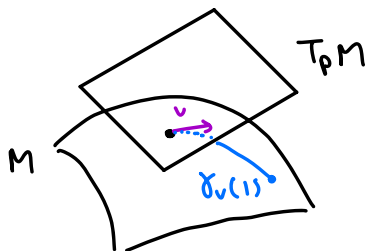


Lec 1. Chap 5. Jacobi Fields

$$\exp_p: T_p M \rightarrow M$$

$$\exp_p v = \gamma_v(1)$$

γ_v is a geodesic with $\gamma_v(0)=p$, $\gamma'_v(0)=v$.



$$d(\exp_p)_v: T_v(T_p M) \rightarrow T_q M$$

$\cong$
 $\mathbb{R}^n$

Gauss lemma: $d(\exp_p)$ is an isometry in the **radial direction**:

$$\langle d(\exp_p)_v(v), d(\exp_p)_v(w) \rangle = \langle v, w \rangle.$$

Let $v, w \in T_p M$. What about $d(\exp_p)_v(w)$? This is given by Jacobi fields. Jacobi field connect geodesics with curvature.

Defⁿ. Let $\gamma: [0, a] \rightarrow M$ be a geodesic in M . A vector field J along γ is called a **Jacobi field** if

$$J''(t) + R(\gamma'(t), J(t)) \gamma'(t) = 0. \quad (*)$$

where

$$J''(t) = \nabla_{\gamma'(t)} \nabla_{\gamma'(t)} J(t).$$

Rmk. Since $\nabla_x Y|_p$ needs Y to be defined in a nbh of p , $\nabla_{\gamma'(t)} J(t)$ is an abuse of notation since J is only defined along γ . Actually, what we mean is

$$J''(t) = \frac{D^2 J}{dt^2} = \frac{D}{dt} \frac{D}{dt} J(t).$$

Coordinate expression of $(*)$:

Let (e_i) be an ON basis at one point of curve γ . Parallel transport gives an ON frame along γ . Let

$$J(t) = \sum_{i=1}^n J^i(t) e_i(t)$$

Now

$$\begin{aligned} \nabla_{\gamma'(t)} J(t) &= \sum_{i=1}^n \left\{ J^i(t) \nabla_{\gamma'(t)} \overset{\text{parallel}}{e_i(t)} + [\nabla_{\gamma'(t)} J^i(t)] e_i(t) \right\} \\ &= \sum_{i=1}^n \frac{dJ^i}{dt} e_i \end{aligned}$$

And

$$J''(t) = \sum_{i=1}^n \frac{d^2 J^i}{dt^2} e_i.$$

Now,

$$\begin{aligned} R(\gamma'(t), J(t)) \gamma'(t) &= \sum_{i=1}^n J^i(t) R(\gamma'(t), e_i) \gamma'(t) \\ &= \sum_{i=1}^n J^i(t) \sum_{j=1}^n \langle R(\gamma'(t), e_i) \gamma'(t), e_j \rangle e_j \\ &= \sum_i \sum_j J^i(t) \underbrace{\langle R(\gamma'(t), e_i) \gamma'(t), e_j \rangle}_{:= a_{ij}(t)} e_j \end{aligned}$$

Thus (*) is equivalent to

$$\boxed{\frac{d^2 J^j}{dt^2} + a_{ij}(t) J^i(t) = 0.} \quad j=1, \dots, n.$$

This is a 2nd order ODE (linear). Thus $J(t)$ is uniquely determined by $J(0)$, $J'(0)$. The solution space is a $2n$ -dim vector space. There are $2n$ linearly independent Jacobi fields along $\gamma(t)$ since (*) is linear in J .

A Trivial Solution:

$$J(t) = \gamma'(t) \quad J(t) = t \gamma'(t)$$

This is indeed a solution if we plug this into (*).

When M^n has constant sectional curvature K , then

$$R(X, Y)Z = K(\langle X, Z \rangle Y - \langle Y, Z \rangle X) \quad (**)$$

pf. Lem 3.4, p.96:

$$\begin{aligned} \langle R(X, Y)W, Z \rangle &= K(\langle X, W \rangle \langle Y, Z \rangle - \langle Y, W \rangle \langle X, Z \rangle) \\ &= \langle K(\langle X, W \rangle Y - \langle Y, W \rangle X), Z \rangle \end{aligned}$$

Therefore

$$R(X, Y)W = K(\langle X, W \rangle Y - \langle Y, W \rangle X)$$

So this is equivalent to (**). $\square$

Now assume $J(t) \perp \gamma'(t)$, and $|\gamma'(t)| = 1$. Then by above

$$R(\gamma'(t), J)\gamma'(t) = KJ(t).$$

Therefore

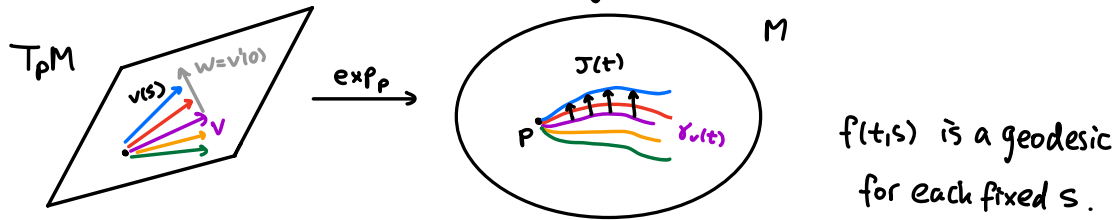
$$J''(t) + KJ(t) = 0.$$

Assume $J(0) = 0$, $J'(0) = w$. Then let $w(t)$ be the parallel transport of w along $\gamma(t)$. We have

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t) & K > 0 \\ t w(t) & K = 0 \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t) & K < 0 \end{cases}.$$

Lec 2.

The Jacobi field are variation field of geodesics:



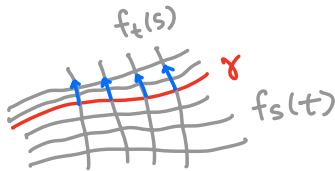
Prop. Let $\gamma: [0, a] \rightarrow M$ be a geodesic, J a Jacobi field along γ s.t. $J(0) = 0$, $J'(0) = w$. Denote $p = \gamma(0)$, $v(s)$ a curve in $T_p M$ with $v(0) = \gamma'(0)$, $v'(0) = w$. Let $f: [0, a] \times (-\varepsilon, \varepsilon) \rightarrow M$ be a geodesic variation defined on

$$f(t, s) = \exp_p t v(s).$$

Then

$$J(t) = \frac{\partial f}{\partial s}(t, 0) = (d \exp_p)_{tv}(tw).$$

Rmk.



$f_s(t)$ means fixing $s \in (-\varepsilon, \varepsilon)$ and let t vary and similarly for $f_t(s)$. Then

$$\frac{\partial f}{\partial s}(t, 0) := f'_t(s) \text{ evaluated at } s=0.$$

pf. Show that $\partial f / \partial s$ satisfies the Jacobi Equation:

$$\nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) \frac{\partial f}{\partial t} \stackrel{w.s.}{=} 0. \quad (*)$$

Observation:

- $\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} = 0$ since $f_s(t)$ is a geodesic for each $s \in (-\varepsilon, \varepsilon)$.
- Note $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$ in $T_*([0, a] \times (-\varepsilon, \varepsilon))$, thus

$$f_* \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = \left[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right] = 0.$$

Therefore:

$$\nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) \frac{\partial f}{\partial t}$$

$$= \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial t}$$

$$= \nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} = 0.$$

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$$

Therefore $\partial f / \partial s$ is a Jacobi field. Now want to show

$$J(t) = d(\exp_p)_{tv} tw$$

if $J(0) = 0$, $J'(0) = w$. Let $p = \gamma(0)$, $v = \gamma'(0)$. Then

$$\exp_p tv = \gamma(t).$$

Let $f(t, s) = \exp_p tv(s)$, where $v(s) = v + sw$. Then $f(t, s)$ is a geodesic variation. By result above, $\partial f / \partial s$ is a Jacobi field. Now

$$\frac{\partial f}{\partial s}(t, 0) = (d\exp_p)_{tv} tw$$

(check this). Then $\frac{\partial f}{\partial s}(0, 0) = 0$. Now

we can evaluate at $(0, 0)$ first then take cov. div. since ∇ is in $\downarrow$ the s direction, not in t direction.

$$\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} \Big|_{(0,0)} = \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} \Big|_{(0,0)} = \frac{d}{dt} \Big|_{t=0} \exp_p tv = \frac{d}{dt} \Big|_{t=0} \gamma(t) = \gamma'(0) = v$$

$$= \frac{\partial}{\partial s}(v(s))(0, 0) = w = J'(0).$$

So by uniqueness of Jacobi fields (same initial condition), we have

$$J(t) = \frac{\partial f}{\partial s}(t, 0) = (d\exp_p)_{tv} tw.$$

□

Prop. If $J(t) = (d\exp_p)_{tv} tw$ is the Jacobi field along geodesic $\gamma(t)$, with

$\gamma(0) = p$, $\gamma'(0) = v$. Assume $|w| = 1$, $|v| = 1$. Then

$$|J(t)|^2 = t^2 - \frac{1}{3} \langle R(v, w)v, w \rangle t^4 + o(t^4)$$

pf. Taylor Expand at $t=0$: $|J(0)|=0$.

Since covariant derivative is compatible with metric,

$$\langle J(t), J(t) \rangle' \Big|_{t=0} = 2 \langle J'(t), J(t) \rangle \Big|_{t=0} = 0. \quad (J(0)=0).$$

$$\begin{aligned} \langle J(t), J(t) \rangle'' \Big|_{t=0} &= 2 \langle J''(t), J(t) \rangle + 2 \langle J'(t), J'(t) \rangle \Big|_{t=0} \\ &= 2|w|^2 = 2. \end{aligned}$$

$$\begin{aligned} \langle J(t), J(t) \rangle''' \Big|_{t=0} &= 2 \langle J'''(t), J(t) \rangle + 6 \langle J''(t), J'(t) \rangle \Big|_{t=0} \\ &= 6 \langle J''(t), J'(t) \rangle \Big|_{t=0} \\ &= 0 \quad (J''(0)=0 \text{ by Jacobi Eqn}). \end{aligned}$$

Finally,

$$\begin{aligned} \langle J(t), J(t) \rangle^{(4)} &= [\langle J(t), J(t) \rangle^{(3)}]' \\ &= 8 \langle J'''(t), J'(t) \rangle + 2 \langle J'''(t), J(t) \rangle + 6 \langle J''(t), J''(t) \rangle \end{aligned}$$

$$\langle J(t), J(t) \rangle^{(4)} \Big|_{t=0} = 8 \langle J'''(t), J'(t) \rangle \Big|_{t=0}$$

Differentiate Jacobi Eqn:

$$J'''(t) + (\nabla_{J'(t)} R)(J'(t), J(t)) J'(t) + R(J'(t), J'(t)) J'(t) = 0$$

$$\Rightarrow J'''(0) = -R(v, w)v$$

Thus

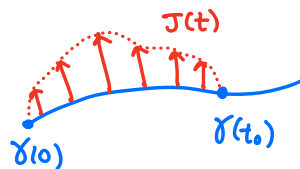
$$\langle J(t), J(t) \rangle^{(4)} \Big|_{t=0} = -\frac{8}{4!} \langle R(v, w)v, w \rangle.$$

□

Lec 3.

Defⁿ. Let $\gamma: [0, a] \rightarrow M$ be a geodesic. The point $\gamma(t_0)$ is **conjugate** to $\gamma(0)$ along γ if there is a Jacobi field J along γ , not identically zero, with $J(0) = J(t_0) = 0$.

The maximum number of such linearly independent Jacobi field is called the **multiplicity** of the conjugate point $\gamma(t_0)$.

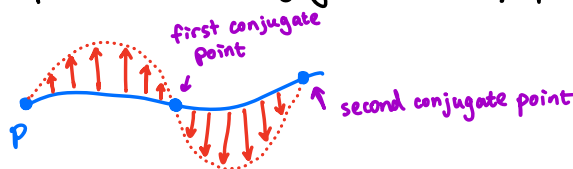


Rmk. Jacobi field $J(t)$ is uniquely determined by $J(0)$ and $J'(0)$. So if $\dim M = n$, then linearly independent Jacobi field is $\leq 2n$. Since we want $J(0) = 0$, so multiplicity $\leq n$. But $J(t) = t\gamma'(t)$ is a Jacobi field st. $J(t_0) \neq 0 \forall t_0$. Thus multiplicity $\leq n-1$.

Ex. • $\mathbb{R}^n$ has no conjugate point: $R=0$ in $\mathbb{R}^n$, so Jacobi equation implies $J''(t) = 0$. So $J(t)$ is linear in t . Thus $J(t_0) \neq 0 \forall t_0 > 0$.

• S^n . In this case $J(t) = (\sin t)w(t)$, where $w(t)$ parallel, $\langle \gamma'(t), w(t) \rangle = 0$.
 $-p = \gamma(\pi) \in S^n$ conjugate to $p = \gamma(0) \in S^n$ with multiplicity $n-1$.

Defⁿ. The set of (first) conjugate points to the point $p \in M$ for all geodesics starting at p is called **conjugate locus** of p . This is denoted by $C(p)$.



Ex: • $p \in \mathbb{R}^n$, $C(p) = \emptyset$ (Actually true when sectional curvature $K \leq 0$).
 • $p \in S^n$, $C(p) = \{-p\}$.

Prop. Let $\gamma: [0, a] \rightarrow M$ be a geodesic, $\gamma(0) = p$. Then $\gamma(t_0) = q$ conjugate to p along $\gamma \Leftrightarrow v_0 = \gamma'(0)$ is a critical point of $\exp_p$, i.e. $(d\exp_p)_{v_0}$ singular. Moreover, multiplicity of $q = \dim \ker (d\exp_p)_{v_0}$.

pf. Since $J(0) = 0$, $J'(0) = w$, we have shown that

$$J(t) = (d\exp_p)_{tv}(tw)$$

where $v = \gamma'(0)$. $J(t_0) = 0$ implies $(d\exp_p)_{v_0}(t_0 w) = 0$.

Linearly indep $w \Leftrightarrow J'(0) = w$ distinct $\Leftrightarrow$ Linearly indep $J(t)$ s.t. $J(t_0) = 0$.

Thus $\text{mult}(p) = \dim \ker (d\exp_p)_{v_0}$. $\square$

Prop. Let J be a Jacobi field along a geodesic $\gamma: [0, a] \rightarrow M$. Then

$$\langle J(t), \gamma'(t) \rangle = \langle J'(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle$$

pf. $\frac{d}{dt} \langle J(t), \gamma'(t) \rangle = \langle \frac{DJ}{dt}, \gamma' \rangle + \langle J, \frac{D}{dt} \gamma' \rangle = \langle \frac{DJ}{dt}, \gamma' \rangle$.

Similarly

$$\frac{d^2}{dt^2} \langle J(t), \gamma'(t) \rangle = \langle J''(t), \gamma'(t) \rangle.$$

$$= - \langle R(\gamma'(t), J(t)) \gamma'(t), \gamma'(t) \rangle = 0$$

(antisymmetry of Riemann curvature).

So $\langle J, \gamma' \rangle$ is linear. Check the rest. $\square$

Cor. If $J(0) = J(a) = 0$, then $\langle J(t), \gamma'(t) \rangle = 0$.

Cor. The space of Jacobi field with $J(0) = 0$ and $\langle J(t), \gamma'(t) \rangle = 0$ has $\dim \leq n$.

pf. With $J(0) = 0$, we have $\dim = n$. $\langle J(t), \gamma'(t) \rangle = 0 \Rightarrow$ one less. $\square$

Prop. Let $\gamma: [0, a] \rightarrow M$ be a geodesic. Let $v_1 \in T_{\gamma(0)}M$ and $v_2 \in T_{\gamma(a)}M$. If $\gamma(a)$ is not conjugate to $\gamma(0)$ along γ , then $\exists!$ Jacobi field J with $J(0) = v_1$, $J(a) = v_2$. [∂ -condition].

Pf. Uniqueness: Suppose J_1, J_2 satisfies ∂ -condition, then $J_1 - J_2$ a Jacobi field with $(J_1 - J_2)(\gamma(0)) = (J_1 - J_2)(\gamma(a)) = 0$. But $\gamma(0)$ not conjugate to $\gamma(a) \Rightarrow J_1 - J_2 = 0$. Thus $J_1 = J_2$.

Existence: Let

$$\mathcal{J} = \{ J \text{ Jacobi fields along } \gamma(t) \text{ with } J(0) = 0 \}.$$

Then $\dim \mathcal{J} = n$. Let $\theta: \mathcal{J} \rightarrow T_{\gamma(a)}M$ be the map $\theta(J) = J(a)$.

θ is linear between v.s with same dim. θ is injective since $\gamma(0)$ is not conjugate to $\gamma(a)$. Thus θ is surjective. Thus $\exists$ Jacobi field $\bar{J}_1$ s.t. $\bar{J}_1(a) = v_2 \in T_{\gamma(a)}M$, and $\bar{J}_1(0) = 0$. Now consider

$$\tilde{\mathcal{J}} = \{ J \text{ Jacobi fields along } \gamma(t) \text{ with } J(a) = 0 \}$$

define $\tilde{\theta}$ analogously and apply same argument. Get $\bar{J}_2$ s.t. $\bar{J}_2(0) = v_1$, $\bar{J}_2(a) = 0$. Then

$$J = \bar{J}_1 + \bar{J}_2$$

is the desired Jacobi field. □

Cor. (p. 119, do Carmo, 3.10).

Lec 4. Chap 6. Isometric Immersions

$$M^n \xrightarrow{f} \bar{M}^{n+m}$$

immersion

If $\bar{M}$ is Riemannian, it induces a Riemannian metric

$$v_1, v_2 \in T_p \bar{M}, \quad \langle v_1, v_2 \rangle = \langle (df)_p v_1, (df)_p v_2 \rangle$$

Since f is immersion, the metric on $\bar{M}$ is positive definite.

If

$$f^* g_{\bar{M}} = g_M.$$

f is called an **isometric immersion**.

Relation between curvatures of M^n and $\bar{M}^{n+m}$?

Answer: 2nd Fundamental Form.

Induced Riemannian Connection:

$$\nabla_v w = (\bar{\nabla}_{\bar{v}} \bar{w})^T$$

$\nabla_v w$ ← ∇ : connection on M
 $\bar{\nabla}_{\bar{v}} \bar{w}$ ← $\bar{\nabla}$: connection on $\bar{M}$
 $\bar{v}, \bar{w}$: extensions of v, w on $\bar{M}$

← projection onto tangent space $T_p M \subset T_p \bar{M}$

[check: Well-defined]

$$\text{Def}^n. \text{ Let } B(v, w) = \bar{\nabla}_{\bar{v}} \bar{w} - (\bar{\nabla}_{\bar{v}} \bar{w})^T \in (T_p M)^\perp$$

where we use the identification $T_p \bar{M} = T_p M \oplus (T_p M)^\perp$.

This is called the **2nd - fundamental form**.

Prop. $B(v, w)$ is symmetric, therefore is a symmetric tensor. $B: T_p M \times T_p M \rightarrow (T_p M)^\perp$.

pf. $B(fv, w) = f B(v, w)$. So tensorial in v variable.

$$\begin{aligned} B(v, w) - B(w, v) &= \bar{\nabla}_{\bar{v}} \bar{w} - \nabla_v w - \bar{\nabla}_{\bar{w}} \bar{v} + \nabla_w v \\ &= [\bar{v}, \bar{w}] - [v, w] = [\bar{v}, \bar{w}] - [\bar{v}, \bar{w}]^T = 0 \end{aligned}$$

↑
evaluate at a point
on submanifold M .

Other forms of 2nd - fundamental form: $\eta \in T_p M^\perp$. Define

$$\mathbb{I}_\eta: T_p M \times T_p M \rightarrow \mathbb{R}$$

$$\mathbb{I}_\eta(v, w) = \langle B(v, w), \eta \rangle.$$

This is a (0,2) - tensor.

Let $N_1, \dots, N_m$ be an ON basis of $T_p M^\perp$. Then

$$B(v, w) = \sum_{i=1}^m \mathbb{I}_{N_i}(v, w) N_i.$$

Let $S_\eta: T_p M \rightarrow T_p M$, s.t.

$$\langle S_\eta(v), w \rangle = \mathbb{I}_\eta(v, w).$$

Prop. For $p \in M$, $\eta \in (T_p M)^\perp$ fixed, let N be a local extension of η normal to M . Then

$$S_\eta(v) = -(\bar{\nabla}_v N)^T.$$

(This is differential of Gauss map when $M^n \hookrightarrow \mathbb{R}^{n+1}$).

pf. $\forall v \in T_p M$,

$$\begin{aligned} \langle S_\eta(v), w \rangle &= \mathbb{I}_\eta(v, w) = \langle B(v, w), \eta \rangle \quad \swarrow \text{tangential} \\ &= \langle \bar{\nabla}_v \bar{w} - \bar{\nabla}_w v, \eta \rangle \\ &= \langle \bar{\nabla}_v \bar{w}, N \rangle \\ &= \bar{v} \langle \bar{w}, N \rangle - \langle w, \bar{\nabla}_v N \rangle \\ &= -\langle (\bar{\nabla}_v N)^T, w \rangle \end{aligned}$$

□

Mean curvature of M in $\bar{M}^{n+m}$ is

$$H = \sum_{i=1}^m (\text{tr } \mathbb{I}_{N_i}) N_i.$$

$H=0$ is called a **minimal submanifold**.

Thm (Gauss Equation). Let $p \in T_p M$, $v, w \in T_p M$ orthonormal. Then

$$K(v, w) = \bar{K}(v, w) + \langle B(v, v), B(w, w) \rangle - |B(v, w)|^2.$$

Apply to $S^n \hookrightarrow \mathbb{R}^{n+1}$. (In this case Gauss map is Id).

Thus $S_\eta(v) = -v$. Thus $\mathbb{I}_\eta(v, w) = -\langle v, w \rangle$. Thus

$B(v, w) = -\langle v, w \rangle \vec{N}$ is unique normal direction.

Therefore

$$K_{S^n}(v, w) = 1 \Rightarrow K_{S^n} \equiv 1.$$

Rmk: $\bar{\nabla}_{\bar{X}} \bar{\gamma}(p)$ only depends on value of $\bar{X}$ at point p and value of $\bar{\gamma}$ in a nbhd of p . But can choose $\gamma: [0, 1] \rightarrow M$ st. $\gamma' = X$ at p . Then

$$\bar{\nabla}_{\bar{X}} \bar{\gamma} = \bar{\nabla}_{\gamma'} \bar{\gamma} = \frac{D}{dt} \bar{\gamma}$$

But the later only depends on the value of $\bar{\gamma}$ on γ , since $\gamma \subset M$, $\bar{\gamma}|_\gamma = \gamma|_\gamma$. So $\bar{\nabla}_{\bar{X}} \bar{\gamma}(p)$ depends on $X(p)$ and value of γ on int. curves of X .

Lec 5.

Gauss Equation: Let $M^n \hookrightarrow \bar{M}^{n+m}$ be an isometric immersion, $x, y \in T_p M$ orthonormal vectors. Then

$$K(x, y) = \bar{K}(x, y) + \langle B(x, x), B(y, y) \rangle - |B(x, y)|^2.$$

pf. $K(x, y) - \bar{K}(x, y)$

last 2 terms cancel since Y is tangential to M and $(\bar{\nabla}_{[\bar{x}, \bar{y}]} \bar{X})^T = \nabla_X Y$

$$= \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X, Y]} X, Y \rangle - \langle \bar{\nabla}_Y \bar{\nabla}_X X - \bar{\nabla}_X \bar{\nabla}_Y X + \bar{\nabla}_{[\bar{X}, \bar{Y}]} \bar{X}, Y \rangle$$

Now

$$\begin{aligned} \bar{\nabla}_Y \bar{\nabla}_X X &= \bar{\nabla}_Y (\nabla_X X + B(x, x)) \\ &= \bar{\nabla}_Y (\nabla_X X + \sum_{i=1}^m \langle B(x, x), e_i \rangle e_i) \quad \text{where } e_i \text{ o.n. basis of } T_p^\perp M. \\ &= \nabla_Y \nabla_X X + B(Y, \nabla_X X) + \sum_{i=1}^m Y \langle B(x, x), e_i \rangle e_i \\ &\quad + \sum_{i=1}^m \langle B(x, x), e_i \rangle \bar{\nabla}_Y e_i \end{aligned}$$

Thus

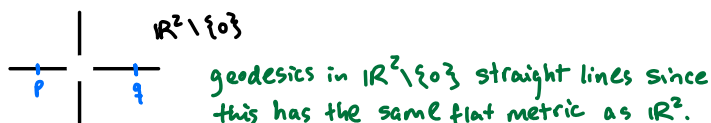
$$\begin{aligned} \langle \bar{\nabla}_Y \bar{\nabla}_X X, Y \rangle &= \langle \nabla_Y \nabla_X X, Y \rangle + \sum_{i=1}^m \langle B(x, x), e_i \rangle \langle \bar{\nabla}_Y e_i, Y \rangle \\ &= \langle \nabla_Y \nabla_X X, Y \rangle - \langle B(x, x), B(Y, Y) \rangle. \end{aligned}$$

Do other terms similarly. □

Chap 7. Complete Manifolds.

Question: Given $p, q \in M^n$, Can we find a geodesic connecting them?

Answer: No if M^n is not complete as metric space!



Defⁿ. A Riem. manifold is **geodesically complete** if for all $p \in M$, the $\exp_p$ is defined for all $v \in T_p M$. i.e., any geodesic $\gamma(t)$ starts from p is defined for all $t \in \mathbb{R}$.

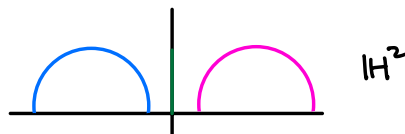
E.g. $\mathbb{R}^n, S^n$. Complete.

E.g. $\mathbb{H}^2 = \{(x, y) : y > 0\}$, $ds^2 = \frac{dx^2 + dy^2}{y^2}$ is complete

Although some geodesics hit the boundary, it takes infinitely long to reach the boundary.

But if we change the metric to $ds^2 = \frac{dx^2 + dy^2}{y}$, then $\mathbb{H}^2$ is incomplete: takes finite time to reach boundary:

$$d((0,0), (0,1)) = \int_0^1 \frac{1}{\sqrt{y}} dy < \infty.$$



Thm (Hopf-Rinow, 1931). Let M be a connected Riem. manifold, $p \in M$,
TFAE:

- (a) $\exp_p$ is defined for all of $T_p M$.
- (b) The closed and bounded sets of M are compact (Heine-Borel property).
- (c) M is complete as a metric space.
- (d) M is geodesically complete.

In addition, any statement above implies

$$(f) \quad \forall q \in M, \exists \text{ geo } \gamma \text{ joining } p, q \text{ s.t. } L(\gamma) = d(p, q).$$

pf. (a) $\Rightarrow$ (f): Next time.

(a) $\Rightarrow$ (b): Let $A \subset M$ closed and bounded. Then $A \subset B_r(p)$ for some big r .

By (f), $\exp_p$ is surjective. Thus $B_r(p) \subset \exp_p \overline{B_r(0)}$ where $\overline{B_r(0)} \subset T_p M$ compact. Thus A is a closed subset in compact $\exp_p \overline{B_r(0)}$, hence compact.

(b) $\Rightarrow$ (c): Proved in analysis.

(c) $\Rightarrow$ (d) read do Carmo.

lec 6.

Hopf - Rinow:

(a) $\Rightarrow$ (f): Let $B_\delta(p)$ be a normal ball of p . If $q \in B_\delta(p)$, done. So assume $q \notin B_\delta(p)$. $\partial B_\delta(p)$ is compact, so there exists $x_0 \in \partial B_\delta(p)$ s.t.

$$d(q, x_0) = d(q, \partial B_\delta(p)) \leftarrow \begin{array}{l} x_0 \text{ is where the cts function} \\ d(q, \cdot) : \partial B_\delta(p) \rightarrow \mathbb{R} \\ \text{attains minimum} \end{array}$$

$\exists v \in T_p M, \gamma$ s.t. $\exp_p t_0 v = x_0$. Let $\gamma(t) : \exp_p tv$,

where $\gamma(t)$ defined $\forall t \in \mathbb{R}$ by (a). Denote $d(p, q) = p$.

Claim: $\gamma(p) = q$.

We show this by continuity method: Suffices

to show:

$$d(q, \gamma(s)) = p - s \quad \forall s \in [0, p] \quad (*)$$

Let $A = \{s \in [0, p] : (*) \text{ holds}\}$.

A is closed since everything is continuous, preim of eqs. $0 \in A$.

$\hookrightarrow A = \psi^{-1}(p)$ where $\psi = d(q, \gamma(\cdot)) + \cdot : [0, p] \rightarrow \mathbb{R}$.

Claim I: Suppose $\gamma(t_0) = x_0$. Then $t_0 \in A$.

pf. $p - t_0 = d(p, q) - d(p, \gamma(t_0))$. By Δ -ineqn: $d(q, \gamma(t_0)) \geq d(p, q) - d(p, \gamma(t_0)) \geq p - t_0$.

Let c be any curve connecting p, q . Then

$$L(c) \geq t_0 + d(q, \gamma(t_0)) \leftarrow \begin{array}{l} \text{1st part: geo in } B_\delta(p) \text{ minimal} \\ \text{2nd part: choice of } x_0 = \gamma(t_0). \end{array}$$

Take inf: $p = d(p, q) \geq t_0 + d(q, \gamma(t_0))$.

Thus $\leq \& \geq \Rightarrow =$.

//

Now show if $s_0 \in A$, and $t_0 \leq s_0 < p$, then $s_0 + t'_0 \in A$ for some $t'_0 > 0$.

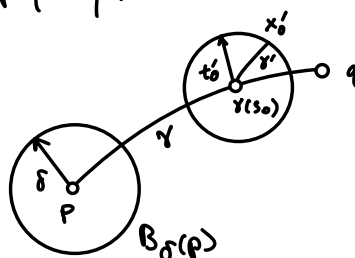
We have $d(\gamma(s_0), q) = p - s_0$. Choose normal ball at $\gamma(s_0)$ with rad t_0 , choose x'_0 s.t. $d(x'_0, q) = d(\partial B_{t'_0}(\gamma(s_0)), q)$.

Claim II: $x'_0 = \gamma(s_0 + t'_0)$. Then we are done.

pf. Δ -ineq: $d(p, x'_0) \geq d(p, q) - d(x'_0, q)$

$d(\gamma(s_0), q) = t'_0 + d(x'_0, q)$ follows exactly pf of Claim I.

plug in Δ -ineq.



$$\begin{aligned}
 d(p, x'_0) &\geq d(p, q) - d(x'_0, q) \\
 &= p - d(x'_0, q) \rightarrow d(x'_0, q) = d(\gamma(s_0), q) - t'_0 \\
 &= p - s_0 - t'_0 \\
 &= p - (p - s_0 - t'_0) \\
 &= s_0 + t'_0 = L(\gamma|_{[0, s_0]}) + L(\gamma'|_{[0, t'_0]})
 \end{aligned}$$

But

$$s_0 + t'_0 = d(p, \gamma(s_0)) + d(\gamma(s_0), x'_0) \geq d(p, x'_0)$$

Thus $\gamma \cup \gamma'$ is a curve realizing distance: $L(\gamma \cup \gamma') = d(p, x'_0)$.

So $\gamma \cup \gamma'$ must be a smooth geodesic without corner. //

Finally: Exercise: If $\tau \in A$, then $[0, \tau] \subset A$.

(use def and Δ -ineq)

Therefore we conclude that $[0, p] \subset A$.

□

Cor. If M is compact, then M is complete.

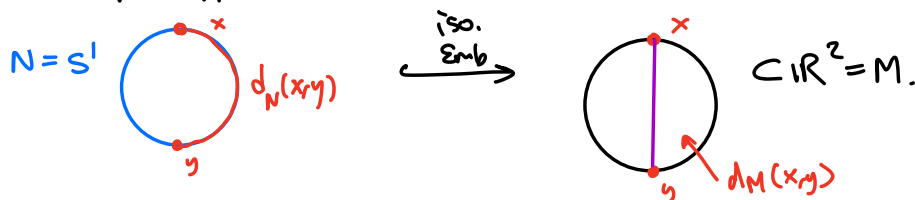
pf. Compact metric space is complete.

□

Cor. A closed submanifold of a complete R.M. is complete in the induced metric.

(Hw). $N \subset M$, M as above, then N complete.

Hint $d_N \geq d_M$:



1926 1981
Thm (Cartan - Hadamard). Let M^n be a complete, simply connected R.M. with $K(p, \sigma) \leq 0$ for all $p \in M$, $\sigma \in T_p M$. Then M^n is diffeomorphic to $\mathbb{R}^n$. In fact $\exp_p: T_p M \rightarrow M$ is a diffeomorphism.

Cor. S^2 , $S^2 \times \mathbb{R}$ has no metric with $K \leq 0$.

Pf. They are simply connected but not $\mathbb{R}^n$. □

If $f: \tilde{M} \rightarrow M$ is a differentiable covering map and M simply connected, then f is a diffeomorphism.

Defⁿ. $f: \tilde{M} \rightarrow M$ is a **covering map** if f is cts and onto, s.t. $\forall p \in M$, $\exists U_p$ a nbhd of $p \subset M$, s.t.

$$f^{-1}(U_p) = \bigsqcup_i \tilde{U}_i$$

where each $\tilde{U}_i \subset \tilde{M}$ open, s.t. $f|_{\tilde{U}_i}$ is a homeo.

lem. Suppose $\tilde{M}, M$ are connected R.M, $\tilde{M}$ complete, and $\pi: \tilde{M} \rightarrow M$ is local isometry. Then M is also complete, and π is a covering map.

Week 4 lecture:

Hadamard Theorem

Hopf - Poincaré thm: If M is complete, then ^① any two points can be connected by a minimal geod. ^② $\exp_p$ is onto. $g = \exp_p$ to $\gamma'(0)$

Inverse is not true.

Ex: $\mathbb{R}_+ = (0, \infty)$.

$\gamma(t)$ $f = \gamma(t_0)$

If q is in a normal neighborhood of p , $\exp_p$ is a diffeo in normal neighborhood, $d\exp_p$ is isomorphism.

Thm: (Cartan - Hadamard theorem)

1926

1881

Let M^n be a complete simply connected Riem mfd with $K(p, \sigma) \leq 0$ for all $p \in M$, $\sigma \in T_p M$.

Then M^n is diffeo to $\mathbb{R}^n$. In fact

$\exp_p: T_p M \rightarrow M$ is a diffeo.

show $\exp_p$ is a covering map.

Lemma: Suppose $\tilde{M}$ and M are connected Riem mfd's

with $\tilde{M}$ complete and $\pi: \tilde{M} \rightarrow M$ is local isomorphism.

Then M is complete and π is a covering map

Proof: Local isometry maps geod to geod.

① Show M is complete.

② Show π is onto

③ π is a covering map.

① Show M is complete: Since π is a local isometry, $\exists p$ s.t. $\pi^{-1}(p)$ is not empty.

Show any unit speed geod γ starting from p is defined for all $t \in \mathbb{R}$.

Let $\tilde{p} \in \pi^{-1}(p)$, $\tilde{v} = \pi_x^{-1}(\gamma'(0)) \in T_{\tilde{p}}\tilde{M}$.

Let $\tilde{\gamma}$ be the geod $\tilde{\gamma}(0) = \tilde{p}$, $\tilde{\gamma}'(0) = \tilde{v}$.

Since π is a local isometry, π map $\tilde{\gamma}$ to γ .

Since $\tilde{M}$ is complete $\tilde{\gamma}$ is defined for all $t \in \mathbb{R}$.

γ is defined for all $t \in \mathbb{R}$.

② π is onto. Fix any $\tilde{p} \in \tilde{M}$. Let $p = \pi(\tilde{p})$.

Given any $q \in M$. Since M is complete, there is a minimal normal geod γ connecting p, q .

Let $\tilde{\gamma}$ be the geod

with $\tilde{\gamma}(0) = \tilde{p}$, $\tilde{\gamma}'(0) =$

$\pi_x^{-1}(\gamma'(0))$.

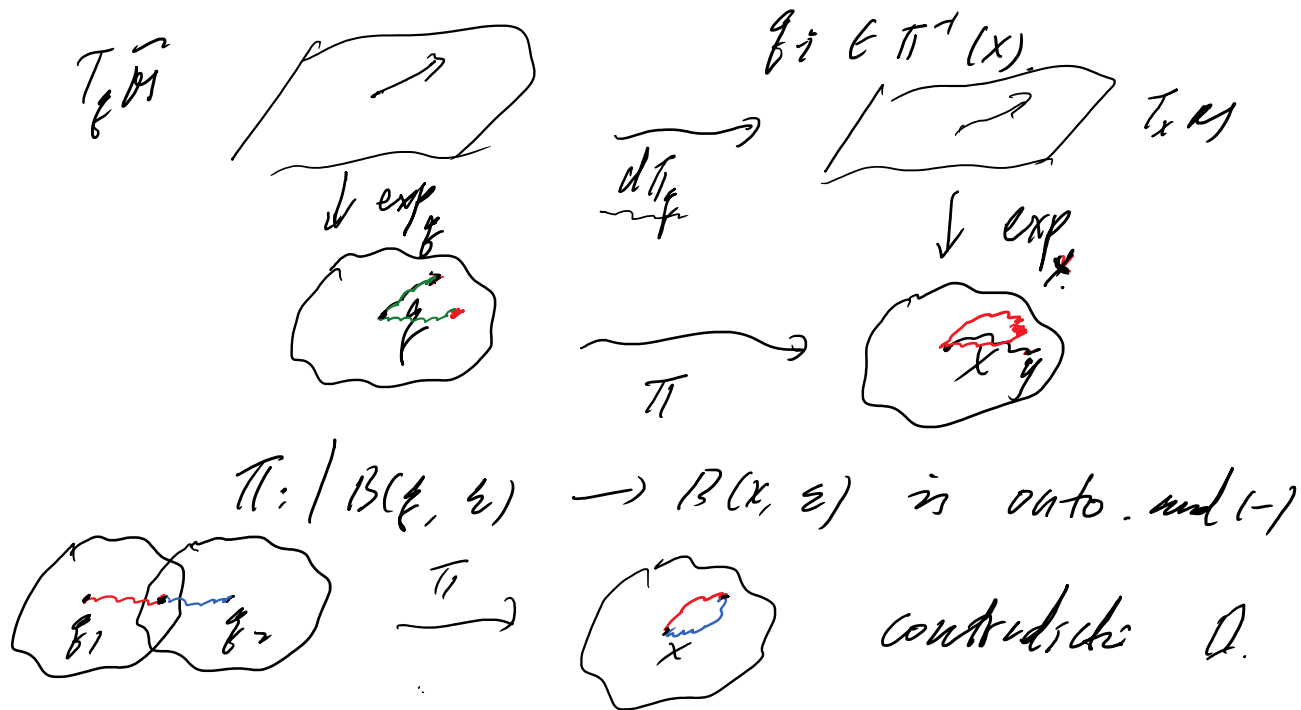
$\pi(\tilde{\gamma}) = \gamma$

$q = \pi(\tilde{\gamma}(t))$ where $t = d(p, q)$



$$\pi(\tilde{x}) = x, \quad \tilde{g} = \pi(\tilde{\gamma}(\tilde{d}_0)), \text{ where } d_0 = d(p, \tilde{g}).$$

- ③ π is a covering map. $\forall x \in M$, let $B_x(\varepsilon)$ be a normal neighborhood of x . Show $B(x, \varepsilon)$ is evenly covered. Let $\tilde{g} \in \pi^{-1}(x)$. Show ① $\pi: B(\tilde{g}, \varepsilon) \rightarrow B(x, \varepsilon)$ is a diffeo. ② $B(\tilde{g}_1, \varepsilon) \cap B(\tilde{g}_2, \varepsilon) = \emptyset$ if $\tilde{g}_1 \neq \tilde{g}_2$.



Proof of Hadamard theorem: Since M has $K \leq 0$, then $\text{Clp} = \emptyset$ (Homework, P117, #3). $\exp_p$ is a local diffeo. Equip $T_p M$ with pull back metric so $\exp_p$ is a local isometry. By Gauss lemma, straight line start from 0 are geodesic. Hence $T_p M$ is

complete. By Lemma $\exp_p$ is a covering map.
Hence $\exp_p$ is a diffeo. $\square$

Lec 9. chap 8, Spaces of constant (sectional) Curvature

Thm. If M^n is a complete R.M. w/ constant sectional curvature K , the universal cover $\tilde{M}$ (with covering metric) is isometric to:

$$\mathbb{H}^n (K=-1) \quad \mathbb{R}^n (K=0) \quad S^n (K=1).$$

Recall: (M^n, g) , $\tilde{g} = \lambda g$, $\lambda > 0$. Then the sectional curvature of $\tilde{g}$ is given by

$$K_{\tilde{g}} = \frac{1}{\lambda} K_g.$$

Therefore for space with constant curvature, K can always be scaled.

Model space $(\mathbb{H}^n, \mathbb{R}^n, S^n)$ has maximal isometry:

$$\dim(\text{Isom}(M^n)) \leq n + \dim(O(n)) = n + \frac{n(n-1)}{2} = n \frac{(n+1)}{2}.$$

In model space, equality is achieved.

$$\text{Isom}(\mathbb{R}^n) = \mathbb{R}^n \ltimes O(n)$$

$$\text{Isom}(S^n) = O(n+1).$$

$\mathbb{H}^n$: 4-models:

① Upper half plane: $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$

$$g = (dx_1^2 + \dots + dx_n^2) / x_n^2.$$

② Poincaré Disk: $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 < 1\}.$

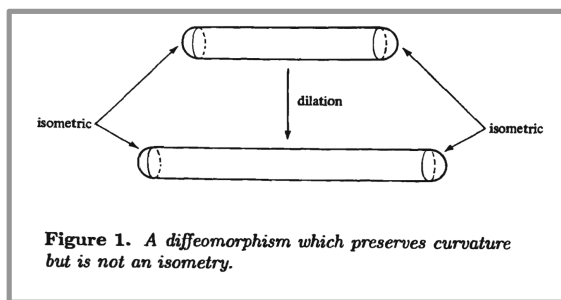
$$g_{ij} = \frac{4 \delta_{ij}}{(1 - |x|^2)^2}.$$

.....

Q: If $f: (M, g) \rightarrow (\tilde{M}, \tilde{g})$ a diffeomorphism preserving sectional curvature, is f an isometry?

This completely fails when $M, \tilde{M}$ have constant sectional curvature. Any diffeo preserves the constant curvature, but not isometric.

E.g. In dim 2:



lem. Isometry is uniquely determined by its value and differential at a point.
 More precisely, let $f_1, f_2 : M \rightarrow N$ be two (local) isometry st. $\exists p \in M$
 with $f_1(p) = f_2(p)$, $df_1|_p = df_2|_p$, then $f_1 = f_2$.

Lec 10.

Does curvature tensor determine metric? A: Yes, under suitable way.

$M^n, \bar{M}^n, p \in M, \bar{p} \in \bar{M}$. $I: T_p M \rightarrow T_{\bar{p}} \bar{M}$ a linear isometry.

Let $B(p, r)$ be a normal nbhd of p .

Defⁿ. $\varphi: B(p, r) \rightarrow B(\bar{p}, r)$ by $\varphi = \exp_{\bar{p}} \circ I \circ \exp_p^{-1}$.

$$\begin{array}{ccc} T_p M & \xrightarrow{I} & T_{\bar{p}} \bar{M} \\ \exp_p^{-1} \uparrow & & \uparrow \exp_{\bar{p}} \\ B(p, r) & \xrightarrow{\varphi} & B(\bar{p}, r) \end{array}$$

If r is sufficiently small, $B(\bar{p}, r)$ is a normal ball. Then φ is a diffeo.

For $q \in B(p, r)$, $\exists!$ normal geodesic $\gamma: [0, t] \rightarrow B(p, r)$ s.t. $\gamma(0) = p$, $\gamma(t) = q$.

Let P_t be parallel transport from $\gamma(0)$ to $\gamma(t)$. Similarly for $\bar{P}_t$.

Define $I_t: T_q M \rightarrow T_{\bar{q}} \bar{M}$, $\bar{q} = \varphi(q)$.

$$I_t = \bar{P}_t \circ I \circ (P_t)^{-1}$$

Thm (Cartan, local version). Suppose for all $q \in B(p, r)$, we have

$$\langle R(x, y)u, v \rangle = \langle \bar{R}(I_t(x), I_t(y))I_t(u), I_t(v) \rangle$$

for all $x, y, u, v \in T_q M$. Then φ is an isometry and $d\varphi = I_t$ ($d\varphi_p = I$).

pf. $\forall q \in B(p, r)$, let $\gamma: [0, l] \rightarrow B(p, r)$ be the normal geod s.t. $\gamma(0) = p$, $\gamma(l) = q$.

$\forall v \in T_p M$. Let J be the Jacobi field along γ s.t. $J(0) = 0$, $J(l) = v$.

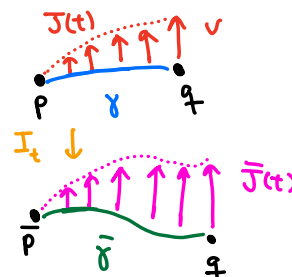
Let $\bar{J}(t) = I_t(J(t))$. $\bar{J}(0) = 0$, $|\bar{J}(l)| = |J(l)|$

Since I_t is an isometry by def.

Claim: $\bar{J}(t)$ is also a Jacobi field.

pf of claim: let $e_1, \dots, e_n$ be a basis of $T_p M$

that is O.N. and let $e_i(t)$ be the parallel transport



Cor (Space of Constant Curvature has lots of isometry).

If $M^n, \bar{M}^n$ has same constant curvature ($\Rightarrow$ same Riem. cur. tensor). Let $p \in M$, $\bar{p} \in \bar{M}$, $(e_i)_i \in T_p M$, $(\bar{e}_i)_i \in T_{\bar{p}} \bar{M}$ O.N. bases, then $\exists$ nbhd V of M of p , $\bar{V}$ of $\bar{M}$ of $\bar{p}$, and an isometry $f: V \rightarrow \bar{V}$ s.t. $f(p) = \bar{p}$, $df_p(e_i) = \bar{e}_i$.

Thm. If M^n is a complete R.M. w/ constant sectional curvature K , the universal cover $\tilde{M}$ (with covering metric) is isometric to:

$$\mathbb{H}^n (K=-1) \quad \mathbb{R}^n (K=0) \quad S^n (K=1).$$

pf. When $K = -1, 0$. For $\mathbb{H}^n(\mathbb{R}^n)$, the normal nbhd is the whole manifold.

By Cartan thm, $\exists$ locally isometry $\mathbb{H}^n(\mathbb{R}^n) \rightarrow \tilde{M}$. Since $\mathbb{H}^n(\mathbb{R}^n)$ is complete, thus f is a covering map. Since $\tilde{M}$ is simply connected, f is a diffeo. Hence f is an isometry.

Defⁿ: **Spaceform**: Complete R.M. w/ constant sectional curvature.

Prop. If M^n is a space form w/ curvature K , then

$$M^n \cong_{\text{isometric}} \begin{cases} S^n / \pi_1(M) & K=1 \\ \mathbb{R}^n / \pi_1(M) & K=0 \\ \mathbb{H}^n / \pi_1(M) & K=-1 \end{cases}$$

$\pi_1(M)$ acts totally discontinuously (i.e., $\forall x \in \tilde{M}$, $\exists U^{\text{open}} \ni x$ s.t. $\forall \gamma \in \pi_1(M)$, $\gamma \neq e$, $\gamma U \cap U = \emptyset$) on $\tilde{M}$. So $\pi_1(M) \leq \text{Isom}(\tilde{M})$. Then

$$\pi: \tilde{M} \rightarrow \tilde{M} / \pi_1(M)$$

is a covering map.

Notation: $\text{Isom}(\mathbb{M}_K^n)$, isometry gp of a space form w/ $\dim=n$, $\text{curv}=K$.

Determine $\text{Isom}(\mathbb{M}_K^n)$.

Lec 11. Variation of Energy.

Defⁿ.

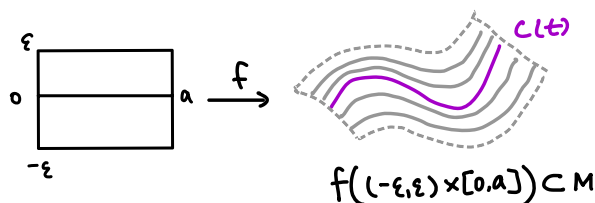
$c: [0, a] \rightarrow M$ a piecewise differentiable curve. A **variation of c** is a continuous map $f: (-\varepsilon, \varepsilon) \times [0, a] \rightarrow M$ st.

(1) $f(0, t) = c(t)$

(2) There exists a subdivision

$$0 = t_0 < t_1 < \dots < t_{k+1} = a$$

st. $f|_{(-\varepsilon, \varepsilon) \times (t_i, t_{i+1})}$ is differentiable

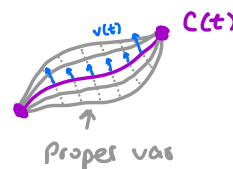


A variation is **proper** if $f(s, 0) = c(0)$, $f(s, a) = c(a)$ for $s \in (-\varepsilon, \varepsilon)$.

Then we write $f(s, t) = f_s(t)$ with $f_0(t) = c(t)$.

$$v(t) = \frac{\partial f}{\partial s}(0, t)$$

is called a **variation field of f** .



Rmk. A good variation is

$$f(s, t) = \exp_{c(t)} s v(t)$$

Then f is proper iff $v(0) = v(a) = 0$.

$$L(f_s(t)) = \int_0^a \left| \frac{\partial f}{\partial t}(s, t) \right| dt = L(s)$$

Energy function:

$$E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s, t) \right|^2 dt.$$

Cauchy-Schwarz ineqn:

$$\left(\int_0^a f g dt \right)^2 \leq \left(\int_0^a f^2 dt \right) \left(\int_0^a g^2 dt \right).$$

Put $f = 1$, $g = \left| \frac{\partial f}{\partial t} \right|$ gives

$$L(s)^2 \leq a E(s).$$

Equality iff $\left| \frac{\partial f}{\partial t} \right|$ is constant.

For fixed constant speed curve, minimizing L is the same as minimizing E .

lem. Let $p, q \in M$, $\gamma: [0, a] \rightarrow M$ a minimal geodesic joining p, q , then for all $c: [0, a] \rightarrow M$ joining p, q ,

$$E(\gamma) \leq E(c)$$

Equality holds iff c is a minimizing geodesic.

pf. Since γ is a geodesic, $|\gamma'(t)|$ is constant. Then by Cauchy Schwarz,
 $a E(\gamma) = L(\gamma)^2 \leq L(c)^2 \leq a E(c)$.

If equality holds, then

$$L(c)^2 = a E(c)$$

So $|c'(t)|$ is constant. Also, $L(\gamma) = L(c)$. So c is a minimizing geodesic. $\square$

Prop (1st Variation Formula). Let $c: [0, a] \rightarrow M$ be differentiable (p.w. diff then add 'em up) curve, $f: (-\varepsilon, \varepsilon) \times [0, a] \rightarrow M$ a differentiable variation of c .

If E is the energy function of f : $E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s, t) \right|^2 dt$. Then

$$\frac{1}{2} E'(0) = - \int_0^a \langle v(t), \nabla_{c'(t)} c'(t) \rangle dt + \langle v(t), c'(t) \rangle \Big|_0^a.$$

pf. $E(s) = \int_0^a \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle dt$.

Thus

$$\frac{d}{ds} E(s) = 2 \int_0^a \left\langle \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle dt \quad \text{and} \quad \left[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right] = 0$$

$$= 2 \int_0^a \left\langle \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle dt$$

$$= 2 \int_0^a \left\{ \frac{d}{dt} \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle - \left\langle \frac{\partial f}{\partial s}, \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} \right\rangle \right\} dt$$

$$\stackrel{s=0}{=} \langle v(t), c'(t) \rangle \Big|_0^a - \int_0^a \langle v(t), \nabla_{c'(t)} c'(t) \rangle dt.$$

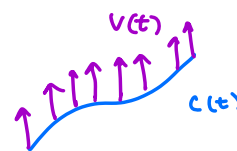
$\square$

Lec 13.

$c: [0, a] \rightarrow M$ a smooth curve, $V(t)$ a variation field

Energy function: $E: \{\text{curves}\} \rightarrow \mathbb{R}$:

$$E(c) = \int_0^a |c'(t)|^2 dt.$$



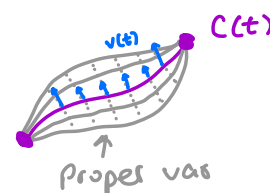
1st variation formula:

$$\frac{1}{2} E'(0) = - \int_0^a \langle V(t), \nabla_{c'(t)} c'(t) \rangle dt + \langle V(t), c'(t) \rangle \Big|_0^a.$$

If c is piecewise smooth, add up:

$$\frac{1}{2} E'(0) = - \int_0^a \langle V(t), \nabla_{c'(t)} c'(t) \rangle dt + \langle V(t), c'(t) \rangle \Big|_0^a$$

$$- \sum_{i=1}^k \langle V(t_i), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) \rangle.$$



Prop. A piecewise smooth curve $c: [0, a] \rightarrow M$ is a geodesic iff for any proper variation f of c , we have $\frac{dE}{ds}(0) = 0$. i.e., geodesics are critical points of the energy for all proper variation (proper: $V(0) = V(a) = 0$)

$$E: \Omega_{p,q} \rightarrow \mathbb{R} \quad \Omega_{p,q} = \{\text{all piecewise smooth curve connecting } p, q\}.$$

pf. ($\Rightarrow$) By def, geodesic is always smooth. Then

$$\frac{1}{2} E'(0) = - \int_0^a \langle V(t), \nabla_{c'(t)} c'(t) \rangle dt + \langle V(t), c'(t) \rangle \Big|_0^a = 0.$$

0 by properness
0 since geodesic

($\Leftarrow$) $\frac{dE}{ds}(0) = 0$ means $E'(0) = 0$ for all variation field $V(t)$. Thus pick

$$V(t) = g(t) \nabla_{c'(t)} c'(t)$$

where $g: [0, a] \rightarrow \mathbb{R}$ is s.t.

$$g(t) > 0 \text{ when } t \neq t_i$$

$$g(t_i) = 0$$



Therefore if we plug in $v(t)$ defined above,

$$\begin{aligned} \frac{1}{2} E'(0) &= - \int_0^a \langle v(t), \nabla_{c'(t)} c'(t) \rangle dt + \overset{0 \text{ by properness}}{\langle v(t), c'(t) \rangle} \Big|_0^a \\ &\quad - \sum_{i=1}^k \overset{0 \text{ by choice of } g}{\langle v(t_i), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) \rangle} \\ &= - \int_0^a g(t) \langle \nabla_{c'(t)} c'(t), \nabla_{c'(t)} c'(t) \rangle dt \end{aligned}$$

Since $g(t) > 0$ on each piece (t_i, t_{i+1}) , So $|\nabla_{c'(t)} c'(t)|^2 = 0$ on each piece.
So $\nabla_{c'(t)} c'(t) = 0$ on $(t_i, t_{i+1}) \forall i$. Thus on each piece c is a geodesic.
Thus we know now

$$\frac{1}{2} E'(0) = - \sum_{i=1}^k \langle v(t_i), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) \rangle.$$

Now chose $\bar{v}(t) \Rightarrow t$.

$$\bar{v}(t_i) = \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-)$$

$\forall t_i$, and $\bar{v}(0) = \bar{v}(a) = 0$. Then

$$\frac{1}{2} E'(0) = - \sum_{i=0}^k \left| \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) \right|^2 = 0$$

Hence

$$\frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) = 0$$

for all i . So $c \in C^1$. Since $\nabla_{c'(t)} c'(t) = 0 \forall t = t_i$, so $c \in C^\infty$ and is a geod
by uniqueness of geod. $\square$

$\rightarrow$ Solution is completely determined by $\gamma(0), \gamma'(0)$.

Prop (2nd Variation Formula). Let $\gamma: [0, a] \rightarrow M$ be a geodesic and $f: (-\epsilon, \epsilon) \times [0, a] \rightarrow M$ a smooth variation. Let E be the energy function, then

$$\begin{aligned} \frac{1}{2} \frac{d^2 E}{ds^2}(0) &= - \int_0^a \langle v(t), \frac{D^2 v}{dt^2} + R(\gamma', v) \gamma' \rangle dt \\ &\quad + \left\langle \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial s}, \gamma'(t) \right\rangle \Big|_{(0,0)}^{(0,a)} + \left\langle v(t), \frac{Dv}{dt}(t) \right\rangle \Big|_0^a. \end{aligned}$$

pf. By our proof of the 1st variation formula, we have

$$\frac{1}{2} \frac{d}{ds} E(s) = - \int_0^a \left\langle \frac{\partial f}{\partial s}, \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} \right\rangle dt + \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle \Big|_{(s,0)}^{(s,a)}.$$

Thus

→ 0 when $s=0$

$$\frac{1}{2} \frac{d^2 E}{ds^2}(s) = - \int_0^a \left\langle \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial s}, \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial s}, \nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} \right\rangle dt$$

$$+ \left\langle \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle \Big|_0^a + \left\langle \frac{\partial f}{\partial t}, \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial s} \right\rangle \Big|_0^a.$$

$$\stackrel{(s=0)}{=} - \int_0^a \left\langle \frac{\partial f}{\partial s}, \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) \frac{\partial f}{\partial t} \right\rangle dt + \dots$$

$$= - \int_0^a \left\langle \frac{\partial f}{\partial s}, \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) \frac{\partial f}{\partial t} \right\rangle dt + \dots$$

$\underbrace{\qquad\qquad\qquad}_{[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}] = 0}$

= what we want (eval at $s=0$).

□

Lec 14.

2nd Variation Formula:

$$\frac{1}{2} E''(0) = - \int_0^a \langle v(t), \frac{D^2 v}{dt^2} + R(Y', v) Y' \rangle dt + \left\langle \nabla_{v(t)} \frac{\partial f}{\partial s}, Y'(t) \right\rangle \Big|_{(0,0)}^{(0,a)} + \langle v(t), v'(t) \rangle \Big|_0^a.$$

For proper variation, $v(0) = v(a) = 0$,

$$\frac{1}{2} E''(0) = - \int_0^a \langle v(t), \frac{D^2 v}{dt^2} + R(Y', v) Y' \rangle dt.$$

Eg. For S^n , $K \equiv 1$. So
 $Ric(v) = \sum K(e_i, v)$.
 Take a O.N.B.
 $(e_i = v, e_2, \dots, e_n)$.
 $K(e_1, v) = K(v, v) = 1$.
 $K(e_j, v) = 0, j \neq 1$.
 $Ric_{S^n} = n-1$.

Application of 2nd Variation Formula:

we don't divide $n-1$:

$$Ric(v) = \sum_{i=1}^n K(e_i, v) \downarrow (e_i)_{i \in I} \text{ O.N.B.}$$

Thm (Bonnet-Myers). Let M^n be a complete R.M. Suppose $Ric_p(v) \geq (n-1)H > 0$. Then M is compact and $\text{diam}(M) \leq \pi/\sqrt{H} = \text{diam } S_H^n$, where S_H^n is a n -dim simply connected manifold with constant sectional curvature H .

Cor. If M^n has $Ric \geq (n-1)H > 0$, then $\pi_1(M)$ is finite.

pf. Let $\tilde{M}^n$ be its universal cover, equipped with the covering metric, i.e., the covering $\pi: \tilde{M} \rightarrow M$ is made to be a local isometry. Thus $Ric_{\tilde{M}} \geq (n-1)H > 0$.

Apply Bonnet-Myers Thm: $\tilde{M}$ is compact $\xRightarrow[\text{topology}]{}$ $\pi_1(M)$ is finite. $\square$

Cor. T^n has no metric with $Ric \geq 0$.

pf. $\pi_1(T^n) = \mathbb{Z}^n$ which is infinite. $\square$

The result of B.M. is optimal in the following sense:

o If we weaken the assumption to $K \geq 0$, claim of B.M. is not true.

Counter Example: $z = x^2 + y^2$: 

maximal diameter sphere Thm (SY cheng)

o The estimate of B.M. is achieved by S^n . $\text{diam } M = \frac{\pi}{\sqrt{H}}$ iff $M^n \stackrel{\text{isometric}}{\cong} S^n$.

pf of Bonnet-Myers Thm.

Only need to show $\text{diam } M \leq \pi/\sqrt{H}$, then Hopf-Rinow implies M is compact.

We'll show: $\forall p, q \in M$, $\exists$ a curve from p to q with length $\leq \pi/\sqrt{H}$.

Let $p, q \in M$, $\gamma: [0,1] \rightarrow M$ a minimal geod from p to q (Hopf-Rinow). WTS:

$$L(\gamma) \leq \pi/\sqrt{H}. \quad \forall p, q \in M.$$

Contradiction: If not, then $\exists p, q \in M$, γ min geod of (p, q) , s.t. $L(\gamma) > \frac{\pi}{\sqrt{H}}$.

Since γ is minimal, $E''(0) \geq 0$ for any proper vector field $V(t)$ along γ .

Let $e_n = \gamma'(0)/L(\gamma)$, which is unit. Extend this to $(e_1, \dots, e_n)$ be a O.N. B.

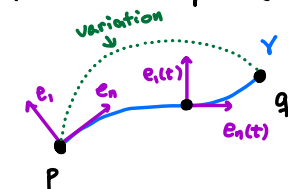
at $T_{\gamma(0)}M = T_pM$. Let $(e_1(t), \dots, e_n(t))$ be the parallel transport of $(e_1, \dots, e_n)$

along γ , which is a O.N. frame along γ .

Let $V_i(t) = \sin(\pi t)e_i(t)$.

Then

$$\begin{aligned} \frac{1}{2} E_i''(0) &= - \int_0^1 \left\langle \sin(\pi t)e_i, \underbrace{-\pi^2 \sin(\pi t)e_i}_{\frac{D^2}{dt^2} e_i = 0} + L(\gamma)^2 \sin(\pi t) R(e_n, e_i)e_n \right\rangle dt \\ &= - \int_0^1 \sin \pi t \left\{ -\pi^2 \sin(\pi t) + L(\gamma)^2 K(e_n, e_i) \right\} dt. \end{aligned}$$



Thus

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^{n-1} E_i''(0) &= - \int_0^1 \sin \pi t \left\{ -(n-1)\pi^2 \sin(\pi t) + L(\gamma)^2 \text{Ric}(e_n) \right\} dt \\ &< -(n-1) \int_0^1 \sin(\pi t) \left\{ -\pi^2 + \left(\frac{\pi}{\sqrt{H}} \right)^2 H \right\} dt = 0. \end{aligned}$$

But by assumption,

$$\frac{1}{2} \sum_{i=1}^{n-1} E_i''(0) > 0$$

which is the desired contradiction. $\square$

Thm (Weinstein). Let f be an isometry of a compact oriented R.M. M^n . If M has $K > 0$ and f preserves the orientation if n even and reverse it if n odd. Then f has a fixed point : $\exists p$ s.t. $f(p) = p$.

Rmk: Theorem remains true when f is a conformal diffeomorphism.

(open) Q: true if f is a diffeo? $\Rightarrow$ Hopf conjecture.

lem. Let $A \in O(n-1)$. Suppose $\det A = (-1)^n$. Then at least one eigenvalue of A is 1 (i.e., $\exists v \neq 0$ s.t. $A(v) = v$).

pf. $n = \text{even}$, then $n-1$ is odd. Thus we have odd # of real eigenvalue (complex values come in pairs). Since $A \in O(n-1)$, if λ a value of A , $|\lambda| = 1$. Thus at least 1 eigenvalue is 1, otherwise $\det A = -1$.

$n = \text{odd}$ case similar.

□

Lec 15.

any compact manifold is complete

Thm (Weinstein). Let f be an isometry of a compact oriented R.M. M^n . If M has $K > 0$ and f preserves the orientation if n even and reverse it if n odd. Then f has a fixed point : $\exists p$ st. $f(p) = p$.

Pf. By contradiction. Suppose $d(q, f(q)) > 0 \forall q \in M$. Since M is compact, $\exists p \in M$ st. $d(p, f(p)) \leq d(q, f(q)) \forall q \in M$ and $d(p, f(p)) > 0$. Let γ be a minimal geodesic connecting p and $f(p)$. $\gamma: [0, l] \rightarrow M$.

Let $A: T_p M \rightarrow T_p M$ be $A = P \circ df_p$ where P is the parallel transport from $T_{f(p)} M$ to $T_p M$ along γ . Then A is an isometry.

Claim: $df(\gamma'(0)) = (f \circ \gamma)'(0) = \gamma'(l)$.

Pf. By Δ -ineqn and γ minimizing:

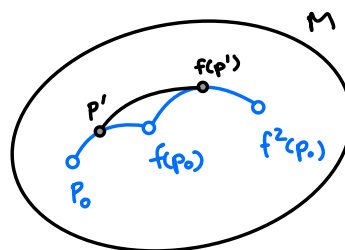
$$\begin{aligned} d(p', f(p')) &\leq d(p', f(p)) + d(f(p), f(p')) \\ &= d(p', f(p)) + d(p, p') \\ &= d(p, f(p)). \end{aligned}$$

Thus $d(p', f(p')) = d(p, f(p))$ since $d(p, f(p))$ is minimal. Therefore

$$d(p', f(p')) = d(p', f(p)) + d(f(p), f(p')).$$

Δ -ineqn is eqn $\Rightarrow \Delta$ is degenerate. So $\gamma \cup f(\gamma)$ is a geodesic. This

proves the claim. //



Now we can restrict A to $\langle \gamma'(0) \rangle^\perp \subset T_p M$. Then $A \in O(n-1)$. $\det A = (-1)^n$.

By linear algebra lemma last time: $\exists e \in T_p M$ st. $e \perp \gamma'(0)$ and $A(e) = e$.

Let $e(t)$ be the parallel transport along γ .

Let β be the geodesic st. $\beta(0) = p$, $\beta'(0) = e$. Then

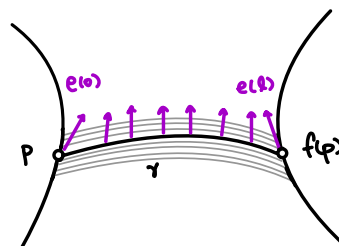
$$(f(\beta))'(0) = (f \circ \beta)'(0) = df(\beta'(0)) = df(e) = P^{-1} \circ A(e) = P^{-1}(e) = e(l).$$

Let $h(s,t) = \exp_{\gamma(t)}(se(t))$. Then

$$\left. \frac{\partial h}{\partial s} \right|_{s=0} = e(t).$$

By the 2nd variation formula:

$$\begin{aligned} \frac{1}{2} E''(0) &= - \int_0^l \langle e(t), e''(t) + R(\gamma', e) \gamma' \rangle dt \\ &\quad + \langle e'(t), e(t) \rangle \Big|_0^l + \langle \gamma'(t), \nabla_{e(t)} \frac{\partial h}{\partial s} \rangle \Big|_{(0,0)}^{(0,l)} \\ &= - \int_0^l K(e, \gamma') dt < 0 \end{aligned}$$



Since the curvature is positive. This shows that there exists a curve c in the variation, s.t.

$$L(c)^2 \leq l E(c) < l E(\gamma) = L(\gamma)^2.$$

This is the desired contradiction. □

Cor (Synge's Thm). Let M^n be a compact with $K > 0$.

(1) If M^n is orientable and n even, then M is simply connected.

(2) If n is odd, then M^n is orientable.

Cor of Synge's Thm: $\mathbb{R}P^2 \times \mathbb{R}P^2$ has no metric with $K > 0$.

Pf. $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Hopf Conjecture: $S^2 \times S^2$ has no metric with $K > 0$.

Pf of Synge's Thm: p. 206 of do Carmo. □

Lec 16.

Chap 10. Index lemma & Rauch Comparison.

Defⁿ. Given $\gamma: [0, l] \rightarrow M$ a normal geodesic ($|\dot{\gamma}|=1$). Then define the index form of γ is

$$I_{\gamma}(V, W) = \int_0^l [\langle V'(t), W'(t) \rangle - \langle R(\gamma', V) \gamma', W \rangle] dt \quad (*)$$

where V, W are piecewise differentiable vector field along $\gamma(t)$.

Rmk: Note that the 1st term in $(*)$ is

$$\langle V', W' \rangle = \langle V', W \rangle' - \langle V'', W \rangle.$$

Thus we have the following 2 observations:

- o When V is proper, then $I(V, V) = \frac{1}{2} E''(0)$.
- o When J is a Jacobi field, $I(J, W) = \langle J', W \rangle|_0^l$, only depend on the boundary value.
In particular, if J is a proper Jacobi field, $I(J, J) = 0$. When $J(0) = 0$, we have $I(J, J) = \langle J', J \rangle(l) = \frac{1}{2} \langle J, J \rangle'(l)$.

Now read $(*)$ as it is:

- o If $V \parallel \gamma'(t)$, $I(V, V) \geq 0$ ($V = f(t) \gamma'(t)$).

Basic Index lemma. Assume γ has no conjugate point. Then $I(J, J) \leq I(V, V)$ for all Jacobi field J and V any piecewise differentiable vector field along γ s.t. $J \perp \gamma'(t)$, $V \perp \gamma'(t)$, $J(0) = V(0) = 0$, $J(l) = V(l)$. Moreover, $I(J, J) = I(V, V)$ iff $V = J$.
i.e., Jacobi field minimizes index.

(Consequently, for any proper V and $V \perp \gamma'(t)$, $I(V, V) \geq 0$ if γ has no conj. pt.).

↑ Assuming this

Inverse direction: Let J be the unique Jacobi field along γ s.t. $J(0) = 0$, $J(l) = V(l)$.

Then $V - J$ is proper and thus $I(V - J, V - J) \geq 0$. So

$$I(V, V) - 2I(V, J) + I(J, J) \geq 0.$$

Now by Rmk above:

$$I(V, J) = \langle J', V \rangle(L) = \langle J', J \rangle(L) = I(J, J).$$

So

$$I(V, V) \geq I(J, J).$$

Cor. When γ has no conjugate pt, γ is minimal among all nearby curves. And if γ has a conjugate point inside, then γ is not minimal. Precisely, if $\gamma(t_0)$ is the first conjugate point of $\gamma(0)$ along γ . Then

(a) $I(V, V) > 0$ for all nontrivial vector field V st. $V(0) = 0$, $V(t) = 0$, $t < t_0$.

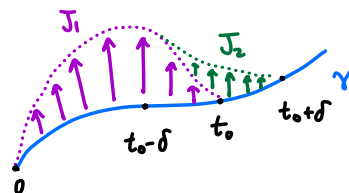
($\Rightarrow \gamma|_{[0, t]}$ is locally minimizing among all curves with same endpoint, for $I(V, V) = \frac{1}{2} E''(0) > 0$ by Rmk).

(b) $\gamma|_{[0, t]}$ is not minimal for $t > t_0$.

pf. (a) By index lemma, $I(V, V) \geq 0$. Since only Jacobi field on $\gamma|_{[0, t]}$ with $J(t) = 0$ is $J \equiv 0$, and $V \neq J$, then $I(V, V) > 0$ ($I(V, V) = I(J, J)$ iff $V = J$ by index lemma).

(b) let $\delta > 0$ be small and let

$$V = \begin{cases} J_1 & t \in [0, t_0 - \delta] \\ J_2 & t \in [t_0 - \delta, t_0 + \delta] \\ 0 & t \in [t_0 + \delta, t] \end{cases}$$



where J_1 is a Jacobi field st. $J_1(t_0) = J_1(0) = 0$. J_2 a Jacobi field st.

$J_2(t_0 - \delta) = J_1(t_0 - \delta)$ and $J_2(t_0 + \delta) = 0$. Then

$$I(V, V)|_{\gamma|_{[t_0 - \delta, t_0 + \delta]}} < I(J_1, J_1)|_{\gamma|_{[t_0 - \delta, t_0 + \delta]}}$$

by index lem, and
 $\downarrow$ extend J_1 by 0 on $[t_0, t_0 + \delta]$

Thus

$$\frac{1}{2} E''(0) = I(V, V)|_{\gamma|_{[0, t]}} < I(J_1, J_1)|_{\gamma|_{[0, t]}} = 0.$$

proper Jacobi field extended by 0
 $\downarrow$ Rmk gives the claim.

Therefore, $\gamma|_{[0, t]}$ is not minimal.

□

Application: Local Hessian Comparison.

Defⁿ. Let $f \in C^2(M)$. $\text{Hess } f$ is a symmetric 2-tensor

$$\begin{aligned}\text{Hess } f(X, Y) &= \langle \nabla_X \text{grad } f, Y \rangle \\ &\stackrel{(\text{Ex})}{=} XYf - (\nabla_X Y)f.\end{aligned}$$

Fix $p \in M$. The distance function $p(x) = d(p, x)$.

$p(x)$ is smooth on $M \setminus \{p, \text{cut locus of } p\}$.

lec 17.

Recall:

Defⁿ. Given $\gamma: [0, L] \rightarrow M$ a normal geodesic ($|\dot{\gamma}|=1$). Then define the

index form of γ is

$$I_{\gamma}(V, W) = \int_0^L [\langle V'(t), W'(t) \rangle - \langle R(\gamma', V) \gamma', W \rangle] dt \quad (*)$$

where V, W are piecewise differentiable vector field along $\gamma(t)$.

Basic Index lemma. Assume γ has no conjugate point. Then $I(J, J) \leq I(V, V)$ for all Jacobi field J and V any piecewise differentiable vector field along γ s.t. $J \perp \gamma'(t)$, $V \perp \gamma'(t)$, $J(0) = V(0) = 0$, $J(L) = V(L)$. Moreover, $I(J, J) = I(V, V)$ iff $V = J$.
i.e., Jacobi field minimizes index.

Pf. Let $(e_i)_{i=1}^{n-1}$ be an O.N. basis of $(\gamma'(0))^{\perp}$. Then parallel transport to get $(e_i(t))_{i=1}^{n-1}$ an O.N. basis of $(\gamma'(t))^{\perp}$. Let J_i be the Jacobi fields with $J_i(0) = 0$, $J_i(L) = e_i(L)$ [Existence and uniqueness of Jacobi field since $\gamma(L)$ not conjugate to $\gamma(0)$ along γ]. Then $J_i(t)$'s are linearly independent for $0 \leq t \leq L$ [Otherwise $\sum c_i J_i = 0$ at some $t_0 < L$ and c_i 's not all zero, we have to a conjugate point].

Write

$$V(t) = \sum_{i=1}^{n-1} f_i(t) J_i(t) \quad \text{for } t > 0.$$

Then

$$J(t) = \sum_{i=1}^{n-1} f_i(L) J_i(t) \quad \text{for } t > 0. \quad (**)$$

Since the expression on RHS of $(**)$ is a Jacobi field with same boundary condition as $J(t)$ [$\exists$ and ! of Jacobi field]. Now

$$V'(t) = \underbrace{\sum_{i=1}^{n-1} f_i'(t) J_i(t)}_A + \underbrace{\sum_{i=1}^{n-1} f_i(t) J_i'(t)}_B$$

derivative on vectors
are all $\nabla_{\gamma'} = \frac{D}{dt}$.

Then

$$I(V, V) = \int_0^L \langle A, A \rangle + 2\langle A, B \rangle - \langle R(\gamma', V) \gamma', V \rangle dt. \quad \textcircled{2}$$

Shut up and calculate :)

$$\int_0^L \langle B, B \rangle dt = \int_0^L \sum_{i,j} f_i f_j \langle J'_i, J'_j \rangle dt$$

You'll see why we do this term first very soon :)

int by part

$$= \sum_{i,j} f_i f_j \langle J'_i, J'_j \rangle \Big|_0^L - \sum_{i,j} \int_0^L \langle J''_i, J'_j \rangle f_i f_j dt - \sum_{i,j} \int_0^L \langle J'_i, J''_j \rangle (f'_i f_j + f_i f'_j) dt$$

$J_i(0) = 0$

$$= \sum_{i,j} f_i(L) f_j(L) \langle J'_i, J'_j \rangle(L) - \sum_{i,j} \int_0^L \langle R(\gamma', J_i) \gamma', J_j \rangle f_i f_j dt$$

= $\langle J, J' \rangle$

Jacobi Eqn

put them inside

$$- \sum_{i,j} \int_0^L \underbrace{\langle f_i J'_i, f'_j J'_j \rangle}_B + \underbrace{\langle f_j J'_j, f'_i J'_i \rangle}_A dt$$

$$= \langle J, J' \rangle(L) + \int_0^L \langle R(\gamma', V) \gamma', V \rangle dt - \int_0^L \langle B, A \rangle dt$$

$$- \sum_{i,j} \int_0^L \langle J'_i, J'_j \rangle f'_i f_j dt$$

Claim: $\langle J'_i, J'_j \rangle = \langle J'_j, J'_i \rangle$

pf. $\frac{d}{dt} [\langle J'_i, J'_j \rangle - \langle J'_j, J'_i \rangle] = 0$ by product rule and Jacobi Equation.

Since both terms are the same at $t=0$, we get the claim. ///

$$= \langle J, J' \rangle(L) + \int_0^L \langle R(\gamma', V) \gamma', V \rangle dt - 2 \int_0^L \langle B, A \rangle dt.$$

Putting everything in $\textcircled{2}$, we get

$$I(V, V) = \int_0^L \langle A, A \rangle dt + \langle J, J' \rangle(L) = \int_0^L \underbrace{\langle A, A \rangle}_{\geq 0} dt + \overset{\substack{\text{observation last} \\ \text{time}}}{I(J, J)}$$

$\Rightarrow I(V, V) \geq I(J, J)$. And equality iff $A=0 \Leftrightarrow f_i$ all constant $\Leftrightarrow V=J$. □

lec 18.

Chap 13, §2 of do Carmo.

Defⁿ.

$$\text{Cut}(p) = \left\{ q \mid \begin{array}{l} \text{If } \gamma \text{ is a normal minimal geodesic connecting } p, q, \gamma(t_0) = q, \\ \text{then } \gamma|_{[0, t_0 + \epsilon]} \text{ is not minimal } \forall \epsilon > 0 \end{array} \right\}$$

This is called the **cut locus** of p .

↑ closed and measure zero

Prop. $\gamma(t_0)$ is a cut point of $p = \gamma(0)$ along γ and $\gamma|_{[0, t_0]}$ is minimal $\Leftrightarrow$

Either $q = \gamma(t_0)$ is the 1st conjugate point of p along γ or $\exists$ another minimal geodesic $\sigma \neq \gamma$ connecting p, q .

Thm. $p(x) = d(p, x)$ is smooth on $M^n \setminus \{p, \text{Cut}(p)\}$.

Defⁿ. $(\text{Hess } p)(v, w) = \langle \nabla_v \text{grad } p, w \rangle$

Lec 19.

$$\text{Index: } I_\gamma(V, V) = \int_0^l \langle V', V' \rangle - \langle R(\gamma', V) \gamma', V \rangle dt$$

(*) Normalise V to unit vector first. So multiply by $|V|^2$.

(*) $\langle R(\gamma', V) \gamma', V \rangle = K(\gamma', V) \cdot |V|^2$ when $V \perp \gamma'$.

Index Comparison (Hessian Comparison).

Let $\gamma: [0, l] \rightarrow M^n$, $\tilde{\gamma}: [0, l] \rightarrow \tilde{M}^{n+k}$ ($k \geq 0$) be normed geodesics. $J, \tilde{J}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ s.t. $J(0) = 0 = \tilde{J}(0)$, $|J(l)| = |\tilde{J}(l)|$, $J \perp \gamma'$ and $\tilde{J} \perp \tilde{\gamma}'$.

Assume $\tilde{\gamma}$ has no conjugate point, and $\tilde{K}(\tilde{\gamma}', \tilde{X}) \geq K(\gamma', X)$ for all $X \in (\gamma')^\perp$ and $\tilde{X} \in (\tilde{\gamma}')^\perp$. Then

parallel direction contribution to K is zero.

$$I_{\tilde{\gamma}}(\tilde{J}, \tilde{J}) \leq I_\gamma(J, J) = \langle J, J' \rangle(l)$$

Hess $\tilde{\rho}_{\tilde{\gamma}(l)}(\tilde{J}, \tilde{J})$ Hess $\rho_{\gamma(l)}(J, J)$.

pf. Let $(e_i)_{i=1}^n$ and $(\tilde{e}_i)_{i=1}^{n+k}$ be O.N. frame parallel along $\gamma, \tilde{\gamma}$, s.t. $e_1 = \gamma'$, $\tilde{e}_1 = \tilde{\gamma}'$.

Given a vector field V along γ , write $V(t) = \sum_{i=1}^n f_i(t) e_i$. Define $\phi: \mathcal{X}(\gamma) \rightarrow \mathcal{X}(\tilde{\gamma})$

$$\phi(V) = \sum_{i=1}^n f_i \tilde{e}_i.$$

Then $\langle V, W \rangle = \langle \phi(V), \phi(W) \rangle$ and

$$\phi(V)' = \phi(V')$$

because basis are parallel. By ☺,

$$I(J, J) \geq I(\phi(J), \phi(J)) \geq I(\tilde{J}, \tilde{J})$$

↑ curvature assumption ↑ index lemma
 $\phi(J)(l) = \tilde{J}(l)$

□

Rauch Comparison Theorem.

Let $\gamma: [0, l] \rightarrow M^n$, $\tilde{\gamma}: [0, l] \rightarrow \tilde{M}^{n+k}$ ($k \geq 0$) be normed geodesics. $J, \tilde{J}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ s.t. $J(0) = 0 = \tilde{J}(0)$, $|J(l)| = |\tilde{J}(l)|$, $J \perp \gamma'$ and $\tilde{J} \perp \tilde{\gamma}'$, and

$|\tilde{J}'(0)| = |J'(0)|$. Assume $\tilde{\gamma}$ has no conj. pt. and $\tilde{K}(\tilde{\gamma}', \tilde{X}) \geq K(\gamma', X) \forall \tilde{X}, X$, then $|J| \geq |\tilde{J}|$.

pf. Consider $|J(t)|^2 / |\tilde{J}(t)|^2$. Then

$$\lim_{t \rightarrow 0} \frac{|J(t)|^2}{|\tilde{J}(t)|^2} \stackrel{\substack{\uparrow \\ \text{L'Hospital} \\ \times 2}}{=} \frac{|J'(0)|}{|\tilde{J}'(0)|} = 1.$$

Thus it suffices to show $\frac{d}{dt}(|J(t)|^2 / |\tilde{J}(t)|^2) \geq 0 \quad \forall t$.

By quotient rule: WTS

$$\frac{d}{dt}(|J(t)|^2 / |\tilde{J}(t)|^2) = \langle J', J \rangle |\tilde{J}|^2 - \langle \tilde{J}, \tilde{J}' \rangle |J|^2 \geq 0 \quad \text{☺}$$

If $J(t_0) = 0$, then the statement is trivially true. At $t = t_0$, let

$$V(t) = \frac{J(t)}{|J(t_0)|} \quad \tilde{V}(t) = \frac{\tilde{J}(t)}{|\tilde{J}(t_0)|}.$$

Then

$$\frac{\langle J'(t_0), J(t_0) \rangle}{\langle J(t_0), J(t_0) \rangle} = \langle V'(t_0), V(t_0) \rangle = I_{t_0}(V, V).$$

Similarly for $\tilde{J}$. Now $V, \tilde{V}$ are Jacobi fields, $V(0) = \tilde{V}(0) = 0$, $|\tilde{V}(t_0)| = |V(t_0)| = 1$.

By index comparison,

$$I(\tilde{V}, \tilde{V}) \leq I(V, V)$$

This is exactly ☺

□

Lec 20.

Rauch Comparison Theorem.

Let $\gamma: [0, l] \rightarrow M^n$, $\tilde{\gamma}: [0, l] \rightarrow \tilde{M}^{n+k}$ ($k \geq 0$) be normed geodesics. $J, \tilde{J}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ s.t. $J(0) = 0 = \tilde{J}(0)$, $|J(l)| = |\tilde{J}(l)|$, $J \perp \gamma'$ and $\tilde{J} \perp \tilde{\gamma}'$, and $|\tilde{J}'(0)| = |J'(0)|$. Assume $\tilde{\gamma}$ has no conj. pt. and $\tilde{K}(\tilde{\gamma}', \tilde{X}) \geq K(\gamma', X) \quad \forall \tilde{X}, X$, then $|J| \geq |\tilde{J}|$.

Cor. If M^n has $K_M \leq H$, then conjugate radius of $M \geq \pi/\sqrt{H}$, and is infinite when $H \leq 0$. In particular, if $K \leq 0$, there is no conjugate point.

$$\text{Conj. Radius} = \max \left\{ r \in \mathbb{R} : \exp_q: B_r(0) \subset T_q M \rightarrow M \right. \\ \left. \text{has maximal rank } \forall q \in M \right\}.$$

