell. \end{aligned}$$

Therefore we obtain that

$$R_{ijk}^{\ell} = \partial_j\Gamma_{ik}^{\ell} - \partial_i\Gamma_{jk}^{\ell} + \Gamma_{ik}^{m}\Gamma_{jm}^{\ell} - \Gamma_{jk}^{m}\Gamma_{im}^{\ell}. \tag{5.12}$$

Now recall that Γ_{ij}^k is given by (5.11). Another very important tensor associated with the Riemann curvature tensor is the ***Ricci tensor***. Consider the map

$$R(X, \cdot)Z : \mathfrak{X}(M) \to \mathfrak{X}(M).$$

One defines the Ricci tensor at every point $p \in M$ to be

$$\mathrm{Ric}_p(v, w) = \mathrm{Tr}\big[R_p(v, \cdot)w\big]$$

for $v, w \in T_pM$. Note that at every point p we have the map $R_p(v,\cdot)w : T_pM \to T_pM$. Therefore the expression above makes sense. Since at every point Ric is just a (0,2)-tensor, locally we can expand it as $\mathrm{Ric} = R_{ij}\,dx^i \otimes dx^j$. To find out what R_{ij} is, we note that

$$R_{ij} = \mathrm{Ric}\big(\partial_i, \partial_j\big) = \mathrm{Tr}\, R(\partial_i, \cdot)\partial_j =: \mathrm{Tr}\, L$$

and

$$L(\partial_k) = R(\partial_i, \partial_k)\partial_j = R_{ikj}^{\ell}\frac{\partial}{\partial x^\ell}.$$

Therefore, we see that

$$R_{ij} = \mathrm{Tr}\, L = \sum_k R_{ikj}^{k} = R_{ikj}^{k}$$

using the Einstein summation convention.

One can also define a new tensor

$$R(X, Y, Z, W) = \langle R(X,Y)Z, W\rangle = g(R(X,Y)Z, W).$$

Then at every point this is a (0,4)-tensor. Its coefficient is given by

$$R_{ijk\ell} = R(\partial_i, \partial_j, \partial_k, \partial_\ell) = \langle R(\partial_i, \partial_j)\partial_k, \partial_\ell\rangle = \langle R_{ijk}^{m}\partial_m, \partial_\ell\rangle = R_{ijk}^{m}\langle\partial_m, \partial_\ell\rangle = g_{m\ell}R_{ijk}^{m}.$$

The following properties are proven in [2, Chap4, Prop2.5]:

Proposition 5.24. We have

1. $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$.
2. $R(X, Y, Z, W) = -R(Y, X, Z, W)$.
3. $R(X, Y, Z, W) = -R(X, Y, W, Z)$.
4. $R(X, Y, Z, W) = R(Z, W, X, Y)$.

Proof. These properties are probably more obvious and easier to prove using coefficients. That is, to prove 1 for example, we prove $R_{ijks} + R_{jkis} + R_{kijs} = 0$, using $R_{ijks} = R_{ijk}^{\ell} g_{\ell s}$ and the properties of the coefficients R_{ijk}^{ℓ}. Similarly for the rest. □

For ***sectional curvature***, read [2, Chap4, Sec3]. I find this introduction boring and a little unmotivated...

5.4 A Few Words about Tensor Fields

Using the language from the beginning of this notes, a smooth (k, l)-tensor field T on a manifold M is a smooth section of the tensor bundle $T^{(k,l)}M$, i.e., $T \in \Gamma(T^{(k,l)}M)$. If $\omega^1, ..., \omega^k \in \mathfrak{F}(M)$ be covector fields, and $X_1, ..., X_l \in \mathfrak{X}(M)$ be vector fields, then T can be identified with a multilinear map

$$T : \underbrace{\mathfrak{F}(M) \times \cdots \times \mathfrak{F}(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l} \to C^\infty(M).$$

For a tensor field T, it is easy to check that this map is multilinear over $C^\infty(M)$. The next result shows that the converse is also true:

Lemma 5.25. *Every multilinear map*

$$T : \underbrace{\mathfrak{F}(M) \times \cdots \times \mathfrak{F}(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l} \to C^\infty(M).$$

is induced by a (k, l)-tensor field T iff it is multilinear over $C^\infty(M)$.

Proof. The proof is given in [1, Lemma 12.24]. The lemma is proved only for covariant tensor fields in [1], but the same argument works for mixed tensor fields. □

The previous result may be viewed as a characterization of smooth tensor fields on M. However, in the previous sections, despite we call $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ the *curvature tensor* all the time, it does not fit the characterization of the previous lemma, since its codomain is $\mathfrak{X}(M)$, not $C^\infty(M)$. So how do we justify calling it a tensor if it is not a tensor field? Actually the justification is given by the following lemma:

Lemma 5.26. *Let V be a finite dimensional vector space. Then there is a basis independent (natural) isomorphism between $T^{(k+1,l)}(V)$ and the space S of multilinear maps*

$$\underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to V.$$

Proof. We define a map $\Phi : S \to T^{(k+1,l)}(V)$ as follows. For every $f \in S$, and for every $\varphi_1, ..., \varphi_{k+1} \in V^*, v_1, ...v_l \in V$, set

$$\Phi(f)(\varphi_1, ..., \varphi_{k+1}, v_1, ..., v_l) = \varphi_{k+1}(f(\varphi_1, ..., \varphi_k, v_1, ..., v_l)).$$

This map is clearly linear. Now we pick a basis $e_1, ..., e_n$ of V and its dual basis $\varepsilon^1, ..., \varepsilon^n$ of V^*. To show injectivity, note that if $\Phi(f) = 0$, then

$$\varphi_{k+1}(f(\varphi_1, ..., \varphi_k, v_1, ..., v_l)) = 0$$

for every $\varphi_1, ..., \varphi_{k+1} \in V^*, v_1, ...v_l \in V$. Put $w = f(\varphi_1, ..., \varphi_k, v_1, ..., v_l) \in V$ and set $\varphi_{k+1} = \varepsilon^i$ for $i = 1, ..., n$, we get $w^i = 0$ for $i = 1, ..., n$. Therefore $w = 0 \in V$, giving $f = 0$. This proves injectivity.

For surjectivity, let $\alpha \in T^{(k+1,l)}(V)$. By multilinearity, we see that $\Phi(f) = \alpha$ is equivalent to

$$\Phi(f)(\varepsilon^{i_1}, ..., \varepsilon^{i_{k+1}}, e_{j_1}, ..., e_{j_l}) = \alpha(\varepsilon^{i_1}, ..., \varepsilon^{i_{k+1}}, e_{j_1}, ..., e_{j_l})$$

for all $1 \leq i_1, ..., i_{k+1}, j_1, ..., j_l \leq n$. One can check that setting

$$f(\varepsilon^{i_1}, ..., \varepsilon^{i_k}, e_{j_1}, ..., e_{j_l}) = \sum_{\lambda=1}^{n} \alpha(\varepsilon^{i_1}, ..., \varepsilon^{i_k}, \varepsilon^{\lambda}, e_{j_1}, ..., e_{j_l})e_\lambda \in V$$

gives the desired $f \in S$. This concludes the proof. Note that also in the proof we chose a basis of V, but the isomorphism is still basis independent. $\square$

Remark 5.27. Note that under this isomorphism, when $k = 0$, then we have an identification of all linear maps $V \to V$ with $T^{(1,1)}(V)$.

Therefore, the lemma above gives the following result:

Corollary 5.28. *Every multilinear map*

$$T : \underbrace{\mathfrak{F}(M) \times \cdots \times \mathfrak{F}(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l} \to \mathfrak{X}(M).$$

is induced by a $(k + 1, l)$*-tensor field* T *iff it is multilinear over* $C^\infty(M)$.

We have checked that $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is multilinear in $C^\infty(M)$. Therefore, this induces a (1,3)-tensor field, which we call the Riemann curvature tensor (field).

6 | Geodesics

6.1 Basic Definitions

Definition 6.1. A parametrized curve $\gamma : I \to M$ is called a ***geodesic at*** $t_0 \in I$ if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$$

at $t_0 \in I$. If γ is a geodesic at every $t \in I$, then we say γ is a ***geodesic***.

This definition is the one given in [2]. In the lecture, we had a slightly different approach to geodesics, which I'll briefly summarize below: Let (M^n, g) be a Riemannian manifold and $\gamma : I \to M$ a smooth curve. We define a ***tangent vector field*** X ***along*** γ to be a map $X : I \to TM$ such that $X(t) \in T_{\gamma(t)}M$. We call X smooth if the map $X : I \to TM$ is a smooth map between manifolds. Then we define the operator $\nabla_{d/dt}X$, which is just DX/dt. How do we define such an operator $\nabla_{d/dt}$? It make sense that if $X(t)$ is a vector field along γ, then we define

$$\nabla_{\frac{d}{dt}} X(\gamma(t)) = \nabla_{\frac{d\gamma}{dt}} X.$$

Moreover, just like any derivatives, it should satisfy the Libniz rule. Therefore if we write $X = \xi^i(t)\partial_i$, and $d\gamma/dt = (dx^i/dt)\partial_i$ where $\varphi \circ \gamma(t) = (x^1(t), ..., x^n(t))$ with φ a local chart, then by Libniz rule,

$$\nabla_{\frac{d}{dt}}(\xi^i \partial_i) = \frac{d\xi^i}{dt}\frac{\partial}{\partial x^i} + \xi^i \nabla_{\frac{d\gamma}{dt}} \frac{\partial}{\partial x^i}$$

and recall that

$$\nabla_{\frac{d\gamma}{dt}} \frac{\partial}{\partial x^i} = \frac{dx^i}{dt}\Gamma^k_{ij}\frac{\partial}{\partial x^k}.$$

Therefore, we get that

$$\nabla_{\frac{d}{dt}}(\xi^i \partial_i) = \sum_k \left(\frac{d\xi^k}{dt} + \Gamma^k_{ij}\frac{dx^i}{dt}\xi^j\right)\frac{\partial}{\partial x^k}$$

which is exactly $\frac{D}{dt}(\xi^i \partial_i)$. Then we define a smooth tangent vector field X along γ to be ***parallel*** iff

$$\nabla_{\frac{d}{dt}} X = 0.$$

Remark 6.2. The reason we define parallel vector field X along a curve to be those such that $D/dt(X) = 0$ is very simple. In $\mathbb{R}^n$, a curve $\gamma : I \to \mathbb{R}^n$ can be parametrized by $\gamma(t) = (\gamma^1(t), ..., \gamma^n(t))$, and a vector field along γ can be parametrized by $v(t)$, where for each $t \in I$, we have a vector $v \in \mathbb{R}^n$. Then parallel simply means

$$\frac{d}{dt}v(t) = 0.$$

In curved space, $D/dt(X(t)) = 0$ is the generalization of this equation, for if $M = \mathbb{R}^n$, then $g_{ij} = \delta_{ij}$ and $\Gamma_{ij}^k = 0$ for all i, j, k, thus this reduces to the equation above.

We already mentioned that given $\gamma : I \to M$ and an initial data $X(a) = v \in T_{\gamma(a)}M$ for $a \in I$, there is a unique vector $X(b) \in T_{\gamma(b)}M$ that is obtained from $X(a)$ via parallel transport. This vector $X(b)$ can be obtained by solving the equation $DX/dt = 0$ with initial value $X(a) = v$. Therefore, we may define a map $P_\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M$ by

$$P_\gamma(v) = X(b).$$

Note that by existence and uniqueness of the solution, P_γ is an isomorphism between vector spaces. But P_γ is also an isometry between inner product spaces. To see this, just

Homework 6.3. Show the following identity holds true:

$$\frac{d}{dt}\langle X, Y\rangle = \langle \nabla_{\frac{d}{dt}} X, Y\rangle + \langle X, \nabla_{\frac{d}{dt}} Y\rangle.$$

Solution. Note that we are using the Levi-Civita connection with connection coefficients Γ_{ij}^k. This is compatible with the metric g by construction. Then this is a direct consequence of Proposition 5.18. □

Using this, if X, Y are parallel vector fields extending v, w along γ, then we have $\nabla_{d/dt}X = \nabla_{d/dt}Y = 0$. Hence

$$\langle X, Y\rangle(a) = \langle X, Y\rangle(b).$$

For the special case that $\gamma(a) = \gamma(b) = p$, i.e., γ is a ***loop*** based at $p \in M$, then $P_\gamma : T_pM \to T_pM$. Since P_γ preserves the inner product in T_pM, it is a member of the ***orthogonal transformations***, i.e., $P_\gamma \in O(T_pM)$. Assuming T_pM has an orientation, is $P_\gamma \in SO(T_pM)$? i.e., does it also preserves orientation? This question is answered by the following proposition:

Proposition 6.4. If $[\gamma] \in \pi_1(M)$ represents the trivial class, then $P_\gamma \in SO(T_pM)$. The converse is not necessarily true.

Remark 6.5. For a Riemannian manifold (M, g) and a point $p \in M$, we may define its ***holonomy group*** centered at p by

$$\mathrm{Hol}(M, g, p) = \{P_\gamma : \gamma \text{ a piecewise smooth loop based at } p\}.$$

As the name suggests, this is a group, with operation given by $P_{\gamma_1} \circ P_{\gamma_2} = P_{\gamma_1 \sharp \gamma_2}$ where $\gamma_1 \sharp \gamma_2$ is the concantenation of γ_1 and γ_2, and $(P_\gamma)^{-1} = P_{\gamma^{-1}}$ with identity being P_c where $c : I \to M$ is the constant loop based at p. We may also define the ***reduced holonomy group***

$$\mathrm{Hol}_o(M, g, p) = \{P_\gamma \in \mathrm{Hol}(M, g, p) : [\gamma] = 1 \in \pi_1(M)\}.$$

By remark above, the reduced holonomy group preserves orientation of T_pM.

Definition 6.6. Let $\gamma : I \to M$ be a smooth curve. Then γ is called a ***geodesic*** if $\nabla_{\frac{d}{dt}}\dot{\gamma} = 0$ in I.

Note that we previously calculated $\nabla_{d/dt} = D/dt$ on a vector field. Using those results, the geodesic equation in local coordinate $\varphi \circ \gamma = (x^1(t), ..., x^n(t))$ for a local chart φ is given by a system of ODEs

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \tag{6.1}$$

for $k = 1, ..., n$. The set of equations (6.1) is called the ***geodesic equation***. One more comment we want to make is that for a geodesic, its tangent vector are of constant length, for

$$\frac{d}{dt}\langle\dot{\gamma}, \dot{\gamma}\rangle = 2\langle\nabla_{\frac{d}{dt}}\dot{\gamma}, \dot{\gamma}\rangle = 0$$

since $\nabla_{d/dt}\dot{\gamma} = 0$ by definition. Therefore, for $t \in I$, the length of the geodesic $\gamma|_{[0,t]}$ is given by

$$L(\gamma|_{[0,t]}) = \int_0^t |\gamma'(t)|dt = |\gamma'(0)|t.$$

We may generalize the definition of parallel vectors to a vector field, not just vector fields along some given curve.

Definition 6.7. Let $X \in \mathfrak{X}(M)$. Then X is called ***parallel*** on M (or on an open set U of M) if $\nabla X = 0$. i.e., if for all $v \in T_pM$ and $p \in M$ (or $p \in U$), we have $\nabla_v X = 0$.

Notice that this definition makes sense, since every $v \in T_pM$ is the tangent vector $\gamma'(t_0)$ of some curve $\gamma : I \to M$ with $\gamma(t_0) = p$. Therefore, this definition is just to require X to be parallel transported by all possible curves in M (or in U). Equivalently, this is equivalent to $\nabla_Y X = 0$ for all $Y \in \mathfrak{X}(M)$ (or $Y \in \mathfrak{X}(U)$).

Note that if X is parallel on M, then X must be of constant length. To see this, note that along any curve γ, we have

$$\frac{d}{dt}\langle X, X\rangle = \langle\nabla_{\dot{\gamma}}X, X\rangle + \langle X, \nabla_{\dot{\gamma}}X\rangle = 0$$

since X is parallel along γ so $\nabla_{\dot{\gamma}}X = 0$. Therefore, for any two points $p, q \in M$, choosing $\gamma : I \to M$ such that $\gamma(0) = p, \gamma(1) = q$, the above equation gives $\langle X, X\rangle(p) = \langle X, X\rangle(q)$.

6.1.1 Supplement: Covariant Derivative of Tensor Fields

So far, we only defined covariant derivative of a vector field. But it is also possible to define covariant derivative of arbitrary tensor fields. The detailed construction is given in [4, p.95-98]. Here we will only do it for 1-forms and (0,2)-tensor fields, and we will do some explicit calculations.

Let $\omega \in \Omega^1(M)$. Then one can define a map $\nabla_Y\omega \in \Omega^1(M)$ for $Y \in \mathfrak{X}(M)$. The construction is similar to everything we did so far: assume $\nabla_Y\omega$ exists, we think about what properties we want it to have, and then use this to deduce an expression of $\nabla_Y\omega$, and then define it using the expression we derived.

Suppose now we have already defined how to take ∇_Y of a general tensor field, then we want the ***Libniz rule***:

$$\nabla_Y(X \otimes \omega) = (\nabla_Y X) \otimes \omega + X \otimes \nabla_Y \omega$$

for $X, Y \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$. We now define the ***contraction operator*** $C(\omega \otimes X) = \omega(X)$. The reason that C is called contraction operator is due to the following calculation using local coefficients:

$$C(\omega \otimes X) = C(\omega_i X^j dx^i \otimes \partial_j) = (\omega_i dx^i)(X^j \partial_j) = \omega_i X^j dx^i(\partial_j) = \omega_i X^j \delta^i_j = \omega_i X^i.$$

and we want ∇_Y ***to commute with contraction.*** Finally, we want, for $f \in C^\infty(M) \cong \Omega^0(M)$, we have

$$\nabla_Y f = Y(f).$$

Using these 3 properties that we desired, it is easy to define what $\nabla_Y \omega$ is using the calculation below:

$$\begin{aligned} Y\omega(X) = YC(\omega \otimes X) = \nabla_Y C(\omega \otimes Y) &= C(\omega \otimes \nabla_Y X + X \otimes \nabla_Y \omega) \\ &= \omega(\nabla_Y X) + (\nabla_Y \omega)(X). \end{aligned}$$

Therefore, we define, for $\omega \in \Omega^1(M)$,

$$\nabla_Y \omega(X) := Y\omega(X) - \omega(\nabla_Y X). \tag{6.2}$$

Similarly, one may define a covariant 2-tensor $\nabla_Z \eta$ for every covariant 2-tensor field $\eta \in \Gamma(T^*M \otimes T^*M)$ and $Z \in \mathfrak{X}(M)$. We can again define contraction operator by

$$C(\eta \otimes X \otimes Y) = \eta(X, Y).$$

Again one sees that

$$\begin{aligned} C(\eta_{ij} X^k Y^\ell dx^i \otimes dx^j \otimes \partial_k \otimes \partial_\ell) &= (\eta_{ij} dx^i \otimes dx^j)(X^k \partial_k, Y^\ell \partial_\ell) \\ &= \eta^{ij} X^k X^\ell \delta^i_k \delta^j_\ell = \eta_{ij} X^i Y^j. \end{aligned}$$

Again we want Libniz rule, commutation with contraction, and $\nabla_Y f = Yf$. Therefore,

$$\begin{aligned} \nabla_Z \eta(X, Y) &= Z\eta(X, Y) \\ &= C(\nabla_Z \eta \otimes X \otimes Y + \eta \otimes \nabla_Z X \otimes Y + \eta \otimes X \otimes \nabla_Z Y) \\ &= \nabla_Z \eta(X, Y) + \eta(\nabla_Z X, Y) + \eta(X, \nabla_Z Y). \end{aligned}$$

Hence now we define

$$\nabla_Z \eta(X, Y) := Z\eta(X, Y) - \eta(\nabla_Z X, Y) - \eta(X, \nabla_Z Y). \tag{6.3}$$

Proposition 6.8. The metric tensor g is parallel.

Proof. Since ∇ is compatible with g, Corollary 5.19 gives that

$$\nabla_Z g(X, Y) := Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0.$$

This concludes the proof. □

Now we give some explcit calculations. First we study the case where ∇_Y acts on a vector field $X \in \mathfrak{X}(M)$. Note that $\nabla_Y X$ is not $C^\infty(M)$-linear in X since it follows the Libniz rule, which is not functional linear, but it is $C^\infty(M)$-linear in Y. Therefore, we may view $\nabla X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $(\nabla X)(Y) = \nabla_Y X$, which may be identified with a (1,1)-tensor field by Lemma 5.25 and Corollary 5.28. Now we want to calculate some coefficients. Write

$$\nabla X = (\nabla_i X^j) dx^i \otimes \partial_j$$

where $\nabla_i X^j \in \mathbb{R}$. Then we have that

$$\begin{aligned}
\nabla_i X^j &= \nabla X(\partial_i, dx^j) = dx^j(\nabla_{\partial_i} X) \\
&= dx^j\Big((\partial_j X^k)\partial_k + X^k \Gamma^\ell_{ik}\partial_\ell\Big) \\
&= \partial_i X^k \delta^j_k + X^k \Gamma^\ell_{ik}\delta^j_\ell \\
&= \partial_i X^j + \Gamma^j_{ik} X^k.
\end{aligned}$$

Similarly, we can consider, for $\omega \in \Omega^1(M)$, the map $\nabla\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ defined by $(\nabla\omega)(Y, X) = (\nabla_Y \omega)(X)$. Therefore, one may write

$$\nabla\omega = (\nabla_i \omega_j) dx^i \otimes dx^j$$

for $\nabla_i \omega_j \in \mathbb{R}$.

Homework 6.9. Show that $\nabla_i \omega_j = \partial_i \omega_j - \Gamma^k_{ij}\omega_k$.

Solution. Using (6.2), we have that

$$\begin{aligned}
\nabla_i \omega_j &= \nabla\omega(\partial_i, \partial_j) \\
&= \nabla_{\partial_i}\omega(\partial_j) \\
&= \partial_i \omega(\partial_j) - \omega(\nabla_{\partial_i}\partial_j) \\
&= \partial_i \omega_j - \omega(\Gamma^k_{ij}\partial_k) = \partial_i\omega_j - \Gamma^k_{ij}\omega_k.
\end{aligned}$$

This concludes the proof. □

6.2 The Exponential Map

Applying the existence and uniqueness theorem of linear ODEs to the geodesic equation (6.1), one can reach the following conclusion:

Lemma 6.10. *Let $p \in M$. Then there exists an open neighborhood V of M containing p, positive number $\delta, \eta > 0$, and a smooth mapping*

$$\gamma : (-\delta, \delta) \times U \to M \qquad U = \big\{(q, v) : q \in V, v \in T_q M, |v| < \eta\big\}$$

such that the curve $t \mapsto \gamma(t, q, v) = \gamma_{q,v}(t), t \in (-\delta, \delta)$, is the unique geodesic of M such that $\gamma_{q,v}(0) = q$ and $\gamma'_{q,v}(0) = v$, for each $q \in V, v \in T_q M$ with $|v| < \eta$.

The proof is essentially due to the theory of ODEs, which is proven explicitly in [4, Thm 4.31]. Then a similar proof can be deduced by mimicing the proof of [4, Prop 5.19]. This result can be used to define the exponential map, after we prove the following result:

Homework 6.11. A geodesic reparametrized by a constant scaling factor remains a geodesic. More specifically, if $\gamma_{q,v}(t)$ is defined on the interval $(-\delta, \delta)$, then the geodesic $\gamma_{q,av}(t)$ for some positive real number a is defined on the interval $(-\delta/a, \delta/a)$ and

$$\gamma_{q,av}(t) = \gamma_{q,v}(at).$$

Solution. Let $h : (-\delta/a, \delta/a) \to M$ be the curve defined by $h(t) = \gamma_{q,v}(at)$. Then we have $h(0) = q$ and $h'(0) = a\gamma'_{q,v}(0) = av$ since for $f \in C^\infty(M)$ we have

$$\begin{aligned} h'(0)f &= h_*\left(\left.\frac{d}{dt}\right|_{t=0}\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} (f \circ h) \\ &= \left.\frac{d}{dt}\right|_{t=0} \big(f \circ \gamma_{q,v}(at)\big) \\ &= (f \circ \gamma_{q,v})(0) \cdot a = a\gamma'_{q,v}(0)f. \end{aligned}$$

The calculation above (replacing zero by some arbitrary t) shows that $h'(t) = a\gamma'_{q,v}(at)$. Therefore, we have that

$$\frac{D}{dt}\left(\frac{dh}{dt}\right) = \nabla_{h'(t)}h'(t) = a^2\nabla_{\gamma'_{q,v}(at)}\gamma'_{q,v}(at) = 0$$

where, for the first equality, we extend $h'(t)$ to a neighborhood of $h(t)$ in M, and in the last step we used the fact that $\gamma'_{q,v}$ is a geodesic, so it parallel transport its own tangent vector. Therefore h is a geodesic passing through q with velocity av at the instant $t = 0$. By uniqueness, we have that

$$h(t) = \gamma_{q,v}(at) = \gamma_{q,av}(t).$$

This concludes the proof. □

This allows us to refine the lemma above:

Proposition 6.12. Given $p \in M$, there exist a neighborhood V of M, a number $\varepsilon > 0$ and a smooth map $\gamma : (-2, 2) \times U \to M$, where $U = \big\{(q, v) : q \in V, v \in T_qM, |v| < \varepsilon\big\}$, such that the map $t \mapsto \gamma(t, q, v) = \gamma_{q,v}(t)$ with $t \in (-2, 2)$ is the unique geodesic of M satisfying $\gamma_{q,v}(0) = q$ and $\gamma'_{q,v}(0) = v$, for every $q \in V$ and $v \in T_qM$ with $|v| < \varepsilon$.

Proof. The geodesic $\gamma_{q,v}(t)$ is defined for $|t| < \delta$ and $|v| < \eta$. From the previous homework problem, $\gamma_{q,\delta v/2}(t)$ is defined for $|t| < 2$. Taking $\varepsilon < \delta\eta/2$, we obtain that the geodesic $\gamma_{q,t}(v)$ is defined for $|t| < 2$ and $|v| < \varepsilon$. □

This allows us to define the exponential map:

Definition 6.13. Let $p \in M$ and $U \subset TM$ be the open set given by the previous proposition. Then the ***exponential map*** $\exp : U \to M$ is defined by

$$\exp(q, v) = \gamma_{q,v}(1) = \gamma_{q,v/|v|}(|v|)$$

with $(q, v) \in U$. We further define the restriction $\exp_q : B_\varepsilon(0) \subset T_qM \to M$ of exp to an open subset $B_\varepsilon(0)$ of $T_q(M)$ by

$$\exp_q(v) = \exp(q, v).$$

Proposition 6.14. Given $q \in M$, there exists $\varepsilon > 0$ such that $\exp_q : B_\varepsilon(0) \subset T_qM \to M$ is a diffeomorphism of $B_\varepsilon(0)$ onto an open subset of M.

Proof. We calculate $(\exp_q)_{*,0} : T_0(T_qM) \cong T_qM \to T_qM$. Note that for $v \in T_qM$, the map $t \mapsto tv$ is a curve in T_qM passing through zero at $t = 0$ and whose tangent vector at $t = 0$ is v. Therefore, we have

$$(\exp_q)_{*,0}(v) = \frac{d}{dt}\bigg|_{t=0} \exp_q(tv) = \frac{d}{dt}\bigg|_{t=0} \gamma_{q,tv}(1) = \frac{d}{dt}\bigg|_{t=0} \gamma_{q,v}(t) = v.$$

Hence $(\exp_q)_{*,0} : T_qM \to T_qM$ is the identity. Therefore it is a local diffeomorphism on a neighborhood of 0 by the inverse function theorem. □

Lemma 6.15 (Gauss's Lemma). *Let $p \in M$ and $v \in T_pM$ such that* $\exp_p$ *is defined. Let $w \in T_v(T_pM) \cong T_pM$. Then we have*

$$\langle (\exp_p)_{*,v}(v), (\exp_p)_{*,v}(w)\rangle = \langle v, w\rangle.$$

Proof. Let $w = w_T + w_N$, where w_T is parallel to v and w_N is orthogonal to v. Then notice that $w_T = \frac{\langle w_T, v\rangle}{|v|^2} v$. Since $(\exp_p)_{*,v}$ is linear, we get that $(\exp_p)_{*,v}(w) = (\exp_p)_{*,v}(w_T) + (\exp_p)_{*,v}(w_N)$. And notice that

$$\begin{aligned}(\exp_p)_{*,v}(v) &= \frac{d}{dt}\bigg|_{t=0} \exp_p(v + tv) \\ &= \frac{d}{dt}\bigg|_{t=0} \gamma_{p,v+tv}(1) \\ &= \frac{d}{dt}\bigg|_{t=0} \gamma_{p,v}(1+t) = \gamma'_{p,v}(1).\end{aligned}$$

Therefore, we get that

$$\begin{aligned}\langle (\exp_p)_{*,v}(v), (\exp_p)_{*,v}(w_T)\rangle &= \langle \gamma'_{p,v}(1), \frac{\langle w_T, v\rangle}{|v|^2}\gamma'_{q,v}(1)\rangle \\ &= \frac{\langle w_T, v\rangle}{|v|^2}|\gamma'_{q,v}(1)|^2 = \langle w, v\rangle\end{aligned}$$

where we used the fact that since $\gamma'_{q,v}(1)$ is the parallel transport of v by the geodesic $\gamma_{q,v}$, we have $|\gamma'_{q,v}(1)|^2 = |v|^2$, and $\langle w_T, v\rangle = \langle w, v\rangle$. Therefore, all it remains is to check that

$$\langle (\exp_p)_{*,v}(v), (\exp_p)_{*,v}(w_N)\rangle = 0.$$

Since $\exp_p v$ is defined, there exists $\varepsilon > 0$, such that $\exp_p tv(s)$ is defined for $0 \le t \le 1, -\varepsilon < s < \varepsilon$, where $v(s)$ is a curve in T_pM with $v(0) = v, v'(0) = w_N$, and $|v(s)|$ is constant for all s. Now define the map $f : [0,1] \times (-\varepsilon, \varepsilon) \to M$, $f(t,s) = \exp_p tv(s)$. Now, for each fixed t, notice that $f(t,s) : (-\varepsilon, \varepsilon) \to M$ is a curve, and for each fixed s, the map $f(t,s) : [0,1] \to M$ is a curve (**in fact a geodesic, see figure on the next page**). Now define $\partial f/\partial s(t_0, s_0)$ to be the tangent vector of the curve $f(t_0, s) : (-\varepsilon, \varepsilon) \to M$ at

$s = s_0$, and similarly for $\partial f/\partial t(t_0, s_0)$. Then note that for any smooth function g with correct domain, we have

$$\frac{\partial f}{\partial s}(1,0)g = f(1,s)_{*,(1,0)}\left(\frac{d}{ds}\bigg|_{s=0}\right)g = \frac{d}{ds}\bigg|_{s=0} g(\exp_p(v(s))) = (\exp_p)_{*,v}(w_N)g$$

since recall that $F : M \to N$, then $F_{*,x}(v)$ can be calculated by finding a curve $\varphi : I \to M$ with $\varphi(t_0) = x, \varphi'(t_0) = v$ and we get $F_{*,x}(v) = (F \circ \varphi)'(t_0)$. Here we set $F = \exp_p$ and $\varphi(s) = v(s)$. This shows that

$$\frac{\partial f}{\partial s}(1,0) = (\exp_p)_{*,v}(w_N).$$

Similarly, by letting $F = \exp_p$ and $\varphi = tv(0) = tv$ gives

$$\frac{\partial f}{\partial t}(1,0)g = f(t,0)_{*,(1,0)}\left(\frac{d}{dt}\bigg|_{t=1}\right)g = \frac{d}{dt}\bigg|_{t=1} g(\exp_p(tv)) = (\exp_p)_{*,v}(v)g.$$

So this shows that

$$\frac{\partial f}{\partial t}(1,0) = (\exp_p)_{*,v}(v).$$

Therefore, we want to show that

$$0 = \langle (\exp_p)_{*,v}(w_N), (\exp_p)_{*,v}(v)\rangle = \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle(1,0).$$

But note that by compatibility, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle = \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle + \langle \frac{\partial f}{\partial t}, \frac{D}{dt}\frac{\partial f}{\partial t}\rangle &= \langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \\ &= \frac{1}{2}\frac{d}{ds}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle = \frac{1}{2}\frac{d}{ds}|v(s)|^2 = 0.\end{aligned}$$

since $\partial f/\partial t$ is the tangent vector of the geodesic $t \mapsto f(t, s_0)$ for each fixed s_0. Therefore $(D/dt)(\partial f/\partial t) = 0$. The last equality is because we assumed that $|v(s)|$ is constant. Now since

$$\lim_{t\to 0}\frac{\partial f}{\partial s}(t,0) = \lim_{t\to 0}(\exp_p)_{*,tv}(tw_N) = 0,$$

we conclude that

$$\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle(1,0) = 0.$$

This concludes the proof. □

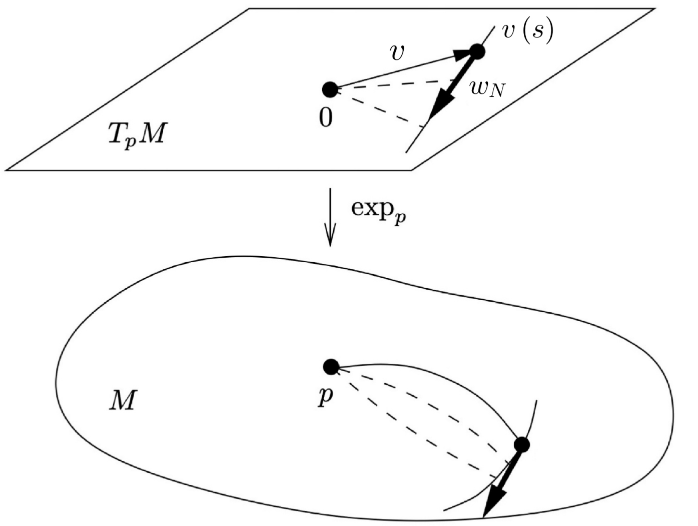


Figure 6.1: An illustration of the proof above. What we actually did is parametrized a circle from $[0, 1] \times (-\varepsilon, \varepsilon)$ to T_pM via the map $(t, s) \mapsto tv(s)$. Then this map composed with $\exp_p$ gives the $f(t, s)$ map. Note that for fixed s, as we travel along t direction, we have a geodesic $\exp_p(tv(s))$, which are the image of the radial curves of the circle in T_pM under $\exp_p$.

Now we can prove the local minimizing property of geodesics:

Proposition 6.16. Let $B_\varepsilon(p) = \exp_p B_\varepsilon(0)$ be the geodesic ball at p and let $\gamma : [0, 1] \to B_\varepsilon(p)$ be a geodesic starting at p, where $\varepsilon > 0$ is chosen such that $\exp_p : B_\varepsilon(0) \to B_\varepsilon(p)$ is a diffeomorphism. Then if $c : [0, 1] \to M$ is any differentiable curve joining $\gamma(0)$ and $\gamma(1)$, we have

$$L(\gamma) \le L(c).$$

Proof. WLOG we may assume c is contained in $B_\varepsilon(p)$. If we can prove this statement, then for a general $c : [0, 1] \to M$ joining endpoints of γ, just take $t \in [0, 1]$ be the first point such that $c(t) \in \partial B_\varepsilon(p)$. Note that by Gauss's lemma, $B_\varepsilon(p)$ is also a ball centered at p with radius ε. Then we have

$$L(c) \ge L(c|_{[0,t]}) \ge \varepsilon \ge L(\gamma).$$

Now suppose $c \subset B_\varepsilon(p)$. Since $\exp_p : B_\varepsilon(0) \to B_\varepsilon(p)$ is a diffeomorphism, we may lift γ and c to $\tilde{\gamma}$ and $\tilde{c}$ in T_pM. Then, we have that $\gamma(t) = \exp_p \tilde{\gamma}(t) = \exp_p tv$ where $v = \gamma'(0)$ and $c(t) = \exp_p \tilde{c}(t)$. Write $\tilde{c}(t) = r(t)w(t)$, where $w(t) \in T_pM$ with $|w(t)| \equiv 1$ and the function $r : [0, 1] \to \mathbb{R}$ is the length of $\tilde{c}$ with value $|\tilde{c}(t)|$ at t. Note that

$$\langle w'(t), w(t)\rangle = \frac{1}{2}\frac{d}{dt}\langle w(t), w(t)\rangle = 0$$

since $|w(t)| = 1$. Now write $w_r = r'(t)w(t)$ and $w_n = r(t)w'(t)$. Then

$$|\tilde{c}'(t)|^2 = |r'(t)w(t) + r(t)w'(t)|^2 = |w_r|^2 + |w_n|^2 \ge |w_r|^2 = |r'(t)|^2.$$

Hence

$$L(\tilde{c}) = \int_0^1 |\tilde{c}'(t)|dt \ge \int_0^1 |r'(t)|dt = L(\tilde{\gamma}).$$

Now by Gauss's lemma, we have

$$|c'(t)| = |(\exp_p)_* \tilde{c}'(t)| = |(\exp_p)_*(w_r(t)) + (\exp_p)_*(w_n(t))| \geq |w_r| = |\gamma'(t)|.$$

This further implies that

$$L(c) = \int_0^1 |c'(t)| dt \geq \int_0^1 |\gamma'(t)| dt = L(\gamma).$$

This concludes the proof. □

One might wonder if it is possible to define $\exp_p$ on the entire T_pM. This is answered by the following theorem [2, Chap 7, Thm 2.8]:

Theorem 6.17 (Hopf-Rinow)**.** *Let M be a Riemannian manifold and $p \in M$. Then the following are equivalent:*

1. $\exp_p$ *is defined on all of T_pM.*
2. *The closed and bounded sets of M are compact.*
3. *M is complete as a metric space.*
4. *M is geodesically complete, i.e., every geodesic can be extended to a geodesic defined on $\mathbb{R}$.*

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